A Mixed Dimer Model for Exceptional Type Quiver Representations and Cluster Algebras

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Quiver Representations

Quivers are directed graphs, and a representation of a quiver is a realization of that graph as a commutative diagram of vector spaces (except not necessarily commutative).

The dimension vector of a representation is the list of dimensions of each of the vector spaces in the diagram.

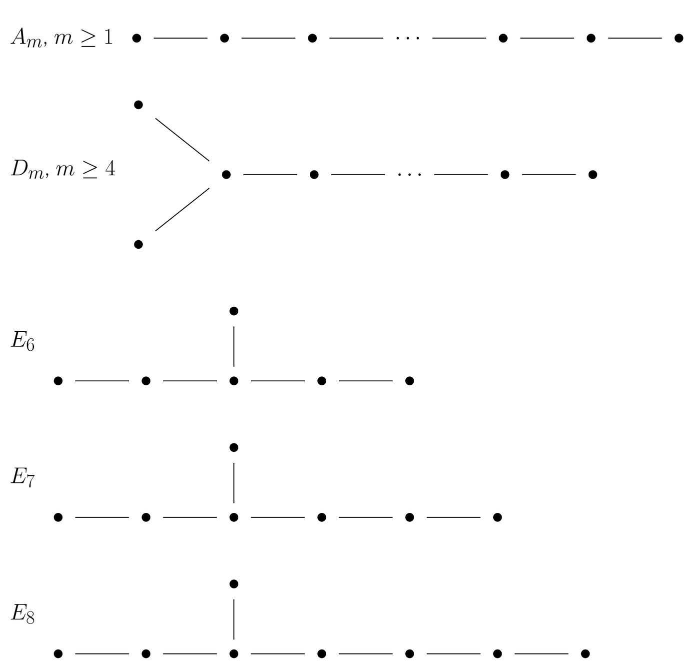

Figure 1. Dynkin Quivers

Many familiar notions from linear algebra generalize to quiver representations. Any quiver representation can be factored as a ⊕ of "prime" (indecomposable) representations.

F-polynomials

The F-polynomial of a quiver representation encodes information about its subrepresentations. The goal of this project is to find a combinatorial rule for F-polynomials in type E.

$$F_M(\underline{u}) = \sum_{\underline{e}} \chi(Gr_{\underline{e}}(M))\underline{u}^{\underline{e}}$$

F-polynomials show up in cluster algebras, where they can be used to give explicit formulae for cluster variables in terms of the cluster variables of an initial seed.

Dynkin quiver fun facts

- Dynkin quivers are exactly the ones having finitely many "prime" representations.
- The dimension vectors of the "prime" representations are the positive roots of the corresponding root system.
- Distinct indecomposable representations have distinct dimension vectors.
- To reconstruct an indecomposable representation from its dimension vector, you can choose each of the linear maps "at random". This strategy works with probability 1.

