ON TRIANGULATIONS OF ORDER POLYTOPES FOR SNAKE POSETS

MOLLY BRADLEY, SOGOL CYRUSIAN, ALEISTER JONES, MARIO TOMBA,
AND KATHERINE TUNG

Abstract. Motivated by recent work by von Bell et al., we study triangulations of the order polytopes of generalized snake posets. We impose a partial ordering on the regular triangulations of this order polytope, and conjecture that it is a lattice and that it induces a good orientation as defined by Kalai. We explore the geometry of the secondary polytope of the order polytope and make progress towards proving that all two-dimensional faces are quadrilaterals, pentagons, and hexagons. We also investigate the "twist group", as introduced by von Bell et al., which acts on the set of regular triangulations. We prove the twist action is free and give an eigenbasis for the action of the twist group on the secondary polytope. Lastly, we show that a subset of the triangulations admit the same number of bistellar flips. Our results make partial progress towards proving certain conjectures posed by von Bell et al.

Contents

1. Introduction

Triangulations of point configurations have been studied for centuries as a natural method of breaking up a given space into smaller pieces which are easier to understand. While triangulations have applications in many fields, including computer science and algebra, we are primarily concerned with their usefulness in the field of combinatorics. For example, the nth Catalan number

$$\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n},$$

which counts many combinatorial objects including parenthesizations of the product of $n+1$ factors and Dyck paths of length $2n$, also counts triangulations of a convex $(n+2)-$gon. We refer the reader to the book by De Loera, Rambau and Santos [DRS10] for more background and history.

Another connection between geometry and combinatorics can be found in the study of order polytopes. Order polytopes of finite posets were first introduced by Richard Stanley in [Sta86], motivated by a link between the geometric structure of these objects and the combinatorial structure of finite posets. Triangulations of order polytopes are currently an area of interest.

Much of our work is motivated by a 2022 paper of von Bell, Braun, Hanely, Serhiyenko, Vega, Vindas-Meléndez, and Yip, titled *Triangulations, Order Polytopes, and Generalized Snake Posets* [Bel+22]. In this paper, the authors study a family of generalized snake posets $P(\mathbf{w})$ and the triangulations of the corresponding order polytope $\mathcal{O}(Q_{\mathbf{w}})$, where $Q_{\mathbf{w}}$ is the poset of meet-irreducibles of a modification of $P(\mathbf{w})$. In one section of their paper, the authors classify the circuits of the vertex set of these order polytopes. In another section, they introduce a "twist" action on triangulations of $\mathcal{O}(Q_{\mathbf{w}})$. They prove several initial results about this action, namely that twists preserve regular triangulations and twists commute with bistellar flips.

Their paper concludes with several conjectures on a specific class of generalized snake posets, which is denoted by $\mathcal{V}$. For further detail on this class, see Definition 2.14. In particular, our paper focuses on the following three conjectures:

Conjecture 1.1 ([Bel+22], Conjecture 6.1)**.** For $\mathbf{w} \in \mathcal{V}$, the flip graph of regular triangulations for $\mathcal{O}(Q_{\mathbf{w}})$ is k-regular, where k is the dimension of the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$.

Conjecture 1.2 ([Bel+22], Conjecture 6.4)**.** For $\mathbf{w} \in \mathcal{V}$, all triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ are regular.

Conjecture 1.3 ([Bel+22], Conjecture 6.5)**.** The number of regular triangulations of $\mathcal{O}(S_k)$ is $2^{k+1} \cdot \mathrm{Cat}(2k+1)$.

In working towards proving these conjectures, we pose some of our own conjectures about triangulations of order polytopes of snake posets. In Section 4, we make several remarks about the secondary polytope $\Sigma_{\mathcal{O}(S_k)}$ which lead us to the following conjecture about its dual polytope.

Conjecture 1.4. The dual polytope of $\Sigma_{\mathcal{O}(S_k)}$ is a flag simplicial complex.

In Section 3.2, we define a partial ordering on triangulations of $\mathcal{O}(S_k)$ that satisfies interesting properties. Specifically, we have the following conjectures, which we have verified for $k = 1, 2, 3$.

Conjecture 1.5. For any snake poset S_k, the set of triangulations of $\mathcal{O}(S_k)$ under the partial ordering given in Definition 3.2 is a lattice.

Conjecture 1.6. The orientation on the flip graph induced by the partial ordering is a *good* orientation in the sense of Kalai [Kal88]: every face of the polytope contains a unique sink vertex.

In studying the secondary polytope $\Sigma_{\mathcal{O}(S_k)}$, it is natural to consider the 2-dimensional faces, as these correspond to certain cycles in the bistellar flip graph. This leads us to our final conjecture:

Conjecture 1.7. The 2-dimensional faces of $\Sigma_{\mathcal{O}(S_k)}$ are quadrilaterals, pentagons and hexagons.

We prove four main theorems which relate to the conjectures above. In Section 4, we classify all of the squares which appear in the secondary polytope via the following theorem, where the notion of "commuting" is defined in Definition 4.3.

Theorem 1.8. Let Z_1 and Z_2 be circuits of $\mathcal{O}(Q_{\mathbf{w}})$, where $\mathbf{w} \in \mathcal{V}$. Then Z_1 and Z_2 commute at $\mathcal{T}$ if and only if $\mathcal{T}$ can be flipped at Z_1 and Z_2 and at least one of the following hold:

(i) Z_1 and Z_2 appear on different maximal simplices in $\mathcal{T}$, or
(ii) Z_1 and Z_2 share no vertex

We also make progress towards similar theorems for pentagonal and hexagonal faces. In Section 5, we make partial progress towards the valence-regularity of the flip graph of $\mathcal{O}(S_k)$.

Theorem 1.9. Let $\mathcal{T}$ be a triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ obtained by applying one flip to the canonical triangulation $\mathcal{T}_{\mathbf{w}}$, where $\mathbf{w} = w_0 w_1 \cdots w_n \in \mathcal{V}$. Then, $\mathcal{T}$ admits $n+1$ flips.

In the context of stating their Conjecture 1.3, the authors of [Bel+22] implicitly conjecture that nonidentity elements of the twist groups has no fixed triangulations. We give a proof of the following statement in Section 6.

Theorem 1.10. The twist group acts freely on the regular triangulations of $\mathcal{O}(Q_{\mathbf{w}})$.

Finally, in Section 7, we construct an eigenbasis for the twist group that arises from a combinatorial interpretation of $P(\mathbf{w})$.

Theorem 1.11. Let $\mathbf{w} = \varepsilon w_1 \cdots w_n \in \mathcal{V}$. Let $V \cong \mathbb{R}^{n+1}$ be the linear subspace of $\mathbb{R}^{2n+6}$ parallel to the affine subspace containing the secondary polytope $\Sigma_{\mathcal{O}(Q_{\mathbf{w}})}$. Then the twist group acts on V in some eigenbasis $v_1, v_2, \ldots, v_{n+1}$ in which each v_i corresponds with a letter ε or w_i in $\mathbf{w}$ and each twist τ_i negates exactly the basis elements that correspond to the w_i in the ladder that τ_i reflects.

1.1. **Main Results and Organization.** We organize this report as follows. In Section 2.1, we provide background on triangulations, circuits and flips, in Section 2.2 we provide background on order polytopes and snake posets, in Section 2.3, we characterize the circuits of $\mathcal{O}(Q_{\mathbf{w}})$, and in Section 2.4 we provide background on the twist group. In Section 3, we enumerate the circuits of $O(S_k)$ and we define a partial ordering on triangulations of $\mathcal{O}(Q_{\mathbf{w}})$, conjecturing that this partial ordering satisfies certain interesting properties. In Section 4, we study the geometry of the secondary polytope of $O(S_k)$, introducing several conjectures and making progress towards proving Conjecture 1.7. In Section 5, we prove that a subset of triangulations of $\mathcal{O}(S_k)$ admit $2k+1$ flips, and precisely describe which flips are possible. In Section 6, we prove that the twist action on the set of regular triangulations of order polytopes $\mathcal{O}(Q_{\mathbf{w}})$ is free. In Section 7, we analyze the geometric interpretation of the twist action and we describe its eigenbasis. Finally, in Section 8, we discuss work in progress on towards some of the conjectures above and discuss further directions.

2. Background

2.1. **Triangulations, circuits and flips.** In this section, we briefly explain the basic concepts that are needed for this problem such as simplicial complexes, triangulations and circuits, among others. We follow the exposition given in [DRS10].

A *d-simplex* is the convex hull of $d+1$ affinely independent points in $\mathbb{R}^n$ for some n. Simplices generalize the concept of a triangle. For instance, a 1-simplex is a line segment, a

2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on. More importantly, simplices are the basic elements of simplicial complexes.

Definition 2.1. A *(geometric) simplicial complex* Δ is a set of simplices in $\mathbb{R}^n$ satisfying two properties: (1) if $\sigma \in \Delta$ and τ is a subsimplex of σ, meaning τ is the convex hull of a subset of the vertices of σ), then $\tau \in \Delta$, and (2) if a pair of simplices in $\mathcal{T}$ intersect, then they intersect exactly at a shared face of each.

We say that $\sigma \in \Delta$ is a *face* and, if σ is a maximal simplex, we say that it is a *facet*.

A particular case of a simplicial complex is the triangulation of a given point configuration, which will be the main object of study of our paper.

Definition 2.2. Given a point configuration $\mathcal{A} \subseteq \mathbb{R}^d$ with convex hull $\text{conv}(A)$, a *triangulation* of $\mathcal{A}$ is a collection $\mathcal{T}$ of d-simplices such that

(i) The union of all simplices in $\mathcal{T}$ is $\text{conv}(\mathcal{A})$.
(ii) The collection $\mathcal{T}$ forms a geometric simplicial complex.
(iii) Every simplex in $\mathcal{T}$ has vertices only from $\mathcal{A}$.

We say that $\mathcal{T}$ is *unimodular* if every simplex has normalized volume one, and we say that it is *regular* if there are heights $h_1, ..., h_d \in \mathbb{R}$ such that the projection of the upper convex hull of $\hat{A} = \left\{ \begin{bmatrix} a_1 \\ h_1 \end{bmatrix}, \ldots, \begin{bmatrix} a_d \\ h_d \end{bmatrix} \right\} \subseteq \mathbb{R}^{d+1}$ to $\mathbb{R}^d$ is $\mathcal{T}$.

Often we might identify a triangulation with its maximal simplices, since these encode all the information of the triangulation. Further, we might sometimes only specify the *vertices* of the maximal simplices instead of the convex hull of them.

Example 2.3. Consider the polytope in $\mathbb{R}^3$ defined by the convex hull of the point configuration $\mathcal{A} = \{(1,0,0),(0,1,0),(0,-1,0),(0,0,1),(0,0,-1)\}$. A triangulation of this polytope consists of the maximal simplices:

$$\sigma_1 = \text{conv}\{(1,0,0),(0,1,0),(0,-1,0),(0,0,1)\}$$
$$\sigma_2 = \text{conv}\{(1,0,0),(0,1,0),(0,-1,0),(0,0,-1)\}$$

Example 2.4. Let $\mathcal{A} = \{A, B, C, D, E, F\} \subset \mathbb{R}^2$ where

$$A = (1,0), B = \Big(\frac{1}{2}, \frac{\sqrt{3}}{2}\Big), C = \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\Big), D = (-1,0), E = \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big), F = \Big(\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big).$$

Let $\mathcal{T}_1 = \{ABC, CDE, EFA, ACE\}$. This triangulation is regular: we may place vertices A, C, E at height 2 and vertices B, D, F at height 1. Then the projection of the convex hull of $\hat{A}$ to $\mathbb{R}^2$ is $\mathcal{T}_1$.

Definition 2.5. A point configuration $\mathcal{A} = \{\underline{a}_j : j \in J\} \subseteq \mathbb{R}^d$ is said to have *corank one* if it has a unique affine dependence $\sum_{j \in Z} \lambda_j \underline{a}_j = \underline{0}$ with $\sum_{j\in Z} \lambda_j = 0$, up to multiplication by a constant.

If $\mathcal{A}$ has corank one, we can partition J into the following three sets

$$J_+ = \{j \in J : \lambda_j > 0\}, \quad J_0 = \{j \in J : \lambda_j = 0\}, \quad J_- = \{j \in J : \lambda_j < 0\}$$

The pair (J_+, J_-) is called the *Radon partition* of J, and these are the only subsets of J whose relative interiors intersect at a point.

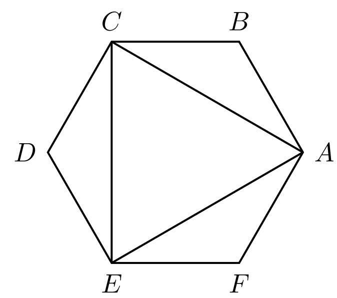


FIGURE 1. An illustration of $\mathcal{T}_1$, a triangulation of a regular hexagon.

Definition 2.6. A subset $Z \subseteq J$ is called a *circuit* if it is a minimal affinely dependent set. The partition of Z into the pair (Z_+, Z_-) such that $\operatorname{conv}(Z_+) \cap \operatorname{conv}(Z_-) \neq \varnothing$ is called an *oriented circuit*.

Note that, as in the definitions above, we sometimes abuse notation and refer to the circuit Z as the set points of $\mathcal{A}$ indexed by Z.

Lemma 2.7 ([DRS10], Lemma 2.4.2)**.** Let $\mathcal{A}$ be a point configuration of corank one with index set $J = J_+ \cup J_0 \cup J_-$. Then, there are only two triangulations of $\mathcal{A}$, namely:

$$\mathcal{T}^+ = \{J \setminus \{j\} : j \in J_+\} \quad \text{and} \quad \mathcal{T}^- = \{J \setminus \{j\} : j \in J_-\},$$

where we have only specified the maximal simplices of each triangulation.

Remark. Note that the lemma above implies that, in particular, there are only two triangulations of a given circuit. That is, given a circuit Z with oriented circuit (Z_+, Z_-), the only two triangulations of Z are $\mathcal{T}_Z^+ = \{Z \setminus \{z\} \ : \ z \in Z_+\}$ and $\mathcal{T}_Z^- = \{Z \setminus \{z\} \ : \ z \in Z_-\}$. We will use this fact in various sections of this paper.

These results are particularly interesting since they guarantee that one can move between triangulations of a larger point configuration A by swapping between triangulations of the circuits that appear in it. This is the notion of a *bistellar flip*. However, to precisely define a flip, we need some more background.

Definition 2.8. Given a simplicial complex Δ and a face $\sigma \in \Delta$, the *link* of σ in Δ is the simplicial complex

$$\operatorname{lk}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \varnothing\}.$$

Given two simplicial complexes Δ and Δ', their *join* is $\Delta * \Delta' = \{\sigma \cup \sigma' : \sigma \in \Delta, \sigma' \in \Delta'\}$

We can now precisely define the flip of a triangulation. We note that there are several different equivalent definitions of a bistellar flip, but here we use the definition given in [DRS10, Theorem 4.4.1].

Definition 2.9. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangulations of a point configuration $\mathcal{A}$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ *differ by a flip* if and only if there is a circuit Z of $\mathcal{A}$ such that

- (i) They contain, respectively, the two triangulations $\mathcal{T}_Z^+$ and $\mathcal{T}_Z^-$ of Z.
- (ii) All the maximal simplices of $\mathcal{T}_Z^+$ and $\mathcal{T}_Z^-$ have the same link L in $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.
- (iii) Removing the subcomplex $\mathcal{T}_Z^+ * L$ from $\mathcal{T}_1$ and replacing it with $\mathcal{T}_Z^- * L$ gives $\mathcal{T}_2$.

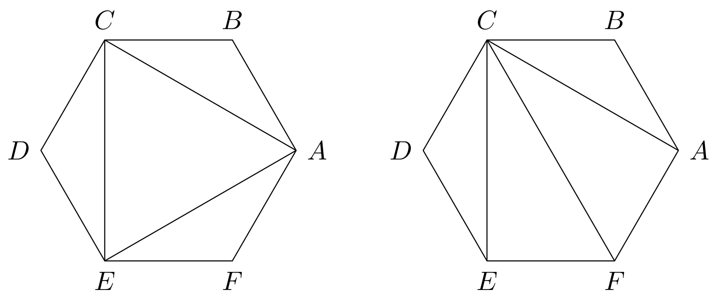


FIGURE 2. The triangulation $\mathcal{T}_2$ (right) obtained by flipping $\mathcal{T}_1$ (left) at circuit $ACEF$.

