PROBLEM 2: FREE RESOLUTIONS AND HILBERT SERIES FOR SKEW SPECHT IDEALS

CASEY APPLETON, ZACHARY CHANCE MEDLIN, AND MARIO TOMBA

Abstract. Given a diagram D with n boxes, the Specht ideal associated to D is the ideal generated by the Specht polynomials for all bijective fillings of the diagram D with the numbers from 1 to n. Specht ideals have arisen naturally over the years in the study of graph theory, subspace arrangements, and optimization, albeit with different names. Their study from a combinatorial commutative algebra perspective was pioneered by Yanagawa and collaborators, including Murai, Shibata, and Watanabe. However, these authors have only studied Specht ideals for Young diagrams corresponding to partitions. In this report, we examine the $\mathfrak{S}_n$-equivariant Hilbert series and free resolutions for Specht ideals of more general diagrams, including two-row skew shapes and certain generalizations of hook partitions. We conjecture $\mathfrak{S}_n$-equivariant resolutions for these cases as well as representation stability for certain families of diagrams, and present some partial results providing evidence of the conjectures.

Contents

1. Introduction 2
1.1. Main Results and Organization 2
2. Specht modules and ideals for arbitrary diagrams 2
2.1. Tabloids and polytabloids 3
2.2. Specht polynomials 5
2.3. Specht ideals 6
3. Commutative Algebra toolbox 6
4. Two-row ribbons 9
4.1. Description of the resolution 9
4.2. Exactness of the complex 15
4.3. $\mathfrak{S}_n$-equivariant Hilbert series 16
5. Hooks and their generalizations 18
6. Skew shapes from $(d, d, 1)$ 24
6.1. $(d, d, 1)/\mu$ 24
7. Characteristic dependence 25
Acknowledgements 26
References 26

1. Introduction

Given a diagram D of n boxes in $\mathbb{N} \times \mathbb{N}$, one may associate to it an ideal in the polynomial ring $\Bbbk[x_1, \dots, x_n]$ over a field $\Bbbk$ called a *Specht ideal*. This ideal is generated by all the Specht polynomials for bijective fillings of the diagram D with the numbers from 1 to n. These ideals have appeared in many guises over the years, including in the study of:

- algebra and combinatorics of subspace arrangements [ZGS14; Bro+16; BPS05]
- graph theory [LL81; Lov94; Loe95]
- combinatorial Hilbert schemes [Woo05; DK24]
- symmetric systems of equations [MRV21]

One key property of Specht ideals is that they are stable under the action of the symmetric group $\mathfrak{S}_n$ that permutes the variables. This allows one to leverage the representation theory of the symmetric group to understand homological information about the ideals, such as their Hilbert series and free resolutions. This perspective has been successful in the study of Specht ideals in their own right from a combinatorial commutative algebra perspective by several authors, including:

- Galetto [Gal20] found $\mathfrak{S}_n$-equivariant minimal free resolutions for ideals generated by all squarefree monomials of degree d (i.e. where the diagram D corresponds to a two row shape consisting of 2 disjoint horizontal strips).
- Raicu and Murai [MR22] described the $\mathfrak{S}_n$-module structure of $\mathrm{Tor}_i(I_\lambda, \Bbbk)$ for a more general class of $\mathfrak{S}_n$-stable monomial ideals, including for Specht ideals corresponding to diagrams of any number of disconnected horizontal strips.
- For two-row partitions, the $\mathfrak{S}_n$-module structure of $\mathrm{Tor}_i(I_\lambda, \Bbbk)$ was described in [ZGS14], with explicit maps for the minimal free resolutions constructed in [SY23b].
- The $\mathfrak{S}_n$-equivariant minimal free resolution for partitions of the form $(d, d, 1)$ were explicitly constructed in [SY23a].

