A Combinatorial Characterization of Newton Polytopes of Dual Schubert Polynomials

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Dual Schubert Polynomials

In the (strong) Bruhat order on the symmetric group S_n, let the edge $u \lessdot ut_{ab}$ have weight

$$m(u \lessdot ut_{ab}) \coloneqq x_a + x_{a+1} + \cdots + x_{b-1},$$

and let the saturated chain $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$ have weight

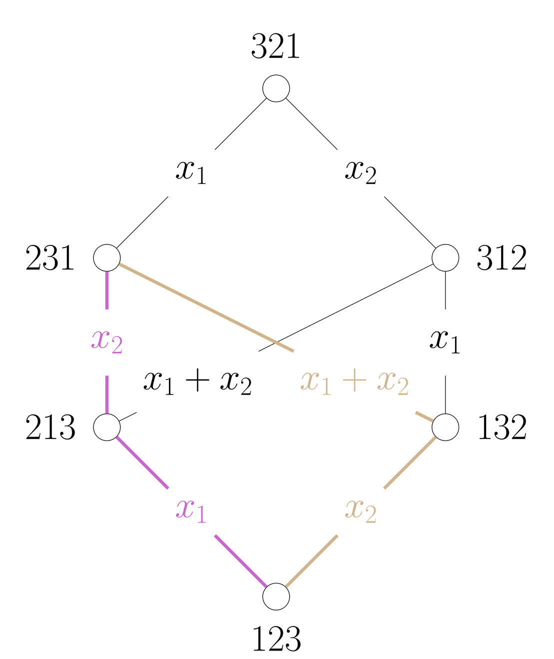
$$m_C \coloneqq m(u_0 \lessdot u_1) m(u_1 \lessdot u_2) \cdots m(u_{\ell-1} \lessdot u_\ell).$$

Definition [BGG73, PS09]

For $w \in S_n$, the *dual Schubert polynomial* D^w is defined by

$$D^w(x_1, \ldots, x_{n-1}) \coloneqq \frac{1}{\ell(w)!} \sum_C m_C(x_1, \ldots, x_{n-1}),$$

where $\ell(w)$ denotes the Coxeter length of w, and the sum is over all saturated chains C from id to w.



$$D^{231} = \frac{1}{2!}(x_1 x_2 + x_2(x_1 + x_2)).$$

Newton Polytopes

- For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let x^α denote the monomial $x^\alpha \coloneqq x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{R}[x_1, \ldots, x_n]$. We call α the *exponent vector* of x^α.
- Let $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial. The *support* of f, denoted $\mathrm{supp}(f)$, is the set of exponent vectors α of the nonzero terms of f.

Definition

The *Newton polytope* of a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$, denoted $\mathrm{Newton}(f)$, is the convex hull of $\mathrm{supp}(f)$ in $\mathbb{R}^n$.

- [MTY19] A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ has *saturated Newton polytope (SNP)* if $\mathrm{supp}(f)$ consists of all integer points in $\mathrm{Newton}(f)$.
- Example: Since $D^{231} = x_1 x_2 + 0.5x_2^2 = x^{(1,1)} + 0.5x^{(0,2)}$, $\mathrm{Newton}(D^{231})$ is the line segment from $(1,1)$ to $(0,2)$ in $\mathbb{R}^2$. There are no integer points on this line segment besides the endpoints, so D^{231} has SNP.
- Non-example: the polynomial $f = x_1^2 + x_2x_3 + x_2x_4 + x_3x_4$ has SNP but f^2 does not.

Previous Results on SNP

Many polynomials with algebraic combinatorial significance are now known to have SNP, such as

- Schur polynomials [Rad52],
- resultants [GKZ90],
- cycle index polynomials and Reutenauer's symmetric polynomials and Stembridge's symmetric polynomials and symmetric Macdonald polynomials [MTY19],
- key polynomials and Schubert polynomials [FMD18], and
- double Schubert polynomials [CRMM23].

Work of Huh, Matherne, Mészáros, and St. Dizier [HMMSD22] proved Lorentzian-ness, which implies SNP, for dual Schubert polynomials. We offer the first elementary proof of SNP for dual Schuberts by fully characterizing their supports.

Main Theorem (ATZ '24)

The support of the dual Schubert polynomial D^w is

$$\mathrm{supp}(D^w) = \sum_{(a,b) \in \mathrm{Inv}(w)} \{e_a, e_{a+1}, \ldots, e_{b-1}\},$$

where the right-hand side is a Minkowski sum of sets of elementary basis vectors. The sum is over pairs of indices (a,b) for which there is an inversion in w.

