k-triangulations on Surfaces

Saskia Solotko, Katherine Tung, Philip Yang, Yuchong Zhang
Mentor: Pasha Pylyavskyy
TA: Joe McDonough
UMN Twin Cities REU

August 1, 2024

k-triangulation definition

Definition (k-triangulation of a convex polygon)

A k-triangulation of a convex n-gon is a maximal set of edges such that no $k + 1$ pairwise intersect.

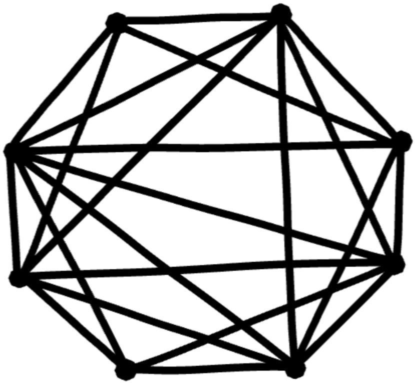


Figure 1: A 2-triangulation of the 8-gon. Note there are no 3-crossings.

k-stars: relevant for k-triangulations

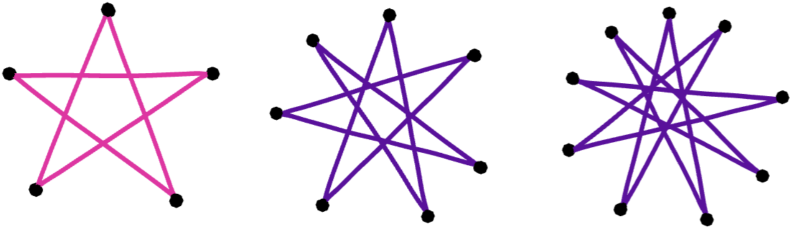


Figure 2: A 2-star, a 3-star, and a 4 star

A k-star consists of $2k + 1$ vertices and $2k + 1$ edges of length k.

k-triangulations as complexes of k-stars

Theorem (Pilaud-Santos '04)

Any k-triangulation of the n-gon contains exactly $n-2k$ k-stars, $k(n-2k-1)$ k edges of length $> k$, and $k(2n-2k-1)$ total edges.

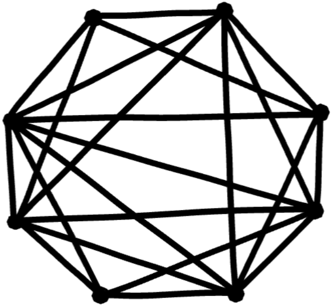


Figure 3: A 2-triangulation of the 8-gon has 6 edges of length > 2 and 22 total edges.

Definition

A k-relevant angle of a k-triangulation consists of edges of length $\geq k$ and has no intermediate "bisector" edges.

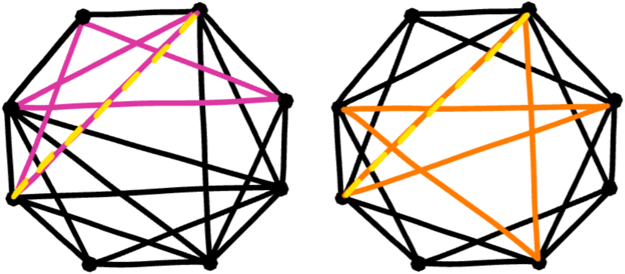
Theorem (Pilaud-Santos '04)

In a k-triangulation, every k-relevant angle is contained in a unique k-star:

- *length $> k$ (relevant) edges: in exactly 2 k-stars*
- *length $= k$ (boundary) edges: in exactly 1 k-star*
- *length $< k$ (irrelevant) edges: in exactly 0*

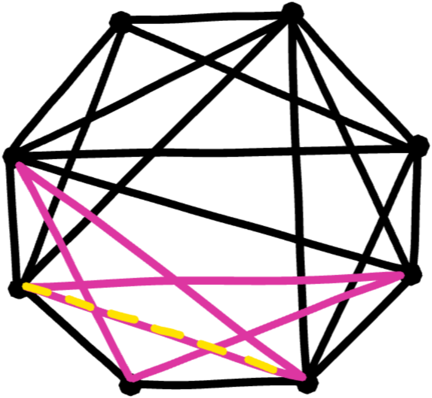
length $> k$ edge: in exactly 2 k-stars

Ex: length 3 edge, $k = 2$-triangulation



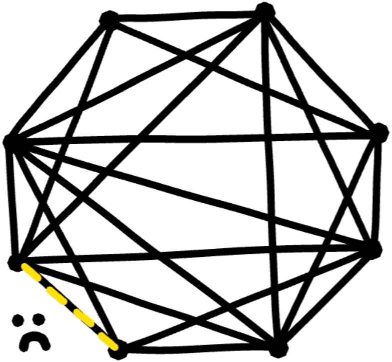
length $= k$ edge: in exactly 1 k-star

Ex: length 2 edge



length $< k$ edge: in 0 k-stars

Ex: length 1 edge



k-triangulations on surface

Let $\mathcal{S}$ denote a surface and $\overline{\mathcal{S}}$ its universal cover with natural projection $\pi : \overline{\mathcal{S}} \to \mathcal{S}$.

Definition

A k-triangulation T on a surface S with marked points on boundaries is a maximal set of edges such that $\pi^{-1}(T)$ is $(k+1)$-crossing free.

For convenience, we say $\pi^{-1}(T)$ is a k-triangulation of $\overline{\mathcal{S}}$ denoted $\overline{T}$.

