Multitriangulations of the Half-Cylinder

Saskia Solotko, Katherine Tung, Philip Yang, Yuchong Zhang
Mentor: Pasha Pylyavskyy
TA: Joe McDonough
UMN Twin Cities REU

January 14, 2025

k-triangulation definition

Definition (k-triangulation of a convex polygon)

A k-triangulation of a convex n-gon is a maximal set of edges such that no $k+1$ pairwise intersect.

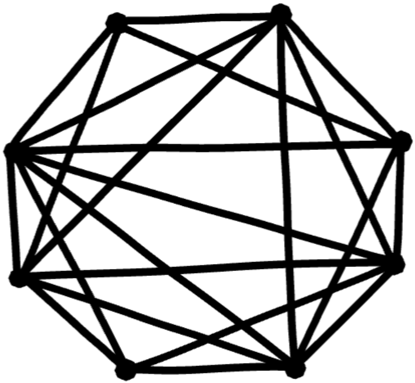


Figure 1: A 2-triangulation of the 8-gon. Note there are no 3-crossings.

k-stars: relevant for k-triangulations

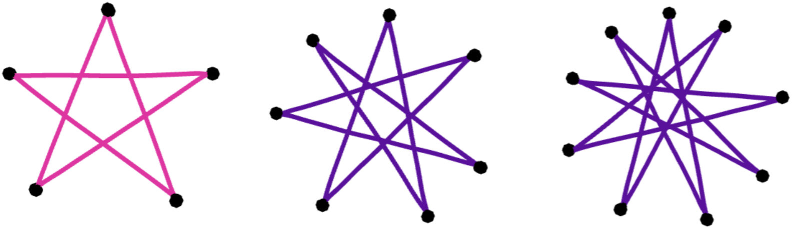


Figure 2: A 2-star, a 3-star, and a 4 star

A k-star consists of $2k+1$ vertices and $2k+1$ edges of length k.

k-triangulations as complexes of k-stars

Theorem (Pilaud-Santos '04)

Any k-triangulation of the n-gon contains exactly $n - 2k$ k-stars, $k(n - 2k - 1)$ k edges of length $> k$, and $k(2n - 2k - 1)$ total edges.

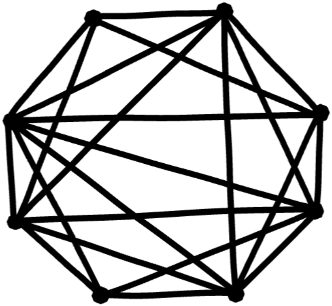


Figure 3: A 2-triangulation of the 8-gon has 6 edges of length > 2 and 22 total edges.

Definition

A k-relevant angle of a k-triangulation consists of edges of length $\geq k$ and has no intermediate "bisector" edges.

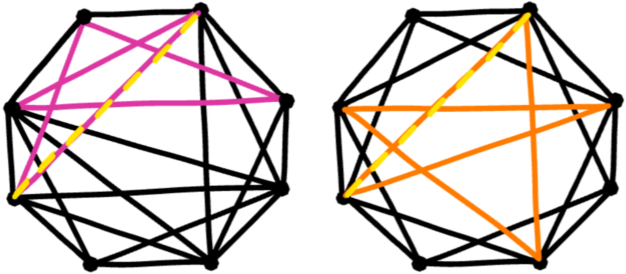
Theorem (Pilaud-Santos '04)

In a k-triangulation, every k-relevant angle is contained in a unique k-star:

- *length $> k$ (relevant) edges: in exactly 2 k-stars*
- *length $= k$ (boundary) edges: in exactly 1 k-star*
- *length $< k$ (irrelevant) edges: in exactly 0*