Despite there being some success in understanding free resolutions of skew Specht ideals by Galetto and Raicu–Murai, there has been no systematic study of Specht ideals for diagrams which do not correspond to partitions. In this report, we initiate the study of free resolutions of Specht ideals for general diagrams. We particularly focus our attention on two-row skew shapes and diagrams arising from permuting rows of hook shapes.

1.1. Main Results and Organization.

In Section 2, we give background on the representation theory of $\mathfrak{S}_n$, Specht polynomials for arbitrary diagrams, and Specht ideals. In Section 3, we explain the notions from commutative algebra that we will need throughout the paper, including Hilbert series and free resolutions. In Section 4, we describe a conjectural free resolution for two-row ribbons (Conjecture 4), and we make partial progress towards proving it. In Section 4.3, we investigate the graded pieces of the $\mathfrak{S}_n$-equivariant Hilbert series for two-row ribbons. In Section 5, we work on generalizing the Eagon-Northcott complex to certain not necessarily skew diagrams and we show the $\mathfrak{S}_n$-equivariant structure of the free modules in the minimal free resolution for hooks (Proposition 5.2). Finally, in Section 6 we conjecture a minimal free resolution for certain cases that generalize the $(d, d, 1)$ case proved by Shibata and Yanagawa.

2. Specht modules and ideals for arbitrary diagrams

We begin by introducing the notion of Specht modules for arbitrary diagrams. These have previously been studied in, for instance, [RS95a; RS95b; RS98; Liu10].

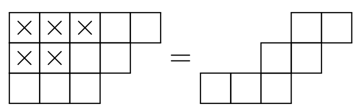
Definition 2.1. A *diagram* D is a finite subset of $\mathbb{N} \times \mathbb{N}$. Associate $\mathbb{N} \times \mathbb{N}$ with the boxes of a top-left justified infinite grid, where (i, j) denotes the box in the ith row from the top and the jth column from the left.

Given a diagram D with n boxes, a *tableau* T of shape D is a bijective labeling of the boxes of D with positive integers $\{1, \ldots, n\}$.

We denote by $\mathrm{Tab}(D)$ the set of tableaux of shape D. A tableau is *standard* if the numbers strictly increase along the rows and down the columns. We denote by $\mathrm{SYT}(D)$ the set of standard tableaux of shape D.

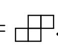
Examples of diagrams include:

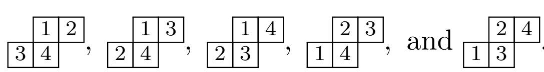
- Given a positive integer n, a *partition* of n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of integers such that $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 1$ and $\sum_i \lambda_i = n$, and we write $\lambda \vdash n$. We often represent a partition λ by its *Young diagram*, which is a collection of boxes with left-justified rows having λ_i boxes in row i.
- Given a partition μ such that $\mu_i \leq \lambda_i$ for each i, we represent the *skew shape* λ/μ as the diagram of boxes obtained by overlapping the Young diagrams of λ and μ and removing the common boxes. For example, the skew shape $(5, 4, 3)/(3, 2)$ is represented below. Observe that if $\mu = \varnothing$, then $\lambda/\mu = \lambda$.



Throughout this report, we will often focus our attention on a particularly nice class of skew shapes called *ribbons*.

Definition 2.2. A *ribbon* (also sometimes called a *skew/rim hook* or *border strip*) is a skew shape with no 2×2 boxes. Given a composition of n, $\alpha = (\alpha_1, \ldots, \alpha_k)$, we denote by $\mathrm{Ribb}(\alpha) = \mathrm{Ribb}(\alpha_1, \ldots, \alpha_k)$ the unique ribbon that has α_i boxes in row i.

Example 2.3. Consider the skew shape $(3, 2)/(1) = \mathrm{Ribb}(2, 2) =$ . The standard Young tableaux of this shape are:



Definition 2.4. The *Specht module* $\mathcal{S}_D$ for a diagram D is the $\Bbbk$-linear span of $\mathrm{Tab}(D)$, where $\mathrm{Tab}(D)$ is naturally a $\mathfrak{S}_n$-module via left multiplication.