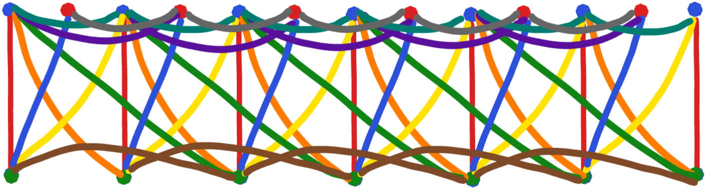


Figure 4: A 2-triangulation on the universal cover of the $(2+1)$-annulus

Results for 2-triangulations on the $(n+0)$-annulus

Let T denote a 2-triangulation on the $(n+0)$ annulus corresponding to $\overline{T}$ on the universal cover.

Lemma

Every such T has exactly one edge of length $2n$.

Definition

An angle of T is 2-relevant if it contains at least one edge of length > 2 and $< 2n$ and additionally has no intermediate "bisector" edges.

Theorem (STYZ 24)

Every 2-relevant angle of T is contained in a unique 2-star.

star decomposition example

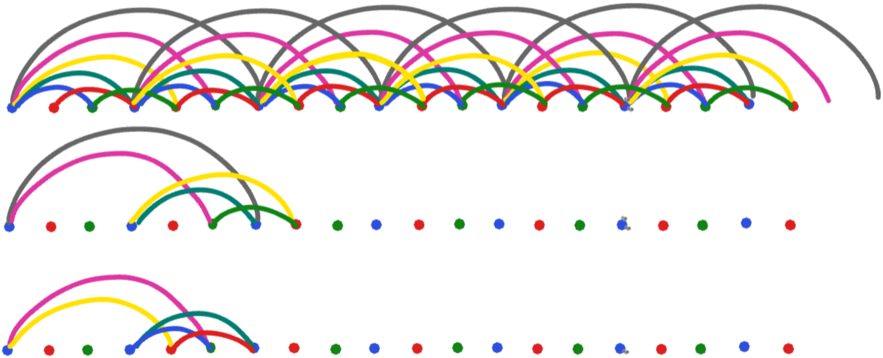


Figure 5: A 2-triangulation on the universal cover of the $(3+0)$-annulus and the stars that comprise it

Theorem (STYZ 24)

There is a bijection between 2-triangulations of the $(n+0)$-annulus and 2-triangulations of the $4n$-gon invariant under rotation by π/n.

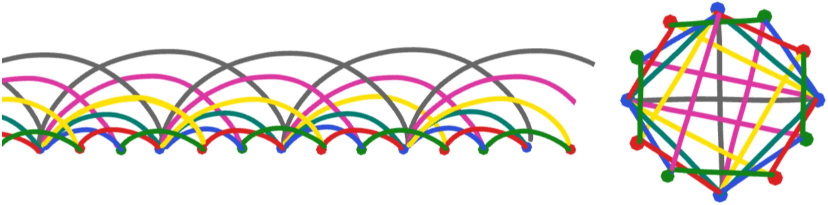


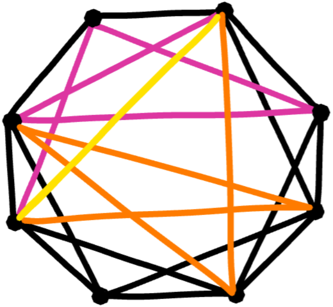
Figure 6: A 2-triangulation on the universal cover of the $(3+0)$-annulus and the corresponding 2-triangulation of the $4n$-gon

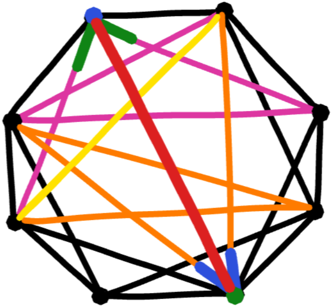
Corollary

For $k=2$, any k-triangulation of the $(n+0)$-annulus contains exactly $n-1$ k-stars, $k(n-1)$ k-relevant edges, and $k(2n-1)$ edges.

Lemma (Pilaud-Santos '04)

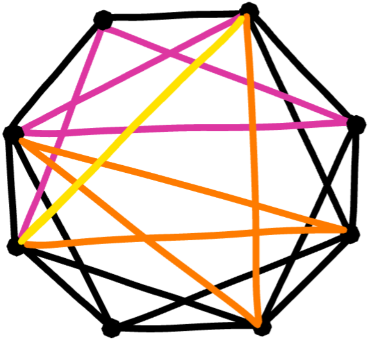
A pair of stars in T has a unique bisector edge.

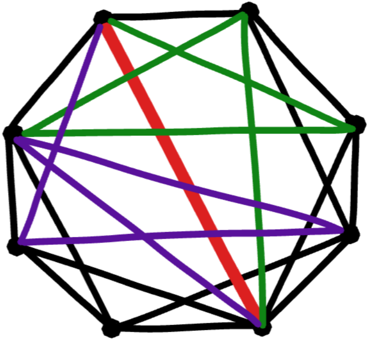




Theorem (Pilaud-Santos '04)

For any k-relevant edge $e \in T$ there is a unique flip edge f such that $(T \setminus e) \cup \{f\}$ is a k-triangulation.





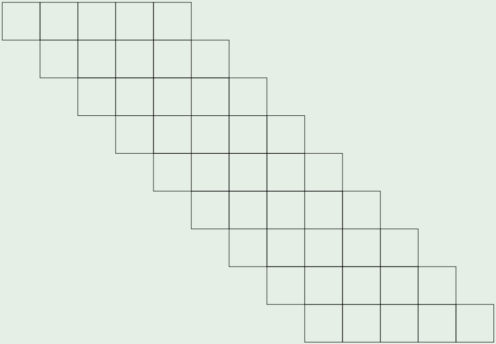
Cylindrical Polyominoes

Definition (STYZ '24+)

A *cylindrical polyomino* $\mathbb{Y}$ *of type* (n, k) is an infinite skew Young diagram (reflected along the y-axis) with a box $\square$ centered at every point in $\{(i, j) \in \mathbb{Z}^2 \mid k \leq j - i \leq kn\} \subseteq \mathbb{Z}^2$.