F-polynomial Example for $Q = 0 \longleftarrow 1 \longrightarrow 2$

$$F_{(1,1,1)}(u_0, u_1, u_2) = 1 + u_0 + u_2 + u_0u_2 + u_0u_1u_2$$

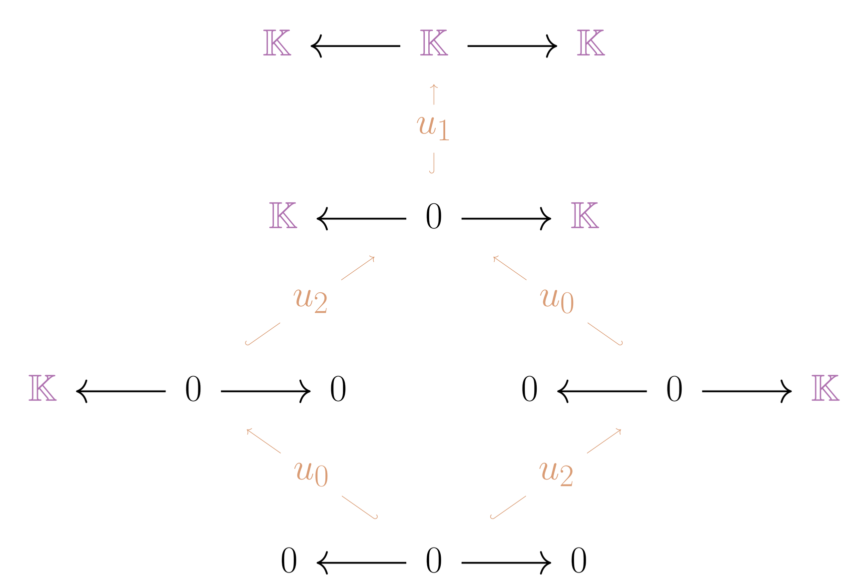

Figure 2. A_3 F-polynomial Example

Subrepresentations of Type A Indecomposables

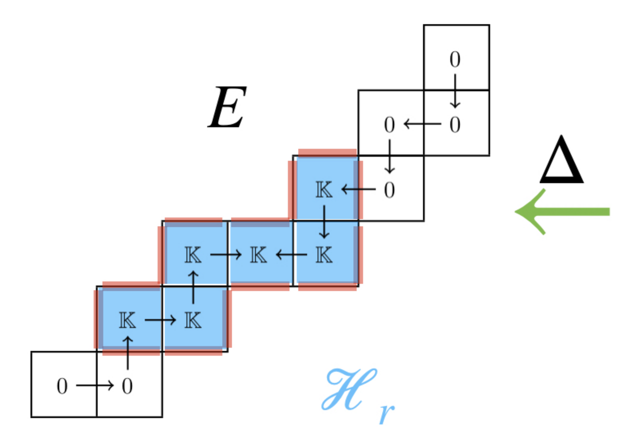

Figure 3. Subrepresentations for $\underline{dim}(M) = (1, \ldots, 1)$

Sub-dimension Vector Conditions

For type A_n quivers Q, the term $u_0^{e_0} \cdots u_{n-1}^{e_{n-1}}$ appears in the F-polynomial for $\underline{d}$ if and only if

1. $0 \leq \underline{e} \leq \underline{d}$
2. $i \to j \implies e_i - e_j \leq \max(d_i - d_j, 0)$

It was proven in [Tran '09] that the same characterization works for Type D_n Quivers with the extra condition that

- Q avoids having too many *type D_n critical arrows* $(d_i, e_i) \to (d_j, e_j)$.

Type D_n critical arrows are $(1,1) \to (2,1)$ or $(2,1) \to (1,0)$.

This condition works for type E_6 quiver representations with all dimension vector entries ≤ 2, leaving $\underline{d} = (1,2,3,2,2,1), (1,2,3,1,2,1)$ as the remaining cases.

For $\underline{d} = (1,2,3,2,2,1)$, a similar characterization can be given:

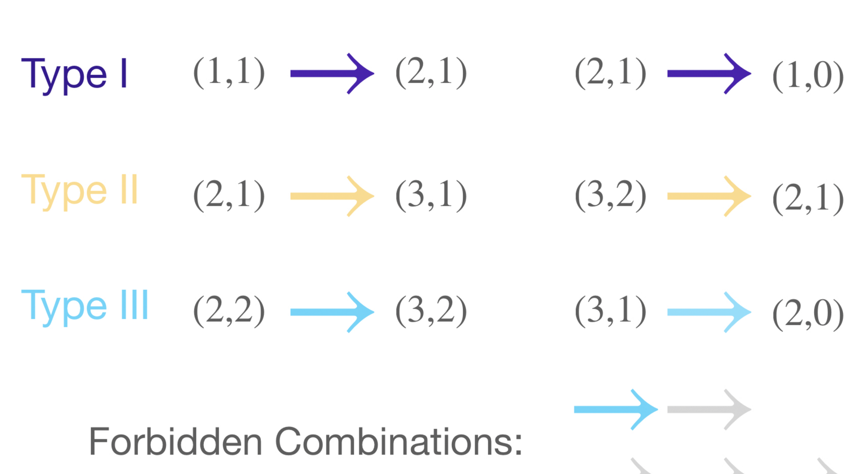

Figure 4. Each of the gray arrows can be any of the 3 types of critical arrows

A_3 F-polynomial Example continued

$$\underline{dim}(M) = (1,1,1)$$

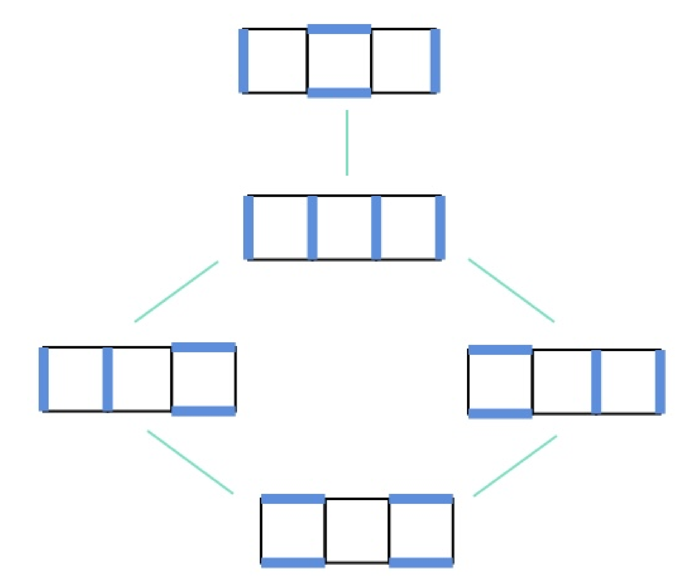

Figure 5. A_3 Perfect Matching Lattice

Symmetric Differences of Perfect Matchings

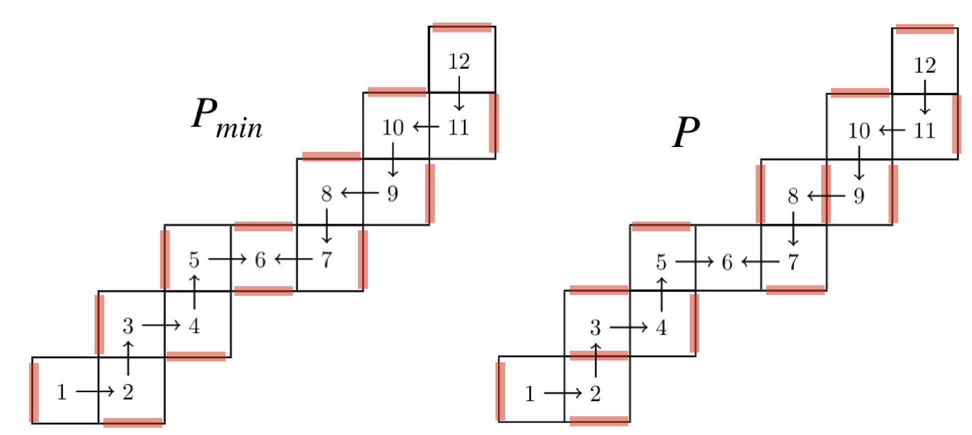

Figure 6. Perfect Matchings

F-polynomial Coefficients of type E_6 quivers

For dimension vectors of indecomposable representations of E_6 quivers aside from $\underline{d} = (1,2,3,2,2,1), (1,2,3,1,2,1)$, **the known combinatorial rule for type D_n works verbatim.**

Conjecture:

Coefficients greater than 1 in the F-polynomial for $\underline{d} = (1,2,3,2,2,1)$ correspond to the following cycles in the mixed dimer configuration.

