Differential Powers of Semigroup and Polynomial Rings

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Table of Contents

Introduction/Motivation

In our research, we are concerned with the algebraic properties of *affine semigroups*, which we visualize as integer lattices in $\mathbb{R}^d$.

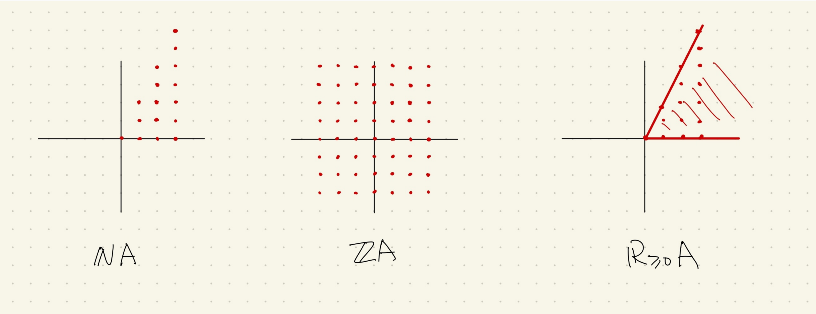
We leverage the combinatorial structure of affine semigroups to obtain new results about algebraic objects defined over them.

Definition (Affine Semigroup)

Let $A = [\mathbf{a}_1 \cdots \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$ be a matrix whose columns have $\mathbb{Z}$-span $\mathbb{Z}^d$. The set

$$\mathbb{N}A = \{k_1\mathbf{a}_1 + \cdots + k_n\mathbf{a}_n \mid k_1, \ldots, k_n \in \mathbb{N}\}$$

is the *affine semigroup* generated by A.



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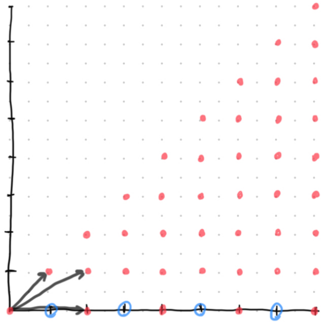
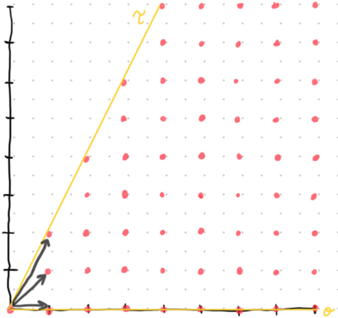
We suppose that the cone $\mathbb{R}_{\geq 0}A$ is *strongly convex*, meaning that $\mathbf{0}$ is a face of $\mathbb{R}_{\geq 0}A$, and is *saturated*, meaning $\mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d = \mathbb{N}A$, *i.e.* the semigroup $\mathbb{N}A$ has no holes.



The semigroup generated by $A=\begin{bmatrix}2&2&1\\0&1&1\end{bmatrix}$, which is not saturated.



The semigroup generated by $A=\begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}$, with its maximal proper faces τ, σ

Semigroup Rings

Definition (Semigroup Rings)

For a matrix $A \in \mathbb{Z}^{d\times n}$, the *semigroup ring* of A is the ring

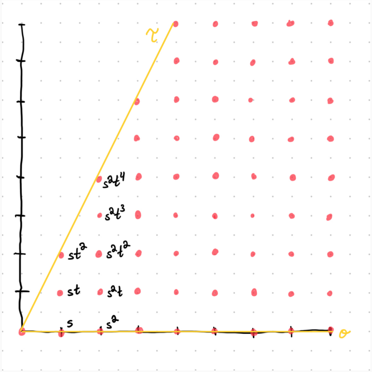
$$R = \mathbb{C}[\mathbb{N}A] = \mathbb{C}[t^{\mathbf{a}_1}, \dots, t^{\mathbf{a}_n}],$$

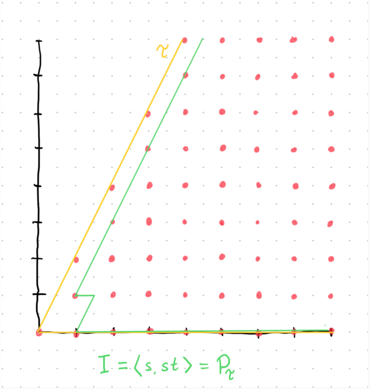
where $t^{\mathbf{b}} = t_1^{b_1} \cdots t_d^{b_d}$.

