Differential Powers of Semigroup and Polynomial Rings

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Introduction/Motivation
In our research, we are concerned with the algebraic properties of *affine semigroups*, which we visualize as integer lattices in $\mathbb{R}^d$.

We leverage the combinatorial structure of affine semigroups to obtain new results about algebraic objects defined over them.
Definition (Affine Semigroup)

Let $A = [a_1 \cdots a_n] \in \mathbb{Z}^{d \times n}$ be a matrix whose columns have $\mathbb{Z}$-span $\mathbb{Z}^d$. The set

$$\mathbb{N} A = \{ k_1 a_1 + \cdots + k_n a_n \mid k_1, \ldots, k_n \in \mathbb{N} \}$$

is the affine semigroup generated by $A$. 

\[ \mathbb{N} A \quad \mathbb{Z} A \quad \mathbb{R}_{\geq 0} A \]
Definition (Affine Semigroup)

Let \( A = [a_1 \cdots a_n] \in \mathbb{Z}^{d \times n} \) be a matrix whose columns have \( \mathbb{Z} \)-span \( \mathbb{Z}^d \). The set

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\]

is the affine semigroup generated by \( A \).

We suppose that the cone \( \mathbb{R}_{\geq 0} A \) is strongly convex, meaning that \( \mathbf{0} \) is a face of \( \mathbb{R}_{\geq 0} A \), and is saturated, meaning \( \mathbb{R}_{\geq 0} A \cap \mathbb{Z}^d = \mathbb{N}A \), i.e. the semigroup \( \mathbb{N}A \) has no holes.
The semigroup generated by $A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$, which is not saturated.

The semigroup generated by $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, with its maximal proper faces $\rho$, $\sigma$. 
Definition (Semigroup Rings)

For a matrix $A \in \mathbb{Z}^{d \times n}$, the semigroup ring of $A$ is the ring

$$R = \mathbb{C}[\mathbb{N}A] = \mathbb{C}[t^{a_1}, \ldots, t^{a_n}],$$

where $t^b = t_1^{b_1} \cdots t_d^{b_d}$.

Many algebraic objects corresponding to $R$ can be analyzed purely in terms of the semigroup $\mathbb{N}A$:

- Monomials $t^a \in R \iff$ lattice points $a \in \mathbb{N}A$
- Multiplication in $R \iff$ addition in $\mathbb{N}A$
- Monomial prime ideals $P \subseteq R \iff$ faces $F$ of $\mathbb{R}_{\geq 0}A$
Semigroup Rings

$I = \langle s, st \rangle = P_\tau$
Note that the *polynomial ring* $S = \mathbb{C}[x_1, \ldots, x_n]$ is the semigroup ring corresponding to the identity matrix. In particular, the faces associated to $S$ are subsets $F$ of the variables $\{x_1, \ldots, x_n\}$. 
The semigroup ring $R$ lives inside the ring of Laurent polynomials $L = \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$, which naturally has differential operators defined in terms of partial derivatives $\partial_i = \frac{\partial}{\partial t_i}$. 


The semigroup ring $R$ lives inside the ring of Laurent polynomials $L = \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$, which naturally has differential operators defined in terms of partial derivatives $\partial_i = \frac{\partial}{\partial t_i}$.

The general definition of differential operators is abstract and difficult to work with. Luckily, in the case of normal affine semigroup rings, differential operators have a nice combinatorial description.
The ring of differential operators $D(R)$ of $R$ are generated by certain operators $D_a$ for $a \in \mathbb{Z}^d$, where $D_a$ acts on $\mathbb{N}A$ by sending a lattice point $b \in \mathbb{N}A$ to the translate $a + b$.
The *order* of the differential operator $D_a$ is determined by the distance of $a$ from the *hyperplanes* corresponding to the faces of $\mathbb{R}_{\geq 0} A$. 

