Equality and Saturated Newton Polytope Property for Postnikov-Stanley Polynomials

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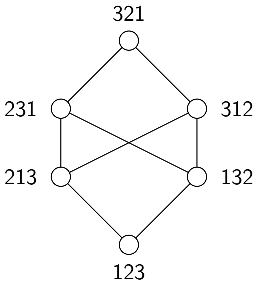
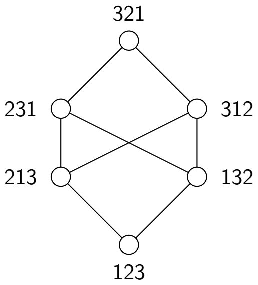
The Symmetric Group S_n

- S_n: permutations of $\{1, \ldots, n\}$
- Bottom element $1\ 2 \cdots n$
- Top element $n\ (n-1) \cdots 1$
- $\mathrm{Inv}(u)$: the set of all inversions (a, b) of u such that $a < b$ and $u(a) > u(b)$



The (Strong) Bruhat Order of S_n

- $\ell(u)$: count of inversions in u
- t_{ab} swaps the numbers in positions a, b (not values a, b)
- Covering relation: $u \lessdot v$ if $v = ut_{ab}$ and $\ell(v) = \ell(u) + 1$
- Interval $[u, w] : \{v \mid u \leq v \leq w\}$



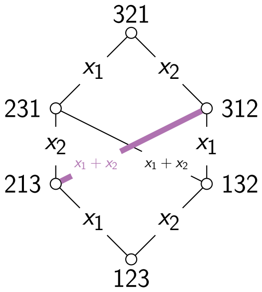
Edge Weights

Definition

For $u \lessdot v$ and $v = ut_{ab}$, the *weight* $m(u \lessdot v)$ is $x_a + x_{a+1} + \cdots + x_{b-1}$.

Example

Since $312 = 213t_{13}$, we have $m(213 \lessdot 312) = x_1 + x_2$.



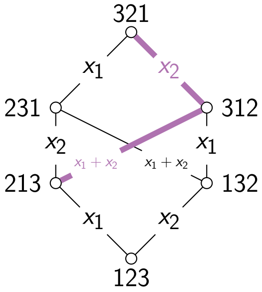
Chain Weights

Definition

Let $u_0 \le u_\ell$ and $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$ be a saturated chain of $[u_0, u_\ell]$. Define the *weight* $m_C(x)$ of the chain C by $\prod_{i=1}^{\ell} m(u_{i-1} \lessdot u_i)$.

Example

For $[213, 321]$, the weight of the saturated chain $213 \lessdot 312 \lessdot 321$ is $(x_1 + x_2) \cdot x_2$.



Skew Dual Schubert Polynomials

Definition (Postnikov-Stanley '09)

For $u \leq w$, the skew dual Schubert polynomial or *Postnikov-Stanley polynomial* D_u^w is defined by

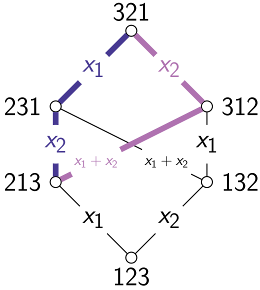
$$D_u^w = \frac{1}{(\ell(w) - \ell(u))!} \sum_{C: u=u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = w} m_C(x).$$

Example

$D_{213}^{321} = \frac{1}{2!}(x_1 x_2 + (x_1 + x_2) \cdot x_2)$

Definition (Bernstein et al '73)

When $u = \mathrm{id}$, D_u^w is called a *dual Schubert polynomial*.



Saturated Newton Polytope (SNP)

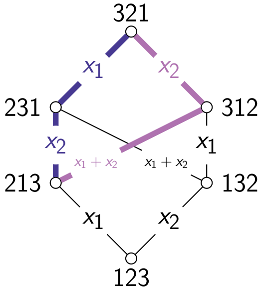
For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Definition

The *Newton polytope* $\mathrm{Newton}(f)$ of $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha$ is the convex hull of its exponent vectors α in $\mathbb{R}^n$.

Example

$D^{321}_{213} = x_1 x_2 + \frac{1}{2} x_2^2 = x^{(1,1)} + \frac{1}{2} x^{(0,2)}$
$\mathrm{Newton}(D^{321}_{213})$ is the segment from $(1,1)$ to $(0,2)$ in $\mathbb{R}^2$



Definition (Monical-Tokcan-Yong '19)

f has *Saturated Newton Polytope (SNP)* if $c_\alpha \neq 0$ for every integer point $\alpha \in \mathrm{Newton}(f)$.