Example

$(n, k) = (3, 2)$



Cylindrical Pipe Dreams

Definition (STYZ '24+)

A *cylindrical pipe dream of type* (n, k) is a tiling of the cylindrical polyomino $\mathbb{Y}$ of type (n, k) by four kinds of pieces  ,  ,  , and  such that

- The pipe dream is *n-cylindrical*, that is, all the piles at the position $(i + rn, j + rn)$ for arbitrary $r \in \mathbb{Z}_{\geq 0}$ is the same as a pile at the position (i, j);
- There is a  tiled at the position $(i, k - i)$ for all $i \in \mathbb{Z}_{\geq 0}$;
- For every pipe, the number of  ,  or  it passes through is $2k + 1$;
- Each pipe connects $(i, kn - i)$ and $(i + kn, -i)$ for some $i \in \mathbb{Z}_{\geq 0}$;
- For every pair of pipes, they do not cross twice, that is, the number of  piles both pipes pass through is no more than 1;
- There is exactly one  in each successive n rows, tiled at the position $(i, kn - i)$ for some $i \in \mathbb{Z}_{\geq 0}$.

Cylindrical Pipe Dreams: Examples

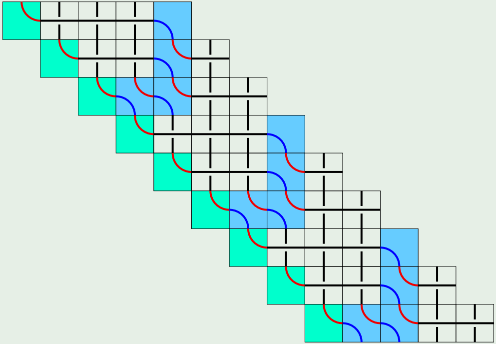
Definition (STYZ '24+)

A *cylindrical pipe dream of type* (n, k) is a tiling using  ,  ,  , and  such that

- n-periodic;
-  tiled at left boundary;
- each pipe has k "";
- each pipe horizontally and vertically crosses kn;
- two pipes don't cross twice;
- one  in each n-period, tiled at right boundary.

Example

$$(n, k) = (3, 2)$$



Cylindrical Pipe Dreams: Correspondence with Multi-Triangulations

Theorem (STYZ '24+)

For $k = 2$, there is a bijection between k-triangulation of the $(n + 0)$-annulus and cylindrical pipe dreams of type (n, k):

- *a length k edge connects i and j: tile*  *at (i, j)*
- *a length kn edge connects i and j: tile*  *at (i, j)*
- *an edge of length between k and kn connects i and j: tile*  *at (i, j)*
- *tile a*  *in every other* 

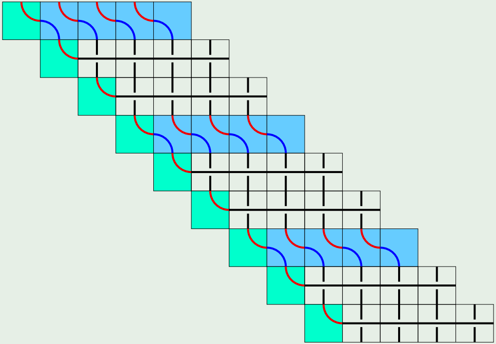
Moreover, in this bijection, each pipe corresponds to a k-star on $\overline{T}$.

Conjecture (STYZ '24+)

The previous theorem can be generalized to arbitrary k.

Cylindrical Pipe Dreams: Correspondence with Multi-Triangulations

Example



Cylindrical Pipe Dreams: Purity

Given a cylindrical pipe dream of type (n, k):

Lemma (STYZ '24+)

Two pipes cross once if and only if they have distance $\leq kn$.

Lemma (STYZ '24+)

There are exactly $2k \cdot (n-1)$ pipes intersects with a given pipe.

Theorem (STYZ '24+)

The number of  *and*  *in each n-period is $k \cdot (n-1)$.*

Corollary (STYZ '24+)

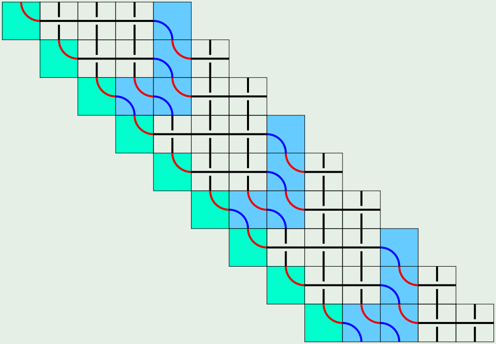
The number of 2-relevant edges in a 2-triangulation of $(n+0)$-annulus is $2n-2$.

Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

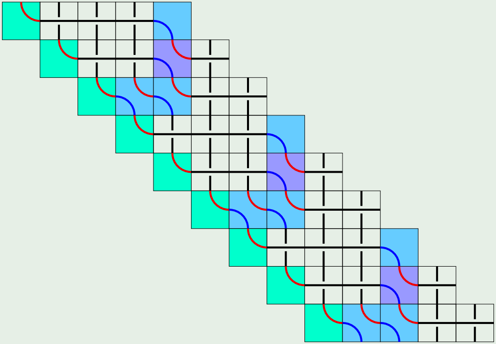


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

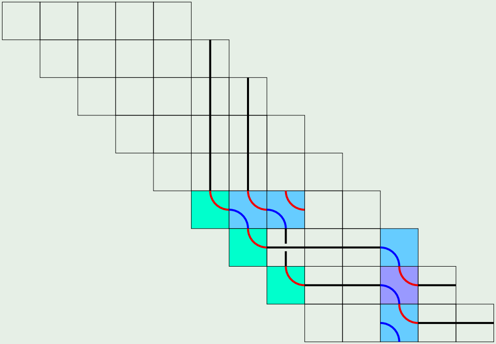


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

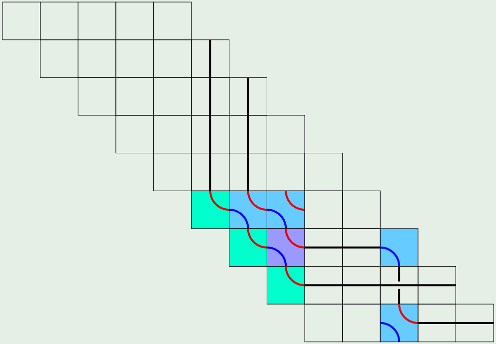


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

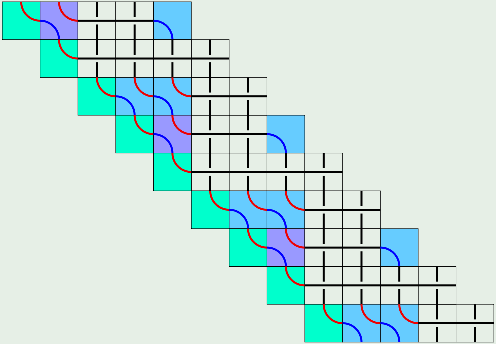


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

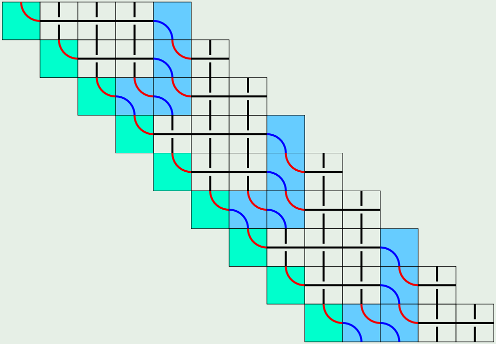


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example

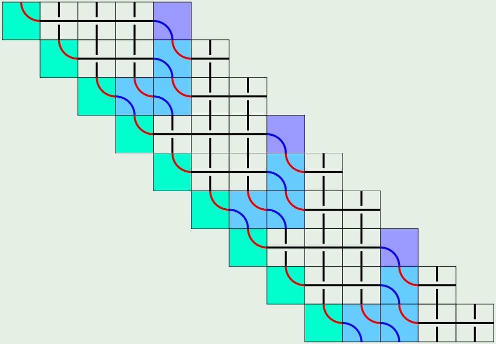


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example

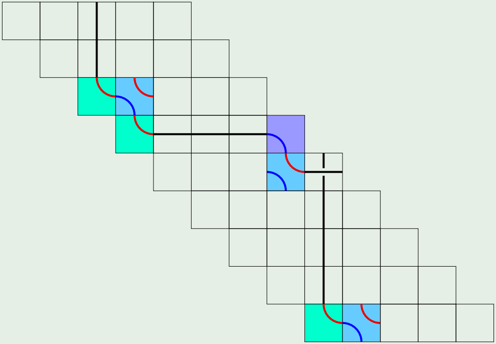


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example

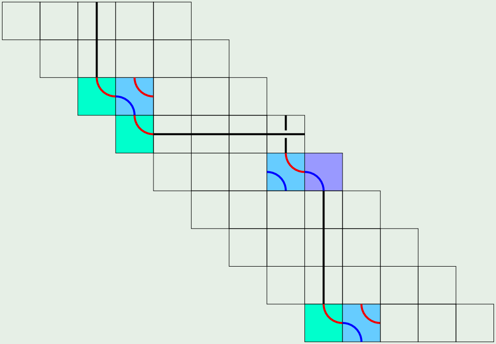


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example

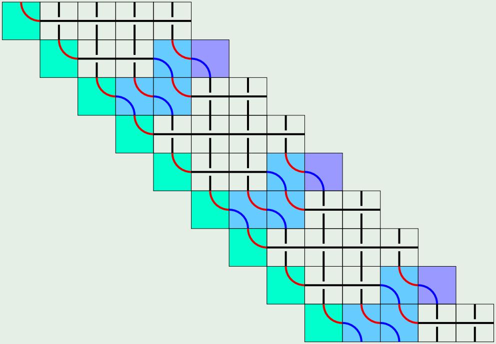


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example



Cylindrical Pipe Dreams: Flip Property

Theorem (STYZ '24+)

Cylindrical pipe dreams of type (n, k) have flip property.

Theorem (STYZ '24+)

2*-triangulations of* $(n + 0)$ *annulus have flip property.*

Conjecture (STYZ '24+)

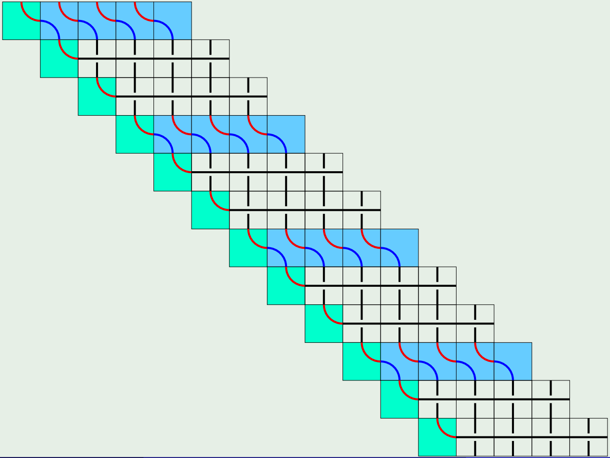
k-triangulations of $(n + 0)$ annulus have flip property.

