Differential Powers in Semigroup and Polynomial Rings

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Introduction

- An ideal generated by monomials is called a **monomial ideal**.
- **Affine semigroups** $\mathbb{N}A$ are finitely generated submonoids of $\mathbb{Z}^d$, where A is a d-dimensional matrix and its columns are generators of the semigroup.
- We call the semigroup $\mathbb{N}A$ **normal** when $\mathbb{N}A = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}A$.
- The affine semigroup ring given by $\mathbb{N}A$ is
$$\mathbb{C}[\mathbb{N}A] := \bigoplus_{a \in \mathbb{N}A} \mathbb{C} \cdot t^a,$$
where $a = (a_1, ..., a_d)$ and $t^a = (t_1^{a_1}, ..., t_d^{a_d})$.
- **Theorem 1 [ST01]:** The **ring of differential operators** $D(R)$ is generated by
$$\{D_b \mid b \in \mathbb{Z}A, \text{ and for all } t^c \in \mathbb{C}[\mathbb{N}A],\ D_b(t^c) = t^{b+c}\}$$
- Let the hyperplanes $h_1 = 0, ..., h_r = 0$ be the boundaries of $\mathbb{N}A$ such that $h_i(\mathbb{N}A) \geq 0$ for all $1 \leq i \leq r$. The **order** of a differential operator D_b is $\sum_{i=1}^{r} \max\{-h_i(b), 0\}$. Let $D^{N-1}(R)$ denote all differential operators of order less than or equal to $N-1$.
- For any monomial ideal $I \subseteq R$, the N**th differential power** of I is
$$I^{\langle N \rangle} := \{f \in R \mid \partial(f) \in I \text{ for all } \partial \in D^{N-1}(R)\}.$$
- We only need to consider a small subset of the differential operators generating $D(R)$ to compute $I^{\langle N \rangle}$.

Example

We can visualize monomials in a monomial ideal as integer points in $\mathbb{Z}^d$ in that the point $(q_1, ..., q_d)$ corresponds to the monomial $t_1^{q_1} \cdots t_d^{q_d}$.

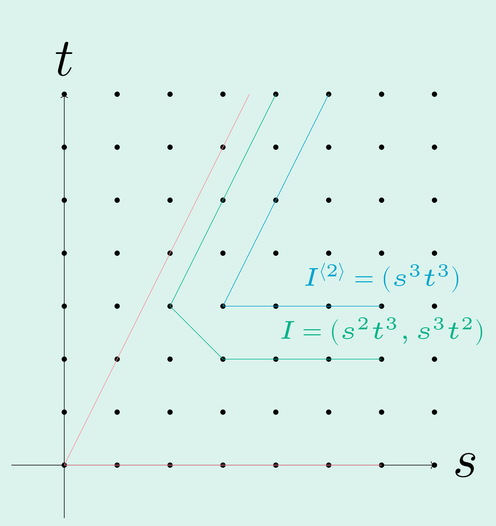


Figure 1. The 2nd differential power of an ideal in the semigroup ring $\mathbb{C}[\mathbb{N}A]$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

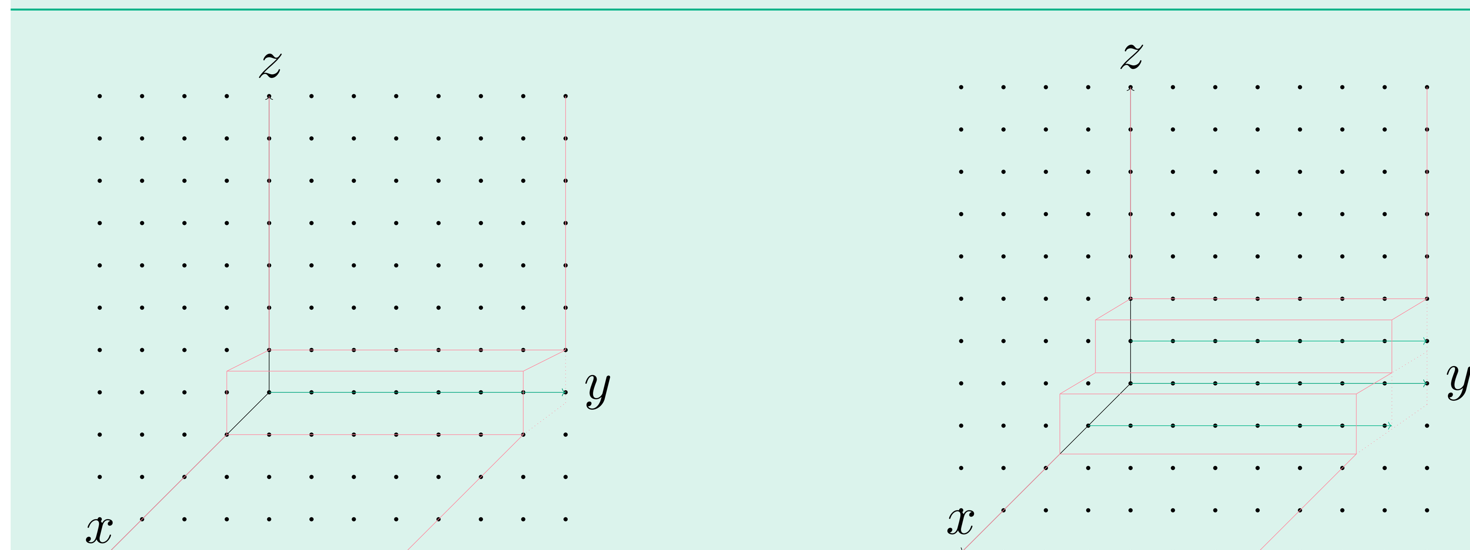
Standard Pairs

- A **standard pair** of a monomial ideal $I \subseteq \mathbb{C}[\mathbb{N}A]$ is (t^ℓ, F), where $t_1^{\ell_1} t_2^{\ell_2} \cdots t_d^{\ell_d} = t^\ell \notin I$, and $F \subseteq \{t_1, ..., t_d\}$ such that $\mathrm{Supp}(t^\ell) \cap F = \emptyset$, and $(t^\ell, F) \nsubseteq (t^k, G)$ for any standard pair (t^k, G).
- A monomial ideal $I \subseteq \mathbb{C}[\mathbb{N}A]$ can be determined by its finite set of standard pairs.