Lastly, we define the *flip graph* of a point configuration $\mathcal{A}$ as the graph whose vertices are the triangulations of $\mathcal{A}$ and where two vertices are connected by an edge if there is a bistellar flip between their corresponding triangulations.

Example 2.10. Continuing the example of the hexagon from 2.4, let $\mathcal{A} = \{A, B, C, D, E, F\}$ consist of the vertices of a regular hexagon. Let $\mathcal{T}_1 = \{ABC, CDE, EFA, ACE\}$. Let $\mathcal{T}_2 = \{ABC, CDE, CEF, FAC\}$. Then $\mathcal{T}_1, \mathcal{T}_2$ differ by a flip: the circuit Z satisfying (i), (ii), and (iii) is $ACEF$.

The subgraph of flip graph induced by regular triangulations is closely related to the *secondary polytope* introduced by Gelfand, Kapranov and Zelevinsky in 1991. Even though the secondary polytope admits a very precise description, we do not need it here and we shall only use the following the following results about it.

Theorem 2.11 ([GKZ08])**.** Let $\mathcal{A}$ be a d-dimensional point configuration. Then, there exists a $(\#\mathcal{A} - d - 1)$-dimensional polytope $\Sigma_{\mathcal{A}}$ whose vertices are in correspondence with the regular triangulations of $\mathcal{A}$ and whose edges correspond to bistellar flips between them. The polytope $\Sigma_{\mathcal{A}}$ is called the *secondary polytope* of $\mathcal{A}$.

2.2. **Order polytopes and snake posets.** The order polytope was first introduced and defined by Richard Stanley in [Sta86]. Here, we will define $\mathcal{O}(P)$ for any poset P using notation from [Bel+22, Section 2.2]. We define an *upper order ideal* $A \subset P$ to be any subset such that $i \in A$ and $j >_P i$ implies $j \in A$. We let $J(P)$ denote the poset of upper order ideals of P, where the partial ordering is by reverse inclusion. For any $A \in J(P)$, we define the *characteristic vector* $\mathbf{v}_A \coloneqq \sum_{i \in A} e_i$, where e_i is the standard basis vector of $\mathbb{R}^d$ and $d = |P|$. We define $\mathcal{O}(P) \coloneqq \mathrm{conv}(V)$, where $V = \{\mathbf{v}_A \colon A \in J(P)\}$.

For any poset P, $\mathcal{O}(P)$ has a *canonical triangulation* $\mathcal{T}$, which satisfies the following property: the maximal simplices of $\mathcal{T}$ are in bijection with maximal chains of $J(P)$, such that given a maximal chain $C = \{A_1, A_2, ...A_n\}$ in $J(P)$, the corresponding maximal simplex of $\mathcal{T}$ is given by $\mathrm{conv}(V)$ where $V = \{\mathbf{v}_A \colon A \in C\}$.

In our project, we study triangulations of a certain family of order polytopes. We begin by defining the family of *generalized snake posets* $P(\mathbf{w})$ as introduced in [Bel+22].

Definition 2.12. [Bel+22, Definition 3.1] For n in $\mathbb{Z}_{\geq 0}$, a *generalized snake word* is a word of the form $\mathbf{w} = w_0 w_1 \cdots w_n$ where $w_0 = \varepsilon$ is the empty letter and w_i is in the alphabet $\{L, R\}$ for $i = 1, ..., n$.

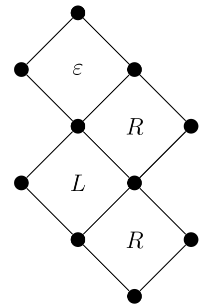


FIGURE 3. The snake poset $S_3 = P(\varepsilon RLR)$.

Definition 2.13. [Bel+22, Definition 3.2] Given a generalized snake word, we define the *generalized snake poset* $P(\mathbf{w})$ recursively in the following way:

- $P(w_0) = P(\varepsilon)$ is the poset on elements $\{0, 1, 2, 3\}$ with cover relations $1 \lessdot 0$, $2 \lessdot 0$, $3 \lessdot 1$, and $3 \lessdot 2$.
- $P(w_0w_1 \cdots w_n)$ is the poset $P(w_0w_1 \cdots w_{n-1})$ with the added cover relations $2n + 3 \lessdot 2n + 1$, $2n + 3 \lessdot 2n + 2$, and

$$\begin{cases} 2n+2 \lessdot 2n-1 & \text{if } n = 1 \text{ and } w_n = L, \text{ or } n \geq 2 \text{ and } w_{n-1}w_n \in \{RL, LR\} \\ 2n+2 \lessdot 2n & \text{if } n = 1 \text{ and } w_n = R, \text{ or } n \geq 2 \text{ and } w_{n-1}w_n \in \{LL, RR\}. \end{cases}$$

We often restrict our attention to a subset of generalized snake posets $\mathcal{V}$, as classified in the following definition.

Definition 2.14. Let $\mathcal{V}$ be the set of generalized snake words $\mathbf{w}$ that contain neither RLR nor LRL.

We are also particularly interested in a special case of generalized snake posets: the *snake poset*. We define a snake poset $S_k = P(\mathbf{w})$ where $\mathbf{w} = w_0w_1 \cdots w_k$ such that $w_i = R$ if i is odd and $w_i = L$ if i is even and not equal to 0. For instance, the snake poset S_3 is represented in Figure 3. We note in particular that $J(S_k) = \hat{P}(\mathbf{w})$ for $\mathbf{w} = \epsilon(RL)(LR)(RL)...$ with k pairs of the form (RL) or (LR), and that $\mathbf{w} \in \mathcal{V}$.

Definition 2.15. Given $\mathbf{w}$ a generalized snake word, let $\hat{P}(\mathbf{w})$ be $P(\mathbf{w})$ with $\hat{0}$ and $\hat{1}$ adjoined. Define $Q_{\mathbf{w}}$ as the poset of meet-irreducibles of $\hat{P}(\mathbf{w})$.

Here we note that by the fundamental theorem of finite distributive lattices, $\hat{P}(\mathbf{w}) \cong J(Q_{\mathbf{w}})$. Thus, every element in $\hat{P}(\mathbf{w})$ is associated with an order filter of $Q_{\mathbf{w}}$. In our project we primarily study the order polytope of $Q_{\mathbf{w}}$, denoted $\mathcal{O}(Q_{\mathbf{w}})$. We denote the canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ by $\mathcal{T}_{\mathbf{w}}$.

2.3. **Characterization of circuits.** In this section, we discuss a result from [Bel+22] which will allow us to characterize the circuits of the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$.

We let $\mathcal{C}(Q_{\mathbf{w}})$ be the set of circuits of the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$, which are the objects we are primarily concerned with. Here we recall that the set of vertices of $\mathcal{O}(Q_{\mathbf{w}})$ is precisely the set of characteristic vectors $\mathbf{v}_A$ corresponding to each $A \in \hat{P}(\mathbf{w})$.

We define the graph $G(\mathbf{w})$ as follows. If $\mathbf{w} = w_0w_1 \cdots w_n$, then the set of vertices of $G(\mathbf{w})$ is $V(G) = \{w_0, w_1, \ldots, w_n\}$. The set of edges of $G(\mathbf{w})$ is $E(G) = \{(w_i, w_i + 1) : i = 0, \ldots, n-1\} \cup \{(w_i, w_{i+2}) \colon w_{i+1}w_{i+2} \in \{RL, LR\}\}$. We denote the set of nonempty connected induced subgraphs of $G(\mathbf{w})$ by $\mathcal{G}(\mathbf{w})$. Here we note that we can identify each induced subgraph uniquely as a subword of $\mathbf{w}$, but not all subwords of $\mathbf{w}$ correspond to elements of $\mathcal{G}(\mathbf{w})$.

If we embed the Hasse diagram of $\hat{P}(\mathbf{w})$ onto the plane such that no two edges cross, we note that each bounded face has degree 4, and we call each such bounded face a *square* of $\hat{P}(\mathbf{w})$. In this notation, there is a one-to-one correspondence between the squares of $\hat{P}(\mathbf{w})$ and the letters of $\mathbf{w}$. We denote by $\mathrm{Sq}(w_i)$ the four elements of $\hat{P}(\mathbf{w})$ which are contained in the i-th bounded face, labelling from top to bottom.

We define a map $\Gamma\colon \mathcal{G}(\mathbf{w}) \to \mathcal{C}(Q_{\mathbf{w}})$ as follows. Given some $H \in \mathcal{G}(\mathbf{w})$, we consider the subword corresponding to H. We consider the set of squares $\mathrm{Sq}(w_i)$ corresponding to each letter w_i in H. For any $A \in \hat{P}(\mathbf{w})$, we say A is compatible with H if A is an element of an odd number of squares $\{\mathrm{Sq}(w_i)\colon w_i \in H\}$. We say $\Gamma(H)$ is precisely the set of all $\mathbf{v}_A$ such that A is compatible with H.

Theorem 2.16. [Bel+22, Theorem 4.5] Let $\mathbf{w} \in \mathcal{V}$ be a generalized snake word of length n. The map $\Gamma\colon \mathcal{G}(\mathbf{w}) \to \mathcal{C}(Q_{\mathbf{w}})$ is a bijection.

In their proof of the above theorem, the authors introduce a convention for assigning signs to the vertices in each circuit. They construct their definition by assigning a sign to each square. Figure 4 shows the signs of vertices in a square corresponding to a square with $\mathrm{sgn}(\mathrm{Sq}(w_i)) = 1$.

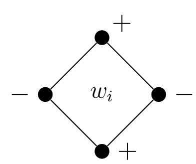


FIGURE 4. Signs of vertices corresponding to a square with $\mathrm{sgn}(\mathrm{Sq}(w_i)) = 1$

If $\mathrm{sgn}(\mathrm{Sq}(w_{i_1})) = -1$, then we swap the positive and negative vertices of the square. This leads us to the following definition.

Definition 2.17. Suppose $H = \{w_{i_1}, \dots, w_{i_\ell}\} \in \mathcal{G}(\mathbf{w})$. We assign signs to the squares as follows: $\mathrm{sgn}(\mathrm{Sq}(w_{i_1})) = 1$. For $j = 2, \dots, \ell$, we have the following:

$$\mathrm{sgn}(\mathrm{Sq}(w_{i_j})) = \begin{cases} \mathrm{sgn}(\mathrm{Sq}(w_{i_{j-1}})) & \text{if } i_j - i_{j-1} = 1 \\ -\mathrm{sgn}(\mathrm{Sq}(w_{i_{j-1}})) & \text{if } i_j - i_{j-1} = 2. \end{cases}$$

We note that when $\mathbf{w} = \varepsilon(RL)(LR)(RL)\cdots$, the poset $Q_{\mathbf{w}} = P(\varepsilon RLRL\cdots) = S_k$. We can therefore characterize the circuits of the vertex set of $\mathcal{O}(S_k)$ by examining the elements of $\mathcal{G}(\mathbf{w})$ for $\mathbf{w} = \varepsilon(RL)(LR)(RL)\cdots$. Given a circuit Z of the vertex set of $\mathcal{O}(S_k)$, we say Z belongs to exactly one of the following four categories:

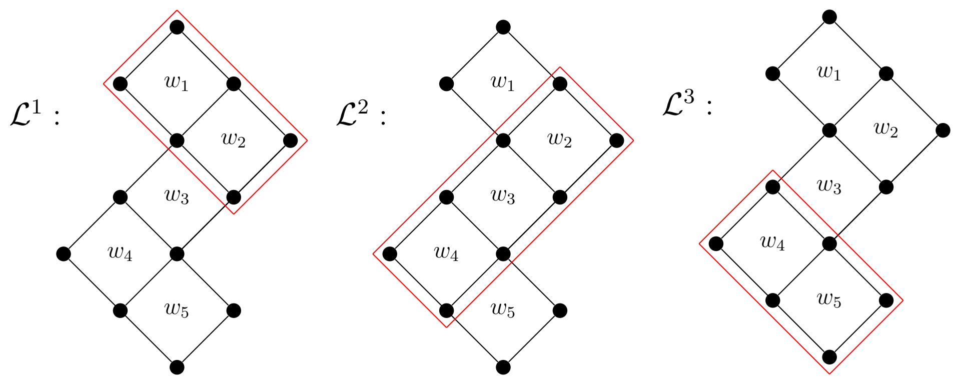
(*i*) Z is a *square* if $Z = \mathrm{Sq}(w_i)$ for some w_i.
(*ii*) Z is a *rectangle* if Z has four elements and is not a square.
(*iii*) Z *avoids a corner* if its corresponding subgraph contains $w_i w_{i+2}$.
(*iv*) Z *contains corners* if it has more than four elements and its corresponding subgraph can be denoted $w_i w_{i+1} \cdots w_{i+r}$ for some r.

2.4. **The twist group.**

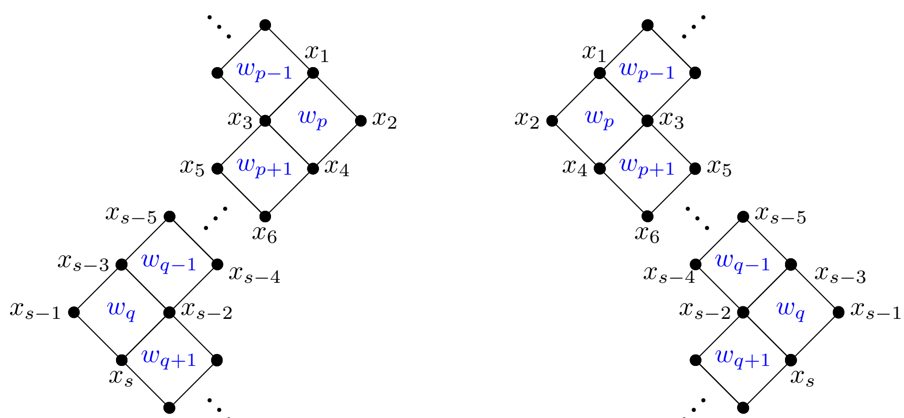
Definition 2.18. Let $\mathbf{w} = w_0w_1\cdots w_n \in \mathcal{V}$, and say w_{i_j} is the jth index such that $w_{i_j}w_{i_{j+1}}w_{i_{j+2}} = RRL$ or LLR. We define the *ladders* $\mathcal{L}^j$ on $\hat{P}(\mathbf{w})$ by

$$\mathcal{L}^1 = \bigcup_{k=0}^{i_1+1} \mathrm{Sq}(w_k),\ \mathcal{L}^j = \bigcup_{k=i_{j-1}+1}^{i_j+1} \mathrm{Sq}(w_i).$$

Visually, the ladders are simply the diagonal lines of squares in the Hasse diagram of $\hat{P}(\mathbf{w})$:



Let V be the set of vertices of $\hat{P}(\mathbf{w})$. Given a ladder, we want to permute these vertices as though we were reflecting them over the center line of the ladder. First, we label the vertices of the ladder $x_1, \ldots, x_s$ as follows, with odd indices on the top line and even indices on the bottom:



Definition 2.19. [Bel+22, Definition 5.3] We define the permutation $\tau_i \in \mathfrak{S}_{|V|}$ of V by

$$\tau_i(v) = \begin{cases} x_{j-1} & \text{if } v = x_j \text{ and } j \text{ even} \\ x_{j+1} & \text{if } v = x_j \text{ and } j \text{ odd} \\ v & \text{otherwise.} \end{cases}$$

Lemma 2.20. [Bel+22, Lemma 5.4] For all $\tau_i, \tau_j \in \mathfrak{S}_{|V|}$, the following properties hold:

(*i*) $\tau_i^2 = 1$

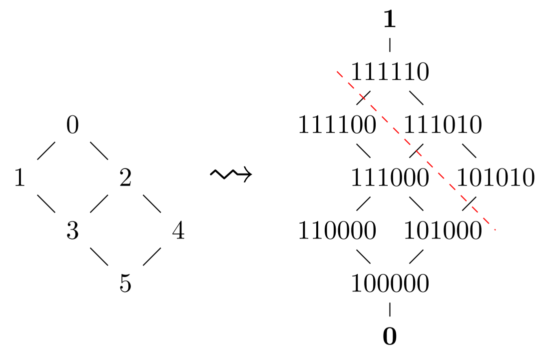
(*ii*) $\tau_i\tau_j = \tau_j\tau_i$

This means the set of all τ_i generate a commutative subgroup of $\mathfrak{S}_{|V|}$. In particular,

Definition 2.21. [Bel+22, Definition 5.5] Let $\mathfrak{T}(\mathbf{w})$ be the subgroup of $\mathfrak{S}_{|V|}$ generated by $\{\tau_i \ : \ 1 \leq i \leq s\}$ where s is the number of ladders in $\hat{P}(\mathbf{w})$. We call $\mathfrak{T}(\mathbf{w})$ the *twist group* of $\hat{P}(\mathbf{w})$. Elements of $\mathfrak{T}(\mathbf{w})$ are called *twists*, and the elements τ_i corresponding to ladder reflections are called *elementary twists*.