Many algebraic objects corresponding to R can be analyzed purely in terms of the semigroup $\mathbb{N}A$:

- Monomials $t^{\mathbf{a}} \in R \Longleftrightarrow$ lattice points $\mathbf{a} \in \mathbb{N}A$
- Multiplication in $R \Longleftrightarrow$ addition in $\mathbb{N}A$
- Monomial prime ideals $P \subseteq R \Longleftrightarrow$ faces F of $\mathbb{R}_{\geq 0}A$

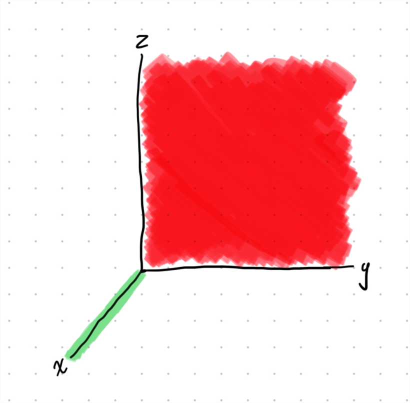
Semigroup Rings





Semigroup Rings

Note that the *polynomial ring* $S = \mathbb{C}[x_1, \ldots, x_n]$ is the semigroup ring corresponding to the identity matrix. In particular, the faces associated to S are subsets F of the variables $\{x_1, \ldots, x_n\}$.



Differential Operators and Powers

The semigroup ring R lives inside the *ring of Laurent polynomials* $L = \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$, which naturally has *differential operators* defined in terms of partial derivatives $\partial_i = \frac{\partial}{\partial t_i}$.

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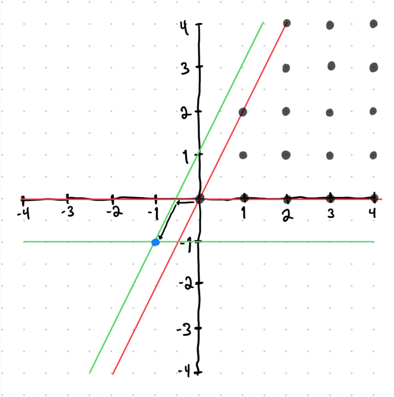
The general definition of differential operators is abstract and difficult to work with. Luckily, in the case of normal affine semigroup rings, differential operators have a nice combinatorial description.

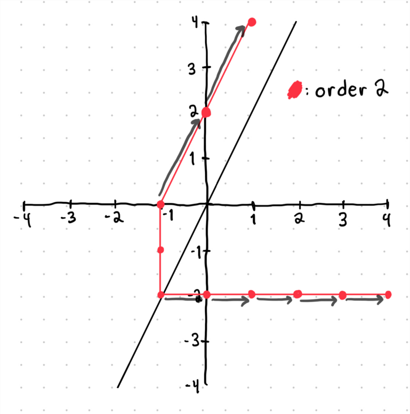
Theorem ([ST01])

The ring of differential operators $D(R)$ of R are generated by certain operators $\mathcal{D}_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^d$, where $\mathcal{D}_{\mathbf{a}}$ acts on $\mathbb{N}A$ by sending a lattice point $\mathbf{b} \in \mathbb{N}A$ to the translate $\mathbf{a} + \mathbf{b}$.



Order of Differential Operators

The *order* of the differential operator $\mathcal{D}_{\mathbf{a}}$ is determined by the distance of $\mathbf{a}$ from the *hyperplanes* corresponding to the faces of $\mathbb{R}_{\geq 0}A$.



Differential Powers

Definition (Differntial Powers)

The N*-th differential power* $I^{\langle N \rangle}$ of ideal $I \subseteq R$ is defined as

$$I^{\langle N \rangle} = \{f \in R : \delta(f) \in I \text{ for all } \delta \in D(R) \text{ of order } < N\}.$$

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Differential powers are closely related to *ordinary powers* $I^N = \{a_1 \cdots a_N \mid a_i \in I\}$ and *symbolic powers* $I^{(N)}$, but are often much easier to compute and comprehend.

Combinatorial Results

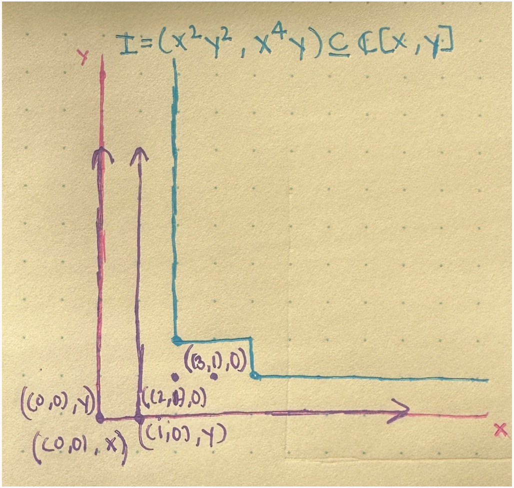
Standard Pairs

Definition (Standard Monomials)

The *standard monomials* of an ideal $I \subseteq \mathbb{C}[x_1, x_2, ..., x_n]$ consists of the set of monomials in $\mathbb{C}[x_1, x_2, ..., x_n]$ not in I.

