On Triangulations of Order Polytopes for Snake Posets

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Triangulations, Circuits, and Flips

- Given a d-dimensional convex polytope $\mathcal{A}$, a **triangulation** of $\mathcal{A}$ is a subdivision of $\mathcal{A}$ into a collection of d-simplices (d-dimensional generalizations of a triangle).
- A triangulation $\mathcal{T}$ is **regular** if there are heights $h_1, ..., h_d \in \mathbb{R}$ such that the projection of the upper convex hull of $\hat{A} = \left\{ \begin{bmatrix} a_1 \\ h_1 \end{bmatrix}, \ldots, \begin{bmatrix} a_d \\ h_d \end{bmatrix} \right\} \subseteq \mathbb{R}^{d+1}$ back down to $\mathbb{R}^d$ is $\mathcal{T}$.
- A subset C of the vertices of $\mathcal{A}$ is a **circuit** if it is affinely dependent. There are exactly 2 ways to triangulate a circuit, and switching between them is called a **bistellar flip**.

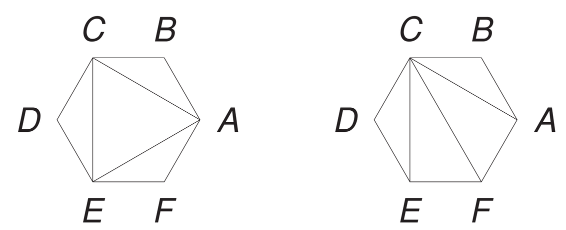


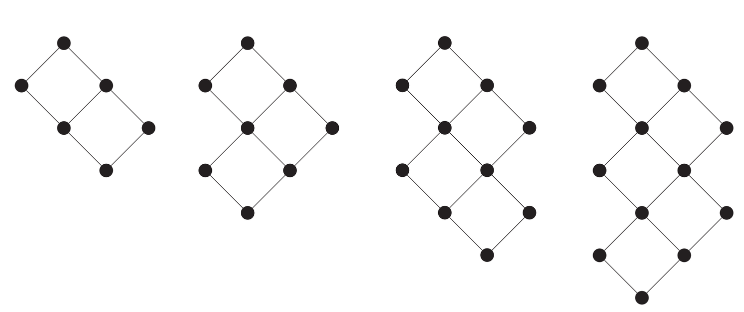
Figure: Triangulation $\mathcal{T}_1$ (left) is obtained by flipping $\mathcal{T}_2$ (right) at circuit $ACEF$.

Theorem ([GKZ08])

Let $\mathcal{A}$ be a d-dimensional convex polytope. Then there exists a $(\#\mathcal{A} - d - 1)$-dimensional polytope called the ***secondary polytope****, denoted $\Sigma_{\mathcal{A}}$, whose vertices are in correspondence with the regular triangulations of $\mathcal{A}$ and whose edges correspond to flips between them.*

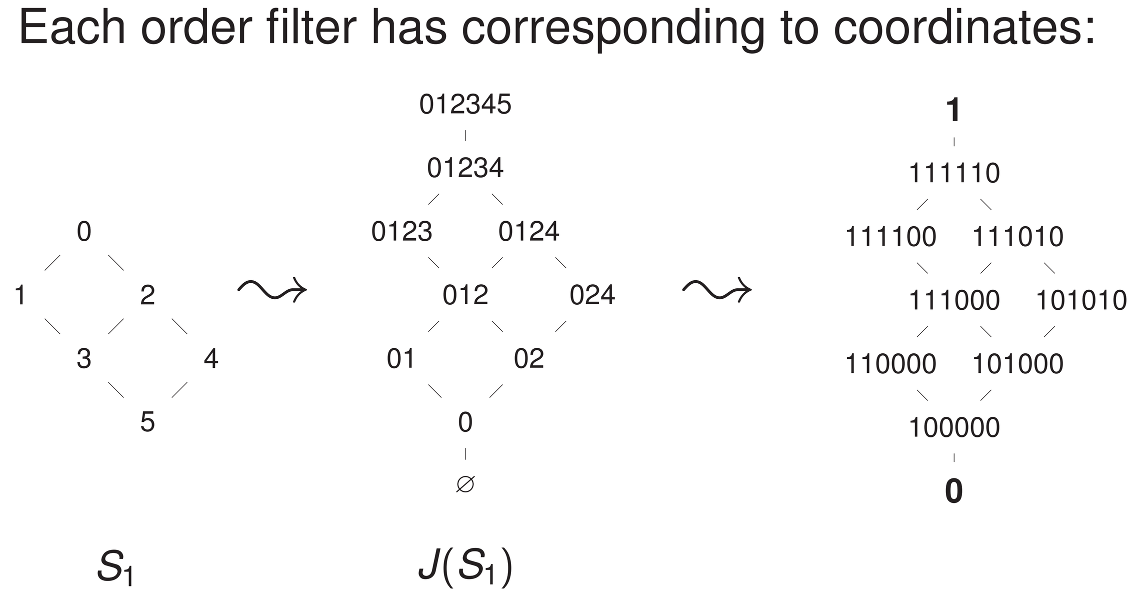
Order Polytopes and Snake Posets

- **Snake posets** are posets of the pattern shown below. The snake poset with $k+1$ squares is denoted S_k.