Cylindrical Pipe Dreams: Regular Cylindrical Pipe Dreams

Definition (STYZ '24+)

Regular cylindrical pipe dream: for every  , there exists a  at the same row.

Example

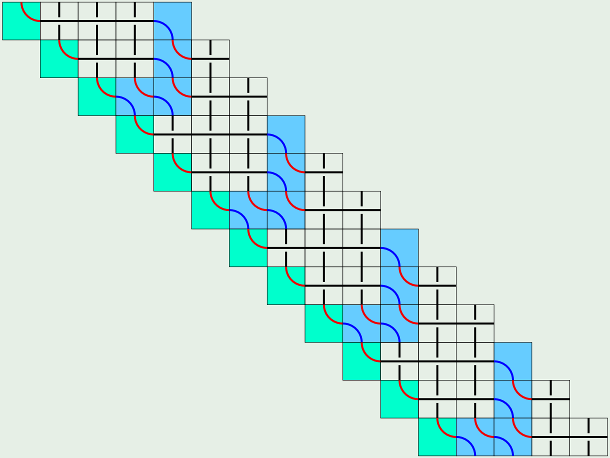


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

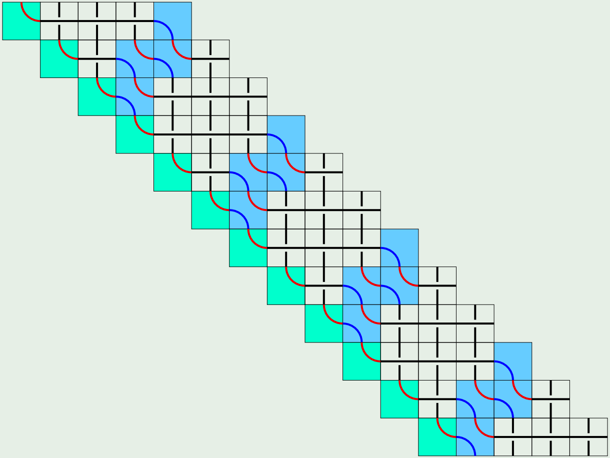


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

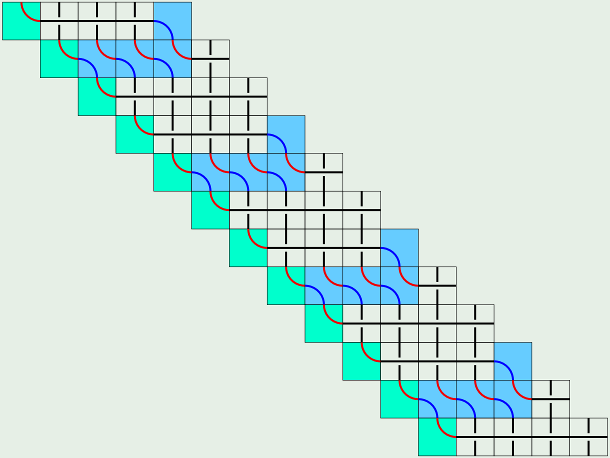


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

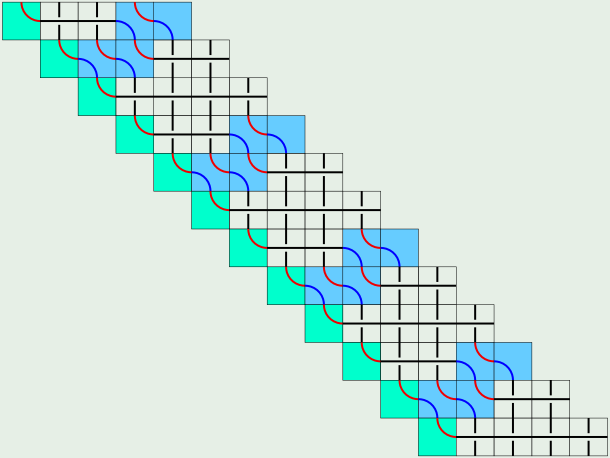


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

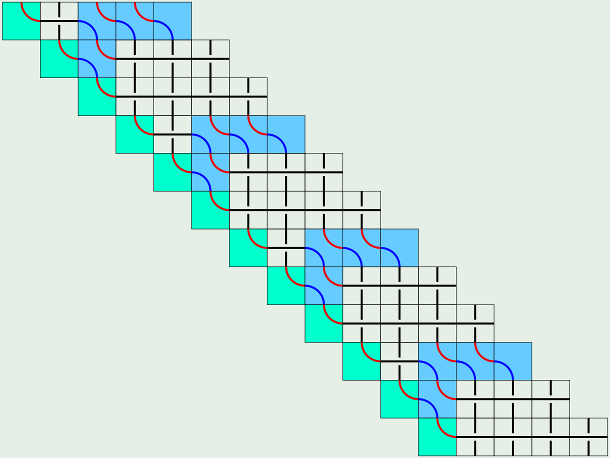


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

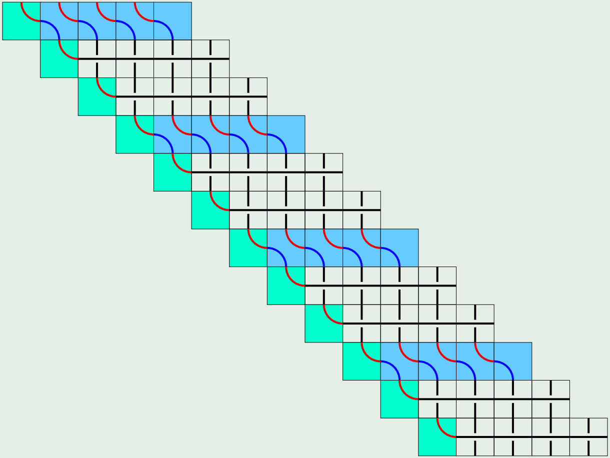


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

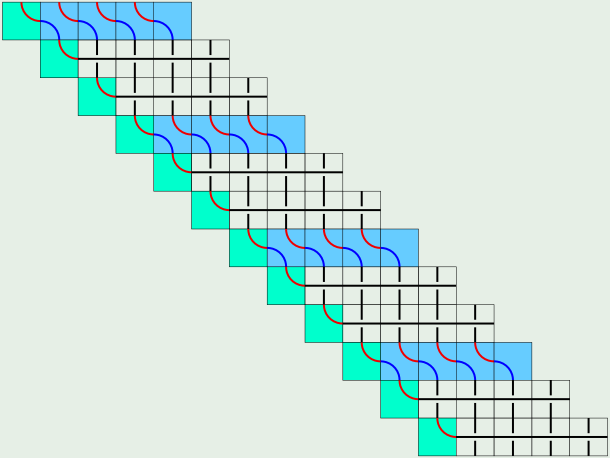


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Theorem (STYZ '24+)

Cylindrical pipe dreams have connected flip graph.

Example

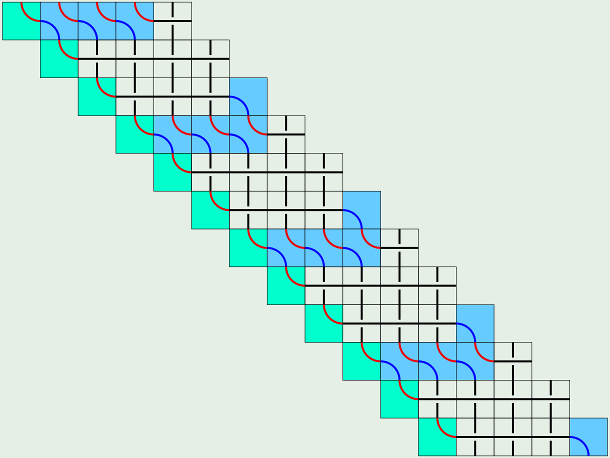


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Theorem (STYZ '24+)

Cylindrical pipe dreams have connected flip graph.

Example

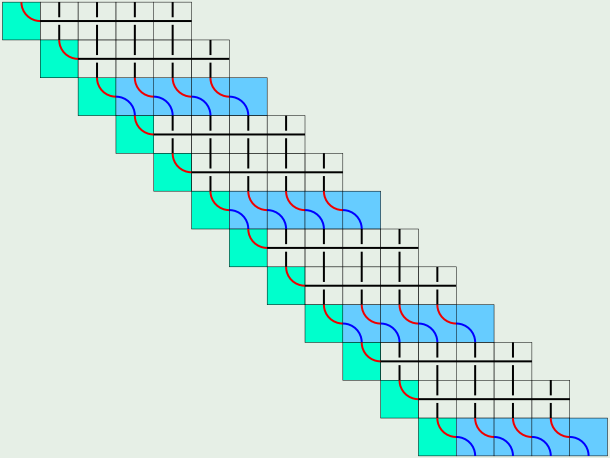