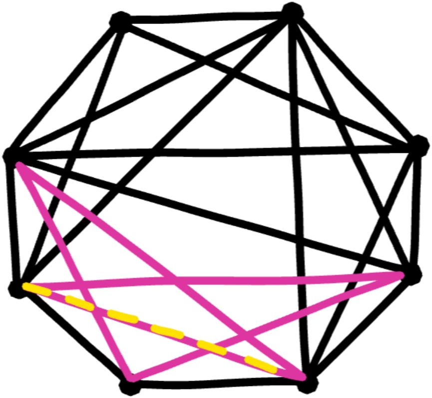
length $> k$ edge: in exactly 2 k-stars

Ex: length 3 edge, $k = 2$-triangulation



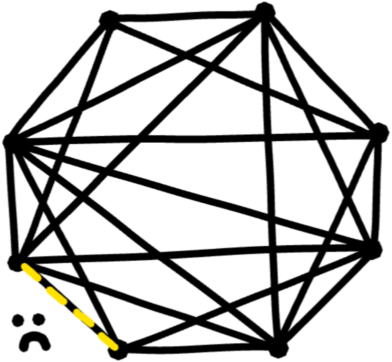
length $= k$ edge: in exactly 1 k-star

Ex: length 2 edge



length $< k$ edge: in 0 k-stars

Ex: length 1 edge



Multitriangulations of the Half-Cylinder

Let $\mathcal{S}$ denote a surface and $\overline{\mathcal{S}}$ its universal cover with natural projection $p : \overline{\mathcal{S}} \to \mathcal{S}$.

Definition

A k-triangulation T on a surface S with marked points on boundaries is a maximal set of edges such that $p^{-1}(T)$ is $(k+1)$-crossing free.

For convenience, we say $p^{-1}(T)$ is a k-triangulation of $\overline{\mathcal{S}}$ denoted $\overline{T}$.

Figure 4: A 2-triangulation of the half cylinder with 3 marked points on a single boundary, i.e. "(3+0) annulus"

Results for 2-triangulations on the $(n + 0)$-annulus

Theorem (STYZ '24+)

2-triangulations of the half cylinder are complexes of 2-stars.

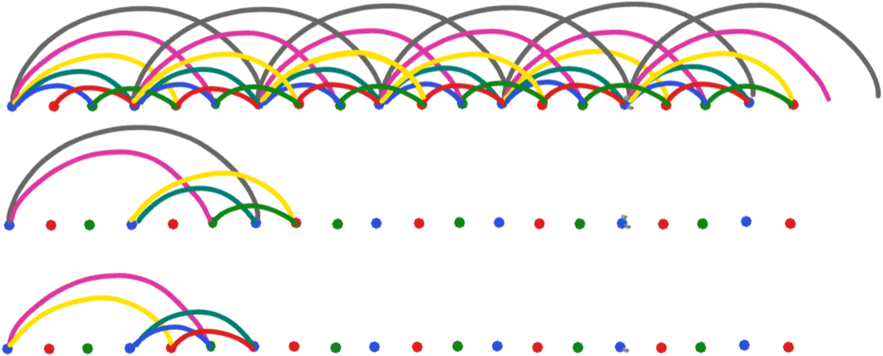


Figure 5: A 2-triangulation on the universal cover of the $(3 + 0)$-annulus and the stars that comprise it

Theorem (STYZ 24)

There is a bijection between 2-triangulations of the $(n+0)$-annulus and 2-triangulations of the $4n$-gon invariant under rotation by π/n.

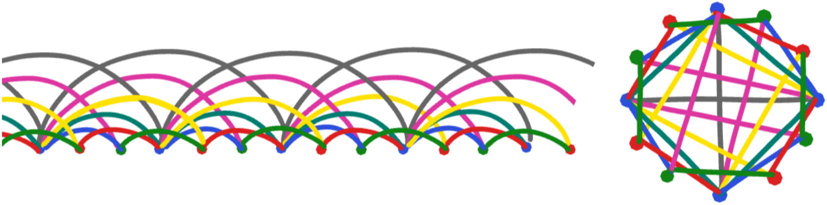


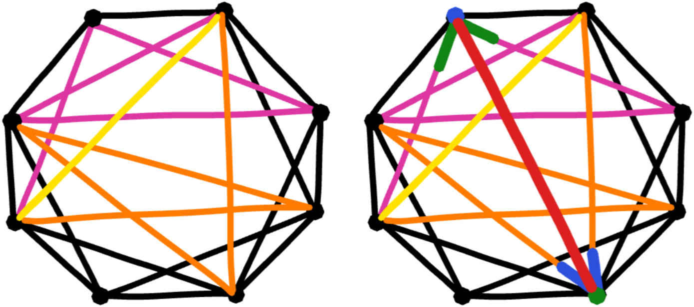
Figure 6: A 2-triangulation on the universal cover of the $(3+0)$-annulus and the corresponding 2-triangulation of the $4n$-gon

Corollary

For $k=2$, any k-triangulation of the $(n+0)$-annulus contains exactly $n-1$ k-stars, $k(n-1)$ k-relevant edges, and $k(2n-1)$ edges.