Definition (Standard Pairs)

A *standard pair* $(\mathbf{a}, Z)$ is an ordered pair containing a vector $\mathbf{a}$ corresponding to a monomial, and a set Z of vectors corresponding to the generators of a face.


I=(x²y², x⁴y) ⊆ ℂ[x,y]
y
x
((3,1),0)
((2,1),0)
((0,0),y)
((0,0),x)
((1,0),y)

Sample Results with Standard Pairs

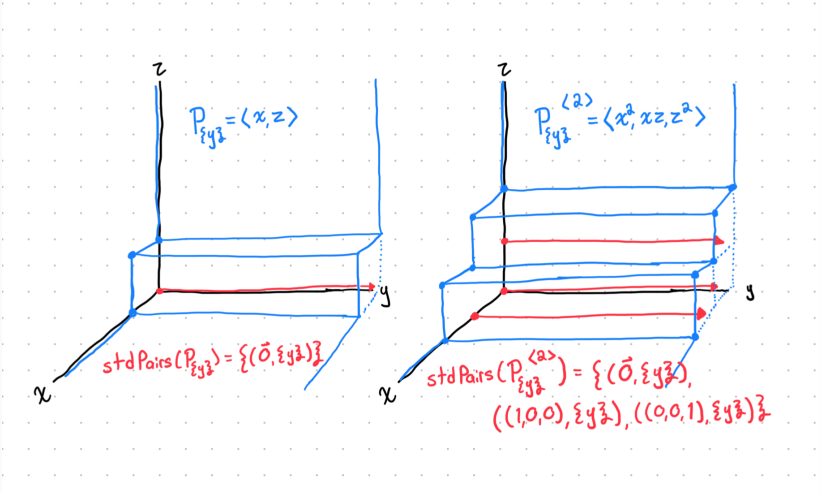


Figure 1: Caption

Theorem

Let P_F be a prime monomial ideal corresponding to a face F, which is given by some subset of the variables $\{x_1, ..., x_n\}$. Then $P_F = \langle x_i \mid x_i \notin F \rangle$, and

$$\mathrm{stdPairs}(P_F^{\langle \ell \rangle}) = \left\{ (\mathbf{a}, F) \mid \sum_{i=1}^{n} a_i < \ell, a_i = 0 \text{ if } x_i \in F \right\}.$$

In terms of differential operators, the inequality $\sum_{i=1}^{n} a_i < \ell$ expresses that the order of $\mathcal{D}_{\mathbf{a}}$ is less than ℓ.

Theorem

Let I be a radical ideal, so that I is given by the intersection of prime ideals $I = P_{F_1} \cap P_{F_2} \cap \cdots \cap P_{F_n}$. Then

$$\mathrm{stdPairs}(I^{\langle \ell \rangle}) = \bigcup_{i=1}^{n} \mathrm{stdPairs}(P_{F_i}^{\langle \ell \rangle}).$$

In essence, this is because the *standard monomials of an intersection of ideals is equal to the union of the standard monomials of each ideal*

Asymptotic Results

Polynomial Ring

Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and $I \subseteq S$ be a monomial ideal.

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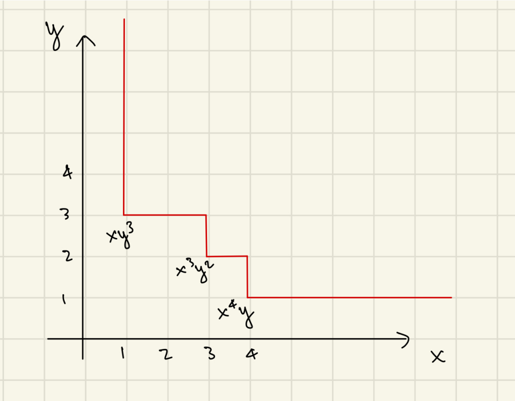
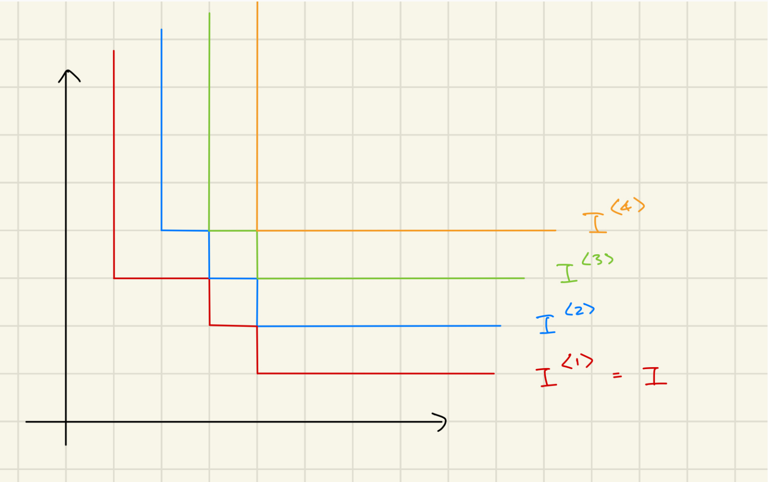