Proof Outline

We say that D^w has *single chain Newton polytope (SCNP)* if there exists a saturated chain C in the interval $[\mathrm{id}, w]$ such that $\mathrm{supp}(m_C) = \mathrm{supp}(D^w)$. Such a saturated chain C is called a *dominant chain* of the interval $[\mathrm{id}, w]$. We show that for each $w \in S_n$, there exists a dominant chain, so D^w has SCNP. We also show that SCNP implies SNP, completing the proof of SNP. It turns out that any dominant chain has weight $\prod_{(a,b) \in \mathrm{Inv}(w)}(x_a + x_{a+1} + \cdots + x_{b-1})$, yielding the characterization in our theorem.

Corollaries of the Main Theorem

Corollary 1. D^w has SNP.

The *generalized permutahedron* $P_n^z(\{z_I\})$ associated to the collection of real numbers $\{z_I\}$ for $I \subseteq [n]$, is given by

$$P_n^z(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \sum_{i=1}^n t_i = z_{[n]} \right\}.$$

Corollary 2. The Newton polytope of D^w is a generalized permutahedron.

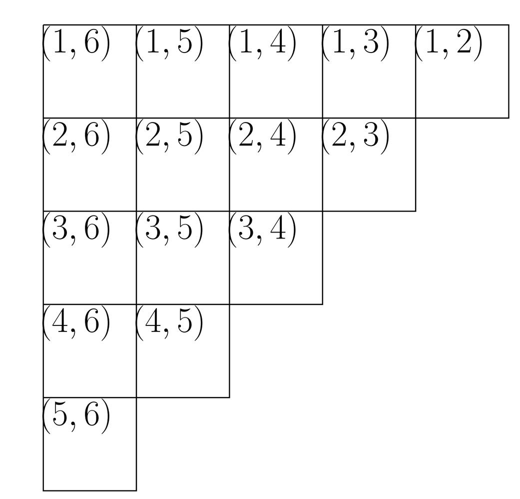
Proposition. ([Mur03, Theorem 4.15], [HMMSD22]) A homogeneous polynomial f has M-convex support if and only if f has SNP and $\mathrm{Newton}(f)$ is a generalized permutahedron.

Corollary 3. D^w has M-convex support.

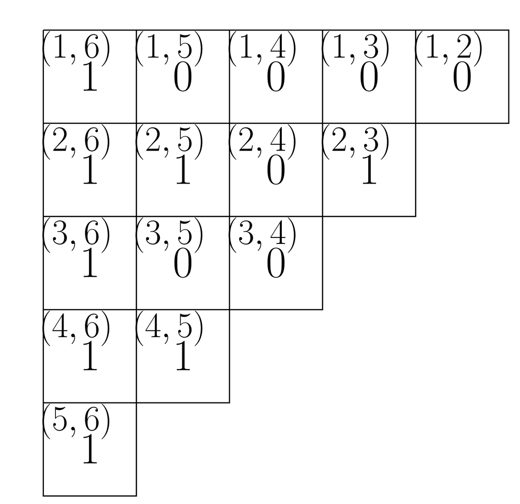
Corollary 4 (ATZ '24). The vertices of $\mathrm{Newton}(D^w)$ are $\{\alpha \in \mathbb{Z}^{n-1}_{\geq 0} \mid x^\alpha$ has coeff. 1 in $\prod_{(a,b) \in \mathrm{Inv}(w)} (x_a + x_{a+1} + \cdots + x_{b-1})\}$.

Characterizing Vertices of $\mathrm{Newton}(D^w)$

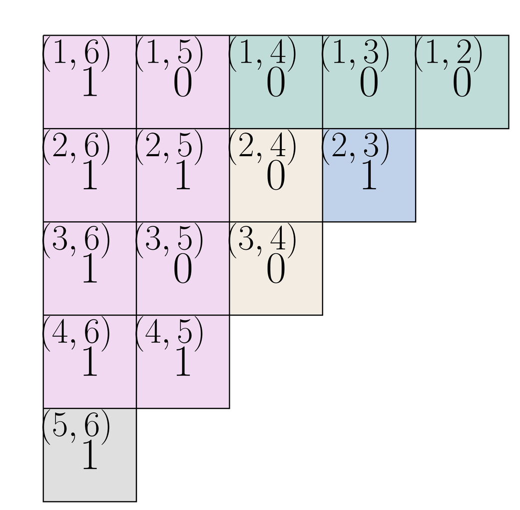
1. Construct a Young diagram of staircase shape $(n-1, n-2, \ldots, 1)$, and label the boxes by the following pairs of inversions: in the ith row of the diagram for $1 \leq i \leq n-1$, label the boxes from left to right by $(i, n), (i, n-1), \ldots, (i, i+1)$.
2. In each box, write a 1 if the inversion pair is in $\mathrm{Inv}(w)$, and a 0 otherwise.
3. Construct $\frac{1}{n+1}\binom{2n}{n}$ tilings of the staircase by $n-1$ rectangles.
4. For each tiling, sum the entries of each rectangle and write the sum at the bottom right corner of the rectangle. Reading the summands from top to bottom gives a vertex of $\mathrm{Newton}(D^w)$.