It is well-known that the Specht modules $\mathcal{S}_\lambda$ where $\lambda \vdash n$ correspond to exactly the irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$.

2.1. **Tabloids and polytabloids.** Let $\Bbbk[\mathfrak{S}_n]$ be the group algebra of the symmetric group $\mathfrak{S}_n$. For any tableau T of shape D, label its rows $R_1, \ldots, R_\ell$ and its columns $C_1, \ldots, C_k$. Define the *row-stabilizer*

$$R_T := \mathfrak{S}_{R_1} \times \mathfrak{S}_{R_2} \times \cdots \times \mathfrak{S}_{R_\ell} \subseteq \mathfrak{S}_n$$

and the *column-stabilizer*

$$C_T := \mathfrak{S}_{C_1} \times \mathfrak{S}_{C_2} \cdots \times \mathfrak{S}_{C_k} \subseteq \mathfrak{S}_n$$

where $\mathfrak{S}(A)$ is the group of permutations for the set A.

Example 2.5. For $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$, we have

$$R_T = \{\epsilon, (1,3), (2,4), (1,3)(2,4)\} \leq \mathfrak{S}_4$$
$$C_T = \{\epsilon, (1,2), (3,4), (1,2)(3,4)\} \leq \mathfrak{S}_4.$$

Given a subset $H \subseteq \mathfrak{S}_n$, define the following sums in the group algebra:

$$H^+ \coloneqq \sum_{\pi \in H} \pi, \qquad H^- = \sum_{\pi \in H} \operatorname{sign}(\pi)\pi.$$

Observe that $C_T^- = C_1^- \cdot C_2^- \cdots C_k^-$.

We say that two tableaux $T, T' \in \operatorname{Tab}(D)$ are *row-equivalent* if $T' = \sigma T$ for some $\sigma \in R_T$. In other words, they are row-equivalent if their rows contain the same elements in a different order.

Definition 2.6. The *tabloid* $\{T\}$ of $T \in \operatorname{Tab}(D)$ is:

$$\{T\} = \{T' \in \operatorname{Tab}(\lambda/\mu) : T' \text{ is row-equivalent to } T\},$$

and the *polytabloid* $e(T)$ is defined by

$$e(T) \coloneqq \sum_{\sigma \in C(T)} \operatorname{sign}(\sigma)\{\sigma T\}.$$

Example 2.7. Consider the tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Then, we have

$$\{T\} = \left\{ \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 2 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 3 & 4 & 1 \\ \hline 2 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 4 & 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \right\},$$

and

$$e(T) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right\} - \left\{ \begin{array}{|c|c|c|}\hline 2 & 4 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \right\}.$$

Remark 2.8. The Specht module $\mathcal{S}_D$ is isomorphic to the $\mathfrak{S}_n$-module

$$\mathcal{S}_D = \operatorname{span}_{\Bbbk}\{e(T) : T \in \operatorname{Tab}(D)\}$$

equipped with the usual $\mathfrak{S}_n$-action that gives it the structure of a $\mathfrak{S}_n$-module.

It is important to remark that $\{e(T) : T \in \operatorname{Tab}(D)\}$ is a $\Bbbk$-linearly dependent set. In the case where D is a skew shape, the linear relations between polytabloids are given by the *Garnir relations*, and there is a known algorithm for expressing any $e(T')$ as a linear combination of $\{e(T) : T \in \operatorname{SYT}(\lambda/\mu)\}$ known as the *straightening algorithm*. For a detailed exposition of this classical algorithm, we refer the reader to [Sag01, Section 2.6]. However, we give the basic definition here.

Definition 2.9. Given a tableau T, let A, B be subsets of the j-th and $(j+1)$-th columns of T. Denote by $G_{A,B}$ the set of permutations of $A \cup B$ that rearrange the elements of $A \cup B$ so that they are increasing down the columns. Then, the *Garnir element associated with* A and B is $g_{A,B} \coloneqq \sum_{\pi \in G_{A,B}} \operatorname{sign}(\pi)\pi$.