By the previous lemma, $\mathfrak{T}(\mathbf{w}) \cong (\mathbb{Z}/2)^s$. In particular, the twist group of $J(S_k)$ is isomorphic to $(\mathbb{Z}/2)^{k+1}$.

Example 2.22. Consider $Q_{\mathbf{w}} = S_1$ and $J(S_1)$ below. The first elementary twist τ_1 reflects the vertices of $J(S_1)$ across the dotted line.



One of the simplices in $\mathcal{T}_{\mathbf{w}}$ is $\{\mathbf{0}, 100000, 110000, 111000, 111100, 111110, \mathbf{1}\}$. Twisting by τ_1, we obtain $\{\mathbf{0}, 100000, 110000, 111010, 111100, 111110, \mathbf{1}\}$.

It's important to note that even if $\mathcal{T}$ is a triangulation of $\mathcal{O}(Q_{\mathbf{w}})$, $\tau(\mathcal{T})$ for $\tau \in \mathfrak{T}(\mathbf{w})$ may not be a triangulation, meaning $\mathfrak{T}(\mathbf{w})$ doesn't necessarily act on the set of triangulations of $\mathcal{O}(Q_{\mathbf{w}})$. However, twisting does preserve circuits:

Lemma 2.23. [Bel+22, Lemma 5.11] If $Z = (Z_+, Z_-)$ is a circuit of $\mathcal{O}(Q_{\mathbf{w}})$, then for any $\tau \in \mathfrak{T}(\mathbf{w})$, $\tau(Z) = (\tau(Z_+), \tau(Z_-))$ is also a circuit.

Another key property of the twist group is that twists and flips commute.

Theorem 2.24. [Bel+22, Theorem 5.6] Suppose $\mathcal{T}$ and $\tau(\mathcal{T})$ are triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ for some $\tau \in \mathfrak{T}(\mathbf{w})$. If $\mathcal{T} = \mathcal{T}_Z^+$ can be flipped at a circuit Z, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{T}_Z^+ \arrow[r, "\text{flip in Z}"] \arrow[d, "twist"] & \mathcal{T}_Z^- \arrow[d, "twist"] \\
\tau(\mathcal{T}_Z^+) = \tau(\mathcal{T}_Z^+)^+_{\tau(Z)} \arrow[r, "\text{flip in } \tau(Z)"] & \tau(\mathcal{T}_Z^-) = \tau(\mathcal{T}_Z^+)^-_{\tau(Z)}
\end{tikzcd}$$

This implies that if τ is a twist and $\mathcal{T}$ and $\tau(\mathcal{T})$ are both triangulations, $\mathcal{T}$ and $\tau(\mathcal{T})$ admit the same number of flips. The action of the twist group on the canonical triangulation is particularly interesting.

Theorem 2.25. [Bel+22, Theorem 5.1] Let $\mathbf{w} \in \mathcal{V}$ have length k. The canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ admits exactly $k + 1$ flips.

Theorem 2.26. [Bel+22, Theorem 5.8] The canonical triangulation $\mathcal{T}_{\mathbf{w}}$ of $\mathcal{O}(Q_{\mathbf{w}})$ is a regular triangulation, and for any twist τ, $\tau(\mathcal{T}_{\mathbf{w}})$ is also a regular triangulation.

Finally, combining Theorem 2.24 and Theorem 2.26, we get

Corollary 2.27. [Bel+22, Corollary 5.12] The component of the flip graph of $\mathcal{O}(Q_{\mathbf{w}})$ containing all regular triangulations admits a $\mathfrak{T}(\mathbf{w})$-action by twists.

3. Circuits and a partial ordering on triangulations

3.1. **Enumeration of circuits.** Since circuits of $J(S_k)$ correspond to connected induced subgraphs as described in Theorem 2.16, it is possible to find a recurrence and a closed formula that enumerates the number of such circuits.

Theorem 3.1. The number of circuits of $J(S_k)$ is precisely $9 \cdot 2^k - 3k - 8$.

Proof. From Theorem 2.16, we know that the number of circuits of the vertex set of $\mathcal{O}(S_k)$ is precisely the number of nonempty connected induced subgraphs of $G(\mathbf{w})$. If $J(S_k)$ corresponds to a snake word of the form $\mathbf{w} = \varepsilon(RL)(LR)\cdots$ where there are k such sets of parentheses, then we can represent any element of $G(\mathbf{w})$ as a subword of $\mathbf{w}$. We write $\mathbf{w} = w_0 \cdots w_{2k}$.

Let C_k be the number of induced subgraphs of $G(\mathbf{w})$. We can verify easily that in the S_0 case we have one induced subgraph, so $C_0 = 1$. We now prove a recursive formula for C_k in terms of C_{k-1}. All of the induced subgraphs which were present in $J(S_{k-1})$ will clearly also be present in $J(S_k)$. We now notice that $J(S_k)$ has two more squares than $J(S_{k-1})$, and the corresponding snake word has two more letters: w_{2k-1} and w_{2k}. The three subwords w_{2k-1}, w_{2k}, and $w_{2k-1}w_{2k}$ will each correspond to induced subgraphs which were clearly not in $J(S_{k-1})$. We also add the set of subwords which can be described as a concatenation of a subword of $J(S_{k-1})$ ending with w_{2k-2} and one of these three new subwords. We let B_{k-1} be the number of induced subgraphs which include the bottom square – in other words, the number of induced subgraphs which correspond to a subwords which contains w_{2k-2}. We see now that $C_k = C_{k-1} + 3 + 3 \cdot B_{k-1}$.

We now prove a recursive formula for B_k in terms of B_{k-1}. We see that $B_0 = 1$. We see that there are two new words in B_k which were not in B_{k-1} – w_{2k} and $w_{2k-1}w_{2k}$. We also see that we can construct all remaining subwords in B_k by adding w_{2k} or $w_{2k-1}w_{2k}$ to any subword in B_{k-1}. So $B_k = 2 + 3 \cdot B_{k-1}$.

Combining these formulas, we solve recursively to get $C_k = 9 \cdot 2^k - 3k - 8$. □

3.2. **A partial order on triangulations.** In this section, we describe a partial order defined on triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ that is defined in terms of circuits and bistellar flips between them.

Definition 3.2. We define a relation ordering on our set of triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ as follows: given two triangulations $\mathcal{T}$ and $\mathcal{T}'$, we say that $\mathcal{T} \lessdot_{\mathcal{O}} \mathcal{T}'$ if there exists a bistellar flip between $\mathcal{T}$ and $\mathcal{T}'$ at some circuit Z such that $\mathcal{T}$ contains the triangulation $\mathcal{T}_Z^-$ and $\mathcal{T}'$ contains $\mathcal{T}_Z^+$. It is clear that the cover relations of this partial ordering will correspond precisely to edges in the bistellar flip graph. Taking the transitive closure of this cover relation, we say that $\mathcal{T} \leq_{\mathcal{O}} \mathcal{T}'$ if there exists some sequence of flips to get from $\mathcal{T}$ to $\mathcal{T}'$ where each flip at some circuit Z exchanges a simplex in $\mathcal{T}_Z^-$ for a simplex in $\mathcal{T}_Z^+$.

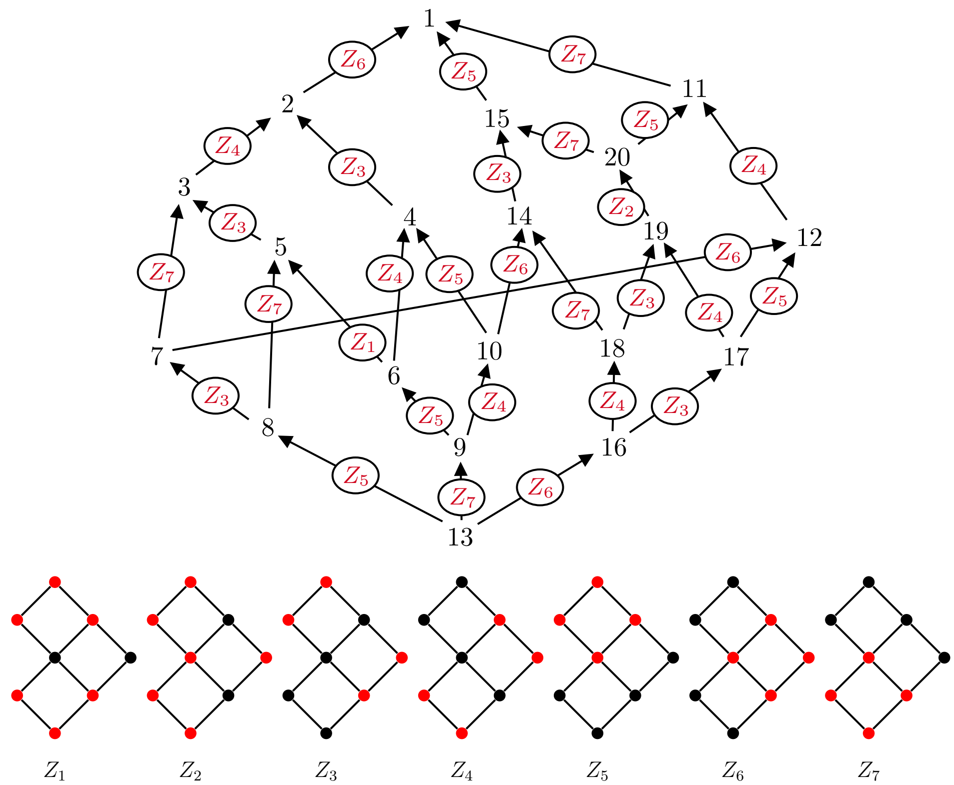


FIGURE 5. The poset of triangulations of $\mathcal{O}(S_1)$. The circuits are enumerated, so an arrow labelled Z_i indicates a flip on circuit Z_i, as labelled in the bottom diagram. The arrow points from the triangulation containing $\mathcal{T}_{Z_i}^-$ to the one containing $\mathcal{T}_{Z_i}^+$. The bottom triangulation (13) is the canonical triangulation.

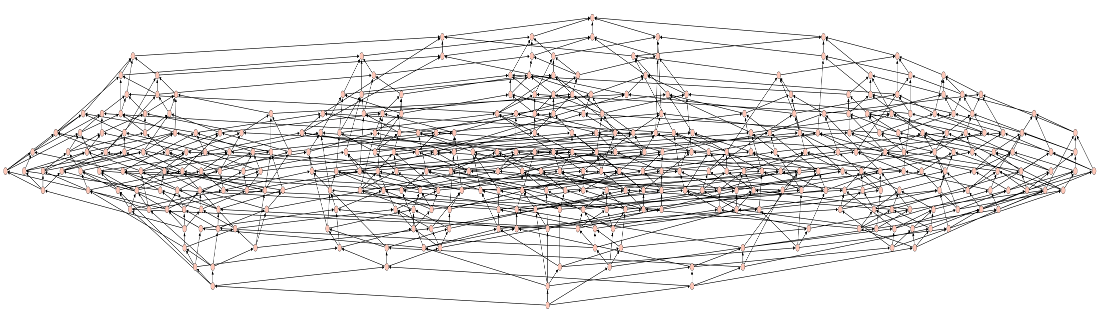


FIGURE 6. The poset of triangulations of $\mathcal{O}(S_2)$.

For example, Figure 5 shows the poset obtained by the ordering above applied to $Q_{\mathbf{w}} = S_1$. It is not trivial that this ordering relation is indeed a partial ordering, which we prove below.

Proposition 3.3. The relation defined in Definition 3.2 is a partial ordering on the set of triangulations of $\mathcal{O}(Q_{\mathbf{w}})$, where $\mathbf{w} = w_0 w_1 \cdots w_n \in \mathcal{V}$.

Proof. It is immediate that the relation is both reflexive and transitive, so we only need to prove that it is antisymmetric.

Let $\mathcal{T}$ be a triangulation of $\mathcal{O}(Q_{\mathbf{w}})$. Each maximal simplex $\omega \in \mathcal{T}$ corresponds to a set of vertices in $\hat{P}(\mathbf{w})$. Since $\hat{P}(\mathbf{w})$ is a ranked poset, for any $\omega \in \mathcal{T}$ we can define a weight $\rho(\omega) = (x_{n+5}, x_{n+4}, \ldots, x_1, x_0)$, where x_i is the number of vertices of rank i in ω. We define a weight function wt on triangulations of $\mathcal{O}(Q_{\mathbf{w}})$.

$$\operatorname{wt}(\mathcal{T}) \coloneqq \sum_{\omega \in \mathcal{T}} \rho(\omega)$$

Since the weights are sequences of non-negative integers, we can order lexicographically. Let $\leq_\ell$ denote lexicographic order. If $\mathcal{T} \lessdot_{\mathcal{O}} \mathcal{T}'$, then $\mathcal{T}' = f_Z(\mathcal{T})$ for some circuit Z such that $\mathcal{T}$ contains $\mathcal{T}_Z^-$ and $\mathcal{T}'$ contains $\mathcal{T}_Z^+$. Following Theorem 2.16, the circuit Z corresponds to a subword $w_{i_1} \cdots w_{i_\ell}$ of $\mathbf{w}$. We know the top vertex $A \in \operatorname{Sq}(w_{i_1})$ will have strictly higher rank than any other vertex in Z and by Definition 2.17, $A \in Z_+$. Hence, every simplex $\omega \in \mathcal{T}$ containing $\mathcal{T}_Z^-$ contains A. In $\mathcal{T}'$, A is contained in exactly one less simplex, and every other vertex of rank greater than or equal to the rank of A is not affected by the flip. Therefore, $\operatorname{wt}(\mathcal{T}') <_\ell \operatorname{wt}(\mathcal{T})$. So for any triangulations $\mathcal{T}, \mathcal{T}'$ with $\mathcal{T} \neq \mathcal{T}'$, if $\mathcal{T} \leq_{\mathcal{O}} \mathcal{T}'$ we have the strict inequality $\operatorname{wt}(\mathcal{T}') < \operatorname{wt}(\mathcal{T})$, so we cannot also have $\mathcal{T}' \leq_{\mathcal{O}} \mathcal{T}$, confirming $\leq_{\mathcal{O}}$ is antisymmetric. □

The above partial ordering seems to satisfy nice properties. In [Kal88], the author describes a useful property that certain orientations of the graph of a polytope satisfy. Let P be a polytope and $G(P)$ be its graph. If O is an acyclic orientation of $G(P)$ then we say that O is a *good orientation* if for every nonempty face F of P, $G(F)$ has exactly one sink. We conjecture that the partial ordering in Definition 3.2 is a lattice, as stated in Conjecture 1.5 and that it has *good orientation*, as stated in Conjecture 1.7.

It is worth remarking that if the above conjectures hold, along with Conjecture 1.1 that the secondary polytope for $\mathcal{O}(S_k)$ is simple, then the orientation induced by our partial ordering satisfies certain hypothesis described in [Her23]. In particular, the orientation satisfies the hypothesis from Theorems 1.1, 1.2 by Hersh and Theorem 7.4 by Preuss in the appendix of [Her23].

If P is a d-dimensional polytope, the *cost* of a vector $v \in P$ is the dot product $\mathbf{c} \cdot v$, where $\mathbf{c} \in \mathbb{R}^d$ is a fixed vector known as the *cost vector*. A cost vector, induces an orientation on the edges of P by following the direction in which the cost of the vertices increases. The directed graph obtained by this orientation on the 1-skeleta of P is denoted by $G(P, \mathbf{c})$.