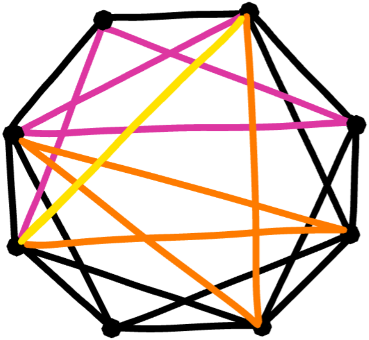
Lemma (Pilaud-Santos '04)

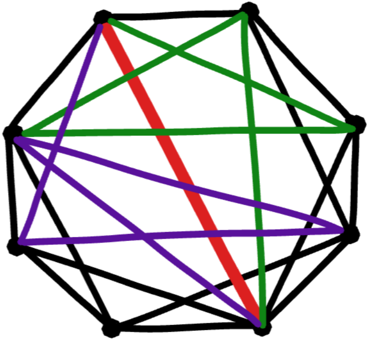
A pair of stars in T has a unique bisector edge.



Theorem (Pilaud-Santos '04)

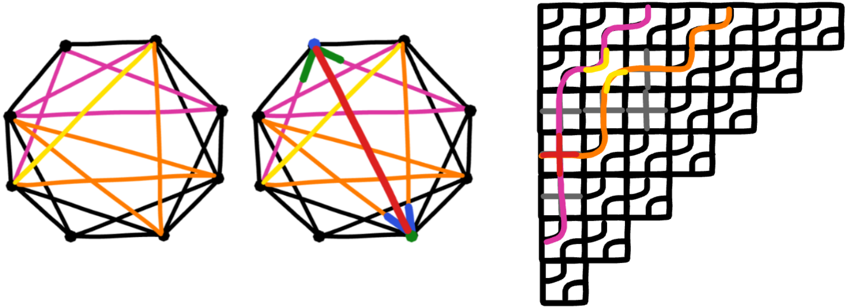
For any k-relevant edge $e \in T$ *there is a unique flip edge* f *such that* $(T \setminus e) \cup \{f\}$ *is a k-triangulation.*





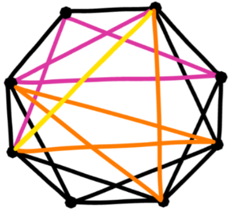
Theorem (Stump '11)

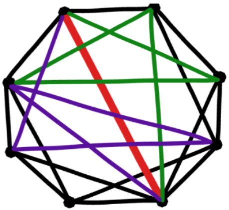
k-triangulations of the n-gon are in canonical bijection with reduced pipe dreams for $\pi_{n,k}$.

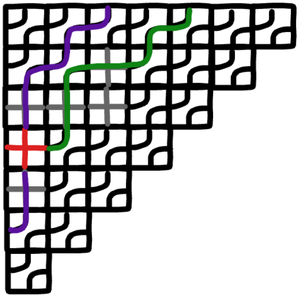


Theorem (Stump '11)

k-triangulations of the n-gon are in canonical bijection with reduced pipe dreams for $\pi_{n,k}$.







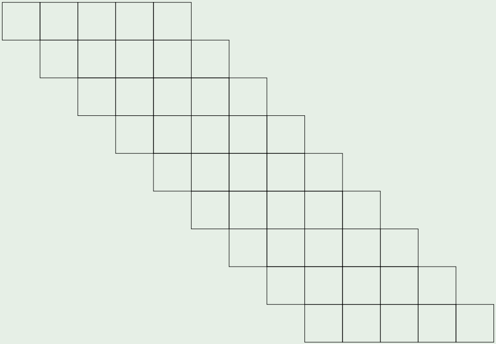
Cylindrical Polyominoes

Definition (STYZ '24+)

A *cylindrical polyomino* $\mathbb{Y}$ *of type* (n, k) is an infinite skew Young diagram (reflected along the y-axis) with a box centered at every point in $\{(i, j) \in \mathbb{Z}^2 \mid k \leq j - i \leq kn\} \subseteq \mathbb{Z}^2$.