In practice, if there is a descent $T_{i,j} > T_{i,j+1}$, we will always take A to be all the elements in column j below $T_{i,j}$ and B to be all the elements in column $j+1$ above $T_{i,j+1}$. We will use the following basic result about Garnir elements, whose proof can be found in [Sag01, Proposition 2.6.3]

Proposition 2.10. *Let T be a tableau and A, B as in Definition 2.9. If $|A \cup B|$ is larger than the number of elements in column j of T, then $g_{A,B}e(T) = 0$.*

Example 2.11. Consider the tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 & 3 & \\ \cline{1-2} 6 & & \\ \cline{1-1}\end{array}.$$

There is a descent $5 > 3$ in the second row. We let $A = \{5, 6\}$ and $B = \{2, 3\}$. Then

$$g_{A,B} = \epsilon - (3, 5) + (3, 6, 5) + (2, 3, 5) - (2, 3, 6, 5) + (3, 6)(2, 5),$$

and one can verify that $g_{A,B}e(T) = 0$.

Remark 2.12. Note that $\{e(T) : T \in \mathrm{SYT}(D)\}$ is a basis for $\mathcal{S}_D$.

2.2. **Specht polynomials.** Following [Pee75], we can also view Specht modules as submodules of the polynomial ring. The polynomial ring $\Bbbk[x_1, \ldots, x_n]$ is a $\Bbbk[\mathfrak{S}_n]$ module via the action $\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Definition 2.13. To any tableau T of shape D with n boxes, define the *Specht polynomial* $f_T \in \Bbbk[x_1, \ldots, x_n]$ to be

$$f_T = C_T^- \prod x_i^{k_i},$$

where k_i is one less than the row in which the box labeled i appears.

Equivalently, the Specht polynomial f_T can be viewed as a product of minors of the Vandermonde matrix.

Definition 2.14. For any n variables $z_1, \ldots, z_n$, we denote by $\mathrm{VD}(z_1, \ldots, z_n)$ the determinant of the $n \times n$ Vandermonde matrix in the variables $z_1, \ldots, z_n$, i.e.,

$$\mathrm{VD}(z_1, \ldots, z_n) \coloneqq \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ z_1^2 & z_2^2 & \cdots & z_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (z_j - z_i).$$

Define Row_D to be a (non-bijective) filling of the shape D such that every cell in the ith row of D contains the value i. For each column C_j, define the following minor of the Vandermonde matrix:

$$[(\mathrm{Row}_D)_j \mid C_j] = \det \mathrm{VD}_{(\mathrm{Row}_d)_j}(x_{c_1}, \ldots, x_{c_k}).$$

Notice that we can break any Specht polynomial f_T up into the following product:

$$f_T = \prod_{j \in [k] // 1 \le k \le \ell} C_k^- x_{1,j}^0 x_{2,j}^1 \cdots x_{\ell,j}^{\ell-1} = \prod [(\mathrm{Row}_D)_j \mid C_j],$$

where $x_{i,j} = x_m$ if m is label for the cell (i, j).

2.3. **Specht ideals.**

Definition 2.15. The *Specht ideal* of shape D is the ideal generated by the Specht polynomials for all fillings of D. That is,

$$I_D \coloneqq \langle f_T : T \in \mathrm{Tab}(D) \rangle \subset \Bbbk[x_1, \dots, x_n].$$