In our setting, we let $\mathbf{c} = \operatorname{wt}(\mathcal{T}_{\mathbf{w}})$ be the cost vector, where $\mathbf{w} = \epsilon(RL)(LR)\cdots$ is the word inducing $Q_{\mathbf{w}} = S_k$. Then, $G(\mathcal{O}(S_k), \mathbf{c})$ is the Hasse diagram of the partial ordering described earlier in this section. The aforementioned theorems from [Her23] rely on the hypothesis that $G(\mathcal{O}(S_k), \mathbf{c})$ is the Hasse diagram of a lattice L (which is our conjecture above), and they conclude that the Möbius function μ_L only takes values $0, 1, -1$ and that every directed path in L has length at most $n - d$, where n is the number of facets and d is the dimension of the polytope. For a more detailed description of these results, we refer the reader to [Her23].

4. Geometry of the secondary polytope

4.1. **Dual polytopes and flag simplicial complexes.** We begin this section by including a picture of the secondary polytope of triangulations of $\mathcal{O}(S_1)$.

In studying a secondary polytope, it is sometimes also natural to consider its dual polytope, defined as follows:

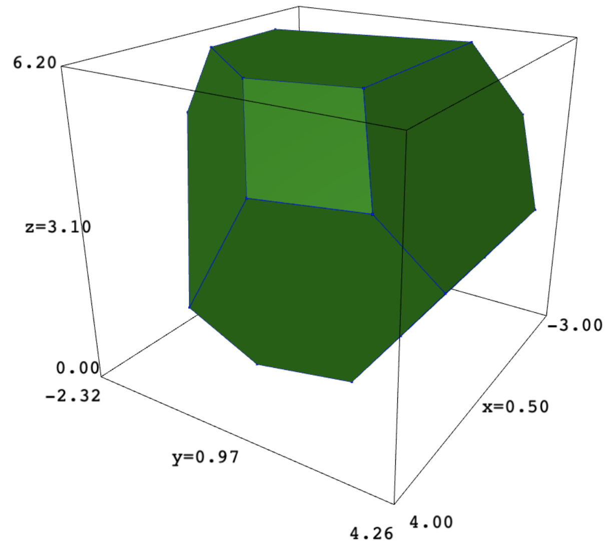


FIGURE 7. $\Sigma_{\mathcal{O}(S_1)}$.

Definition 4.1. Given a polytope $P \subseteq \mathbb{R}^d$, the *polar dual* of P is

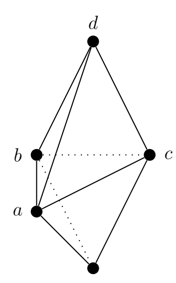
$$P^* := \{\mathbf{c} \in (\mathbb{R}^d)^* : \mathbf{cx} \leq 1 \text{ for all } \mathbf{x} \in P\}$$

We refer to the polar dual of a polytope P simply as the *dual polytope* of P.

In the dual polytope of $\Sigma_{\mathcal{O}(S_k)}$, the facets correspond to triangulations of $\mathcal{O}(S_k)$ and the vertices correspond to facets of $\mathcal{O}(S_k)$. The dual polytope of a simple polytope is always a simplicial complex. In order to better understand the structure of the secondary polytope, it is natural to try and understand the structure of this corresponding simplicial complex. This leads us to the following definition:

Definition 4.2. A *flag simplicial complex* is a simplicial complex with the property that if all facets of some simplex are part of the simplical complex, the simplex itself is as well.

For instance, there cannot be three vertices $\{a, b, c\}$ for which the edges $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$ are all in the complex but the full triangle $\{a, b, c\}$ is not, as in the following example where we take our simplicial complex to be the boundary of a bipyramid.



Our study of the secondary polytopes $\Sigma_{\mathcal{O}(S_k)}$ for $k = 1, 2, 3$ lead us to conjecture that the dual of $\Sigma_{\mathcal{O}(S_k)}$ is always a flag complex, as stated in Conjecture 1.4.

Figure 8 shows the planar embedding of the dual polytope of $\Sigma_{\mathcal{O}(S_1)}$. One can check from the figure that it is a flag simplicial complex, as every 3-cycle of edges in its 1-skeleton actually bounds a triangular boundary face.

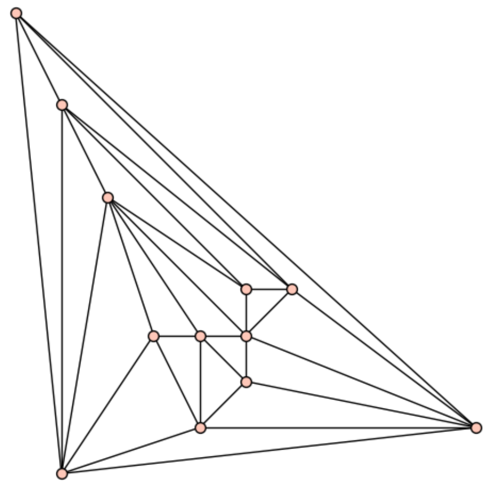


FIGURE 8. Planar embedding of $\Sigma^*_{\mathcal{O}(S_1)}$

4.2. **2D faces of the secondary polytope.** After studying the first few examples of snake posets, we are led to conjecture that the 2D faces of the secondary polytope $\Sigma_{\mathcal{O}(S_k)}$ are all quadrilaterals, pentagons, and hexagons as stated in Conjecture 1.7.

Remark. Conjecture 1.7 bears a strong spiritual similarity to a result of Zelevinsky [Zel06, Prop. 4.6], where the author shows that certain simple polytopes called *nestohedra* have only triangles, quadrangles, pentagons and hexagons as 2-faces. The nestohedra are the polar duals to certain simplicial polytopes whose boundary complexes are what Zelevinsky calls a *nested complex*, and what other authors have called the *complex of nested sets*, associated to a *building set*. Those nestohedra contain as special cases the associahedra and the permutohedra. This similarity may suggest that the secondary polytope of $\mathcal{O}(S_k)$ has a rich geometric structure.

In order to prove this conjecture, we investigate which sequences of flips cause 4-cycles, 5-cycles, and 6-cycles to appear in the bistellar flip graph. We prove a statement that tells us precisely when a given triangulation $\mathcal{T}$ will appear as a vertex of a quadrilateral in the secondary polytope. In Section 8, we make progress towards similar results for pentagons and hexagons. We expect that proving Conjecture 1.7, with careful attention to the direction of arrows in each cycle, should lead to a proof of Conjecture 1.5 that the poset is a lattice via Lemma 2.1 of [Bjö+-6].

4.2.1. *Quadrilaterals.* We begin by introducing notation that we will use throughout this section. If a triangulation $\mathcal{T}$ can be flipped at a circuit Z, we denote the triangulation obtained after flipping at Z by $f_Z(\mathcal{T})$. Further, we denote by $\mathcal{T}_Z$ the triangulation of the circuit Z contained in $\mathcal{T}$. Observe that this notation drops the sign of the triangulation of Z to avoid losing generality.

Definition 4.3. We say that two circuits Z_1, Z_2 *commute at a triangulation* $\mathcal{T}$ if:

(a) $\mathcal{T}$ contains $\mathcal{T}_{Z_1}, \mathcal{T}_{Z_2}$ and can be flipped at these circuits, and
(b) $f_{Z_1}(\mathcal{T})$ contains $\mathcal{T}_{Z_2}$ and can be flipped at Z_2, and $f_{Z_2}(\mathcal{T})$ contains $\mathcal{T}_{Z_1}$ and can be flipped at Z_1, and
(c) $f_{Z_2}(f_{Z_1}(\mathcal{T})) = f_{Z_1}(f_{Z_2}(\mathcal{T}))$

The following proposition relates the notion of commutation of circuits defined above to the flip graph of $\mathcal{O}(Q_{\mathbf{w}})$. In particular, it relates commuting circuits to faces with 4 elements in the flip graph that are oriented in a particularly interesting way following Definition 3.2.

Proposition 4.4. If two circuits Z_1 and Z_2 commute at a triangulation $\mathcal{T}$ of $\mathcal{O}(Q_{\mathbf{w}})$, then there exist four triangulations $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ such that:

- $\mathcal{T}_1$ contains $\mathcal{T}_{Z_1}^-$ and $\mathcal{T}_{Z_2}^-$
- $\mathcal{T}_2$ contains $\mathcal{T}_{Z_1}^-$ and $\mathcal{T}_{Z_2}^+$
- $\mathcal{T}_3$ contains $\mathcal{T}_{Z_1}^+$ and $\mathcal{T}_{Z_2}^-$
- $\mathcal{T}_4$ contains $\mathcal{T}_{Z_1}^+$ and $\mathcal{T}_{Z_2}^+$

and $f_{Z_2}(\mathcal{T}_1) = \mathcal{T}_2$, $f_{Z_1}(\mathcal{T}_2) = \mathcal{T}_4$, and $f_{Z_2}(\mathcal{T}_4) = \mathcal{T}_3$.

$$\begin{tikzcd}
& \mathcal{T}_4 & \\
\mathcal{T}_2 \arrow[ur, "Z_1"] & & \mathcal{T}_3 \arrow[ul, "Z_2"'] \\
& \mathcal{T}_1 \arrow[ul, "Z_2"] \arrow[ur, "Z_1"'] &
\end{tikzcd}$$

Proof. By Definition 4.3, if two circuits Z_1 and Z_2 commute at a triangulation $\mathcal{T}$, then $\mathcal{T}$ can be flipped at Z_1 and Z_2. So $f_{Z_1}(\mathcal{T})$ exists and contains $\mathcal{T}_{Z_1}$ with the opposite sign, and $f_{Z_2}(\mathcal{T})$ exists and contains $\mathcal{T}_{Z_2}$ with the opposite sign. Since Z_1 and Z_2 commute at $\mathcal{T}$, we also know that $f_{Z_1}(\mathcal{T})$ can be flipped at Z_2 and $f_{Z_2}(\mathcal{T})$ can be flipped at Z_1, and in particular $f_{Z_2}(f_{Z_1}(\mathcal{T})) = f_{Z_1}(f_{Z_2}(\mathcal{T}))$. By construction, this triangulation will contain both $\mathcal{T}_{Z_1}$ and $\mathcal{T}_{Z_2}$, each with the opposite sign to how they appear in $\mathcal{T}$. So we see that we have four triangulations $\mathcal{T}$, $f_{Z_1}(\mathcal{T})$, $f_{Z_2}(\mathcal{T})$, and $f_{Z_1}(f_{Z_2}(\mathcal{T}))$ such that there exists one triangulation for every configuration of positive and negative in $\mathcal{T}_{Z_1}$ and $\mathcal{T}_{Z_2}$ and, once these are assigned, it is not hard to check that they satisfy the relations above. $\square$

We now give a sufficient condition for a pair of circuits to be preserved after a flip in either of them, and also show that this sufficient condition implies commutation as well.

Proposition 4.5. If a triangulation $\mathcal{T}$ of $\mathcal{O}(Q_{\mathbf{w}})$ can be flipped at two circuits Z_1, Z_2 such that $Z_1 \cap Z_2 = \varnothing$, then $f_{Z_1}(\mathcal{T})$ and $f_{Z_2}(\mathcal{T})$ can be flipped at Z_2 and Z_1, respectively. Moreover, $f_{Z_1}(f_{Z_2}(\mathcal{T})) = f_{Z_2}(f_{Z_1}(T))$

Proof. We assume that $\mathcal{T}$ can be flipped at Z_1, Z_2 and that it contains $\mathcal{T}_{Z_1}^-$ and $\mathcal{T}_{Z_2}^-$, the other cases following by the same argument. For the first statement, it suffices to show that $f_{Z_1}(\mathcal{T})$ can be flipped at Z_2. Since $Z_1 \cap Z_2 = \varnothing$, then $f_{Z_1}(\mathcal{T})$ contains $\mathcal{T}_{Z_2}^-$. Thus, we only need to prove that any two simplices of $\mathcal{T}_{Z_2}^-$ have the same link in $f_{Z_1}(\mathcal{T})$. Let $\sigma \in \mathcal{T}_{Z_2}^-$, and define

$$B_\sigma = \{\omega \setminus \delta : \omega \in \mathrm{lk}_{\mathcal{T}}(\sigma) \text{ and } \delta \in \omega \text{ for some } \delta \in \mathcal{T}_{Z_1}^-\}$$

By definition of the link, we obtain

$$\begin{aligned}\mathrm{lk}_{f_{Z_1}(\mathcal{T})}(\sigma) = \{\omega \colon \omega \in \mathrm{lk}_{\mathcal{T}}(\sigma) \text{ and } \delta \notin \omega \text{ for all } \delta \in \mathcal{T}_{Z_1}^-\} \\ \cup \bigcup_{\xi \in B_\sigma} \{\xi \cup \delta : \delta \in \mathcal{T}_{Z_1}^+\}\end{aligned}$$

Since $\mathrm{lk}_{\mathcal{T}}(\sigma) = \mathrm{lk}_{\mathcal{T}}(\delta)$ for any $\delta \in \mathcal{T}_{Z_2}^-$, then we have $B_\sigma = B_\delta$ and thus $\mathrm{lk}_{f_{Z_1}(\mathcal{T})}(\sigma) = \mathrm{lk}_{f_{Z_1}(\mathcal{T})}(\delta)$. Therefore, $f_{Z_1}(\mathcal{T})$ can be flipped at Z_2, and by the same argument we conclude that $f_{Z_2}(\mathcal{T})$ can be flipped at Z_1.

We now prove that $f_{Z_2}(f_{Z_1}(\mathcal{T})) = f_{Z_1}(f_{Z_2}(\mathcal{T}))$ by showing that they have the same maximal simplices. Since we can replace Z_1 by Z_2 and viceversa, it suffices to show that $f_{Z_1}(f_{Z_2}(\mathcal{T})) \subseteq f_{Z_2}(f_{Z_1}(\mathcal{T}))$. Let $\sigma \in f_{Z_1}(f_{Z_2}(\mathcal{T}))$.

If σ does not contain any simplex in $\mathcal{T}_{Z_1}^+$ or in $\mathcal{T}_{Z_2}^+$, then since $Z_1 \cap Z_2 = \varnothing$ we must have $\sigma \in \mathcal{T}$ and σ is not affected by either of the flips, so $\sigma \in f_{Z_2}(f_{Z_1}(\mathcal{T}))$.

If σ contains a simplex $\delta_1 \in \mathcal{T}_{Z_1}^+$ but no simplex in $\mathcal{T}_{Z_2}^+$, then for all $\omega \in \mathcal{T}_{Z_1}^-$, we have $(\sigma \setminus \delta_1) \cup \omega \in \mathcal{T}$ and so $(((\sigma \setminus \delta_1) \cup \omega) \setminus \omega) \cup \delta \in f_{Z_1}(\mathcal{T})$ for all $\delta \in \mathcal{T}_{Z_1}^+$. In particular, $(((\sigma \setminus \delta_1) \cup \omega) \setminus \omega) \cup \delta_1 = \sigma \in f_{Z_1}(\mathcal{T})$ and since σ is not changed after flipping at Z_2, we obtain $\sigma \in f_{Z_2}(f_{Z_1}(\mathcal{T}))$.

If σ contains a simplex $\delta_2 \in \mathcal{T}_{Z_2}^+$ but no simplex in $\mathcal{T}_{Z_1}^+$, then a similar argument as above implies that $\sigma \in f_{Z_2}(f_{Z_1}(\mathcal{T}))$.

Lastly, suppose that σ contains $\delta_1 \in \mathcal{T}_{Z_1}^+$ and $\delta_2 \in \mathcal{T}_{Z_2}^+$. Then, $(\sigma \setminus \delta_2) \cup \omega_2 \in f_{Z_1}(\mathcal{T})$ for all $\omega_2 \in \mathcal{T}_{Z_2}^-$. Since $Z_1 \cap Z_2 = \varnothing$, then $\delta_1 \in (\sigma \setminus \delta_2) \cup \omega_2$. Thus $(\sigma \setminus (\delta_1 \cup \delta_2)) \cup (\omega_1 \cup \omega_2) \in \mathcal{T}$ for all $\omega_1 \in \mathcal{T}_{Z_1}^-$ and all $\omega_2 \in \mathcal{T}_{Z_2}^-$. Hence, for all $\xi_1 \in \mathcal{T}_{Z_1}^+$ and $\xi_2 \in \mathcal{T}_{Z_2}^+$, we have $(\sigma \setminus (\delta_1 \cup \delta_2)) \cup (\xi_1 \cup \xi_2) \in f_{Z_1}(f_{Z_2}(\mathcal{T}))$ and choosing $\xi_1 = \delta_1$ and $\xi_2 = \delta_2$ yields $\sigma \in f_{Z_2}(f_{Z_1}(\mathcal{T}))$, finishing the proof. $\square$

In the previous proposition, we gave a sufficient condition for a pair of circuits to commute. However, we can improve this result and prove sufficient and necessary conditions for two circuits to commute at a given triangulation.

