Newton polytopes of dual Schubert polynomials
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Yuchong Zhang
University of Michigan

Collaborators: Serena An, Katherine Tung
Mentor: Shiyun Wang
TA: Meagan Kenney

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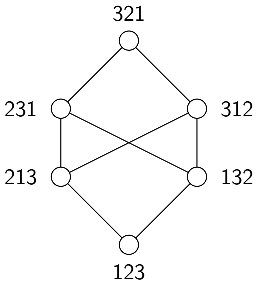
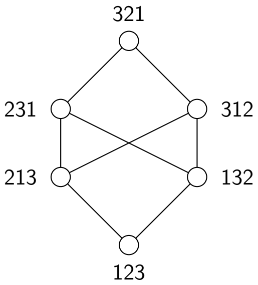
The Symmetric Group S_n

- S_n: permutations of $\{1, \ldots, n\}$
- Bottom element $1\ 2 \cdots n$
- Top element $n\ (n-1) \cdots 1$
- $\mathrm{Inv}(u)$: the set of all inversions (a, b) of u such that $a < b$ and $u(a) > u(b)$



The (Strong) Bruhat Order of S_n

- $\ell(u)$: count of inversions in u
- t_{ab} swaps the numbers in positions a, b (not values a, b)
- Covering relation: $u \lessdot v$ if $v = ut_{ab}$ and $\ell(v) = \ell(u) + 1$
- Interval $[u, w] : \{v \mid u \leq v \leq w\}$



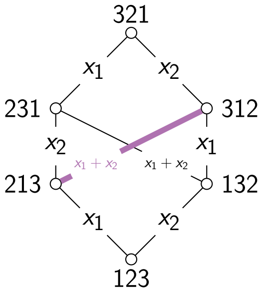
Edge Weights

Definition

For $u \lessdot v$ and $v = ut_{ab}$, the *weight* $m(u \lessdot v)$ is $x_a + x_{a+1} + \cdots + x_{b-1}$.

Example

Since $312 = 213t_{13}$, we have $m(213 \lessdot 312) = x_1 + x_2$.



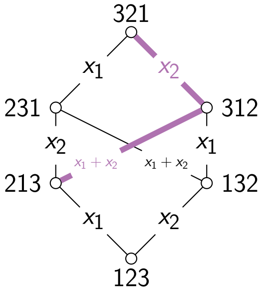
Chain Weights

Definition

Let $u_0 \leq u_\ell$ and $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$ be a saturated chain of $[u_0, u_\ell]$. Define the *weight* $m_C(x)$ of the chain C by $\prod_{i=1}^{\ell} m(u_{i-1} \lessdot u_i)$.

Example

For $[213, 321]$, the weight of the saturated chain $213 \lessdot 312 \lessdot 321$ is $(x_1 + x_2) \cdot x_2$.



Skew Dual Schubert Polynomials

Definition (Postnikov–Stanley '09)

For $u \leq w$, the skew dual Schubert polynomial or *Postnikov–Stanley polynomial* D_u^w is defined by

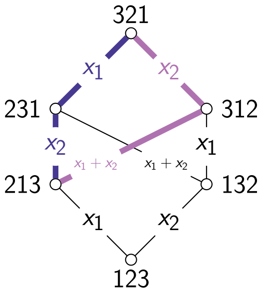
$$D_u^w = \frac{1}{(\ell(w) - \ell(u))!} \sum_{C: u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = w} m_C(x).$$

Example

$D_{213}^{321} = \frac{1}{2!}(x_1 x_2 + (x_1 + x_2) \cdot x_2)$

Definition (Bernstein–Galfand–Galfand '73)

When $u = \mathrm{id}$, D_u^w is called a *dual Schubert polynomial*.

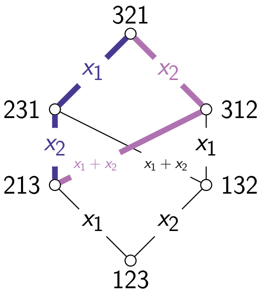


Saturated Newton Polytope (SNP)

For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Definition

The *support* $\mathrm{supp}(f)$ of $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha$ is the set of vectors α such that $c_\alpha \neq 0$.
The *Newton polytope* $\mathrm{Newton}(f)$ of f is the convex hull of $\mathrm{supp}(f)$ in $\mathbb{R}^n$.