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Theorem (STYZ '24+)

Cylindrical pipe dreams have connected flip graph.

Example



Cylindrical Pipe Dreams: Connectedness of Flip Graph

Theorem (STYZ '24+)

2-*triangulations of* $(n+0)$ *annulus have connected flip graph.*

Conjecture (STYZ '24+)

k-triangulations of $(n+0)$ annulus have connected flip graph.

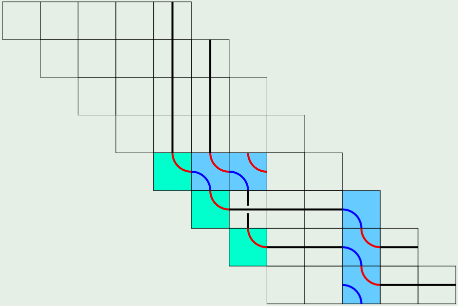
Cylindrical Pipe Dreams: Enumeration

Theorem (STYZ '24+)

There exists a canonical bijection between cylindrical pipe dreams of type (n, k) and nonnegative integer k-tuples valuations $v_0, v_1, \ldots v_{n-1}$ on n vertices such that

$$\sum_{i=0}^{n-1} v_i = (n-1, n-1, \ldots, n-1).$$

Example



Cylindrical Pipe Dreams: Enumeration

Lemma (STYZ '24+)

There are $\binom{2(n-1)}{(n-1)}^k$ *nonnegative k-tuples valuations* $v_0, v_1, \ldots v_{n-1}$ *such that*

$$\sum_{i=0}^{n-1} v_i = (n-1, n-1, \ldots, n-1).$$

Corollary (STYZ 24+)

The number of cylindrical pipe dreams of type (n, k) *is* $\binom{2(n-1)}{(n-1)}^k$.

Corollary (STYZ '24+)

The number of 2*-triangulations of* $(n+0)$ *annulus is* $\binom{2(n-1)}{(n-1)}^2$.