Example

$$(n, k) = (3, 2)$$



Cylindrical Pipe Dreams

Definition (STYZ '24+)

A *cylindrical pipe dream of type* (n, k) is a tiling of the cylindrical polyomino $\mathbb{Y}$ of type (n, k) by four kinds of pieces  ,  ,  , and  such that

- The pipe dream is *n-cylindrical*, that is, all the piles at the position $(i + rn, j + rn)$ for arbitrary $r \in \mathbb{Z}_{\geq 0}$ is the same as a pile at the position (i, j);
- There is a  tiled at the position $(i, k - i)$ for all $i \in \mathbb{Z}_{\geq 0}$;
- For every pipe, the number of  ,  or  it passes through is $2k + 1$;
- Each pipe connects $(i, kn - i)$ and $(i + kn, -i)$ for some $i \in \mathbb{Z}_{\geq 0}$;
- For every pair of pipes, they do not cross twice, that is, the number of  piles both pipes pass through is no more than 1;
- There is exactly one  in each successive n rows, tiled at the position $(i, kn - i)$ for some $i \in \mathbb{Z}_{\geq 0}$.

Cylindrical Pipe Dreams: Examples

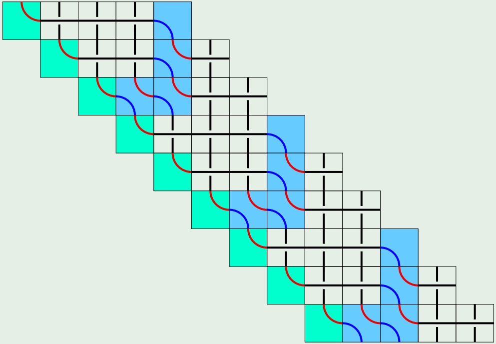
Definition (STYZ '24+)

A *cylindrical pipe dream of type* (n, k) is a tiling using  ,  ,  , and  such that

- n-periodic;
-  tiled at left boundary;
- each pipe has k "";
- each pipe horizontally and vertically crosses kn;
- two pipes don't cross twice;
- one  in each n-period, tiled at right boundary.

Example

$$(n, k) = (3, 2)$$



Cylindrical Pipe Dreams: Correspondence with Multi-Triangulations

Theorem (STYZ '24+)

For $k = 2$, there is a bijection between k-triangulation of the $(n + 0)$-annulus and cylindrical pipe dreams of type (n, k):

- *a length k edge connects i and j: tile*  *at (i, j)*
- *a length kn edge connects i and j: tile*  *at (i, j)*
- *an edge of length between k and kn connects i and j: tile*  *at (i, j)*
- *tile a*  *in every other* 

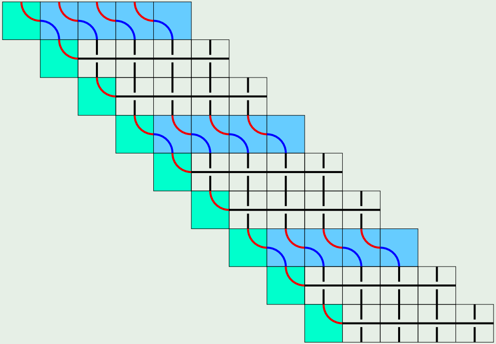
Moreover, in this bijection, each pipe corresponds to a k-star on $\overline{T}$.

Conjecture (STYZ '24+)

The previous theorem can be generalized to arbitrary k.