SNP in Algebraic Combinatorics

Theorem (Rado '52)

Schur polynomials have SNP.

Theorem (Fink-Mézśaros-St. Dizier '18)

Key polynomials and Schubert polynomials have SNP.

Theorem (Monical-Tokcan-Yong '19)

Cycle index polynomials, Reutenauer's symmetric polynomials, Stembridge's symmetric polynomials, and symmetric Macdonald polynomials have SNP.

Theorem (ATZ '24+)

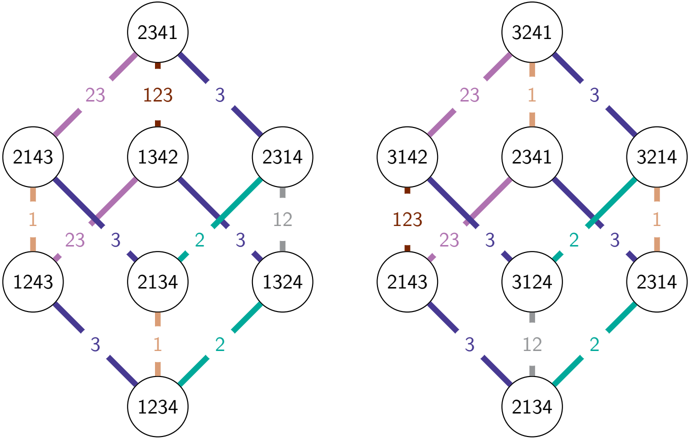
Dual Schubert polynomials have SNP.

Nontrivial Equalities

Definition

An equality between two P-S polynomials is *nontrivial* if the two polynomials are obtained from distinct multisets of saturated chains.

The equality $D_{1234}^{2341} = D_{2134}^{3241}$ is nontrivial.



Nontrivial Equalities, cont.

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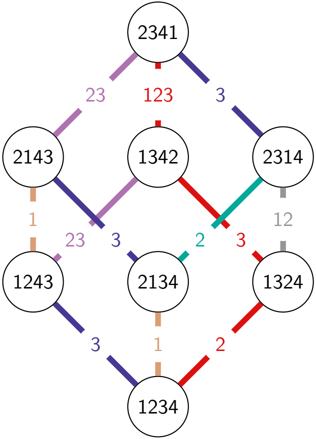
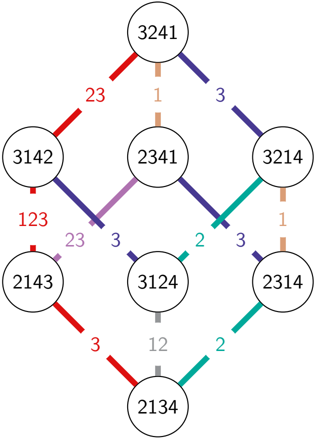


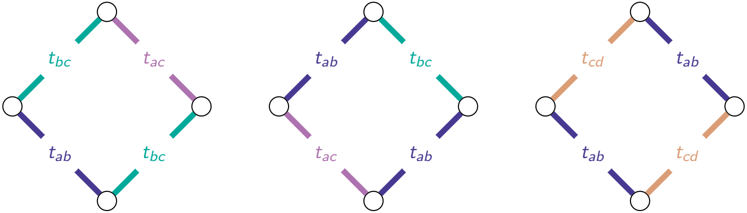


The Rank 2 Case

Lemma

There are no nontrivial equalities between P-S polynomials of rank 2.

All intervals of rank 2 are of one of the following forms (up to a rotation or reflection).

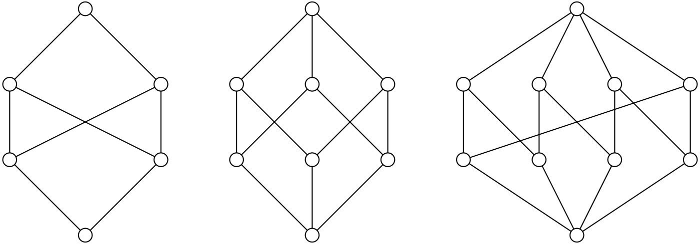


Note that rank 2 intervals contain two parallel sides with the same label u_1. The other two sides u_2, u_3 are either equal or satisfy $u_2 - u_3 = \pm u_1$.

The Rank 3 Case

Theorem (Jantzen '79)

The only rank 3 Bruhat intervals are 2-crowns, 3-crowns, and 4-crowns.



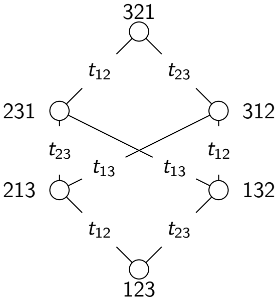
The Rank 3 Case, cont.

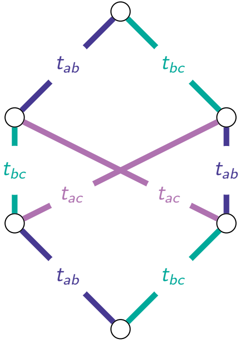
Proposition (ATZ '24+)

There are no nontrivial equalities between P-S polynomials of 2-crowns.

Proposition (ATZ '24+)

There is never equality between P-S polynomials of a 2-crown and a cube.





$$3(x_a + \cdots + x_{b-1})(x_b + \cdots + x_{c-1})(x_a + \cdots + x_{c-1})$$

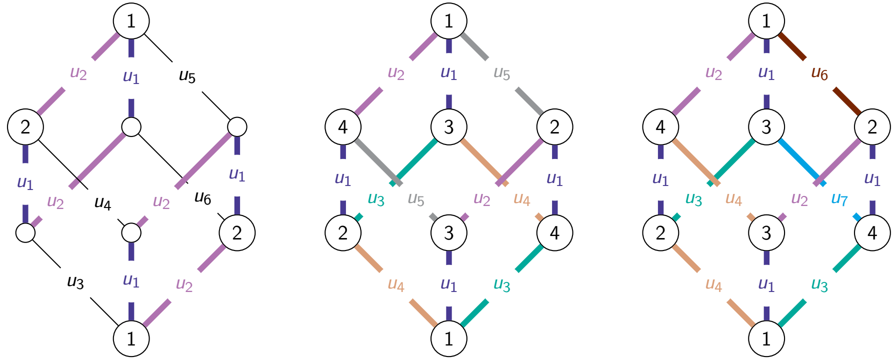
Equalities from Cube Rotations

Lemma (ATZ '24+)

Every cube in the Bruhat order has four parallel edges with the same label.

Theorem (ATZ '24+)

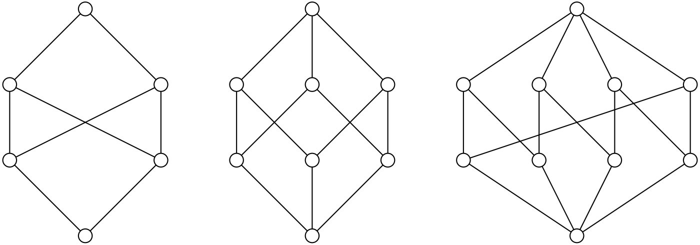
All rotations of a given cube have equal P-S polynomials.



Equality Conjectures

Conjecture (ATZ '24+)

Nontrivial equalities between two rank 3 P-S polynomials only occur between two cubes that are rotations of each other.



Conjecture (ATZ '24+)

Nontrivial equalities between P-S polynomials of the same poset structure only occur when these posets have the same multisets of edge weights.

Chain Weights have SNP

Definition (Postnikov-Stanley '09)

$$D_u^w = \frac{1}{(\ell(w)-\ell(u))!} \sum_{C: u=u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = w} m_C(x).$$

Proposition (ATZ '24+)

Any product of linear factors in $x_1, \ldots, x_n$ with all coefficients nonnegative has SNP.

Single Chain Newton Polytope (SCNP)

Definition (ATZ '24+)

D_u^w has *single-chain Newton polytope (SCNP)* if there exists a saturated chain C in the interval $[u, w]$ such that

$$\mathrm{Newton}(m_C) = \mathrm{Newton}(D_u^w).$$

We call such a C a *dominant chain* of the interval $[u, w]$.

Proposition (ATZ '24+)

If D_u^w has SCNP, then D_u^w has SNP.