Example

$D^{321}_{213} = x_1 x_2 + \frac{1}{2} x_2^2 = x^{(1,1)} + \frac{1}{2} x^{(0,2)}$
$\mathrm{Newton}(D^{321}_{213})$ is the segment from $(1,1)$ to $(0,2)$ in $\mathbb{R}^2$

Definition (Monical–Tokcan–Yong '19)

f has *Saturated Newton Polytope (SNP)* if $\mathrm{supp}(f)$ is the set of integer points in $\mathrm{Newton}(f)$.

SNP in Algebraic Combinatorics

Theorem (Rado '52)

Schur polynomials have SNP.

Theorem (Fink–Mézśaros–St. Dizier '18)

Key polynomials and Schubert polynomials have SNP.

Theorem (Monical–Tokcan–Yong '19)

Cycle index polynomials, Reutenauer's symmetric polynomials, Stembridge's symmetric polynomials, and symmetric Macdonald polynomials have SNP.

Theorem (Huh–Matherne–Mészáros–St. Dizier '19; An–Tung–Z. '24)

Dual Schubert polynomials have SNP.

Chain Weights have SNP

Definition (Postnikov–Stanley '09)

$$D_u^w = \frac{1}{(\ell(w) - \ell(u))!} \sum_{C: u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = w} m_C(x).$$

Proposition (An–Tung–Z. '24)

Any product of linear factors in $x_1, \ldots, x_n$ with all coefficients nonnegative has SNP.

Single-Chain Newton Polytope (SCNP)

Definition (An–Tung–Z. '24)

D_u^w has *single-chain Newton polytope (SCNP)* if there exists a saturated chain C in the interval $[u, w]$ such that

$$\mathrm{supp}(m_C) = \mathrm{supp}(D_u^w).$$

We call such a C a *dominant chain* of the interval $[u, w]$.

Proposition (An–Tung–Z. '24)

If D_u^w has SCNP, then D_u^w has SNP.

Examples and Nonexamples for SCNP

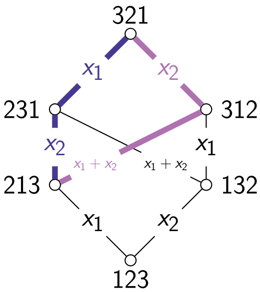
Example

$D^{321}_{213} = \frac{1}{2!}(x_1x_2 + (x_1 + x_2) \cdot x_2)$ has SCNP

$$C := (213 \lessdot 312 \lessdot 321)$$

$$m_C = (x_1 + x_2) \cdot x_2$$

$$\mathrm{supp}(m_C) = \mathrm{supp}(D^{321}_{213})$$



Example

D^{4231}_{1324} does not have SCNP

Dual Schubert Polynomials have SCNP

Definition (An–Tung–Z. '24)

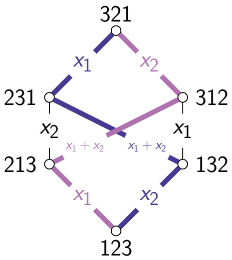
$$u = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_\ell = w$$

is called *greedy* in $[u, w]$ if for all $i \in [\ell]$:
writing $w_{i-1} t_{ab} = w_i$ for $a < b$, there does not exist $w'_{i-1} \lessdot w_i$ with $w'_{i-1} \in [u, w]$ such that

$$w'_{i-1} t_{ab'} = w_i \text{ for } b' > b, \text{ or } \; w'_{i-1} t_{a'b} = w_i \text{ for } a' < a.$$

Example

In $[123, 321]$,
$123 \lessdot 132 \lessdot 231 \lessdot 321$ is greedy
$123 \lessdot 213 \lessdot 312 \lessdot 321$ is also greedy
$123 \lessdot 213 \lessdot 231 \lessdot 321$ is not greedy



Dual Schubert Polynomials have SCNP, cont.