![Graph showing order of differential operators](image)
Differential Powers

Definition (Differential Powers)

The \( N \)-th differential power \( I^{(N)} \) of ideal \( I \subseteq R \) is defined as

\[ I^{(N)} = \{ f \in R : \delta(f) \in I \text{ for all } \delta \in D(R) \text{ of order } < N \}. \]
Definition (Differential Powers)

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Differential powers are closely related to ordinary powers $I^N = \{a_1 \cdots a_N \mid a_i \in I\}$ and symbolic powers $I^{(N)}$, but are often much easier to compute and comprehend.
Combinatorial Results
Definition (Standard Monomials)
The *standard monomials* of an ideal $I \subseteq \mathbb{C}[x_1, x_2, \ldots, x_n]$ consists of the set of monomials in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ not in $I$.

Definition (Standard Pairs)
A *standard pair* $(\mathbf{a}, Z)$ is an ordered pair containing a vector $\mathbf{a}$ corresponding to a monomial, and a set $Z$ of vectors corresponding to the generators of a face.
I = (x^2y^2, x^4y) \subseteq \mathbb{C}[x, y]
Sample Results with Standard Pairs

Figure 1: Caption
Theorem

Let $P_F$ be a prime monomial ideal corresponding to a face $F$, which is given by some subset of the variables $\{x_1, \ldots, x_n\}$. Then $P_F = \langle x_i \mid x_i \notin F \rangle$, and

$$\text{stdPairs}(P_F^{(\ell)}) = \left\{ (a, F) \mid \sum_{i=1}^{n} a_i < \ell, a_i = 0 \text{ if } x_i \in F \right\}.$$ 

In terms of differential operators, the inequality $\sum_{i=1}^{n} a_i < \ell$ expresses that the order of $\mathcal{D}_a$ is less than $\ell$. 
Theorem

Let $I$ be a radical ideal, so that $I$ is given by the intersection of prime ideals $I = P_{F_1} \cap P_{F_2} \cap \cdots \cap P_{F_n}$. Then

$$\text{stdPairs}(I^{(\ell)}) = \bigcup_{i=1}^{n} \text{stdPairs}(P_{F_i}^{(\ell)}).$$

In essence, this is because the standard monomials of an intersection of ideals is equal to the union of the standard monomials of each ideal.
Asymptotic Results
Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and $I \subseteq S$ be a monomial ideal.
Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and $I \subseteq S$ be a monomial ideal. For example, $S = \mathbb{C}[x, y]$ and $I = \langle xy^3, x^3y^2, x^4y \rangle$. 
Theorem

If $\sqrt{I} = \langle x_1 \cdots x_n \rangle$, then there exists $N \in \mathbb{N}$ such that $I^{(m)}$ is principal for all $m \geq N$.

$\sqrt{I} = \langle x_1 \cdots x_n \rangle \iff I$ lies in the interior of the first orthant
Non-example: \( I = \langle x, y \rangle \).
Let \( A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & n \end{bmatrix} \) and \( R \) be the corresponding semigroup ring, that is, \( R = \mathbb{C}[s, st, \ldots, st^n] \). Let \( I \subseteq R \) be a monomial ideal in the interior of \( \mathbb{R}_{\geq 0} A \). For example, \( n = 3 \) and \( I = \langle s^3t^6, s^3t^4, s^4t^2 \rangle \).
The analogues of principal ideals in $R$ is of the form $I_{a,r} = \langle s^{a_1} t^{a_2}, s^{a_1} t^{a_2+1}, \ldots, s^{a_1} t^{a_2+r-1} \rangle$ for $a = (a_1, a_2) \in \mathbb{N}A$ and $r \in \mathbb{N}$. 
Theorem

There exists $N \in \mathbb{N}$ such that $I^{\langle N \rangle}$ is of the shape $I_{a,r}$ for some $a \in \mathbb{N}A$ and $1 \leq r \leq n$.

Lemma

\[
(I_{a,r})^{\langle 2 \rangle} = \begin{cases} 
I_{a+(1,1),n-1}, & \text{if } r = 1; \\
I_{a+(1,1),n}, & \text{if } r = 2; \\
I_{a+(0,1),r-2}, & \text{if } 3 \leq r \leq n.
\end{cases}
\]
Semigroup Ring

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Corollary

There exists $N \in \mathbb{N}$ such that if $n$ is odd, $I^{\langle N \rangle}$ is principal; if $n$ is even, $I^{\langle N \rangle}$ is principal or is generated by 2 elements. The number of generators of $I^m$ for $m \geq N$ is periodic.
Questions?
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