Differential Powers in Prime and Radical Ideals

- The **radical** of an ideal $I \subseteq \mathbb{C}[\mathbb{N}A]$ is
$$\sqrt{I} = \{f \in \mathbb{C}[\mathbb{N}A] \mid \exists \ell \in \mathbb{Z}^+ \text{ such that } f^\ell \in I\}.$$
- When ideals are radical, their Nth differential power is equal to their Nth symbolic power, which have been of considerable interest in algebraic geometry and commutative algebra [Dao+18].
- The following Theorems hold in polynomial rings $\mathbb{C}[x_1, ..., x_n]$.
- **Theorem 2 [Cyr+24]:** Let P_F be a prime monomial ideal corresponding to a face F. Then, $P_F = \langle x_i \mid x_i \notin F \rangle$, and $\mathrm{stdPairs}(P_F^{\langle \ell \rangle}) = \{(x^a, F) \mid \sum_{i=1}^{n} a_i < \ell, a_i = 0 \text{ if } x_i \in F\}$.
- **Theorem 3 [Cyr+24]:** Let I be a radical monomial ideal. Then, $\mathrm{stdPairs}(I^{\langle \ell \rangle}) = \bigcup_{i=1}^{n} \mathrm{stdPairs}(P_{F_i}^{\langle \ell \rangle})$.

Example



(a) $\mathrm{stdPairs}(P_y) = \{(1, y)\}$

(b) $\mathrm{stdPairs}(P_y^{\langle 2 \rangle}) = \{(1, y), (x, y), (z, y)\}$

Figure 2. The standard pairs of the prime monomial ideal $P_y = (x, z) \subseteq C[x, y, z]$ and its second differential power $P_y^{\langle 2 \rangle} = (x^2, xz, z^2)$.

Eventually Periodic Differential Powers

- We first generalize a result in [Ken+21] regarding polynomial rings $\mathbb{C}[x, y]$ to polynomial rings with an arbitrary number of generators.
- **Theorem 4 [Cyr+24]:** Let $I \subseteq \mathbb{C}[x_1, ..., x_n]$ be a radical ideal. Then there exists $N \in \mathbb{N}$ such that for all $m \geq N$, $I^{\langle m \rangle}$ is principal.
- We have an analogous result holding in semigroups rings $\mathbb{C}[\mathbb{N}A]$ where $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & n \end{bmatrix}$.
- **Theorem 5 [Cyr+24]:** Let $I \subseteq \mathbb{C}[t_1, ..., t_d]$ be an ideal. There exists $N \in \mathbb{N}$ such that when $n \geq N$, the number of generators of $I^{\langle n \rangle}$ is periodic.
  - If $2 \nmid n$, $I^{\langle n \rangle}$ can be generated by two elements or is principal.

Example

We give an example of an ideal with differential powers eventually principal in the polynomial ring $\mathbb{C}[x, y]$.

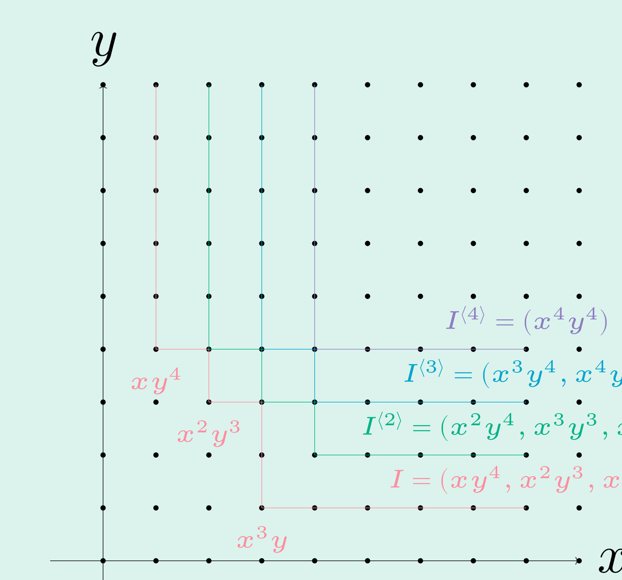


Figure 3. An ideal $I \subseteq \mathbb{C}[x, y]$ with principal differential powers $I^{\langle n \rangle}$ when $n \geq 4$.

References & Acknowledgments

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[Cyr+24] S. Cyrusian, N. Joseph, Z. Medlin, S. Solotko, and M. Yang. Differential Powers in Semigroup and Polynomial Rings. Tech. rep. University of Minnesota, 2024.

[Dao+18] H. Dao et al. Symbolic powers of ideals. In: Springer Proc. Math. Stat. 222 (2018), pp. 387–432.

[Ken+21] J. Kenkel, L. McPherson, J. Page, D. Smolkin, M. Stephenson, and F. Yang. Asymptotic Behavior of Differential Powers. 2021. arXiv: 2111.15653.

[ST01] M. Saito and W. Traves. Differential algebras on semigroup algebras. In: Contemp. Math. 286 (2001), pp. 207–226.