Definition (An–Tung–Z. '24)

The *global weight* $\mathrm{GW}(w)$ of $w \in S_n$ is

$$\mathrm{GW}(w) = \prod_{(a,b)\in \mathrm{Inv}(w)} (x_a + x_{a+1} + \cdots + x_{b-1}).$$

Example

$\mathrm{Inv}(231) = \{(1,3),(2,3)\}$, $\mathrm{GW}(231) = (x_1 + x_2) \cdot x_2$

Theorem (An–Tung–Z. '24)

For all $w \in S_n$, the dual Schubert polynomial D^w has SCNP. Moreover, every greedy chain of $[\mathrm{id}, w]$ is a dominant chain of D^w, and

$$\mathrm{supp}(D^w) = \mathrm{supp}(\mathrm{GW}(w)) = \sum_{(a,b)\in \mathrm{Inv}(w)} \{e_a, e_{a+1}, \ldots, e_{b-1}\}.$$

Newton Polytopes as Generalized Permutahedra

A *generalized permutahedron* $P_n^z(\{z_I\})$, parameterized by collections of real numbers $\{z_I\}$ for $I \subseteq [n]$, is given by

$$P_n^z(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \sum_{i=1}^{n} t_i = z_{[n]} \right\}.$$

Theorem (Postnikov '05)

A polytope is a generalized permutahedron if and only if every edge is parallel to a vector $e_i - e_j$.

Theorem (An–Tung–Z. '24)

For $w \in S_n$, $\mathrm{Newton}(D^w)$ is a generalized permutahedron with

$$z_I = \sum_{(a,b) \in \mathrm{Inv}(w)} \mathbb{1}_{I \supseteq \{a, a+1 \ldots, b-1\}}$$

for all $I \subseteq [n]$.

Vertices of Newton Polytopes

Theorem (An–Tung–Z. '24)

The point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\mathrm{Newton}(D^w)$ if and only if x^α has a coefficient of 1 in $\mathrm{GW}(w)$.

Theorem (An–Tung–Z. '24)

Given a product q of linear factors in $x_1, x_2, \ldots, x_n$ with all coefficients 1, the point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\mathrm{Newton}(q)$ if and only if x^α has a coefficient of 1 in q.

Example

$$\begin{aligned} q &= (x_1 + x_2)(x_2 + x_3)(x_1 + x_3)(x_1 + x_2 + x_3) \\ &= x_1^3 x_2 + x_1^3 x_3 + 2x_1^2 x_2^2 + 4x_1^2 x_2 x_3 + 2x_1^2 x_3^2 \\ &\quad + x_1 x_2^3 + 4x_1 x_2^2 x_3 + 4x_1 x_2 x_3^2 + x_1 x_3^3 + x_2^3 x_3 + 2x_2^2 x_3^2 + x_2 x_3^3 \end{aligned}$$

Vertices: $\{(3,1,0), (3,0,1), (1,3,0), (1,0,3), (0,3,1), (0,1,3)\}$

Vertices of Newton Polytopes (2)

(1, 6)	(1, 5)	(1, 4)	(1, 3)	(1, 2)
(2, 6)	(2, 5)	(2, 4)	(2, 3)	
(3, 6)	(3, 5)	(3, 4)		
(4, 6)	(4, 5)			
(5, 6)				

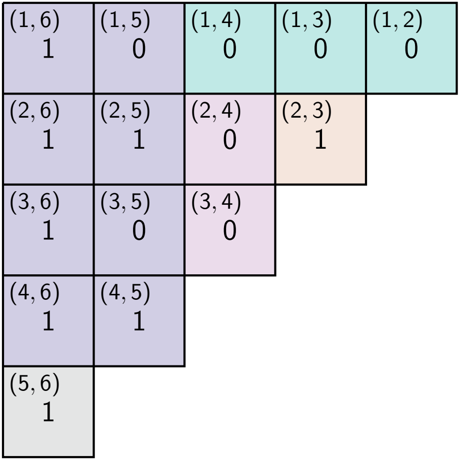
Step 1: Build a staircase Young diagram with $n = 6$.