Often, we will use the well-known fact that the Specht ideal of any skew shape admits a minimal generating set given by the Specht polynomial of standard young tableaux of the given shape. That is,

$$I_{\lambda/\mu} = \langle f_T : T \in \mathrm{SYT}(\lambda/\mu) \rangle.$$

Remark 2.16. There is an important correspondence between Specht modules and Specht ideals. Since $I = I_{\lambda/\mu}$ is a homogeneous ideal, it is *graded* via

$$I_{\lambda/\mu} = \bigoplus_{j \in \mathbb{N}_{\geq 0}} (I_{\lambda/\mu})_j.$$

The symmetric group $\mathfrak{S}_n$ acts on these graded pieces, which are all vector spaces. If I_D is generated in degree d, then the degree d graded piece $(I_D)_d$ is exactly spanned by the Specht polynomials for tableaux of shape D, so it is isomorphic to the Specht module $\mathcal{S}_D$ by . As an alternative way to observe the correspondence, for any diagram D there is an isomorphism

$$I_D \otimes_R \Bbbk \cong \frac{I_D}{\mathfrak{m} \cdot I_D} \cong \mathcal{S}_D,$$

where $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ is the homogeneous maximal ideal of the polynomial ring $R = \Bbbk[x_1, \dots, x_n]$.

3. Commutative Algebra toolbox

In this section, we introduce several key constructions and tools from commutative algebra that we will utilize throughout this report.

Let $R = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring over a field $\Bbbk$, M an R-module. Often, to study the algebraic structure of M, it is useful to gain an understanding of M in terms of various notions of *dimension*, which fall under the umbrella of homological algebra. All R-modules we will consider are *graded*, meaning that $M = \bigoplus_{j \in \mathbb{N}} M_j$, where each M_j is a $\Bbbk$-vector space and $x_i M_j \subseteq M_{j+1}$ for all $1 \leq i \leq n$. The first notion of dimension comes from the grading:

Definition 3.1. Let M be a graded R-module. The *Hilbert series* of M over R is the formal power series

$$\mathrm{HS}_R(M, t) = \sum_{j \in \mathbb{N}} \dim_{\Bbbk} M_j \, t^j.$$

Hilbert series are additive in the following sense: given a degree-preserving short exact sequence of graded R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

then $\mathrm{HS}_R(A, t) - \mathrm{HS}_R(B, t) + \mathrm{HS}_R(C, t) = 0$. This follows from the fact that the dimension of $\Bbbk$-vector spaces is additive.

In this paper, we will consider graded R-modules for which the graded summands M_j are invariant with respect to an action $\mathfrak{S}_n \curvearrowright M$, meaning that they carry representations of the symmetric group, or a module over the group algebra $\Bbbk[\mathfrak{S}_n]$. To keep track of such representations in arbitrary characteristic, we will use the language of Grothendieck rings.

Definition 3.2. Let G be a finite group. The *Grothendieck group* $R_{\Bbbk}(G)$ of virtual $\Bbbk G$-modules is a quotient of the free $\mathbb{Z}$-module whose basis is the set of isomorphism classes $[V]$ of $\Bbbk G$-modules where we mod out by the relations

- $[V] = [V']$ if $V \cong V'$ as $\Bbbk G$-modules, and
- $V_2 = V_1 + V_3$ whenever $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of $\Bbbk G$ modules.

In particular, we have that $[U] + [V] = [U \oplus V]$. The ring multiplication in $R_{\Bbbk}(G)$ is induced by the rule $[U] \cdot [V] \coloneqq [U \otimes_{\Bbbk} V]$, which descends to the quotient.

Observe that equivariant assertions involving $R_{\Bbbk}(G)$ can always be specialized to nonequivariant ones by applying the *dimension homomorphism*

$$R_{\Bbbk}(G) \to \mathbb{Z}, \qquad [V] \mapsto \dim_{\Bbbk} V.$$

Definition 3.3. Let $M = \bigoplus_{j \in \mathbb{N}} M_j$ be a graded R-module with an action of a finite group G such that each M_j is invariant under the action. The *G-equivariant Hilbert series* of M in $R_{\Bbbk}(G)[[t]]$ is

$$\mathrm{Hilb}_{\mathrm{eq}}(M, t) \coloneqq \sum_{i=0}^{\infty} [M_i] t^i.$$

In 1890, Hilbert introduced the notion of free resolutions to capture even finer data about a module over a ring than the Hilbert series.