Polynomial Ring

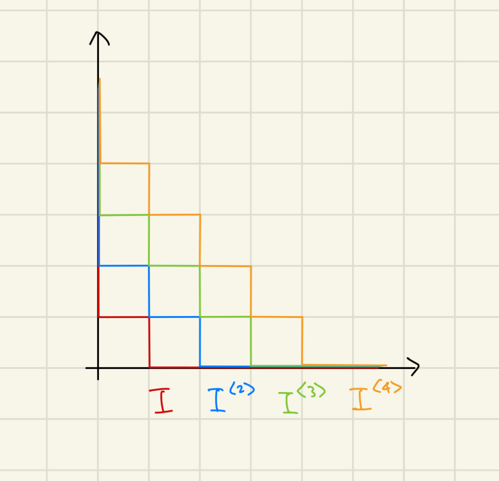


Theorem

If $\sqrt{I} = \langle x_1 \cdots x_n \rangle$, then there exists $N \in \mathbb{N}$ such that $I^{\langle m \rangle}$ is principal for all $m \geq N$.

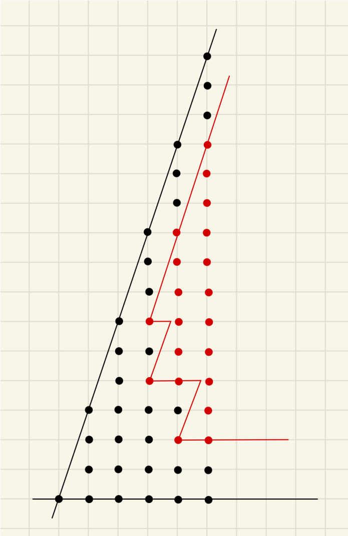
$\sqrt{I} = \langle x_1 \cdots x_n \rangle \Leftrightarrow I$ lies in the interior of the first orthant

Non-example: $I = \langle x, y \rangle$.



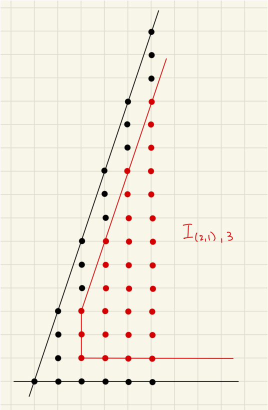
Semigroup Ring

Let $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & n \end{bmatrix}$ and R be the corresponding semigroup ring, that is, $R = \mathbb{C}[s, st, \ldots, st^n]$. Let $I \subseteq R$ be a monomial ideal in the interior of $\mathbb{R}_{\geq 0}A$. For example, $n = 3$ and $I = \langle s^3t^6, s^3t^4, s^4t^2 \rangle$.



Semigroup Ring

The analogues of principal ideals in R is of the form $I_{\mathbf{a},r} = \langle s^{\mathbf{a}_1}t^{\mathbf{a}_2}, s^{\mathbf{a}_1}t^{\mathbf{a}_2+1}, \ldots, s^{\mathbf{a}_1}t^{\mathbf{a}_2+r-1}\rangle$ for $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2) \in \mathbb{N}A$ and $r \in \mathbb{N}$.



Semigroup Ring

Theorem

There exists $N \in \mathbb{N}$ such that $I^{\langle N \rangle}$ is of the shape $I_{\mathbf{a},r}$ for some $\mathbf{a} \in \mathbb{N}A$ and $1 \leq r \leq n$.

Lemma

$$(I_{\mathbf{a},r})^{\langle 2 \rangle} = \begin{cases} I_{\mathbf{a}+(1,1),n-1}, & \text{if } r = 1; \\ I_{\mathbf{a}+(1,1),n}, & \text{if } r = 2; \\ I_{\mathbf{a}+(0,1),r-2}, & \text{if } 3 \leq r \leq n. \end{cases}$$

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Corollary

There exists $N \in \mathbb{N}$ such that if n is odd, $I^{\langle N \rangle}$ is principal; if n is even, $I^{\langle N \rangle}$ is principal or is generated by 2 elements. The number of generators of I^m for $m \geq N$ is periodic.

Questions?

Acknowledgements

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