S_1 S_2 S_3 S_4

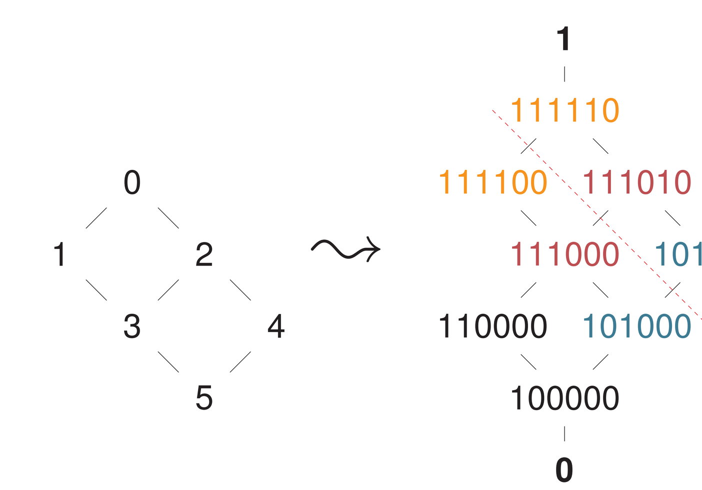
- An order filter of a poset is a subset of elements closed under going upwards. We denote by $J(S_k)$ the poset of order filters of S_k ordered by inclusion. Each order filter has corresponding to coordinates:



- The **order polytope** $\mathcal{O}(S_k)$ is $\text{conv}(\{\mathbf{v}_A : A \in J(S_k)\})$. This is what we're going to triangulate!
- For any poset P, $\mathcal{O}(P)$ has a **canonical triangulation** $\mathcal{T}$, where the maximal simplices of $\mathcal{T}$ are in bijection with maximal chains of $J(P)$.

The Twist Group

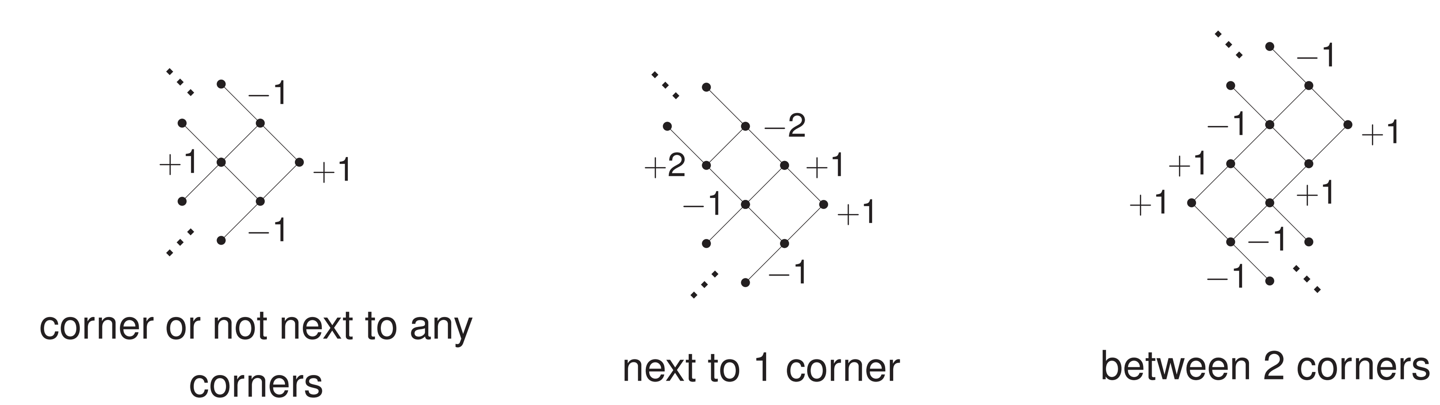
- Consider S_1 and $J(S_1)$ below and the ladder $\mathcal{L}^1$.



- Lemma 5.4 of [Bel+22] implies that the set of all τ_i generate a commutative subgroup of $\mathfrak{S}_{|J(S_k)|}$, hence the following definition.
- The **twist group** $\mathfrak{T}_k \leq \mathfrak{S}_{|J(S_k)|}$ is defined by:
$$\mathfrak{T}_k = \langle \tau_i \mid \tau_i^2 = 1, \tau_i\tau_j = \tau_j\tau_i \rangle \cong (\mathbb{Z}/2\mathbb{Z})^d,$$
where d is the number of ladders.
- **Theorem (Corollary 5.12 [Bel+22])**: the twist group $\mathfrak{T}_k$ acts on the regular triangulations of $\mathcal{O}(S_k)$.