Definition 3.4. A *free resolution* of M over R is an exact sequence of R-modules

$$\mathcal{F}_\bullet : \quad \cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \longrightarrow 0,$$

where $\partial_i \circ \partial_{i+1} = 0$, $\ker \partial_i = \operatorname{im} \partial_{i+1}$, and each F_i is free for $i \geq 0$. A free resolution over R is *minimal* if for each $i \geq 0$, $\operatorname{im} \partial_i \subseteq \langle x_1, \ldots, x_n \rangle F_{i-1}$, that is, no units appear in the matrix of ∂_i.

Free resolutions encode the obstruction of M from being free, in the sense of the following construction. We outline a method for finding a free resolution of M in the case that it is finitely generated. Let $G_0 \subseteq M$ be a finite set of generators of M, and let $F_0 = R^{G_0}$ be the free R-module on these generators. We have a natural surjection $\partial_0 : F_0 \to M$, the kernel of which is the R-module of *linear relations* on the generators G_0 in M.

As F_0 is finitely generated and R is a principal ideal domain, $\ker \partial_0 \subseteq F_0$ is finitely generated as well, so we can find a finite set of generators $G_1 \subseteq \ker \partial_0$. Repeating this process, we define $F_1 = R^{G_1}$ to be the free R-module on the generating set, from which we get a natural surjection $\partial_1 : F_1 \to F_0$ satisfying $\operatorname{im} \partial_1 = \ker \partial_0$. Now, F_1 describes the *relations on the relations* of generators of M. Continuing inductively, we have a free resolution $\mathcal{F}_\bullet$ of M.

The *projective dimension* is the largest $n \in \mathbb{N}$ for which $F_n \neq 0$, if it exists. In general, the process outlined above may not terminate if M is not finitely generated or R is not a polynomial ring. When these conditions are satisfied, however, we have a nice bound on the minimal length of a free resolution:

Theorem 3.5 (Hilbert's Syzygy Theorem)**.** *Suppose $R = \Bbbk[x_1, \ldots, x_n]$ for an algebraically closed field $\Bbbk$. Every finitely generated graded R-module has a free resolution of length $\leq n$, in which each free R-module is finitely generated.*

For $j \geq 0$, let $R(-j)$ denote the free R-module generated in degree j, meaning that as a graded R-module, $R(-j) = \bigoplus_{i \geq j} R_{i-j}$, where R_{i-j} is the degree $(i-j)$-graded summand of R. When M is graded, the free modules F_i in a free resolution may be given a graded structure $F_i = \bigoplus_{j \geq 0} R(-j)^{d_{i,j}}$ for some $d_{i,j} \in \mathbb{N}$ such that the differentials ∂_i are degree-preserving. If the free resolution is minimal, the dimensions $d_{i,j}$ are called the *Betti numbers* of M, and are denoted $\beta_{i,j}(M)$.

When M has an action of a group $G \subseteq GL(n, \Bbbk)$, the action may be extended to the R-modules F_i such that the differentials ∂_i respect the action of G. When one can identify a group that acts on M, one can leverage the representation theory of the group to determine information about the minimal free resolution and Betti numbers of M.

Proposition 3.6. *[Bro+11, Prop. 2.1] Let $A = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$, and let M be a finite A-module. Assume that a finite group G acts on A and M in a degree-preserving way that is compatible with the A-module structure, i.e., $g(am) = g(a)g(m)$ for all $a \in A$ and $m \in M$.*

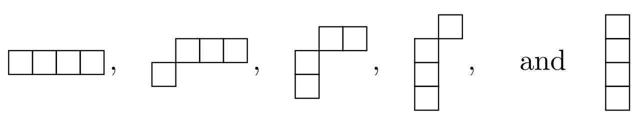
In this setting, there exists a G-equivariant finite free resolution of M over A, where each free module F_i is a $\Bbbk G$-module of the form $A \otimes_{\Bbbk} V_i$ for some finite-dimensional graded $\Bbbk G$-module V_i and all maps being $\Bbbk G$-module morphisms.