Theorem 1.8. Let Z_1 and Z_2 be circuits of $\mathcal{O}(Q_{\mathbf{w}})$, where $\mathbf{w} \in \mathcal{V}$. Then Z_1 and Z_2 commute at $\mathcal{T}$ if and only if $\mathcal{T}$ can be flipped at Z_1 and Z_2 and at least one of the following hold:

(i) Z_1 and Z_2 appear on different maximal simplices in $\mathcal{T}$, or
(ii) Z_1 and Z_2 share no vertex

Proof. We prove the forward direction by contrapositive. Assume that Z_1 and Z_2 occur in the same simplex. Then, the link condition implies that for all $\delta_1 \in \mathcal{T}_{Z_1}^-$ and $\delta_2 \in \mathcal{T}_{Z_2}^-$, there exists some maximal simplex $S \in \mathcal{T}$ such that $\delta_1, \delta_2 \in S$.

Suppose that there exist some subsets $z_1 \subsetneq Z_1$ and $z_2 \subsetneq Z_2$ such that $z_1 \cup z_2$ is a circuit. Since z_1 and z_2 are strictly smaller than Z_1 and Z_2, respectively, then there must exist some simplex σ_1 in $\mathcal{T}_{Z_1}^-$ or $\mathcal{T}_{Z_1}^+$ which contains z_1. Similarly, there must be some simplex σ_2 in $\mathcal{T}_{Z_2}^-$ or $\mathcal{T}_{Z_2}^+$ which contains z_2. Finally, there must be some maximal simplex in $\mathcal{T}_i$ for $i = 1, 2, 3$ or 4 which contains σ_1 and σ_2, which is a contradiction.

Now suppose v is a vertex such that $v \in Z_1 \cap Z_2$. By Proposition 4.4, we can assume without loss of generality that $\mathcal{T}$ contains $\mathcal{T}_{Z_1}^-$ and $\mathcal{T}_{Z_2}^-$. If v is negative in at least one of these circuits, without loss of generality Z_1, then there exists some $\sigma_1 \in \mathcal{T}_{Z_1}^-$ such that $v \notin \sigma_1$. However, there clearly exists some $\sigma_2 \in \mathcal{T}_{Z_2}^-$ such that $v \in \sigma_2$, so these two simplices σ_1 and σ_2 cannot both be contained in some maximal simplex S and we have already reached a contradiction.

Therefore, any $v \in Z_1 \cap Z_2$ must be positive in both circuits. Fix some arbitrary $\delta_1 \in \mathcal{T}_{Z_1}^-$ and $\delta_2 \in \mathcal{T}_{Z_2}^-$ and consider the maximal simplex $S \in \mathcal{T}$ which contains both δ_1 and δ_2. We imagine first applying a flip at Z_1 to $\mathcal{T}$. We choose some $v \in Z_1 \cap Z_2$. We know there exists precisely one simplex $\delta \in \mathcal{T}_{Z_1}^+$ such that δ does not contain v. After flipping at Z_1, we know that the resulting triangulation will have some maximal simplex S' which is our original S but with δ instead of δ_1. In particular, $v \notin S'$. We know that v is also a positive vertex in Z_2, so all simplices in $\mathcal{T}^-_{Z_2}$ contain v. Hence, S' does not contain any simplex in $\mathcal{T}^-_{Z_2}$ and will not be changed when we flip at Z_2.

We now show that if we instead flipped at Z_2 and then flipped at Z_1, the resulting triangulation would not contain S'. Assume for the sake of contradiction that $S' \in f_{Z_1}(f_{Z_2}(\mathcal{T}))$, where f_i represents flipping at circuit Z_i^- and f_i^- is flipping at circuit Z_i^+. By construction, S' contains $\delta \in \mathcal{T}^+_{Z_1}$, so $f_{Z_1}^{-1}(f_{Z_1}(f_{Z_2}(\mathcal{T}))) = f_{Z_2}(\mathcal{T})$ must contain S. However, S contains δ_2, which is in $\mathcal{T}^-_{Z_2}$, which forces a contradiction. Therefore, Z_1 and Z_2 do not commute.

For the converse, we assume that $\mathcal{T}$ can be flipped at Z_1 and Z_2 and that it contains $\mathcal{T}^-_{Z_1}$ and $\mathcal{T}^-_{Z_2}$. Since a flip at a given circuit Z only changes those simplices containing $\mathcal{T}_Z$, it is straightforward to see that if $\mathcal{T}^-_{Z_1}$ and $\mathcal{T}^-_{Z_2}$ are contained in different simplices of $\mathcal{T}$, then $f_{Z_1}(f_{Z_2}(\mathcal{T})) = f_{Z_2}(f_{Z_1}(\mathcal{T}))$. On the other hand, if $Z_1 \cap Z_2 = \varnothing$, then Proposition 4.5 immediately implies that Z_1 and Z_2 commute at $\mathcal{T}$. □

5. Valence-regularity of the bistellar flip graph

We recall Conjecture 1.1. In order to make progress towards proving this conjecture, it is natural to begin by considering a subset of triangulations $\mathcal{T}$ in $\mathcal{O}(Q_{\mathbf{w}})$. This leads us to the following result.

Theorem 1.9. Let $\mathcal{T}$ be a triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ obtained by applying one flip to the canonical triangulation $\mathcal{T}_{\mathbf{w}}$, where $\mathbf{w} = w_0 w_1 \cdots w_n \in \mathcal{V}$. Then, $\mathcal{T}$ admits $n+1$ flips.

Proof. We know from Theorem 2.25 that the canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ admits $n+1$ flips, and these occur precisely at the squares. Let $\mathcal{T}_i$ be the triangulation obtained by applying a flip to $\mathcal{T}_{\mathbf{w}}$ at the circuit $Z_i = \mathrm{Sq}(w_i)$.

Let $Z_j = \mathrm{Sq}(w_j)$ be such that $Z_j \cap Z_i = \varnothing$. By Proposition 4.5, it follows that $\mathcal{T}_i$ can be flipped at Z_j.

Suppose then that $Z_j \cap Z_i \neq \varnothing$ and that $j \neq i+1$ and $j \neq i-1$. Let A be the unique vertex of intersection in $Z_j \cap Z_i$ and let $\sigma \in \mathcal{T}^-_{Z_j}$. Let $\mathcal{T}^-_{Z_1} = \{\delta_1, \delta_2\}$ and $\mathcal{T}^+_{Z_1} = \{\delta_3, \delta_4\}$. Without loss of generality, we might assume that $\sigma \cup \delta_4$ contains Z_i. Hence, no element in $\mathrm{lk}_{\mathcal{T}_i}(\sigma)$ contains δ_4. See Figure 9 for an illustration.

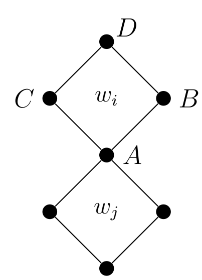


Figure 9. The case $|Z_j \cap Z_i| = 1$, with $\delta_4 = \{B, C, D\}$

Then, the link of σ in $\mathcal{T}_i$ is given by:

$$\mathrm{lk}_{\mathcal{T}_i}(\sigma) = \{S : S \in \mathrm{lk}_{\mathcal{T}_{\mathbf{w}}}(\sigma) \text{ and } \delta_1, \delta_2 \notin S \cup \{A\}\}$$
$$\cup \{(S \setminus \delta_1) \cup (\delta_3 \setminus \{A\}) : S \in \mathrm{lk}_{\mathcal{T}_{\mathbf{w}}} \text{ and } \delta_1 \in S\}.$$

Since all simplices in $\mathcal{T}_{Z_j}^-$ have the same link in $\mathcal{T}_{\mathbf{w}}$, it follows that $\mathcal{T}_i$ admits flips at squares $\mathrm{Sq}(w_j)$ where $j \neq i+1$ and $j \neq i-1$.

It is also clear that $\mathcal{T}_i$ contains $\mathcal{T}_{Z_i}^+$. We now show that $\mathcal{T}_i$ can be flipped at precisely two more circuits, or one more circuit if $i = 0$ or $i = 2k$. We claim that $\mathcal{T}_i$ can be flipped at the rectangles $w_i w_{i+1}$ and $w_{i-1} w_i$. We show this for $w_i w_{i+1}$, and the other case follows by the same argument,

Let Z be the circuit corresponding to the rectangle $w_i w_{i+1}$. In Figure 10, the two sets of red vertices are the two simplices $\sigma_1, \sigma_2 \in \mathcal{T}_Z^-$, where $\sigma_1 = \{A, B, E\}$ and $\sigma_2 = \{B, E, F\}$. We consider the links of each of these two simplices in $\mathcal{T}_i$. We take ω_1 to be a maximal simplex in $\mathcal{T}_i$ which contains σ_1. If ω_1 doesn't contain C or D, then it does not contain any triangulation of $\mathrm{Sq}(w_i)$ and was not changed by the flip at $\mathrm{Sq}(w_i)$, so it is a maximal chain. However, no maximal chain exists which passes through A and B but avoids C, so we have reached a contradiction. If ω_1 contains C, then it contains a simplex in $\mathcal{T}_{Z_i}^-$, so we have reached a contradiction since $\mathcal{T}_i$ does not contain any such simplex. Thus, we must have that $D \in \omega_1$. We now consider σ_2 and take ω_2 to be any maximal simplex in $\mathcal{T}_i$ which contains σ_2. If ω_2 contained A, then it would contain a circuit, so ω_2 does not contain $\mathcal{T}_{Z_i}^+$ since all the simplices in this triangulation contain A. Hence, ω_2 must be a maximal chain, so it must contain D and avoid C. It then readily follows that $\mathrm{lk}_{\mathcal{T}_i}(\sigma_1) = \mathrm{lk}_{\mathcal{T}_i}(\sigma_2)$.

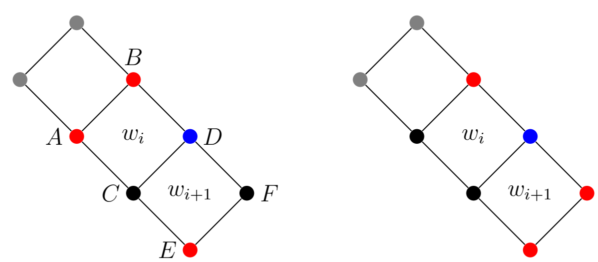


FIGURE 10. The rectangle $w_i w_{i+1}$

We have now proven that $\mathcal{T}_i$ admits at least $2k+1$ flips. It remains to show that $\mathcal{T}_i$ cannot be flipped at any other circuit. We begin by considering any circuit Z which has more than 4 elements. To prove that $\mathcal{T}_i$ cannot be flipped at Z, it suffices to show that Z contains at least one pair of incomparable elements $\{A, B\}$ which have the same sign in Z and are not both contained in $\mathrm{Sq}(w_i)$. To justify this, we begin by noting that since Z has at least six elements, there must exist at least one simplex in $\mathcal{T}_Z^-$ and at least one simplex in $\mathcal{T}_Z^+$ which contains both A and B. We assume for the sake of contradiction that $\mathcal{T}_i$ can be flipped at Z and let ω be a maximal simplex which contains both A and B. If neither A nor B are contained in $\mathrm{Sq}(w_i)$, then we conclude that A and B are both contained in some maximal chain and reach a contradiction. If $B \in \mathrm{Sq}(w_i)$ without loss of generality, then we consider two cases: either ω contains a simplex $\sigma \in \mathcal{T}_{Z_i}^+$ or it does not. If it does not, then ω is a maximal chain and we reach a contradiction. If it does, we begin by noting that there must exist some simplex $\varsigma \in \mathcal{T}_{Z_i}^-$ which contains B. Since $\mathrm{lk}_{\mathcal{T}_i}(\sigma) = \mathrm{lk}_{\mathcal{T}_{\mathbf{w}}}(\varsigma)$, we know there exists some maximal chain in $\mathcal{T}_{\mathbf{w}}$ which contains B and A, and we again reach a contradiction.

It is not hard to check that if Z has more than four elements, it will have two pairs of incomparable elements, where the elements of each pair has the same sign in the circuit. Since there are two such pairs, one of them is not entirely contained in $\mathrm{Sq}(w_i)$.

We now prove that $\mathcal{T}_i$ cannot be flipped at $\mathrm{Sq}(w_{i+1})$ or $\mathrm{Sq}(w_{i-1})$, if these squares exist. If $\mathcal{T}_i$ contains a simplex in $\mathcal{T}_Z^+$ for either of these circuits, then $\mathcal{T}_i$ will have an incomparable pair of elements not entirely contained in $\mathrm{Sq}(w_i)$, so we reach a contradiction. We show that it cannot be flipped at $\mathcal{T}_{Z_{i+1}}^-$ for $Z_{i+1} = \mathrm{Sq}(w_{i+1})$, and the other case follows similarly.

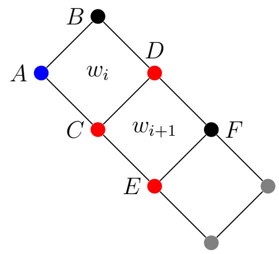
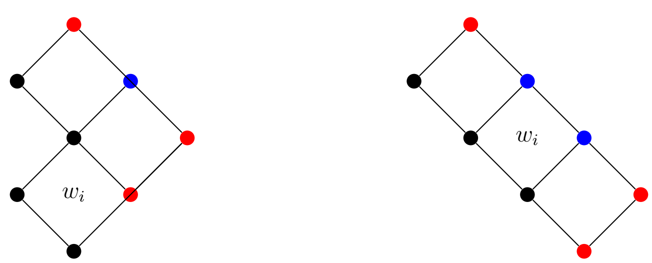


FIGURE 11. The case $Z_{i+1} = \mathrm{Sq}(w_{i+1})$

Let $\sigma_1 = \{D, C, E\}$ and $\sigma_2 = \{D, F, E\}$ be the two simplices of $\mathcal{T}_{Z_{i+1}}^-$ as shown in Figure 11. We know there exists some maximal chain in $\mathcal{T}_{\mathbf{w}}$ which includes $\{B, D, C, E\}$, and since $\{B, C, D\} \in \mathcal{T}_{Z_i}^-$, we conclude that there exists a maximal simplex $\omega_1 \in \mathcal{T}_i$ which contains $\{A, D, C, E\}$. Thus, $\mathrm{lk}_{\mathcal{T}_i}(\sigma_1)$ has some element which contains A. Now assume for the sake of contradiction that there exists a maximal simplex $\omega_2 \in \mathcal{T}_i$ which contains A. Note that ω_2 is therefore not a maximal chain, so it must contain some simplex in $\mathcal{T}_{Z_i}^+$. Then, there exists some maximal chain in $\mathcal{T}_{\mathbf{w}}$ which contains a simplex in $\mathcal{T}_{Z_i}^-$ and also contains $\{E, F\}$, since these are not affected by the flip at $\mathrm{Sq}(w_i)$. However, all simplices in $\mathcal{T}_{Z_i}^-$ contain C, so we have reached a contradiction since C and F are incomparable. Therefore, $\mathrm{lk}_{\mathcal{T}_i}(\sigma_2)$ does not contain any element which contains A, so $\mathrm{lk}_{\mathcal{T}_i}(\sigma_1) \neq \mathrm{lk}_{\mathcal{T}_i}(\sigma_2)$ and so it is not possible to flip at Z_{i+1}.