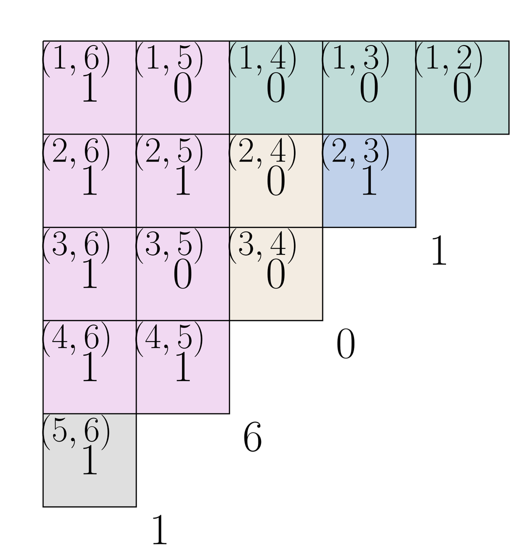
Step 1: Build a staircase Young diagram with $n = 6$.



Step 2: When $w = 253641$, the above boxes are filled with 1's.



Step 3: We consider a tiling by $n-1$ rectangles.



Step 4: We find that $\mathrm{Newton}(D^{253641})$ has vertex $(0, 1, 0, 6, 1)$.

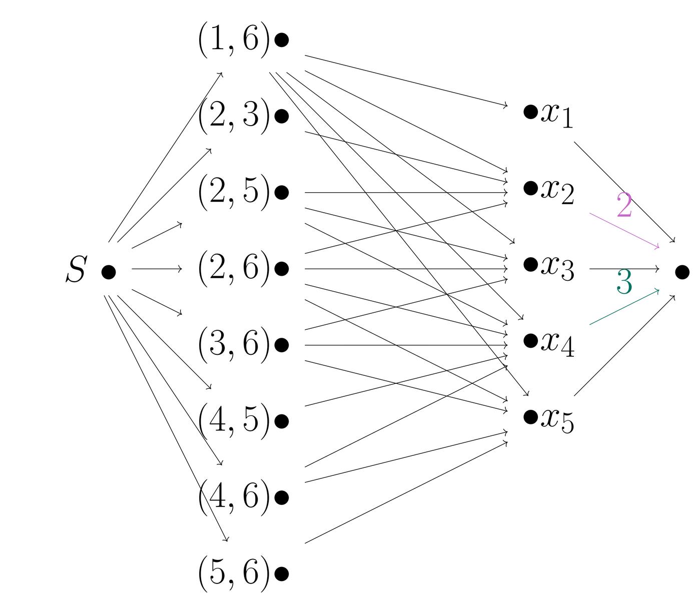
The Vanishing Problem for D^w

Given a Schubert polynomial $\mathfrak{S}_w = \sum_{\alpha \in \mathbb{Z}_{\geq 0}} c_{\alpha,w} x^\alpha$ for $w \in S_n$, Adve, Robichaux, and Yong give a polynomial-time algorithm to determine, given some α, whether $c_{\alpha,w} = 0$ [ARY21]. We prove an analogous result for dual Schubert polynomials.

Theorem (ATZ '24++)

For $w \in S_n$ and $\alpha \in \mathbb{Z}^{n-1}_{\geq 0}$, there is an $O(n^4)$ algorithm to determine whether $\alpha \in \mathrm{supp}(D^w)$.

The idea is to construct a certain bipartite graph with inversions of w as left vertices and variables of D^w as right vertices. The vanishing problem reduces to determining if a flow of $\ell(w)$ units can pass from left to right. Using Dinic's algorithm, this max-flow problem has complexity $O(n^4)$.



The network testing the term $x_1 x_2^2 x_3 x_4^3 x_5$ in D^{253641}.

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Acknowledgements

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