When $\Bbbk G$ is semisimple, this resolution may be chosen minimally. In this case, one has $\Bbbk G$-module isomorphisms

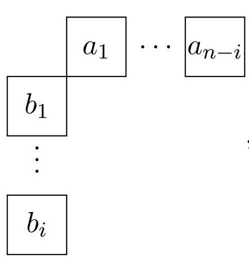
$$Tor_i^A(M, \Bbbk) \cong V_i$$

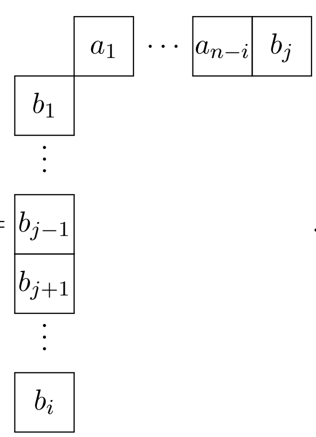
for all i between 0 and the projective dimension of M.

Example 3.7 ($\mathfrak{S}_n$-equivariant Koszul complex)**.** Let $R = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field $\Bbbk$. Consider $R_0 = \Bbbk = R/\langle x_1, \ldots, x_n \rangle$ as an R-module. We construct a minimal $\mathfrak{S}_n$-equivariant free resolution of $\Bbbk$ over R as follows. Let $\mu_0 = (n)$ and $\mu_n = (1^n)$, and for $1 < i < n$, let $\mu_i = (n-i+1, 1^i)/(1)$. For $n = 4$, these correspond to diagrams



For a tableau T of μ_i and some $1 \leq j \leq i$, define a tableau T_j of μ_{i-1} by moving the j^{th} box of the column of T to the right end of the row of T:

if $T =$ 

, then $T_j =$ 

.

Define maps $\partial_i : \mathcal{S}_{\mu_i} \otimes_{\Bbbk} R(-i) \to \mathcal{S}_{\mu_{i-1}} \otimes_{\Bbbk} R(-i+1)$ by the rule $\partial_i(e(T)) = \sum_{j=1}^{i} (-1)^{j+1} e(T_j) \otimes x_{b_j}$ for each tableau T of μ_i. In terms of diagrams, we move boxes from the column of T to the row, multiplying by the corresponding variable and an alternating sign. Letting $\partial_0 : \mathcal{S}_{\mu_0} \otimes_{\Bbbk} R \cong R \to R/\langle x_1, \ldots, x_n \rangle \cong \Bbbk$ be the quotient map, the following sequence of maps turns out to be a minimal $\mathfrak{S}_n$-equivariant free resolution of $\Bbbk$, called the *Koszul complex*:

$$0 \longrightarrow \mathcal{S}_{\mu_n} \otimes_{\Bbbk} R(-n) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} \mathcal{S}_{\mu_0} \otimes_{\Bbbk} R \xrightarrow{\partial_0} \Bbbk \longrightarrow 0.$$

We use the Koszul complex to analyze the $\mathfrak{S}_n$-equivariant Hilbert series of certain ribbon Specht ideals in Section 4.3.

In the following sections, we analyze free resolutions of ideals $I \subseteq R$, considered as R-modules. We note that there is a nice relationship between free resolutions of I and of the quotient R/I: the graded short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

allows one to convert a free resolution of R/I into one of I by replacing the final arrows $R \to R/I \to 0$ with the arrow $I \to 0$, and vice versa. Thus, the Betti numbers of R/I and I are related by a degree shift: $\beta_{i,j}(R/I) = \beta_{i+1,j}(I)$.

The technical details of the following definition are outside the scope of this paper, but it is so intimately related with the structure of free resolutions that it bears mentioning.