Conjecture (STYZ '24+)

The number of k-triangulations of $(n+0)$ annulus is $\binom{2(n-1)}{(n-1)}^k$.

q-analog

Definition

A q-analog is a mathematical expression parameterized by q that generalizes some expression and reduces to it when we take the limit as $q \to 1$.

q-analog

Definition

A q-analog is a mathematical expression parameterized by q that generalizes some expression and reduces to it when we take the limit as $q \to 1$.

Example

- n and $[n]_q = 1 + q + \cdots + q^{n-1}$

q-analog

Definition

A q-analog is a mathematical expression parameterized by q that generalizes some expression and reduces to it when we take the limit as $q \to 1$.

Example

- n and $[n]_q = 1 + q + \cdots + q^{n-1}$
- $n!$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$

q-analog

Definition

A q-analog is a mathematical expression parameterized by q that generalizes some expression and reduces to it when we take the limit as $q \to 1$.

Example

- n and $[n]_q = 1 + q + \cdots + q^{n-1}$
- $n!$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$
- $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$

q-analog

Definition

A q-analog is a mathematical expression parameterized by q that generalizes some expression and reduces to it when we take the limit as $q \to 1$.

Example

- n and $[n]_q = 1 + q + \cdots + q^{n-1}$
- $n!$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$
- $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$
- $$\prod_{1 \le a \le b \le 2nk-2k-1} \frac{a+b+2k}{a+b}$$
  and
  $$\prod_{1 \le a \le b \le 2nk-2k-1} \frac{[a+b+2k]_q}{[a+b]_q}$$
  (secretly q-analog for cardinality of set)

The cyclic sieving phenomenon

Definition (Reiner-Stanton-White '04)

If we have

- a finite set X,
- with a cyclic group $C = \{1, c, c^2, \ldots, c^{n-1}\}$ permuting X,
- and a q-analog polynomial $X(q) \in \mathbb{Z}[q]$,

The cyclic sieving phenomenon

Definition (Reiner-Stanton-White '04)

If we have

- a finite set X,
- with a cyclic group $C = \{1, c, c^2, \ldots, c^{n-1}\}$ permuting X,
- and a q-analog polynomial $X(q) \in \mathbb{Z}[q]$,

such that

$$\forall c^d \in C, |\{x \in X : c^d(x) = x\}| = X\left(\exp\left(\frac{2\pi i d}{n}\right)\right)$$

then $(X, C, X(q))$ is said to exhibit CSP.

Cyclic sieving results

$(X, C, X(q))$ is CSP for

Cyclic sieving results

$(X, C, X(q))$ is CSP for

- (Reiner-Stanton-White '04) $X = k$-element subsets of $\mathbb{Z}_n$, $C = \langle c \rangle = \mathbb{Z}_n$ where c acts on k-element set by adding 1 to each element, $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$

Cyclic sieving results

$(X, C, X(q))$ is CSP for

- (Reiner-Stanton-White '04) $X = k$-element subsets of $\mathbb{Z}_n$, $C = \langle c \rangle = \mathbb{Z}_n$ where c acts on k-element set by adding 1 to each element, $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$
- (Stanton '04) $X =$ set of $n \times n$ ASMs, $C = \langle c \rangle = \mathbb{Z}_4$ where c acts on an ASM by rotating it 90 degrees, $X(q) = q$-analog of enumeration formula for ASMs.

Cyclic sieving results

$(X, C, X(q))$ is CSP for

- (Reiner-Stanton-White '04) $X = k$-element subsets of $\mathbb{Z}_n$, $C = \langle c \rangle = \mathbb{Z}_n$ where c acts on k-element set by adding 1 to each element, $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$
- (Stanton '04) $X =$ set of $n \times n$ ASMs, $C = \langle c \rangle = \mathbb{Z}_4$ where c acts on an ASM by rotating it 90 degrees, $X(q) = q$-analog of enumeration formula for ASMs.
- (Reiner-Stanton-White '04) $X =$ triangulations of $(n+2)$-gon, $C = \langle c \rangle = \mathbb{Z}_{n+2}$ where c is rotation by $\frac{2\pi}{n+2}$ radians, $X(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$ (q-Catalan)

Triangulation example

$X =$ triangulations of 6-gon, $C = \langle \exp(2\pi i/6) \rangle = \mathbb{Z}_6$, $X(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$.

Triangulation example

$X =$ triangulations of 6-gon, $C = \langle \exp(2\pi i/6) \rangle = \mathbb{Z}_6$, $X(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$.

- $X(1) = 14$

Triangulation example

X = triangulations of 6-gon, $C = \langle \exp(2\pi i/6) \rangle = \mathbb{Z}_6$, $X(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$.

- $X(1) = 14$

Triangulation example

X = triangulations of 6-gon, $C = \langle \exp(2\pi i/6) \rangle = \mathbb{Z}_6$, $X(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$.

- $X(1) = 14$

- $X(\exp(2\pi i/6)) = 0$

Triangulation example

X = triangulations of 6-gon, $C = \langle \exp(2\pi i/6) \rangle = \mathbb{Z}_6$, $X(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$.

- $X(1) = 14$

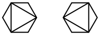
- $X(\exp(2\pi i/6)) = 0$
- $X(\exp(4\pi i/6)) = 2$

Triangulation example

X = triangulations of 6-gon, $C = \langle \exp(2\pi i/6) \rangle = \mathbb{Z}_6$, $X(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$.

- $X(1) = 14$

- $X(\exp(2\pi i/6)) = 0$
- $X(\exp(4\pi i/6)) = 2$

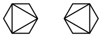


Triangulation example

X = triangulations of 6-gon, $C = \langle \exp(2\pi i/6) \rangle = \mathbb{Z}_6$, $X(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$.