Lastly, we consider the rectangles that share some vertices with $\mathrm{Sq}(w_i)$ and that we have not considered earlier in the proof. In particular, these are the 4-element circuits containing $w_{i-2}w_{i-1}$, or $w_{i-1}w_iw_{i+1}$, or $w_{i+1}w_{i+2}$. In these cases, it is not hard to check that for $\mathcal{T}_Z^-$, the simplices fail to meet the link condition. In the two examples below, each set of red vertices represents a simplex σ in $\mathcal{T}_Z^-$ for some rectangle Z, and the corresponding blue vertices represent elements in $\mathrm{lk}_{\mathcal{T}_i}(\sigma)$ which are not in link of the unique other simplex in $\mathcal{T}_Z^-$. For $\mathcal{T}_Z^+$, neither simplex can appear in $\mathcal{T}_i$ since any simplex contains a pair of incomparable elements which is not contained in $\mathrm{Sq}(w_i)$.



□

6. Freeness of the twist action on regular triangulations

In this section, we focus our attention on Conjecture 1.3. In order to prove that the number of regular triangulations of $\mathcal{O}(S_k)$ is $2^{k+1} \cdot \mathrm{Cat}(2k+1)$, one approach could be to prove that the orbit of each twist in the twist group contains precisely 2^{k+1} elements. In order for this to be true, we need the twist action to be free. With this motivation in mind, we introduce the following series of definitions and propositions, which culminate in a proof of Theorem 1.10. Throughout this section, we assume that $\mathbf{w} \in \mathcal{V}$ as in Definition 2.14.

Definition 6.1. Given a generalized snake poset $\hat{P}(\mathbf{w})$ and a twist $\tau \in \mathfrak{T}(\mathbf{w})$, we define the twisted poset $\tau(\hat{P}(\mathbf{w}))$ to have the covering relations given by $\tau(a) \lessdot \tau(b)$ in $\tau(\hat{P}(\mathbf{w}))$ if and only if $a \lessdot b$ in $\hat{P}(\mathbf{w})$.

Proposition 6.2. Let $\mathcal{T}_{\mathbf{w}}$ be the canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ and let τ be a twist. Then, the maximal simplices of the triangulation $\tau(\mathcal{T}_{\mathbf{w}})$ correspond to the maximal chains of $\tau(J(Q_{\mathbf{w}}))$.

Proof. We begin by noticing that Theorem 2.26 implies that the maximal simplices of $\tau(\mathcal{T}_{\mathbf{w}})$ are given by

$$\{\tau(\sigma) : \sigma \in \mathcal{T}_{\mathbf{w}} \text{ a maximal simplex}\}$$

Let $\sigma = \{\mathbf{v}_{\varnothing}, \mathbf{v}_{\{a_1\}}, \ldots, \mathbf{v}_{\{a_1,\ldots,a_\ell\}}, \mathbf{v}_{S_k}\}$ be a maximal simplex in $\mathcal{T}_{\mathbf{w}}$ indexed by a maximal chain of $J(S_k)$. By definition of the twist action on the elements of $J(S_k)$, we have

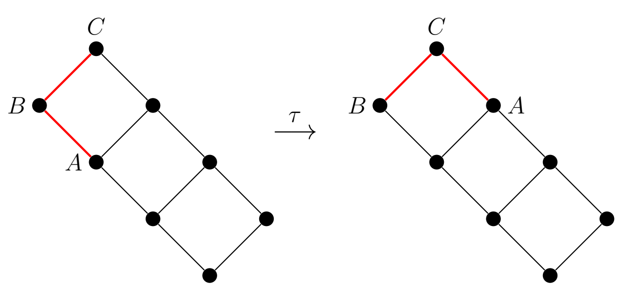
$$\tau(\sigma) = \{\mathbf{v}_{\varnothing}, \mathbf{v}_{\{\tau(a_1)\}}, \ldots, \mathbf{v}_{\{\tau(a_1),\ldots,\tau(a_\ell)\}}, \mathbf{v}_{S_k}\}$$

By the covering relation given in Definition 6.1, $(\varnothing, \{\tau(a_1)\}, \ldots, \{a_1, \ldots, a_\ell\}, S_k)$ is a maximal chain in $\tau(J(S_k))$. Hence, $\tau(\sigma)$ corresponds to a maximal chain of $\tau(J(S_k))$. ☐

Proposition 6.3. Let $Q_{\mathbf{w}}$ be a generalized snake poset and let $\mathcal{T}_{\mathbf{w}}$ be its canonical triangulation. For any twist $\tau \in \mathfrak{T}(\mathbf{w})$, we have $\tau(\mathcal{T}_{\mathbf{w}}) \neq \mathcal{T}_{\mathbf{w}}$

Proof. We assume for the sake of contradiction that there exists some $\tau = \tau_{i_1} \cdots \tau_{i_k}$ such that $\tau(\mathcal{T}_{\mathbf{w}}) = \mathcal{T}_{\mathbf{w}}$.

Let $i = \min\{i_1, ..., i_k\}$. We first consider the case where $i \neq 1$. By construction, τ_i corresponds to twisting the ladder $\mathcal{L}^i$, the "highest" ladder in our poset such that τ acts non-trivially on all of its vertices. We can assume without loss of generality that the topmost square of $\mathcal{L}^i$ corresponds to an L in the word $\mathbf{w} = \varepsilon RLLR\ldots$. Now consider the vertices A, B, C in the topmost square of $\mathcal{L}^i$ as shown in the figure below, where left-hand side is in $J(Q_{\mathbf{w}})$ and the right-hand side is in $\tau(J(Q_{\mathbf{w}}))$. Here we note that since $\mathcal{L}^i$ is the topmost ladder, then $\{A, B, C\}$ are not contained in any other ladders permuted by τ.



We know that simplices of the $\mathcal{T}_{\mathbf{w}}$ correspond to maximal chains of $J(Q_{\mathbf{w}})$. Hence, there exists some simplex of $\mathcal{T}_{\mathbf{w}}$ which contains the set of vertices $\{\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C\}$.

By Proposition 6.2, the maximal simplices of $\tau(\mathcal{T}_{\mathbf{w}})$ correspond to the maximal chains of $\tau(J(Q_{\mathbf{w}}))$. However, in $\tau(J(Q_{\mathbf{w}}))$, the elements A, B are incomparable, which implies that there is no maximal chain that contains both of them. It is not hard to see that if

$i = 1$, the same argument applies, with a few modifications. In particular, A and B will also be incomparable in $\tau(Q_{\mathbf{w}})$. Thus, no maximal simplex in $\tau(\mathcal{T}_{\mathbf{w}})$ contains the vertices $\{\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C\}$, which contradicts $\tau(\mathcal{T}_{\mathbf{w}}) = \mathcal{T}_{\mathbf{w}}$. □

Theorem 1.10. The twist group acts freely on the regular triangulations of $\mathcal{O}(Q_{\mathbf{w}})$.

Proof. Assume for the sake of contradiction that the twist action is not free, so there exists some triangulation $\mathcal{T}$ of $\mathcal{O}(S_k)$ and some twist τ such that $\tau(\mathcal{T}) = \mathcal{T}$. Since the bistellar flip graph is connected, there exists some finite set of flips $f_{Z_1}, f_{Z_2}, ...f_n$ such that $(f_n \circ ... \circ f_{Z_2} \circ f_{Z_1})(\mathcal{T}) = \mathcal{T}_{\mathbf{w}}$, where $\mathcal{T}_{\mathbf{w}}$ denotes the canonical triangulation for $\mathcal{O}(Q_{\mathbf{w}})$. By Theorem 2.26, it follows that every square in the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\tau"] \arrow[d, "f_{Z_1}"] & \mathcal{T} \arrow[d, "f_{Z_1}"] \\
f_{Z_1}(\mathcal{T}) \arrow[r, "\tau"] \arrow[d, "f_{Z_2}"] & f_{Z_1}(\mathcal{T}) \arrow[d, "f_{Z_2}"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
f_{n-1} \circ \cdots \circ f_{Z_1}(\mathcal{T}) \arrow[r, "\tau"] \arrow[d, "f_n"] & f_{n-1} \circ \cdots \circ f_{Z_1}(\mathcal{T}) \arrow[d, "f_n"] \\
\mathcal{T}_{\mathbf{w}} \arrow[r, "\tau"] & \mathcal{T}_{\mathbf{w}}
\end{tikzcd}$$

Therefore, we must have $\tau(\mathcal{T}_{\mathbf{w}}) = \mathcal{T}_{\mathbf{w}}$, so by Proposition 6.3 we have reached a contradiction. □

Corollary 6.4. Each orbit under the twist group action on regular triangulations of $\mathcal{O}(S_k)$ has 2^{k+1} elements.

7. Eigenbasis of the twist action

Viewed in 3D, the actions of the elementary twists τ_1 and τ_2 on the secondary polytope of S_1 are both 180° rotations through axes that pass through the centers of two hexagons. $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$ is just the composition of these two rotations. The axes of rotation pass through the centers of distinct pairs of opposing hexagons.

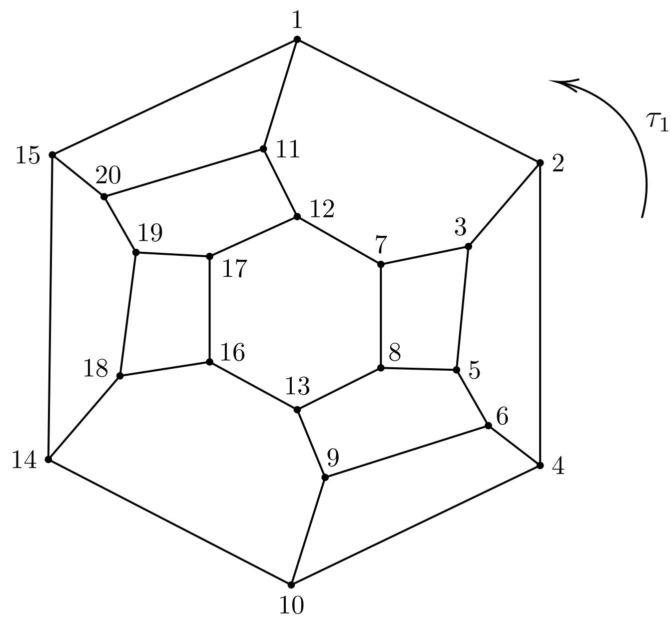


FIGURE 12. Planar embedding of the secondary polytope of S_1 in which the twist τ_1 is a 180° rotation. For example, $\tau_1(\mathcal{T}_{13}) = \mathcal{T}_{12}$.

This is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ as expected. This suggests there may be a nice general description of the action of the twist group on the secondary polytope of any $Q_{\mathbf{w}}$.

Proposition 7.1. Let $\tau_1, \dots, \tau_r$ be commuting diagonalizable operators on a vector space V over a field k, and $v_1, \dots, v_s$ be simultaneous eigenvectors for all of the τ_i, say

$$\tau_i v_j = \lambda_{ij} v_j.$$

Assume also that no two $v_j, v_{j'}$ share the same list of τ_i-eigenvalues, that is, for all $j \neq j'$, there exists some i with $\lambda_{i,j} \neq \lambda_{i,j'}$. Then the $v_1, \dots, v_s$ are linearly independent.

Proof. Create a new operator $\tau = c_1\tau_1 + \cdots + c_r\tau_r$ with coefficients $c_i \in k$, and note that $v_1, \dots, v_r$ are eigenvectors for τ, with v_j having eigenvalue $\sum_{i=1}^r c_i\lambda_{ij}$. By extending the field k if necessary, one can pick the c_i so that these eigenvalues are all distinct (by our assumption about the lists of τ_i-eigenvalues on the v_j being distinct). But then $v_1, \dots, v_s$ are eigenvectors for distinct eigenvalues of τ, and hence linearly independent. $\square$

Since the twists τ_i are involutions, their eigenvalues acting on any space lie in the set $\{+1, -1\}$. And since the elements of the twist group commute, so do the corresponding matrices. This implies the matrices of the elementary twists can be simultaneously diagonalized, i.e. there is an eigenbasis in which all are diagonal. Then the eigenvalues are simply the diagonals. Such a basis is particularly nice because the eigenvalues of the non-elementary twists are easily computed by matrix multiplication, which (because the matrices are diagonal) works out to multiplying the eigenvalues place by place in the order they appear on the diagonal.

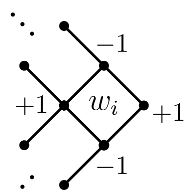
Say $\mathbf{w} \in \mathcal{V}$. $\mathcal{O}(Q_{\mathbf{w}})$ is a $|Q_{\mathbf{w}}|$-dimensional polytope. If $w = \varepsilon w_1 \cdots w_n$, then $|Q_{\mathbf{w}}| = n + 4$ ([Bel+22] Lemma 4.3). The number of vertices in $\mathcal{O}(Q_{\mathbf{w}})$ is $|J(Q_{\mathbf{w}})| = |\hat{P}(w)| = 2n+6$. So by Theorem 2.11, the secondary polytope $\Sigma_{\mathcal{O}(Q_{\mathbf{w}})}$ has dimension $2n+6-(n+4)-1 = n+1$. This is exactly the number of squares in $\hat{P}(\mathbf{w})$. In particular, if $Q_{\mathbf{w}} = S_k$, $\mathbf{w} = \varepsilon(RL)(LR)(RL)\cdots$ with k parentheses, so $n = 2k$.

Theorem 1.11. Let $\mathbf{w} = \varepsilon w_1 \cdots w_n \in \mathcal{V}$. Let $V \cong \mathbb{R}^{n+1}$ be the linear subspace of $\mathbb{R}^{2n+6}$ parallel to the affine subspace containing the secondary polytope $\Sigma_{\mathcal{O}(Q_{\mathbf{w}})}$. Then the twist group acts on V in some eigenbasis $v_1, v_2, \ldots, v_{n+1}$ in which each v_i corresponds with a letter ε or w_i in $\mathbf{w}$ and each elementary twist τ_i negates exactly the basis elements that correspond to the w_i in the ladder that τ_i reflects.

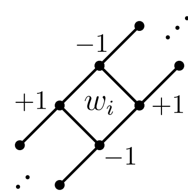
Proof. $\hat{P}(\mathbf{w}) \cong J(Q_{\mathbf{w}})$ is the order filter poset of $Q_{\mathbf{w}}$, so each vertex of $\hat{P}(\mathbf{w})$ is associated with an order filter of $Q_{\mathbf{w}}$, i.e. a vector of length $|Q_{\mathbf{w}}|$. The order polytope $\mathcal{O}(Q_{\mathbf{w}})$ can be expressed as a homogeneous matrix A where the columns correspond to the vertices (order filters) in $\hat{P}(\mathbf{w})$. The top entry of each column is a 1, and the remaining $n+4$ entries are the order filter. [DRS10] Theorem 5.1.10 states that V is the kernel of A.

We build a basis of V as follows. Recall that each coordinate in $\mathbb{R}^{2n+6}$ corresponds to one vertex of $\hat{P}(\mathbf{w})$, so we can define vectors in $\mathbb{R}^{2n+6}$ by assigning a coefficient to each vertex of $\hat{P}(\mathbf{w})$. The basis will contain one element for every letter in $\mathbf{w}$, with v_1 corresponding to $w_0 = \varepsilon$ and v_i corresponding to w_{i-1}. We define the basis element of a letter w_i based on its relationship to corners of $\hat{P}(\mathbf{w})$.