Examples and Nonexamples for SCNP

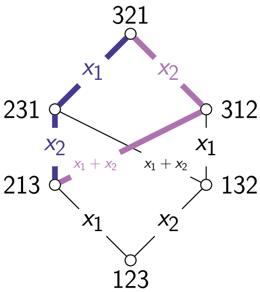
Example

$D^{321}_{213} = \frac{1}{2!}(x_1x_2 + (x_1 + x_2) \cdot x_2)$ has SCNP

$$C := (213 \lessdot 312 \lessdot 321)$$

$$m_C = (x_1 + x_2) \cdot x_2$$

$$\mathrm{Newton}(m_C) = \mathrm{Newton}(D^{321}_{213})$$



Example

D^{4231}_{1324} does not have SCNP

Dual Schubert Polynomials have SCNP

Definition (ATZ '24+)

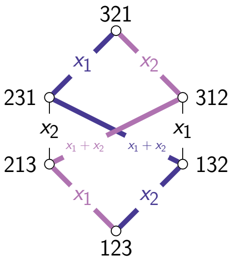
$$u = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_\ell = w$$

is called *greedy* in $[u, w]$ if for all $i \in [\ell]$:
writing $w_{i-1} t_{ab} = w_i$ for $a < b$, there does not exist $w'_{i-1} \lessdot w_i$ with $w'_{i-1} \in [u, w]$ such that

$$w'_{i-1} t_{ab'} = w_i \text{ for } b' > b, \text{ or } \; w'_{i-1} t_{a'b} = w_i \text{ for } a' < a.$$

Example

In $[123, 321]$,
$123 \lessdot 132 \lessdot 231 \lessdot 321$ is greedy
$123 \lessdot 213 \lessdot 312 \lessdot 321$ is also greedy
$123 \lessdot 213 \lessdot 231 \lessdot 321$ is not greedy



Dual Schubert Polynomials have SCNP, cont.

Definition (ATZ '24+)

The *global weight* $\mathrm{GW}(w)$ of $w \in S_n$ is

$$\mathrm{GW}(w) = \prod_{(a,b)\in\mathrm{Inv}(w)} (x_a + x_{a+1} + \cdots + x_{b-1}).$$

Example

$\mathrm{Inv}(231) = \{(1,3),(2,3)\}$, $\mathrm{GW}(231) = (x_1 + x_2) \cdot x_2$

Theorem (ATZ '24+)

For all $w \in S_n$, the dual Schubert polynomial D^w has SCNP. Moreover, every greedy chain of $[\mathrm{id}, w]$ is a dominant chain of D^w, and

$$\mathrm{Newton}(D^w) = \mathrm{Newton}(\mathrm{GW}(w)).$$

Newton Polytopes for Dual Schubert Polynomials

Theorem (ATZ '24+)

The point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\mathrm{Newton}(D^w)$ if and only if x^α has a coefficient of 1 in $\mathrm{GW}(w)$.

Theorem (ATZ '24+)

Given a product q of linear factors in $x_1, x_2, \ldots, x_n$ with all coefficients 1, the point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\mathrm{Newton}(q)$ if and only if x^α has a coefficient of 1 in q.

Example

$$\begin{aligned} q &= (x_1 + x_2)(x_2 + x_3)(x_1 + x_3)(x_1 + x_2 + x_3) \\ &= x_1^3 x_2 + x_1^3 x_3 + 2x_1^2 x_2^2 + 4x_1^2 x_2 x_3 + 2x_1^2 x_3^2 \\ &\quad + x_1 x_2^3 + 4x_1 x_2^2 x_3 + 4x_1 x_2 x_3^2 + x_1 x_3^3 + x_2^3 x_3 + 2x_2^2 x_3^2 + x_2 x_3^3 \end{aligned}$$

Vertices: $\{(3,1,0), (3,0,1), (1,3,0), (1,0,3), (0,3,1), (0,1,3)\}$

Newton Polytopes for Dual Schubert Polynomials, cont.

A *generalized permutahedron* $P_n^z(\{z_I\})$, parameterized by collections of real numbers $\{z_I\}$ for $I \subseteq [n]$, is given by

$$P_n^z(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \sum_{i=1}^{n} t_i = z_{[n]} \right\}.$$

Theorem (ATZ '24+)

For $w \in S_n$, $\mathrm{Newton}(D^w)$ is a generalized permutahedron with

$$z_I = \sum_{(a,b) \in \mathrm{Inv}(w)} \mathbb{1}_{I \supseteq \{a, a+1 \ldots, b-1\}}$$

for all $I \subseteq [n]$.

Conjectures about SCNP and SNP

Conjecture (Pylyavskyy '24+)

For an interval $[u, w]$, if the number of t_{ab} such that $u < t_{ab}u \leq w$ equals $\ell(w) - \ell(u)$, then D_u^w has SCNP.

Conjecture (ATZ '24+)

For $u \in S_n$, there exists $w \in S_n$ such that D_u^w does not have SCNP if and only if u contains a 1324-pattern.

Conjecture (ATZ '24+)

For all Bruhat intervals $[u, w]$, D_u^w has SNP.

Acknowledgments

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