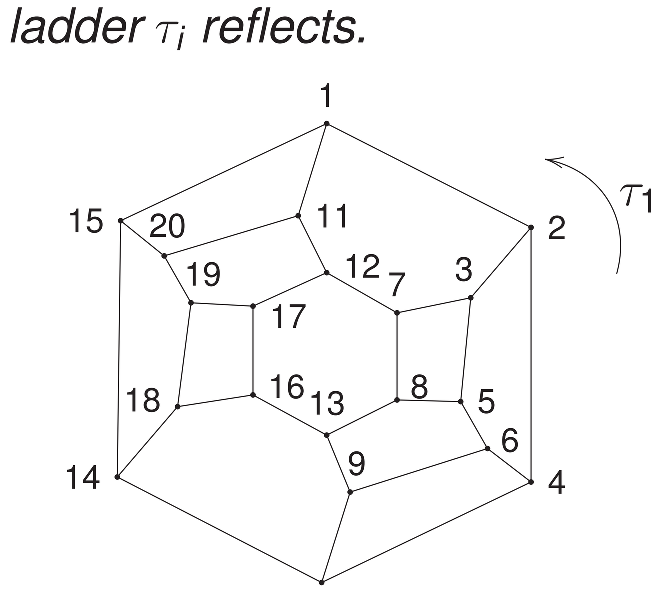
Twist Eigenbasis

- We can define vectors in the space containing $\Sigma_{\mathcal{O}(S_k)}$ by giving a coefficient to each vertex of $J(S_k)$.
- We want a subspace $V \cong \mathbb{R}^{2k+1}$ in which the twists are linear.
- We define one basis element v_i for each square w_i in $J(S_k)$:



Theorem

Let $V \cong \mathbb{R}^{2k+1}$ be the linear subspace of $\mathbb{R}^{4k+6}$ parallel to the subspace containing $\Sigma_{\mathcal{O}(S_k)}$. $v_1, v_2, \ldots, v_{n+1}$ is an eigenbasis of V and each elementary twist τ_i negates exactly the basis elements that correspond to the squares in the ladder τ_i reflects.



On $\Sigma_{\mathcal{O}(S_1)}$ the elementary twists are 180° rotations.

Conjecture ([Bel+22], Conjecture 6.5)

The number of regular triangulations of $\mathcal{O}(S_k)$ is $2^{k+1} \cdot \text{Cat}(2k+1)$.

Theorem

The twist group acts freely on the regular triangulations of $\mathcal{O}(S_k)$.

Corollary

Each orbit under the twist group action on regular triangulations of $\mathcal{O}(S_k)$ has 2^{k+1} elements.

Valence-Regularity of $\mathcal{O}(S_k)$

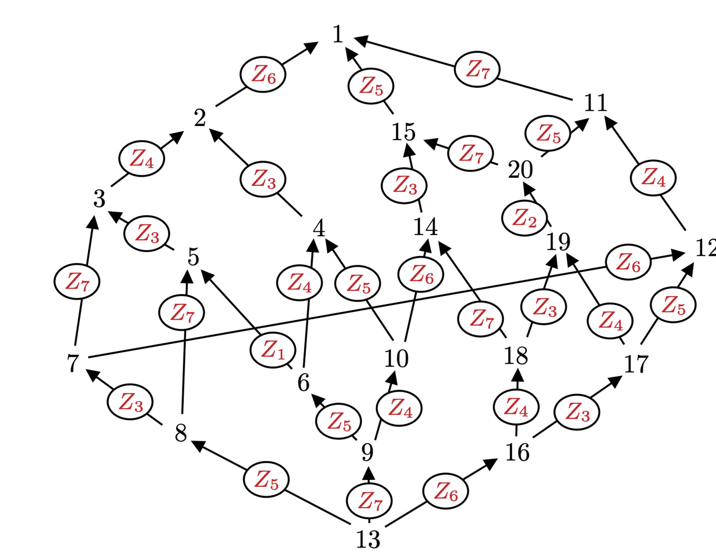
Conjecture 6.1 [Bel+22]: The 1-skeleton of the secondary polytope of $\mathcal{O}(S_k)$ is $(2k+1)$-regular.
Theorem: *Let $\mathcal{T}$ be a triangulation of $\mathcal{O}(S_k)$ obtained by applying one flip to the canonical triangulation $\mathcal{T}_k$. Then, $\mathcal{T}$ admits the same number of flips as $\mathcal{T}_k$, namely $2k+1$ flips.*

Conjecture

The 2-dimensional faces of $\Sigma_{\mathcal{O}(S_k)}$ are squares, pentagons and hexagons

A Poset Structure on Triangulations

The triangulations of $O(S_k)$ admit a nice poset structure. For $k = 1$:



Conjecture: Every face of the poset contains a unique sink vertex, i.e. it is a *good orientation* [Kal88].

Acknowledgments

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References

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