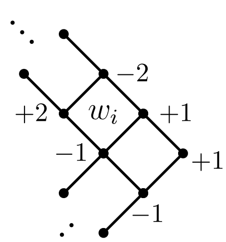
A "corner" of $\hat{P}(\mathbf{w})$ is a square that is contained in 2 ladders. Since $\mathbf{w} \in \mathcal{V}$, no corner is next to/touching another corner. If the square in $\hat{P}(\mathbf{w})$ corresponding to w_i is a corner or is not next to any corners, the corresponding basis element v_{i+1} is



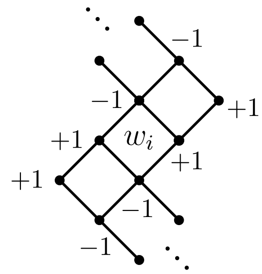
or



or a horizontal reflection of the above, whichever matches the squares surrounding w_i in $\hat{P}(\mathbf{w})$. Every vertex outside of the ladder gets coefficient 0. If w_i is directly next to one corner, v_{i+1} is the appropriate vertical or horizontal reflection of

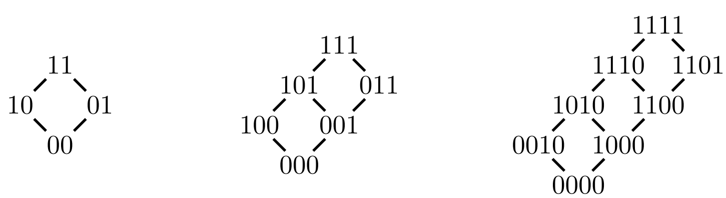


Finally, if w_i is directly in between 2 corners, v_{i+1} is the appropriate horizontal reflection of



To prove $\{v_1, ..., v_{n+1}\} \subset V = \ker A$, we simply need to prove that for each coordinate in the order filters in $\hat{P}(w)$, the products of the coefficients in each v_i by the value of the coordinate (0 or 1) at the corresponding vertex of $\hat{P}(w)$ is 0. We notice first that the total sum of all coefficients in each v_i is 0, ensuring their products with the row of all 1s at the top of A are 0, and that their products with any coordinate that is the same for all vertices with non-zero coefficients in v_i are 0. We also notice that the vertices of any square in $\hat{P}(w)$ share all but 2 coordinates, and similarly that length 2 ladders/sub-ladders share all but 3 coordinates and length 3 ladders/sub-ladders share all but 4. So we can restrict our attention to these coordinates.

We observe that every ladder/sub-ladder of a given length and direction must have the same structure of 0s and 1s on its non-identical coordinates (up to reordering the coordinates/relabelling the vertices of $Q_{\mathbf{w}}$, which does not impact whether v_i is in the kernel because if we reorder the coordinates in $\hat{P}(\mathbf{w})$ we reorder the coordinates in A in the same way):

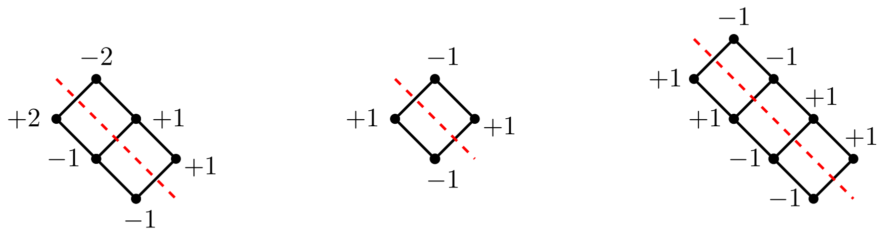


The reflected versions of these ladders/corners have the same coordinate structures reflected in the obvious way.

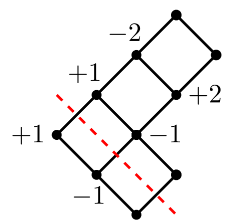
Looking at v_i with w_{i-1} next to one corner, we see that the sum of the products over the first coordinate is $-2(1) + 2(1) + 1(1) - 1(1) + 1(0) - 1(0) = 0$, the sum over the second coordinate is $-2(1) + 2(1) + 1(0) - 1(0) + 1(0) - 1(0) = 0$, and the sum over the third coordinate is $-2(1)+2(0)+1(1)-1(0)+1(1)-1(0) = 0$. So $v_i \in \ker A$. The other two cases for v_i similarly sum to 0 over every coordinate that is not identical on the non-zero vertices of v_i, so $\{v_1, ..., v_{n+1}\} \subset \ker A$.

$\{v_1, ..., v_{n+1}\}$ is a set of $n+1$ vectors in V, so if they are linearly independent they are indeed a basis. Linear independence follows from Proposition 7.1 once we confirm the eigenvalues of the elementary twists are distinct.

We now characterize the twist group using this basis. Say $\hat{P}(\mathbf{w})$ has k ladders, giving it k elementary twists. Say τ_j twists the jth ladder. If the square corresponding to w_i is in the jth ladder, τ_j inverts the corresponding basis element v_{i+1}:



On the other hand, τ_j does not impact any other basis element. In particular, it sends the basis elements corresponding to squares w_i not contained in the jth ladder that nonetheless intersect the jth ladder to themselves:



This completes the proof. $\square$

8. Further thoughts and work in progress

8.1. **Face numbers: f-, h- and γ-vectors.** We have also explored the face numbers of the secondary polytopes $\Sigma_{\mathcal{O}(S_k)}$. Given a d-dimensional polytope P, its *f-vector* is the integer sequence $(f_0, \ldots, f_d)$ where f_i is the number of i-dimensional faces. Since $\Sigma_{\mathcal{O}(S_k)}$ is conjecturally a simple d-dimensional (with $d = 2k+1$), it is appropriate to re-encode its f-vector $(f_0, f_1, \ldots, f_d)$ and f-polynomial $f(t) := \sum_{i=0}^{d} f_i t^i$ to an *h-vector* $(h_0, h_1, \ldots, h_d)$ and h-polynomial $h(t) := \sum_{i=0}^{d} h_i t^i$ via the transformations

$$f(t) = h(t+1) \text{ and } h(t) = f(t-1).$$

The h-vector has provably nonnegative entries, with $h_0 = 1$ and $\sum_{i=0}^{d} h_i = f_0$ and satisfies the *Dehn-Somerville* relations $h_i = h_{d-i}$.

Furthermore, since $\partial\Sigma^*_{\mathcal{O}(S_k)}$ is conjecturally flag, it might be appropriate to re-encode the h-vector into something even more compact, called the *γ-vector* $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$ and γ-polynomial $\gamma(t) := \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i$, via the transformation

$$h(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i} = (1+t)^d \gamma\left(\frac{t}{(1+t)^2}\right).$$

The point of this last transformation is two-fold:

- the γ-vector again only records half as many numbers, so it is more compact, and
- nonnegativity of all of the entries in $\gamma = (\gamma_0, \ldots, \gamma_{\lfloor d/2 \rfloor})$ for flag simplicial spheres is known as the *Charney-Davis-Gal Conjecture*. It was conjectured, first by Charney and Davis [CD95] for $\gamma_{\lfloor d/2 \rfloor}$, and then for the rest of the entries of γ by Gal [Gal05].

For further information about the f and h-vectors, we refer the reader to [Zie95, Section 8]. For a more detailed exposition about the γ-vector, see [Gal05]. Table 1 shows the data we have gathered for some values of k.

k	f-vector $(f_0, f_1, \ldots, f_{2k+1})$	h-vector $(h_0, h_1, \ldots, h_{2k+1})$	γ-vector $(\gamma_0, \ldots, \gamma_k)$
0	$(2, 1)$	$(1, 1)$	(1)
1	$(20, 30, 12, 1)$	$(1, 9, 9, 1)$	$(1, 6)$
2	$(336, 840, 744, 276, 38, 1)$	$(1, 33, 134, 134, 33, 1)$	$(1, 28, 40)$
3	$(6864, 24024, 33184, 22900, 8212, 1430, 96, 1)$	$(1, 89, 875, 2467, 2467, 875, 89, 1)$	$(1, 82, 444, 280)$

TABLE 1. f, h and γ-vector data gathered for $\Sigma_{\mathcal{O}(S_k)}$

Similarly, we have explored the factorizations of the f-polynomial of the facets of $\Sigma_{\mathcal{O}(S_k)}$ to see if these facets are prisms over certain polytopes. Tables 4 and 3 show the f-polynomials of the facets of the secondary polytopes of $\mathcal{O}(S_2)$ and $\mathcal{O}(S_3)$. We remark that all facets with the same number of vertices have the same f-polynomial.

Number of vertices in facet	f-polynomial
25	$(x^2 + 5x + 5)^2$
28	$(x + 2)^2 \cdot (x^2 + 7x + 7)$
36	$(x^2 + 6x + 6)^2$
40	$(x + 2)^2 \cdot (x^2 + 10x + 10)$
64	$x^4 + 19x^3 + 83x^2 + 128x + 64$
88	$x^4 + 24x^3 + 112x^2 + 176x + 88$

TABLE 2. f-polynomials of facets of $\Sigma_{\mathcal{O}(S_2)}$

Number of vertices in facet	f-polynomial
196	$(x + 2)^2 \cdot (x^2 + 7x + 7)^2$
210	$(x^2 + 5x + 5) \cdot (x^4 + 14x^3 + 56x^2 + 84x + 42)$
320	$(x^2 + 5x + 5) \cdot (x^4 + 19x^3 + 83x^2 + 128x + 64)$
400	$(x + 2)^2 \cdot (x^2 + 10x + 10)^2$
416	$(x + 2)^2 \cdot (x^4 + 25x^3 + 129x^2 + 208x + 104)$
528	$(x^2 + 6x + 6) \cdot (x^4 + 24x^3 + 112x^2 + 176x + 88)$
672	$(x + 2)^2 \cdot (x^4 + 36x^3 + 204x^2 + 336x + 168)$
1124	$x^6 + 49x^5 + 505x^4 + 2036x^3 + 3828x^2 + 3372x + 1124$
1180	$x^6 + 53x^5 + 539x^4 + 2152x^3 + 4026x^2 + 3540x + 1180$
1688	$x^6 + 66x^5 + 726x^4 + 3008x^3 + 5724x^2 + 5064x + 1688$

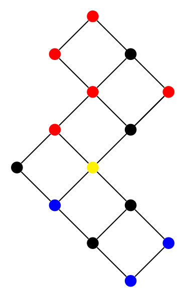
TABLE 3. f-polynomials of facets of $\Sigma_{\mathcal{O}(S_3)}$

Some interesting observations can be drawn from this data. In particular, the facets with 28 vertices in $\Sigma_{\mathcal{O}(S_2)}$ seem to be prisms over 3-associahedra. Similarly, the facets with 210 vertices in $\Sigma_{\mathcal{O}(S_2)}$ seem to be cartesian products of pentagons (2-dimensional associahedra) with 4-dimensional associahedra. It might be interesting to investigate whether the irreducible f-polynomials correspond the order polytopes of some class of posets.

8.2. **Work towards Conjecture 1.7.** In order to prove that all 2D faces of the secondary polytope are quadrilaterals, pentagons, and hexagons, we consider a triangulation $\mathcal{T}$ which can be flipped at two circuits Z_1 and Z_2. From Theorem 1.8, if there does not exist a maximal simplex $\omega \in \mathcal{T}$ $\omega \in \mathcal{T}$ such that for some $\sigma \in \mathcal{T}_{Z_1}$ and some $\delta \in \mathcal{T}_{Z_2}$, $\sigma, \delta \subset \omega$, then $\mathcal{T}$, $f_{Z_1}(\mathcal{T})$, and $f_{Z_2}(\mathcal{T})$ will appear on a quadrilateral face of the secondary polytope.

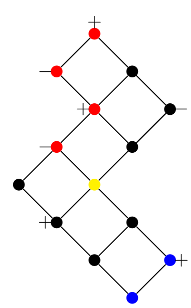
If there does exist some such maximal simplex ω, then we also know from Theorem 1.8 that if $Z_1 \cap Z_2 = \varnothing$, they will still appear on a quadrilateral. Otherwise, we conjecture that if $Z_1 \cap Z_2 = \{x\}$ or $Z_1 \cap Z_2 = Z_1 \setminus \{x\}$ for some vertex x, then $\mathcal{T}$, $f_{Z_1}(\mathcal{T})$, and $f_{Z_2}(\mathcal{T})$ appear be on a pentagonal face. Similarly, we conjecture that if $Z_1 \cap Z_2 = \{x, y\}$ or $Z_1 \cap Z_2 = Z_1 \setminus \{x, y\}$ for some vertices x, y, then $\mathcal{T}$, $f_{Z_1}(\mathcal{T})$, and $f_{Z_2}(\mathcal{T})$ appear be on a hexagonal face. We claim that no other possible configuration of circuits will occur.

We provide reasoning for one example, which can be generalized, where the circuits Z_1 and Z_2 induce a pentagonal cycle in the bistellar flip graph. Let the yellow vertex denote x, the red vertices denote remaining elements in Z_1, and the blue vertices denote remaining elements in Z_2. Let $Z_3 = Z_1 \cup Z_2 \setminus \{x\}$.

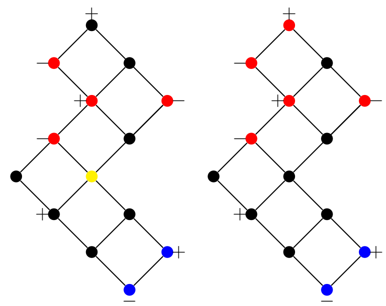


We see that $x \in Z_1^+$ and $x \in Z_2^+$ and we assume for the example that $\mathcal{T}_{Z_1}^-, \mathcal{T}_{Z_2}^- \subset \mathcal{T}$. We know that for all $\sigma \in \mathcal{T}_{Z_1}^-$ and $\delta \in \mathcal{T}_{Z_2}^-$, $x \in \sigma$ and $x \in \delta$. We also know that there exists at least one maximal simplex $\omega_1 \in \mathcal{T}$ such that for some $\sigma_1 \in \mathcal{T}_{Z_1}^-$ and some $\delta_1 \in \mathcal{T}_{Z_2}^-$, $\sigma_1, \delta_1 \subset \omega_1$, and from the above, we also have $x \in \omega_1$. By the link condition, for all remaining $\sigma' \in \mathcal{T}_{Z_1}^-$, there must exist some ω' such that $\sigma' \in \omega'$ and all vertices in $Z_2 \setminus \{x\}$ are also in ω'. Since $x \in \sigma'$, this means that $\delta_1 \in \omega'$. Following this logic, we conclude that for all $\sigma \in \mathcal{T}_{Z_1}^-$ and $\delta \in \mathcal{T}_{Z_2}^-$, there exists some maximal simplex ω such that $\sigma, \delta \subset \omega$.

We label each vertex in our diagram with the sign it has in $Z_3 = (Z_1 \cup Z_2) \setminus \{x\}$. In particular, we note that since x was positive in both circuits, the signs of the vertices in Z_2 will be opposite to the signs of the vertices in Z_3. We now choose some arbitrary $\sigma_1 \in \mathcal{T}_{Z_1}^-$ and $\delta_1 \in \mathcal{T}_{Z_2}^-$ and draw the simplex ω_1 they must appear on in $\mathcal{T}$.

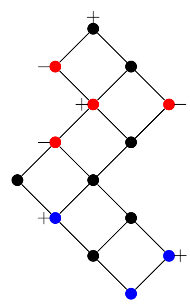


Here we make a key observation: ω_1 almost contains an element of $\mathcal{T}_{Z_3}^+$ or $\mathcal{T}_{Z_3}^-$. In particular, ω_1 contains all but two elements of Z_3. We now imagine flipping at Z_1 and see we produce the following two types of simplices in $f_{Z_1}(\mathcal{T})$: those which contain x and those which do not.

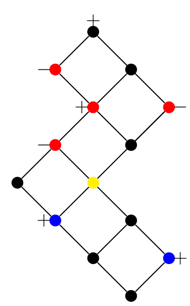


We note that any simplex which doesn't contain x will now contain a simplex in $\mathcal{T}_{Z_3}^+$ because it will be precisely "gaining" the negative vertex it was missing from before. We can characterize all simplices in $\mathcal{T}_{Z_3}^+$ by which positive vertex in Z_3 they do not contain. We thus see that any simplex in $\mathcal{T}_{Z_3}^+$ whose missing positive vertex belongs to Z_2 will be appear in $f_{Z_1}(\mathcal{T})$. Further, it will no longer contain a simplex in $\mathcal{T}_{Z_2}^-$, so it will not change when we flip at Z_2.