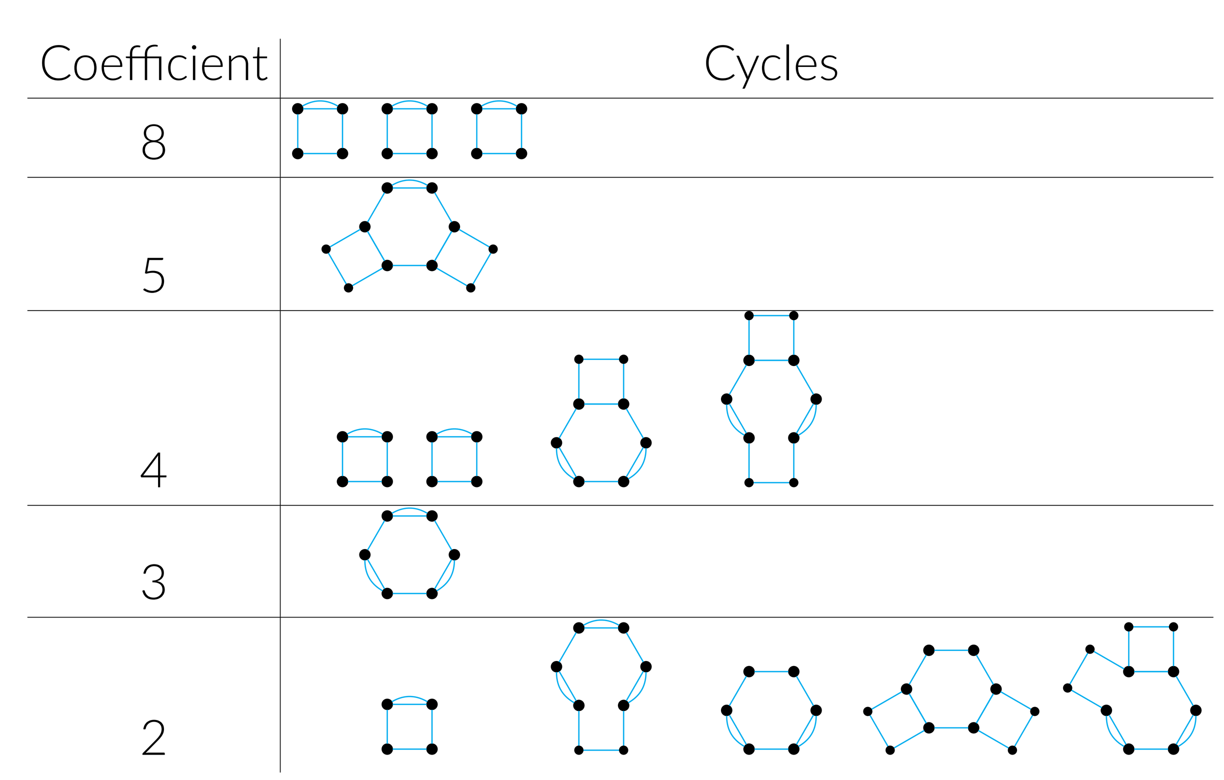

Figure 7

(Mixed) Dimer Models for F-polynomials

For type A_m and D_m quivers, combinatorial rules have been found that construct a planar bipartite graph from the quiver. The F-polynomials are expressed as a weighted sum over (generalized) perfect matchings of this graph called Mixed Dimer configurations.

Constructing Base Graphs for Quivers

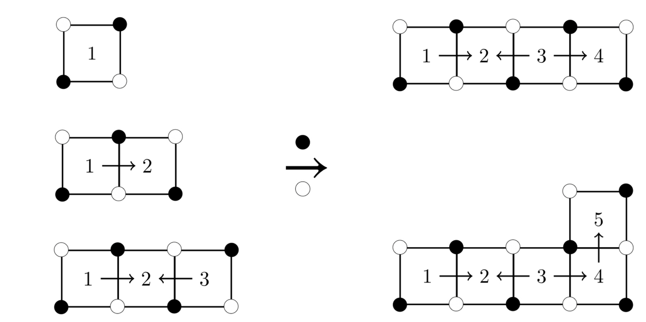

Figure 8. Construction of the planar bipartite "base graph" of an A_5 quiver

Mixed Dimer Configurations

A *Mixed Dimer configuration* for a dimension vector $\underline{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ is a minimal multiset D of edges of the base graph such that every vertex v of a tile i is incident to $\geq d_i$ edges of D.

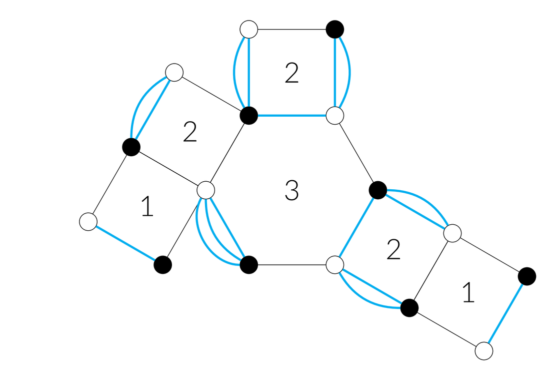

Figure 9. A Mixed Dimer configuration for an E_6 quiver with $\underline{d} = (1,2,3,2,2,1)$.

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Want to know more about the project?

The final report for the project can be found here: