Minimal Infinite Free Resolutions over Certain Families of Numerical Semigroup Algebras

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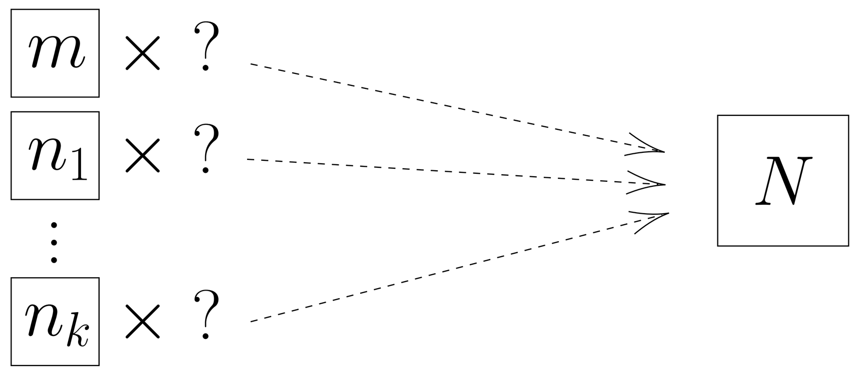
Background

We've got a hankering for exactly N chicken nuggets. The nuggets, however, only come in packs of m, n_1, $n_2, \dots, n_k \in \mathbb{N}$. Using only these, can we satisfy our exact hankering? And if so, what are the different ways we can do it?



Numerical Semigroups

A **numerical semigroup** S is a subset of the non-negative integers $\mathbb{Z}_{\geq 0}$ such that:

1. $0 \in S$
2. S is closed under addition, and
3. $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

A numerical semigroup always has a unique minimal set of generators. We write :

$$S = \langle m, n_1, ..., n_k \rangle = \{a_0 m + a_1 n_1 + ... + a_k n_k : a_i \in \mathbb{Z}_{\geq 0}\}.$$

The smallest generator m is called the **multiplicity** of the semigroup.

Examples

$S = \langle 5, 7, 9 \rangle = \{5, 7, 9, 10, 12, 14, 16, 18, \dots\}$
$R = \langle 6, 12, 13 \rangle = \{6, 12, 13, 18, 19, 24, 25, 26, \dots\}$

The Apéry Set

We can capture much of the additive structure of a semigroup by considering the elements that are in some sense minimal. The **Apéry set** of a semigroup S is given by $\mathrm{Ap}(S) = \{n \in S : n - m \notin S\}$, where m is the multiplicity of S.

$\mathrm{Ap}(S) = \{0, 16, 7, 18, 14\}$, $\mathrm{Ap}(R) = \{0, 13, 26, 39, 52, 65\}$.

EGANS

An **extra-generalized arithmetical numerical semigroup** (EGANS) is a numerical semigroup of the form

$$S = \langle m, mh + \delta, mh + 2\delta, \dots, mh + k\delta \rangle,$$

where $h, k > 0$, $m < mh + k\delta$, and $\gcd(m, \delta) = 1$.

Examples

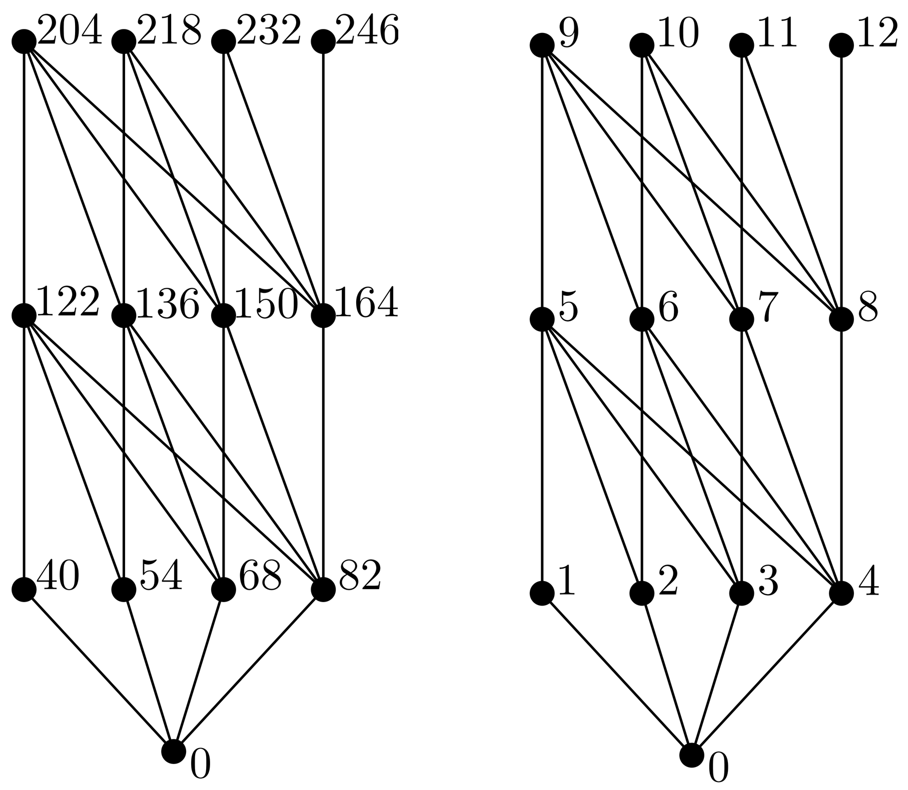
$S = \langle 5, 7, 9 \rangle = \langle 5, 5 \cdot 1 + 2, 5 \cdot 1 + 2 \cdot 2 \rangle$
$T = \langle 13, 47, 42, 37, 32, 27 \rangle = \langle 13, 13 \cdot 4 - 5, \dots, 13 \cdot 4 - 5 \cdot 5 \rangle$