The remaining simplices in $\mathcal{T}_{Z_3}^+$ are those whose missing positive vertex belongs to Z_1. These will be precisely generated once we flip at Z_2, and we can see an example of this below:



We are now in $f_{Z_2}(f_{Z_1}(\mathcal{T}))$. It is clear from our above reasoning that all simplices of $\mathcal{T}_{Z_3}^+$ will appear in $f_{Z_2}(f_{Z_1}(\mathcal{T}))$, further that they will have the same link since their link is contained in the links of $\mathcal{T}_{Z_1}$ and $\mathcal{T}_{Z_2}$ and x does not appear in any of the maximal simplices. We also note the following: there will be some maximal simplices in $f_{Z_2}(f_{Z_1}(\mathcal{T}))$ generated by our series of flips which still contain x. One example appears below:



In particular, these maximal simplices contain simplices in $\mathcal{T}_{Z_1}^+$ and $\mathcal{T}_{Z_2}^+$ which both contain x. If we had flipped at Z_2 and then at Z_1, we would have produced all simplices in $\mathcal{T}_{Z_3}^-$ by the same logic as above. In particular, we see that flipping first at Z_1 either moves the vertex at x to a positive vertex, thereby creating a simplex in $\mathcal{T}_{Z_3}^-$, or creates a new "missing" negative vertex which can then be taken by flipping at Z_1. Further, all resulting maximal simplices which do not contain $\mathcal{T}_{Z_3}^-$ are the same in both cases. It follows from the link condition that these two triangulations $f_{Z_2}(f_{Z_1}(\mathcal{T}))$ and $f_{Z_1}(f_{Z_2}(\mathcal{T}))$ are equivalent up to whether they include $\mathcal{T}_{Z_3}^-$ or $\mathcal{T}_{Z_3}^+$, so we conclude that $f_{Z_3}(f_{Z_2}(f_{Z_1}(\mathcal{T}))) = f_{Z_1}(f_{Z_2}(\mathcal{T}))$.

8.3. **Work towards Conjecture 1.4.** To better understand the structure of the dual of the secondary polytope, we need to find a combinatorial interpretation for the vertices of the dual polytope, as well as for the edges between them. Currently we are trying to find a way to represent the vertices of the dual as some set of simplices that all the triangulations in a given facet of the secondary polytope share. This could provide an understanding of the dual polytope analogue to the simplicial dual of the associahedron.

To show that the dual simplicial polytope is a flag complex, we would need to prove that if we have a set of vertices and there is an edge between every pair, then there must be a face containing all of these vertices. Perhaps, this face would represent a triangulation which shares all the shared simplices. We have collected data on shared simplices in the faces of the secondary polytope for some of the smaller cases.

In the S_1 case, all the quadrilaterals and pentagonal faces have the property that all the triangulations at their vertices intersect at a maximal simplex. However, the hexagonal faces do not all have exactly one simplex in common. In particular, they have four codimension-1 simplices in common.

In higher cases, there are a higher number of low-dimensional simplices shared by elements on the same face. We include the following table of shared simplices on facets of S_2.

Number of vertices in facet	Dimension of shared simplices	Number of shared simplices
25	7	12
28	8	2
36	6	16
40	7	3
64	7	4
88	7	1

TABLE 4. Shared simplices of facets of $\Sigma_{\mathcal{O}(S_2)}$

8.3.1. *Vertices in the dual simplicial polytope.* We have computed the number of facets (i.e. $2k$-dimensional faces) in $\Sigma_{\mathcal{O}(S_k)}$ for the first values of k, obtaining the sequence $2, 12, 38, 96, \ldots$. Since these are the number of vertices in the simplicial boundary complex of the polar dual polytope $\Sigma^*_{\mathcal{O}(S_k)}$ and that simplicial complex $\mathrm{Bd}(\Sigma^*_{\mathcal{O}(S_k)})$ is currently conjectured to be a *flag/clique* complex, it would be very helpful to have some combinatorial objects counted by $2, 12, 38, 96, \ldots$ to model those vertices, and then some sort of pairwise *compatibility* relation among those vertices that tells us when a subset of them forms a simplex in $\mathrm{Bd}(\Sigma^*_{\mathcal{O}(S_k)})$.

Interestingly, OEIS and its email lookup service `superseeker` are suggesting that this sequence *might* continue as follows:

$$a_n = 2(2^{n+2} - 3n - 4) \tag{1}$$

$$(a_0, a_1, a_2, \ldots) = (0, 2, 12, 38, 96, 218, 468, 974, 1992, 4034, 8124, \ldots). \tag{2}$$

However, we have been unable to gather more data to support this conjecture due to computational limitations.

8.4. **Facet normals in the eigenbasis coordinates?** A general difficulty in describing the facets of the secondary polytope $\Sigma_{\mathcal{A}}$ for a point configuration $\mathcal{A} \subset \mathbb{R}^d$ is the following: in seeking facet normal vectors $\mathbf{n}_j$ that define $\Sigma_{\mathcal{A}}$ as an intersection of half-space inequalities

$$\Sigma_{\mathcal{A}} = \bigcap_j \{\mathbf{x} \in \mathbb{R}^{|\mathcal{A}|} : \mathbf{n}_j \cdot \mathbf{x} \leq b_j\},$$

there is no canonical choice when doing this in the ambient space $\mathbb{R}^{|\mathcal{A}|}$. Recall $\Sigma_{\mathcal{A}}$ has dimension $|\mathcal{A}| - (d+1)$, and lives in the subspace $\mathbb{R}^{|\mathcal{A}|-(d+1)}$ which is the kernel of the homogeneous matrix for $\mathcal{A}$. So there are infinitely many choices for each normal vector $\mathbf{n}_j$.

However, if one had some natural $\mathbb{R}$-basis $\{v_i\}_{i=1,2,\ldots,|\mathcal{A}|-(d+1)}$ for this subspace $\mathbb{R}^{|\mathcal{A}|-(d+1)}$ in which $\Sigma_{\mathcal{A}}$ is embedded as a full-dimensional polytope, one could instead use coordinates $\mathbf{x} = (x_1, \ldots, x_{|\mathcal{A}|-(d+1)})$ with respect to this basis. Then the normal vectors $\mathbf{n}_j$ here

$$\Sigma_{\mathcal{A}} = \bigcap_j \{\mathbf{x} \in \mathbb{R}^{|\mathcal{A}|-(d+1)} : \mathbf{n}_j \cdot \mathbf{x} \leq b_j\}$$

will be *unique* up to scaling.

Fortunately, having proven Theorem 1.11, we find ourselves having exactly such a natural $\mathbb{R}$-basis for the space $\mathbb{R}^{n+1}$ containing $\Sigma_{\mathcal{O}(Q_{\mathbf{w}})}$ with $\mathbf{w} = \varepsilon w_1 \cdots w_n \in \mathcal{V}$. The twist group simultaneous eigenbasis $\{v_i\}_{i=1,2,\ldots,n+1}$ seems like a great candidate in which to compute those facet normal vectors $\{\mathbf{n}_j\}$, and see how their coordinates look, when plotted on the $n+1$ "squares" of the generalized snake for $\mathbf{w}$. It might even suggest the mysterious combinatorial objects that should parametrize those facets of $\Sigma_{\mathcal{O}(Q_{\mathbf{w}})}$ (= vertices of $\Sigma^*_{\mathcal{O}(Q_{\mathbf{w}})}$).

8.5. **Work towards Conjecture 1.3.** After proving that the twist action is free, we conjecture that $\mathrm{Cat}(2k+1)$ counts the number of orbits of the twist action on regular triangulations of $\mathcal{O}(S_k)$. We might consider choosing one "representative" for each orbit and finding a bijection between these orbit representatives and a set of objects counted by the Catalan numbers. Using edge-distance from the canonical triangulation, we can pick candidates for our orbit representatives. In the S_1 case, with the exception of orbit #5, everything other choice is well-defined. We had several ideas about how to biject these orbit representatives with various sets of objects counted by the Catalan numbers, including Dyck paths and poset labellings. At this point, it is unclear to us how a bijection such as this one would proceed into higher cases. We include a brief description of our bijection between Dyck paths of length 6 and orbit representatives of triangulations of $\mathcal{O}(S_1)$.

8.5.1. *Bijection between Dyck paths Orbit Representatives in S_1.* We note that the five orbit representatives in S_1 can be described by $\mathcal{T}_{\mathbf{w}}$, the three triangulations obtained by applying a flip at $\mathrm{Sq}(w_1)$, $\mathrm{Sq}(w_2)$, or $\mathrm{Sq}(w_3)$ to $\mathcal{T}_{\mathbf{w}}$, and one of the two triangulations obtained by either applying a flip at $\mathrm{Sq}(w_1)$ and then $\mathrm{Sq}(w_3)$ or vice versa.

We consider Dyck paths of length 3. Let P be a Dyck path. Then, we define three Dyck path "operations".

- (i) If P contains a vertex at $(1,3)$, then $\phi_1(P)$ is the path obtained by moving the vertex $(1,3)$ to $(2,2)$ and applying the minimum number of moves to other vertices until obtaining a valid Dyck path. For example, if $P = NNNEEE$, then $\phi_1(P) = NNEENE$
- (ii) If P contains a vertex at $(0,3)$, i.e. $P = NNNEEE$, then $\phi_2(P)$ is the Dyck path obtained by moving the vertex from $(0,3)$ to $(1,2)$, i.e $\phi_1(P) = NNENEE$.

(*iii*) If P contains a vertex at $(0,2)$, then $\phi_3(P)$ is the path obtained by moving this vertex to $(1,1)$ and applying the minimum number of diagonal moves to other vertices until obtaining a Dyck path. For example, if $P = NNNEEE$, then $\phi_3(P) = NENNEE$

The illustrations below show these operations. Note that the vertex colored red represents the vertex that initiates the move, with corresponding red arrow showing the position after the move. The dashed black arrows represent the minimal move that needs to be made to adjust the path to the new position of the vertex in red so as to obtain a Dyck path.

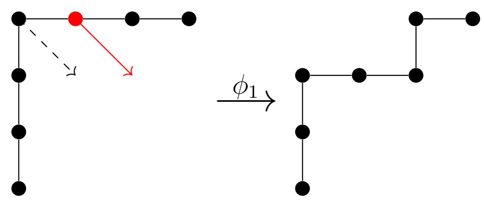


FIGURE 13. Operation (i)

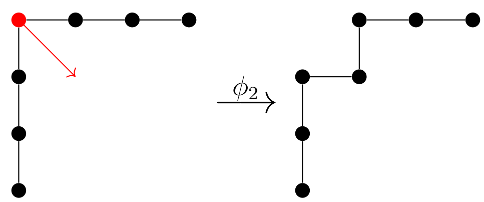


FIGURE 14. Operation (ii)

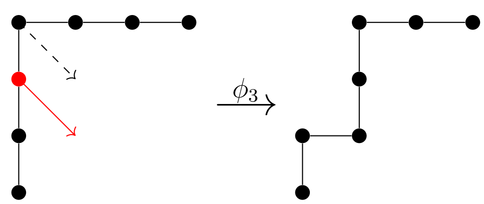


FIGURE 15. Operation (iii)

We will associate ϕ_1 to flipping at $\mathrm{Sq}(w_1)$, ϕ_2 to flipping at $\mathrm{Sq}(w_2)$, and ϕ_3 to $\mathrm{Sq}(w_3)$. Note that this assignment is not arbitrary. The circuits $\mathrm{Sq}(w_1)$, $\mathrm{Sq}(w_2)$, and $\mathrm{Sq}(w_3)$ correspond to the squares in $J(S_1)$ read from top to bottom, and ϕ_1, ϕ_2, ϕ_3 correspond to the moves on the admissible vertices read from top-right to bottom-left

To map our orbit representatives to Dyck paths, we begin by mapping the canonical triangulation to the canonical Dyck path $P = NNNEEE$. With the operations above, the orbit representative corresponding to the flip at $\mathrm{Sq}(w_1)$ maps to $\phi_1(P)$, and so on so forth. For the orbit representative which lies two flips away from $\mathcal{T}_{\mathbf{w}}$, we can imagine that we have applied two operations to our "canonical" Dyck path – we have moved the vertex below the top left one and then we have moved the vertex to the right of the top left one. Both double-operations would result in our fifth and final length-3 Dyck path. In other words, this orbit representative maps to $\phi_1\phi_3(NNNEEE) = NENENE = \phi_3\phi_1(NNNEEE)$

So we have established a bijection between Dyck paths of length 3 and orbit representatives of triangulations by mapping the operation of moving a special Dyck path "vertex" (the top left one and its two closest ones in either direction) to the operation of doing a bistellar flip at one of the three squares $\mathrm{Sq}(w_1), \mathrm{Sq}(w_2), \mathrm{Sq}(w_3)$.

8.6. **Numerical recurrence.** Conjecture 1.3 asserts that the number t_k of (regular) triangulations of $\mathcal{O}(S_k)$ has this formula

$$t_k = 2^{k+1}\mathrm{Cat}(2k+1) = \frac{2^{k+1}}{2k+2}\binom{4k+2}{2k+1}.$$

For $k \geq 1$, one can check that it satisfies this recurrence

$$\frac{t_k}{t_{k-1}} = \frac{8(4k+1)(4k-1)}{(2k+2)(2k+1)},$$

which by clearing denominators, can be written in this equivalent form:

$$(2k+2)(2k+1) \cdot t_k = 8(4k+1)(4k-1) \cdot t_{k-1}. \tag{3}$$

Is there some way to prove (3) in the spirit of some of the proofs of the Catalan formula for triangulations of a polygon?

8.7. **Other symmetries.** Proving Theorem 1.11 leads us to wonder if the secondary polytope of $\mathcal{O}(S_k)$ has any other linear symmetries, or if the linear symmetry group of the secondary polytope of $\mathcal{O}(S_k)$ is isomorphic to $(\mathbb{Z}/2)^{k+1}$.

Acknowledgements

This project was supported in large part by a grant from the D.E. Shaw group, and also by NSF grant DMS-2053288. It was supervised as part of the University of Minnesota School of Mathematics Summer 2024 REU program. The authors would like to thank Vic Reiner for mentoring the project and for his valuable ideas and suggestions. The authors would also like to thank Kaelyn Willingham for his helpful feedback and advice, and Ayah Almousa for co-coordinating the REU.

References

[Bel+22] Matias von Bell et al. "Triangulations, order polytopes, and generalized snake posets". In: *Comb. Theory* 2.3 (2022), Paper No. 10, 34. ISSN: 2766-1334. DOI: 10.5070/c62359166.

[Bjö+-6] Anders Björner et al. "Hyperplane arrangements with a lattice of regions". In: *Discrete & computational geometry.* 5.3 (1990-6). ISSN: 0179-5376.

[CD95] Ruth Charney and Michael Davis. "The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold". In: *Pacific J. Math.* 171.1 (1995), pp. 117–137. ISSN: 0030-8730,1945-5844.

[DRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations.* Vol. 25. Algorithms and Computation in Mathematics. Structures for algorithms and applications. Springer-Verlag, Berlin, 2010, pp. xiv+535. ISBN: 978-3-642-12970-4. DOI: 10.1007/978-3-642-12971-1.

[Gal05] Światosław R. Gal. "Real root conjecture fails for five- and higher-dimensional spheres". In: *Discrete Comput. Geom.* 34.2 (2005), pp. 269–284. ISSN: 0179-5376,1432-0444. DOI: 10.1007/s00454-005-1171-5.

[GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Reprint of the 1994 edition. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. x+523. ISBN: 978-0-8176-4770-4.

[Her23] Patricia Hersh. "Posets arising as 1-skeleta of simple polytopes, the nonrevisiting path conjecture, and poset topology". In: *Discrete & Computational Geometry* (2023), pp. 1–30.

[Kal88] Gil Kalai. "A simple way to tell a simple polytope from its graph". In: *J. Combin. Theory Ser. A* 49.2 (1988), pp. 381–383. ISSN: 0097-3165,1096-0899. DOI: 10.1016/0097-3165(88)90064-7.

[Sta86] Richard P. Stanley. "Two poset polytopes". In: *Discrete Comput. Geom.* 1.1 (1986), pp. 9–23. ISSN: 0179-5376,1432-0444. DOI: 10.1007/BF02187680.

[Zel06] Andrei Zelevinsky. "Nested complexes and their polyhedral realizations". In: *Pure Appl. Math. Q.* 2.3 (2006), pp. 655–671. ISSN: 1558-8599,1558-8602. DOI: 10.4310/PAMQ.2006.v2.n3.a3.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*. Vol. 152. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. x+370. ISBN: 0-387-94365-X. DOI: 10.1007/978-1-4613-8431-1.

Department of Mathematics, University of Pennsylvania, Philadelphia, PA
Email address: mollykb@sas.upenn.edu

Department of Mathematics, University of California, Santa Barbara
Email address: sogol@ucsb.edu

Department of Mathematics, University of Washington, Seattle, WA
Email address: aljones3@uw.edu

Department of Mathematics, Dartmouth College, Hanover, NH
Email address: mario.tomba.morillo.25@dartmouth.edu

Department of Mathematics, Harvard University, Cambridge, MA
Email address: katherinetung@college.harvard.edu