Vertices of Newton Polytopes (3)

$(1,6)$ 1	$(1,5)$ 0	$(1,4)$ 0	$(1,3)$ 0	$(1,2)$ 0
$(2,6)$ 1	$(2,5)$ 1	$(2,4)$ 0	$(2,3)$ 1	
$(3,6)$ 1	$(3,5)$ 0	$(3,4)$ 0		
$(4,6)$ 1	$(4,5)$ 1			
$(5,6)$ 1				

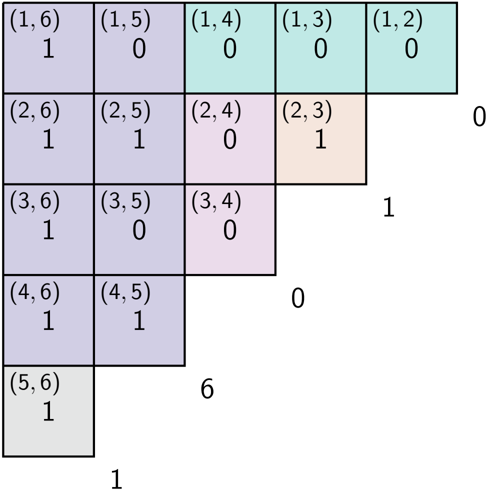
Step 2: When $w = 253641$, the above boxes are filled with 1's.

Vertices of Newton Polytopes (4)



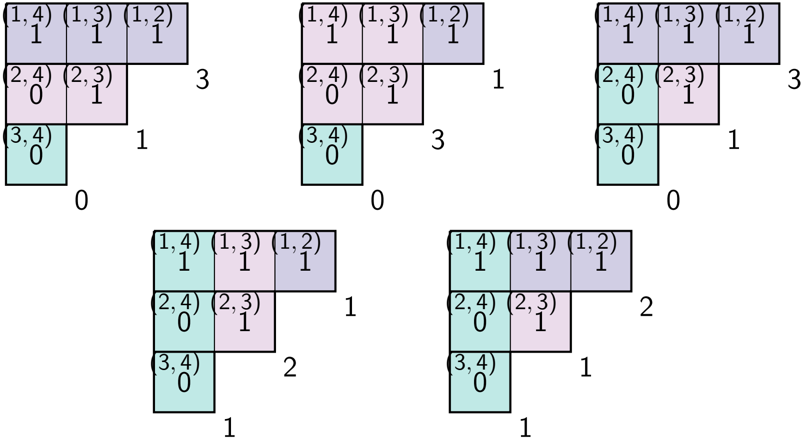
Step 3: We consider a tiling by $n-1$ rectangles.

Vertices of Newton Polytopes (5)



Step 4: We find that $\mathrm{Newton}(D^{253641})$ has vertex $(0, 1, 0, 6, 1)$.

Vertices of Newton Polytopes (6)



$\mathrm{Newton}(D^{4213})$ has vertices $(3,1,0),(1,3,0),(1,2,1),(2,1,1)$.

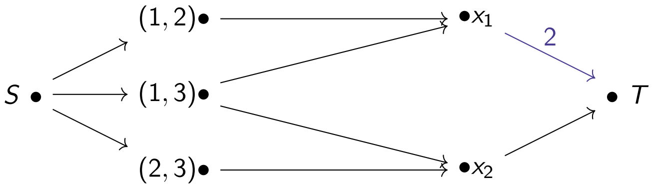
The Vanishing Problem for Dual Schubert Polynomials

Theorem (Adve–Robichaux–Yong '21)

For $w \in S_n$, Schubert polynomial $\mathfrak{S}_w$, and $\alpha \in \mathbb{Z}_{\geq 0}^{n-1}$, there is a polynomial-time algorithm to determine whether $\alpha \in \mathrm{supp}(\mathfrak{S}_w)$.

Theorem (An–Tung–Z. '25+)

For $w \in S_n$ and $\alpha \in \mathbb{Z}_{\geq 0}^{n-1}$, there is an $O(n^4)$ algorithm to determine whether $\alpha \in \mathrm{supp}(D^w)$.



The network testing the term $x_1^2 x_2$ in D^{321}.

Further Results and Conjectures

Theorem (An–Tung–Z. '24)

For all Bruhat intervals $[u, w]$ *in* S_n*,* D_u^w *has SNP, and* $\mathrm{Newton}(D_u^w)$ *is a generalized permutahedron.*

Conjecture (An–Tung–Z. '24)

For $u \in S_n$, there exists $w \in S_n$ such that D_u^w does not have SCNP if and only if u contains a 1324-pattern.

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