The Kunz Poset

The **Apéry poset** of S is $(\mathrm{Ap}(S), \preceq)$, where $a \preceq a'$ if and only if $a' - a \in S$. The **Kunz poset** is the poset obtained by replacing each element of the Apéry poset with its equivalence class in $\mathbb{Z}_m$.

Example

For $U = \langle 13, 40, 54, 68, 82 \rangle = \{0, 13, 26, 39, 40, 53 \dots\}$,



Apéry Poset Kunz Poset

The Semigroup Algebra

With a clever homomorphism, we can transform the additive structure of a semigroup into the multiplicative structure of a ring. Let $\mathbb{K}$ be a field. The homomorphism is clear through an example:

Example

For $S = \langle 5, 7, 9 \rangle$, define

$$\begin{aligned} \varphi :\ \mathbb{K}[y, x_1, x_2] &\to \mathbb{K}[t] \\ y &\mapsto t^5 \\ x_1 &\mapsto t^7 \\ x_2 &\mapsto t^9 \end{aligned}$$

- **The toric ideal of S:** $I_S = \ker(\varphi) = \langle x_1^2 - yx_2, x_2^3 - y^4 x_1, x_1 x_2^2 - y^5 \rangle$
- **The semigroup algebra of S:** $\mathbb{K}[S] = \mathrm{Im}(\varphi) \cong \mathbb{K}[y, x_1, ..., x_k]/I_S$
- $\mathbb{K}[S]$ is graded: $\deg(yx_1) = \deg(t^5 t^7) = 5 + 7 = 12$

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Free Resolutions

A great way to study a ring is to study modules over it. The simplest kinds of modules are free modules, and a **free resolution** is a long exact sequence that constructs a module from free modules. For EGANS semigroups, we provide a combinatorial method of constructing minimal resolutions:

Theorem

Let $S = \langle m, mh + \delta, \dots, mh + k\delta \rangle$ be an EGANS with $k \equiv \alpha \pmod{m}$. There exists a chain map

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & \mathbb{K} & \xleftarrow{\partial_0} & \mathbb{K}[S] & \xleftarrow{\partial_1} & F_1 & \xleftarrow{\partial_2} & F_2 & \longleftarrow & \cdots \\ & & & & \downarrow p_0 & & \downarrow p_1 & & \downarrow p_2 & & \\ 0 & \longleftarrow & \mathbb{K} & \xleftarrow{\partial_0'} & \mathbb{K}[S] & \xleftarrow{\partial_1'} & F_1' & \xleftarrow{\partial_2'} & F_2' & \longleftarrow & \cdots \end{array}$$

that converts the top row, a non-minimal free resolution of $\mathbb{K}$ over $\mathbb{K}[S]$, to the bottom row, a minimal free resolution over $\mathbb{K}[S]$. The chain map above translates words built from the alphabet $\{0, 1, \dots, m - 1\}$ to words built from the alphabet $\{0, 1, \dots, k, m - \alpha\}$ that avoid the patterns

1. (j, k) for any $j \in \{1, 2, \dots, k\}$, and
2. $(j, m - \alpha)$ for any $j \in \{0, 1, \dots, \alpha - 1, m - \alpha\}$.

The free modules above have bases indexed by these words.

Example

For $S = \langle 5, 6, 7 \rangle$, we have the following translations:
$(4) \mapsto x_2^2 \cdot (0) \quad (13) \mapsto y^2 \cdot (02) \quad (133) \mapsto y^4 \cdot (012)$

Resolving the ground field $\mathbb{K}$ affords us substantial information about resolutions of other modules: for any $\mathbb{K}[S]$−module M, the rank of the free module F_i in its free resolution over $\mathbb{K}[S]$ is given by $\dim_{\mathbb{K}}(\mathrm{Tor}_i^{\mathbb{K}[S]}(M, \mathbb{K}))$.

Further Research

This process seems to hint at a more general method of computing free resolutions over numerical semigroup algebras: if one can use the Kunz poset of a numerical semigroup to identify multiplication patterns avoided by generators of its toric ideal, one need only translate short words from the alphabet $\{0, 1, \dots, m - 1\}$ to words in the constructed language, and the rest of the resolution follows.

We have been able to use this observation to conjecture the resolutions of all semigroups with $m = 5$.

References

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