Cylindrical Pipe Dreams: Correspondence with Multi-Triangulations

Example

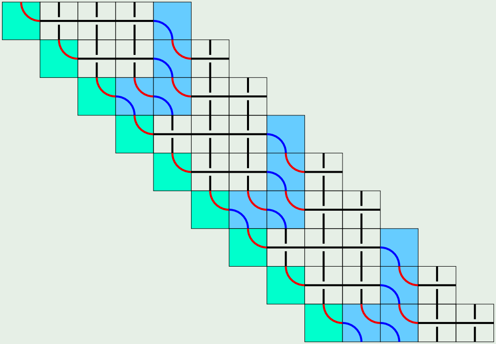


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

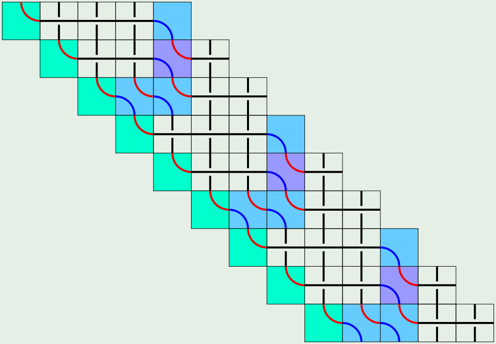


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

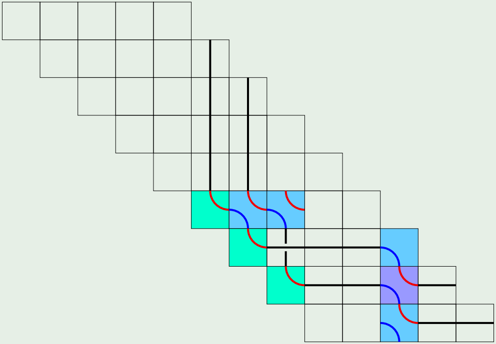


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

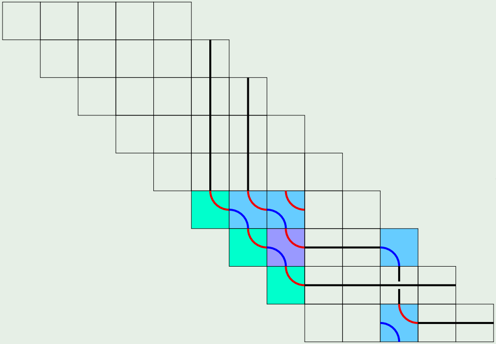


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

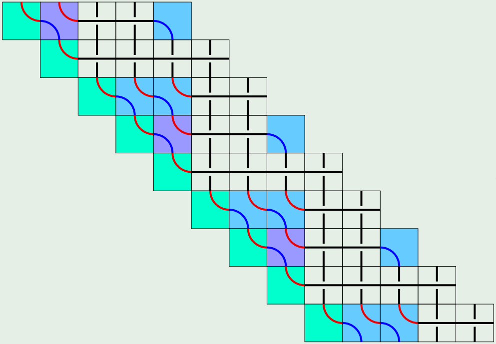


Cylindrical Pipe Dreams: Regular Pipe Flips

Definition (STYZ '24)

Regular pipe flip: A flip for  . Select the two pipes passing through  , identify their intersection  , mutate from  to  for every translation.

Example

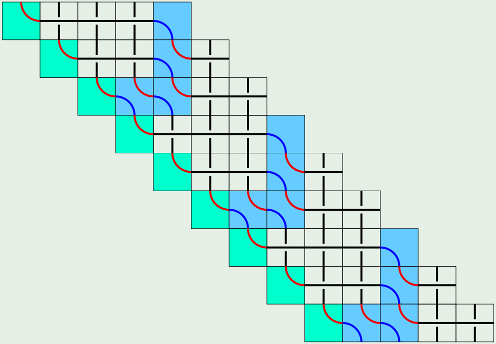
 .

Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example

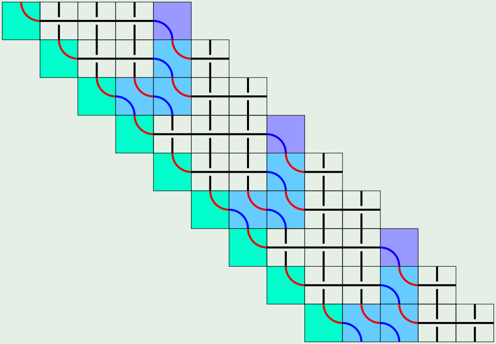


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example

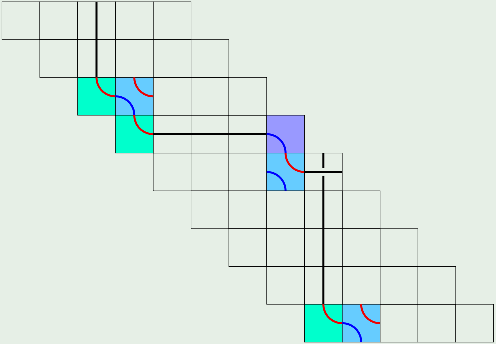


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example

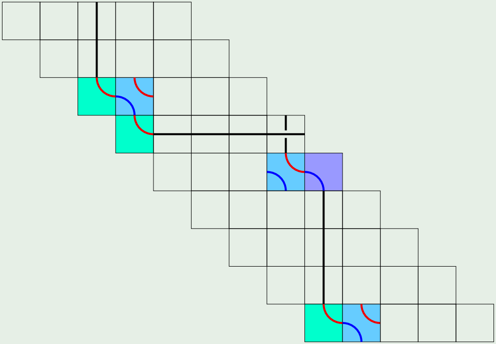


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example

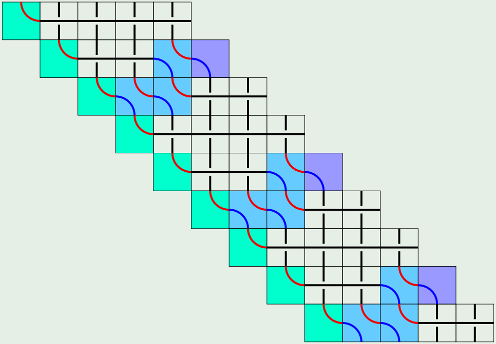


Cylindrical Pipe Dreams: Exceptional Pipe Flips

Definition (STYZ '24+)

Exceptional pipe flip: A flip for  . Select the pipe passing through  and its "$+kn$" translation, mutate from  to their intersection  for every translation.

Example



Cylindrical Pipe Dreams: Flip Property

Theorem (STYZ '24+)

Cylindrical pipe dreams of type (n, k) have flip property.

Theorem (STYZ '24+)

2-*triangulations of* $(n + 0)$ *annulus have flip property.*

Conjecture (STYZ '24+)

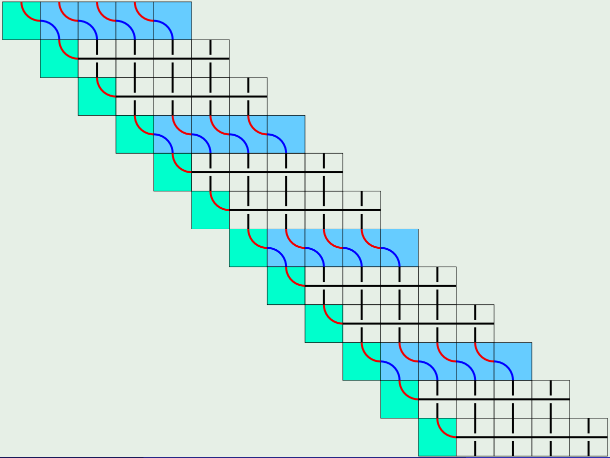
k-triangulations of $(n + 0)$ annulus have flip property.

Cylindrical Pipe Dreams: Regular Cylindrical Pipe Dreams

Definition (STYZ '24+)

Regular cylindrical pipe dream: for every  , there exists a  at the same row.

