Equality and Saturated Newton Polytope Property for Postnikov-Stanley Polynomials

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July 31, 2024
The Symmetric Group $S_n$

- $S_n$: permutations of $\{1, \ldots, n\}$
- Bottom element $1 \ 2 \ \cdots \ n$
- Top element $n \ (n - 1) \ \cdots \ 1$
- $\text{Inv}(u)$: the set of all inversions $(a, b)$ of $u$ such that $a < b$ and $u(a) > u(b)$

![Symmetric Group Diagram]
The (Strong) Bruhat Order of $S_n$

- $\ell(u)$: count of inversions in $u$
- $t_{ab}$ swaps the numbers in positions $a, b$ (not values $a, b$)
- Covering relation: $u \preceq v$ if $v = ut_{ab}$ and $\ell(v) = \ell(u) + 1$
- Interval $[u, w] : \{v \mid u \leq v \leq w\}$
Edge Weights

Definition
For \( u \preceq v \) and \( v = ut_{ab} \), the weight \( m(u \preceq v) \) is \( x_a + x_{a+1} + \cdots + x_{b-1} \).

Example
Since \( 312 = 213t_{13} \), we have \( m(213 \preceq 312) = x_1 + x_2 \).
Chain Weights

Definition

Let $u_0 \leq u_\ell$ and $C = (u_0 < u_1 < \cdots < u_\ell)$ be a saturated chain of $[u_0, u_\ell]$. Define the weight $m_C(x)$ of the chain $C$ by $\prod_{i=1}^{\ell} m(u_{i-1} < u_i)$.

Example

For $[213, 321]$, the weight of the saturated chain $213 < 312 < 321$ is $(x_1 + x_2) \cdot x_2$.
Skew Dual Schubert Polynomials

**Definition (Postnikov-Stanley ’09)**

For $u \leq w$, the skew dual Schubert polynomial or *Postnikov-Stanley polynomial* $D^w_u$ is defined by

$$D^w_u = \frac{1}{(\ell(w) - \ell(u))!} \sum_{C: u = u_0 \leq u_1 \leq \cdots \leq u_\ell = w} m_C(x).$$

**Example**

$$D^{321}_{213} = \frac{1}{2!} \left( x_1 x_2 + (x_1 + x_2) \cdot x_2 \right)$$

**Definition (Bernstein et al ’73)**

When $u = \text{id}$, $D^w_u$ is called a *dual Schubert polynomial*. 

![Diagram showing the relationship between Schubert polynomials and their dual counterparts, with examples and definitions illustrated visually.](link-to-diagram)
Saturated Newton Polytope (SNP)

For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, let $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

**Definition**

The Newton polytope $\text{Newton}(f)$ of $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha$ is the convex hull of its exponent vectors $\alpha$ in $\mathbb{R}^n$.

**Example**

$D_{213}^{321} = x_1 x_2 + \frac{1}{2} x_2^2 = x^{(1,1)} + \frac{1}{2} x^{(0,2)}$

$\text{Newton}(D_{213}^{321})$ is the segment from $(1, 1)$ to $(0, 2)$ in $\mathbb{R}^2$.

**Definition (Monical-Tokcan-Yong '19)**

$f$ has *Saturated Newton Polytope (SNP)* if $c_\alpha \neq 0$ for every integer point $\alpha \in \text{Newton}(f)$. 
### SNP in Algebraic Combinatorics

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Nontrivial Equalities

Definition
An equality between two P-S polynomials is *nontrivial* if the two polynomials are obtained from distinct multisets of saturated chains.

The equality $D_{1234}^{2341} = D_{2134}^{3241}$ is nontrivial.
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The Rank 2 Case

Lemma

There are no nontrivial equalities between P-S polynomials of rank 2.

All intervals of rank 2 are of one of the following forms (up to a rotation or reflection).

Note that rank 2 intervals contain two parallel sides with the same label $u_1$. The other two sides $u_2, u_3$ are either equal or satisfy $u_2 - u_3 = \pm u_1$. 

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The Rank 3 Case

Theorem (Jantzen ’79)

*The only rank 3 Bruhat intervals are 2-crowns, 3-crowns, and 4-crowns.*
The Rank 3 Case, cont.

**Proposition (ATZ ’24+)**

There are no nontrivial equalities between P-S polynomials of 2-crowns.

**Proposition (ATZ ’24+)**

There is never equality between P-S polynomials of a 2-crown and a cube.

\[
3(x_a + \cdots + x_{b-1})(x_b + \cdots + x_{c-1})(x_a + \cdots + x_{c-1})
\]
Equalities from Cube Rotations

**Lemma (ATZ '24+)**

*Every cube in the Bruhat order has four parallel edges with the same label.*

**Theorem (ATZ '24+)**

*All rotations of a given cube have equal P-S polynomials.*
Equality Conjectures

**Conjecture (ATZ '24+)**

Nontrivial equalities between two rank 3 P-S polynomials only occur between two cubes that are rotations of each other.