- $X(1) = 14$

- $X(\exp(2\pi i/6)) = 0$
- $X(\exp(4\pi i/6)) = 2$



- $X(\exp(6\pi i/6)) = 6$

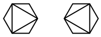
Triangulation example

X = triangulations of 6-gon, $C = \langle \exp(2\pi i/6) \rangle = \mathbb{Z}_6$, $X(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q$.

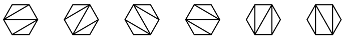
- $X(1) = 14$

- $X(\exp(2\pi i/6)) = 0$

- $X(\exp(4\pi i/6)) = 2$



- $X(\exp(6\pi i/6)) = 6$



Cyclic sieving for 2-triangulation of $(n+0)$-annulus

Definition

Let $M_{2nk,k}$ be the set of k-triangulations of a $2nk$-gon. Let $C = \langle \exp(\pi i/k) \rangle$ act on these triangulations.

Cyclic sieving for 2-triangulation of $(n+0)$-annulus

Definition

Let $M_{2nk,k}$ be the set of k-triangulations of a $2nk$-gon. Let $C = \langle \exp(\pi i/k) \rangle$ act on these triangulations.

Lemma

When $k = 2$, the triple $(M_{2nk,k}, C, M_{2nk,k}(q))$ is CSP.

Cyclic sieving for 2-triangulation of $(n+0)$-annulus

Definition

Let $M_{2nk,k}$ be the set of k-triangulations of a $2nk$-gon. Let $C = \langle \exp(\pi i/k) \rangle$ act on these triangulations.

Lemma

When $k = 2$, the triple $(M_{2nk,k}, C, M_{2nk,k}(q))$ is CSP.

Proof.

It suffices to show that

(a) $M_{2nk,k}(i) = \binom{2(n-1)}{n-1}^2$.

(b) $M_{2nk,k}(-1) = (4n-3)\left(\frac{(4n-4)!}{(2n-1)!(2n-2)!}\right)^2$.

(c) $M_{2nk,k}(-i) = \binom{2(n-1)}{n-1}^2$

□

Cyclic sieving for k-triangulation of $(n+0)$-annulus

Definition

Let $M_{2nk,k}$ be the set of k-triangulations of a $2nk$-gon. Let $C = \langle \exp(\pi i/k) \rangle$ act on these triangulations.

Cyclic sieving for k-triangulation of $(n+0)$-annulus

Definition

Let $M_{2nk,k}$ be the set of k-triangulations of a $2nk$-gon. Let $C = \langle \exp(\pi i/k) \rangle$ act on these triangulations.

Conjecture

The number of k-triangulations of $2kn$-gon invariant under rotation by $\frac{2\pi}{2k} \cdot j$ radians is

$$\prod_{a=1}^{k} \frac{((2n-1)d - \lceil \frac{2a}{m} \rceil)!}{((n-1)d + \lceil \frac{a}{m} \rceil - 1)!} \cdot \frac{(\lceil \frac{2a}{m} \rceil - 1)!}{(nd - \lceil \frac{a}{m} \rceil)!}$$

where $d = \gcd(2k, j)$ and $m = 2k/d$.

Cyclic sieving for k-triangulation of $(n+0)$-annulus

Definition

Let $M_{2nk,k}$ be the set of k-triangulations of a $2nk$-gon. Let $C = \langle \exp(\pi i/k) \rangle$ act on these triangulations.

Conjecture

The number of k-triangulations of $2kn$-gon invariant under rotation by $\frac{2\pi}{2k} \cdot j$ radians is

$$\prod_{a=1}^{k} \frac{((2n-1)d - \lceil \frac{2a}{m} \rceil)!}{((n-1)d + \lceil \frac{a}{m} \rceil - 1)!} \cdot \frac{(\lceil \frac{2a}{m} \rceil - 1)!}{(nd - \lceil \frac{a}{m} \rceil)!}$$

where $d = \gcd(2k, j)$ and $m = 2k/d$.

Corollary

The triple $(M_{2nk,k}, C, M_{2nk,k}(q))$ is CSP.

Future Works: Topology

Let $\Delta_{n,k}$ denote the simplicial complex for the k-triangulations of the $(n, 0)$-annulus.

Conjecture

$\Delta_{n,k}$ is a piecewise linear sphere.

Vertex Decomposable $\Rightarrow$ Shellable $\Rightarrow$ Spheric

Future Works: Topology

Conjecture

The h-polynomial of $\Delta_{n,k}$ is $\left(\sum_{i=0}^{n-1} \binom{n-1}{i}^2 t^i\right)^k$.

	n=1	2	3	4	5	6
k=1	[1]	[1, 1]	[1, 4, 1]	[1, 9, 9, 1]	[1, 16, 36, 16, 1]	[1, 25, 100, 100, 25, 1]
2	[1]	[1, 2, 1]	[1, 8, 18, 8, 1]	[1, 18, 99, 164, 99, 18, 1]		
3	[1]	[1, 3, 3, 1]	[1, 12, 51, 88, 51, 12, 1]			
4	[1]	[1, 4, 6, 4, 1]	[1, 16, 100, 304, 454, 304, 100, 16, 1]			

Acknowledgements

This project was supported in large part by a grant from the D.E. Shaw group, and also by NSF grant DMS-2053288. It was supervised as part of the University of Minnesota School of Mathematics Summer 2024 REU program. The authors would also like to thank Pasha Pylyavskyy and Joe McDonough for their invaluable guidance and expertise throughout the program. The authors also thank Kieran Favazza and Molly MacDonald for insightful discussion on the topic of this report.

Group Photo