Example

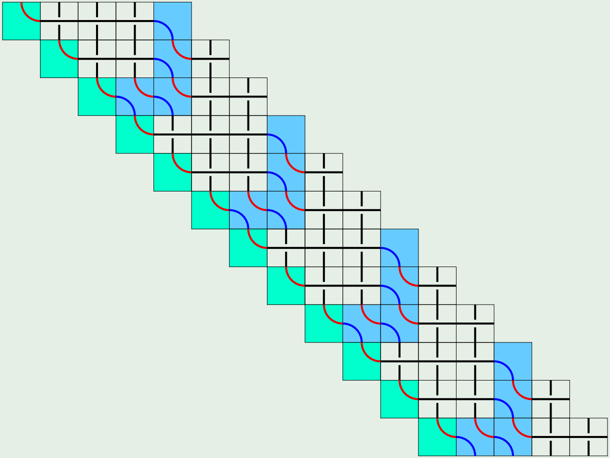


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

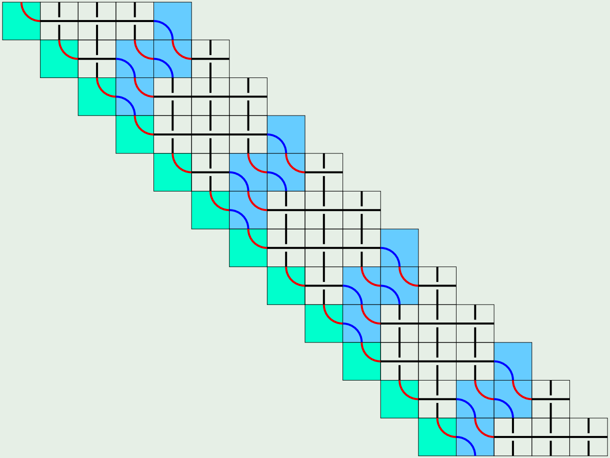


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

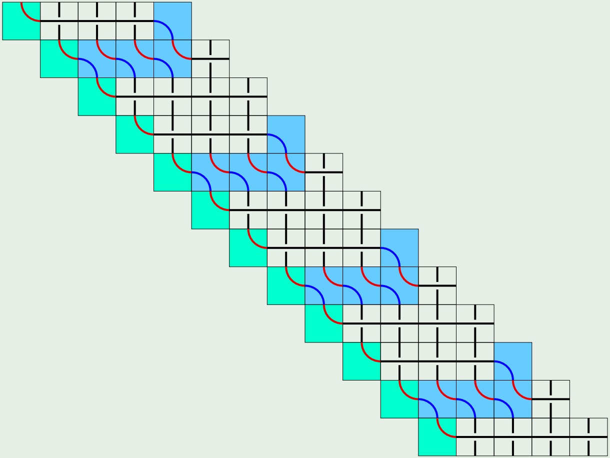


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

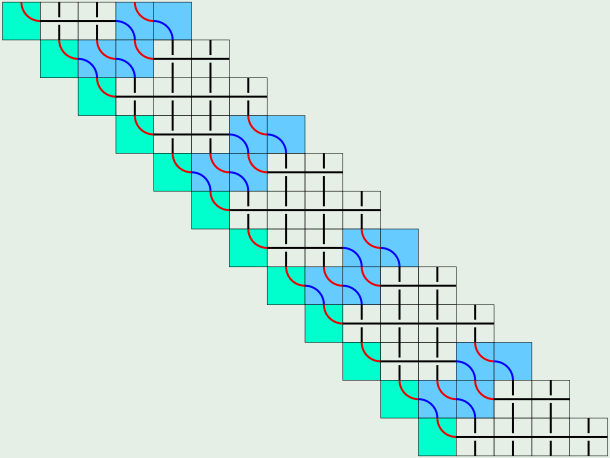


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

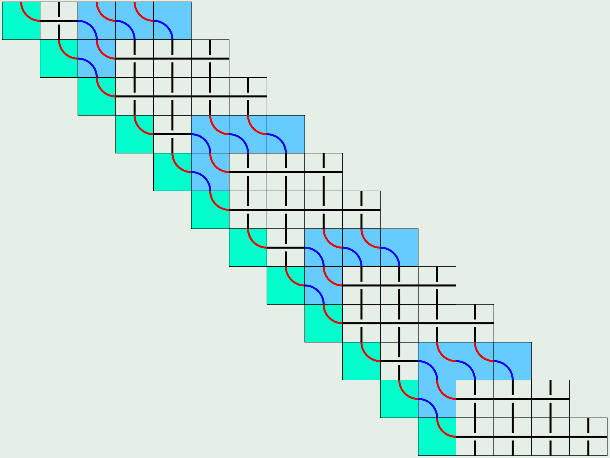


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

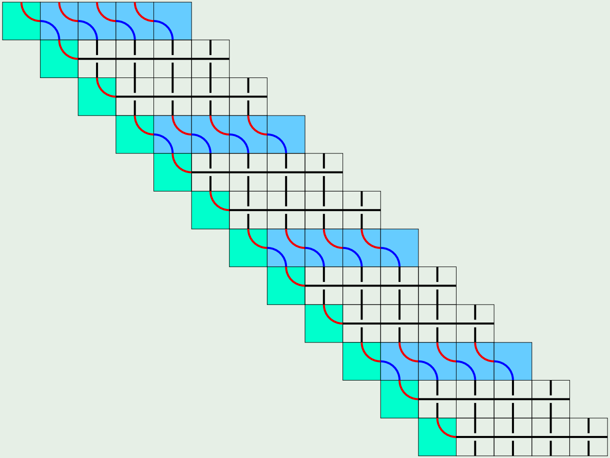


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Lemma (STYZ '24+)

Cylindrical pipe dreams can be flipped to regular cylindrical pipe dreams.