![Diagram of cubes](image)

**Conjecture (ATZ '24+)**

Nontrivial equalities between P-S polynomials of the same poset structure only occur when these posets have the same multisets of edge weights.
Chain Weights have SNP

**Definition (Postnikov-Stanley ’09)**

\[
D_u^w = \frac{1}{(\ell(w) - \ell(u))!} \sum_{C: u = u_0 \prec u_1 \prec \cdots \prec u_\ell = w} m_C(x).
\]

**Proposition (ATZ ’24+)**

Any product of linear factors in \(x_1, \ldots, x_n\) with all coefficients nonnegative has SNP.
Single Chain Newton Polytope (SCNP)

Definition (ATZ '24+)

$D_u^w$ has single-chain Newton polytope (SCNP) if there exists a saturated chain $C$ in the interval $[u, w]$ such that

$$\text{Newton}(m_C) = \text{Newton}(D_u^w).$$

We call such a $C$ a dominant chain of the interval $[u, w]$.

Proposition (ATZ '24+)

If $D_u^w$ has SCNP, then $D_u^w$ has SNP.
Examples and Nonexamples for SCNP

Example

\[ D_{213}^{321} = \frac{1}{2!}(x_1 x_2 + (x_1 + x_2) \cdot x_2) \text{ has SCNP} \]

\[ C := (213 < 312 < 321) \]

\[ m_C = (x_1 + x_2) \cdot x_2 \]

\[ \text{Newton}(m_C) = \text{Newton}(D_{213}^{321}) \]

Example

\[ D_{1324}^{4231} \text{ does not have SCNP} \]
Dual Schubert Polynomials have SCNP

Definition (ATZ '24+)

\[ u = w_0 \preceq w_1 \preceq w_2 \preceq \cdots \preceq w_\ell = w \]

is called greedy in \([u, w]\) if for all \(i \in [\ell]:\)
writing \(w_{i-1} t_{ab} = w_i\) for \(a < b\), there does not exist \(w'_{i-1} \preceq w_i\) with \(w'_{i-1} \in [u, w]\) such that

\[ w'_{i-1} t_{ab'} = w_i \text{ for } b' > b, \text{ or } w'_{i-1} t_{a'b} = w_i \text{ for } a' < a. \]

Example

In \([123, 321]\),
\[
123 \preceq 132 \preceq 231 \preceq 321
\]
is greedy
\[
123 \preceq 213 \preceq 312 \preceq 321
\]
is also greedy
\[
123 \preceq 213 \preceq 231 \preceq 321
\]
is not greedy
Dual Schubert Polynomials have SCNP, cont.

**Definition (ATZ '24+)**

The *global weight* $GW(w)$ of $w \in S_n$ is

$$GW(w) = \prod_{(a,b) \in Inv(w)} (x_a + x_{a+1} + \cdots + x_{b-1}).$$

**Example**

$Inv(231) = \{(1, 3), (2, 3)\}$, $GW(231) = (x_1 + x_2) \cdot x_2$

**Theorem (ATZ '24+)**

*For all $w \in S_n$, the dual Schubert polynomial $D^w$ has SCNP. Moreover, every greedy chain of $[\text{id}, w]$ is a dominant chain of $D^w$, and

$$\text{Newton}(D^w) = \text{Newton}(GW(w)).$$*
Theorem (ATZ ’24+)

The point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\text{Newton}(D^w)$ if and only if $x^\alpha$ has a coefficient of 1 in $\text{GW}(w)$.

Theorem (ATZ ’24+)

Given a product $q$ of linear factors in $x_1, x_2, \ldots, x_n$ with all coefficients 1, the point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\text{Newton}(q)$ if and only if $x^\alpha$ has a coefficient of 1 in $q$.

Example

\[ q = (x_1 + x_2)(x_2 + x_3)(x_1 + x_3)(x_1 + x_2 + x_3) \]
\[ = x_1^3x_2 + x_1^3x_3 + 2x_1^2x_2^2 + 4x_1^2x_2x_3 + 2x_1^2x_3^2 + x_1x_2^3 + 4x_1x_2^2x_3 + 4x_1x_2x_3^2 + x_1x_3^3 + x_2^3x_3 + 2x_2^2x_3^2 + x_2x_3^3 \]

Vertices: \{(3, 1, 0), (3, 0, 1), (1, 3, 0), (1, 0, 3), (0, 3, 1), (0, 1, 3)\}
A generalized permutahedron $P_n^z(\{z_I\})$, parameterized by collections of real numbers $\{z_I\}$ for $I \subseteq [n]$, is given by

$$P_n^z(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \sum_{i=1}^n t_i = z_{[n]} \right\}.$$ 

**Theorem (ATZ '24+)**

For $w \in S_n$, $\text{Newton}(D^w)$ is a generalized permutahedron with

$$z_I = \sum_{(a,b) \in \text{Inv}(w), I \supseteq \{a,a+1\ldots,b-1\}} 1_I$$

for all $I \subseteq [n]$. 
Conjectures about SCNP and SNP

**Conjecture (Pylyavskyy '24+)**
For an interval $[u, w]$, if the number of $t_{ab}$ such that $u < t_{ab}u \leq w$ equals $\ell(w) - \ell(u)$, then $D^w_u$ has SCNP.

**Conjecture (ATZ '24+)**
For $u \in S_n$, there exists $w \in S_n$ such that $D^w_u$ does not have SCNP if and only if $u$ contains a 1324-pattern.

**Conjecture (ATZ '24+)**
For all Bruhat intervals $[u, w]$, $D^w_u$ has SNP.
Acknowledgments

We would like to thank

- Shiyun Wang, our mentor, and Meagan Kenney, our TA for their continuous support and guidance throughout the program
- Pavlo Pylyavskyy for his mentorship and regular check-ins with us
- Casey Appleton and Victor Reiner for helpful conversations
- Ayah Almousa and Victor Reiner for organizing this wonderful REU experience!