Example

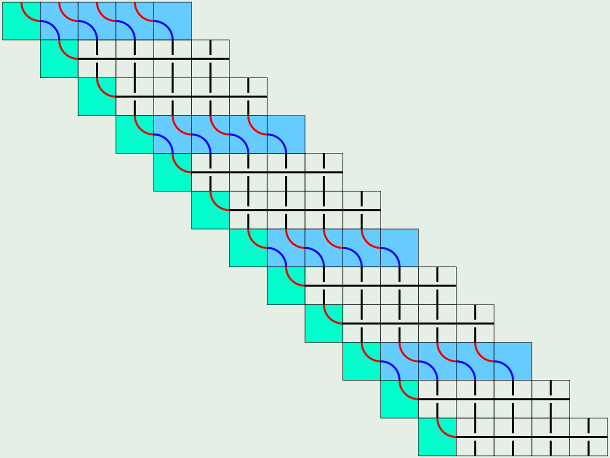


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Theorem (STYZ '24+)

Cylindrical pipe dreams have connected flip graph.

Example

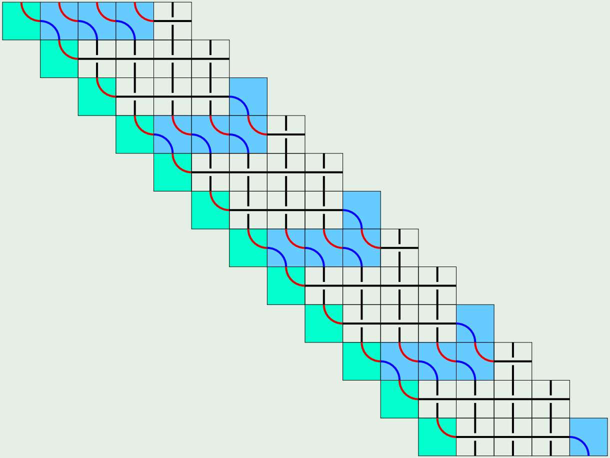


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Theorem (STYZ '24+)

Cylindrical pipe dreams have connected flip graph.

Example

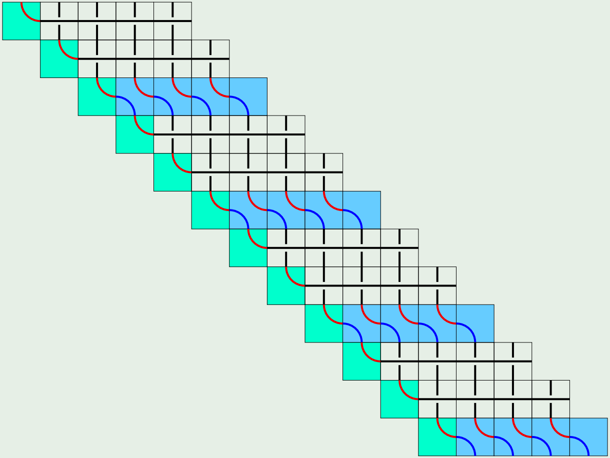


Cylindrical Pipe Dreams: Connectedness of Flip Graph

Theorem (STYZ '24+)

Cylindrical pipe dreams have connected flip graph.

Example



Cylindrical Pipe Dreams: Connectedness of Flip Graph

Theorem (STYZ '24+)

2-triangulations of $(n+0)$ annulus have connected flip graph.

Conjecture (STYZ '24+)

k-triangulations of $(n+0)$ annulus have connected flip graph.

Acknowledgements

This project was supported in large part by a grant from the D.E. Shaw group, and also by NSF grant DMS-2053288. It was supervised as part of the University of Minnesota School of Mathematics Summer 2024 REU program. The authors would also like to thank Pasha Pylyavskyy and Joe McDonough for their invaluable guidance and expertise throughout the program. The authors also thank Kieran Favazza and Molly MacDonald for insightful discussion on the topic of this report.