Definition 3.8. A ring A is *Gorenstein* if it is a Noetherian local ring with finite injective dimension. A *Gorenstein ideal* of A is a perfect ideal I such that A/I is Gorenstein.

By a theorem of Buchsbaum and Eisenbud, we have the following useful fact.

Theorem 3.9. *[BE77, Theorem 1.5] If R is a local ring and $I \subseteq R$ is a Gorenstein ideal, then the minimal free resolution of R/I over R is self-dual.*

4. Two-row ribbons

4.1. Description of the resolution.

In [SY23b], Shibata and Yanagawa found the minimal free resolution of the Specht ideal $I_{(n-d,d)}$, explicitly describing the free modules at each step and the maps between them. Instead of considering two row partitions, in this section we consider two row ribbons. In this section, let $\Bbbk$ be a field of characteristic zero.

Definition 4.1. For a two-row ribbon $\mathrm{Ribb}(k,\ell)$, let $R = \Bbbk[x_1, \ldots, x_{k+\ell}]$. Then, we define the following sequence of free R-modules and maps between them:

$$\mathcal{F}_\bullet^{\mathrm{Ribb}(k,\ell)} : 0 \longrightarrow F_{k+\ell-2} \xrightarrow{\partial_{k+\ell-2}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0, \tag{1}$$

where if $0 \leq i \leq k-1$, we have:

$$F_i = \mathcal{S}_{\mathrm{Ribb}(k-i,\ell,1^i)} \otimes R(-\ell - i),$$

and if $k \leq i \leq k+\ell-2$, we have

$$F_i = \mathcal{S}_{\mathrm{Ribb}(k+\ell-i-1,1^{i+1})} \otimes R(-\ell - i - 1).$$

To define the maps ∂_i, we need some preparation. First, consider $1 \leq i \leq k-1$. In this case, given a basis element $e(T) \otimes 1$, we let $a_1, \ldots, a_{i+1}$ denote the numbers in the first column of T. Then, we define $\partial_i(e(T)\otimes 1)=\sum_{j=1}^{i+1}(-1)^{j-1}e(T^j)\otimes x_{a_j}$ where T^j is obtained by moving the box with label a_j to the right of the first row.

Similarly, $\partial_k(e(T)\otimes 1)=\sum_{j<i}(-1)^{j+i-1}e(T_{j,i})\otimes x_{a_j}x_{a_i}$, where $T_{j,i}$ is the tableau obtained by moving two boxes with labels a_j, a_i in the first column of T to the top-right such that a_i is the last box in the second row of $T_{j,i}$.

Lastly, we consider $k_1\le i\le k+\ell-2$. In this case, we define $\partial_i(e(T)\otimes 1)=\sum_j(-1)^j e(T_j)\otimes x_{a_j}$ where T_j is obtained by moving the box with label a_j from the first column of T to the end of the first non-empty row.

We conjecture the following:

Conjecture 4.2. $\mathcal{F}_\bullet^{\mathrm{Ribb}(k,\ell)}$ *is a minimal free resolution for* $I_{\mathrm{Ribb}(k,\ell)}$

Example 4.3. For the ribbon $\mathrm{Ribb}(2,2)=\ydiagram{1+2,2}$, we have:

$$0\longrightarrow \mathcal{S}_{\ydiagram{1,1,1,1}}\otimes R(-5)\xrightarrow{\partial_2}\mathcal{S}_{\ydiagram{1+1,2,1}}\otimes R(-3)\xrightarrow{\partial_1}\mathcal{S}_{\ydiagram{1+2,2}}\otimes R(-2)\longrightarrow I_{\ydiagram{1+2,2}}\longrightarrow 0.$$

We show how the maps are defined on basis elements. For simplicity of notation, in this example we denote a polytabloid $e(T)$ just by its associated tableau T. For ∂_2, we have:

$$\begin{aligned}\partial_2\left(\begin{ytableau}1\\2\\3\\4\end{ytableau}\otimes 1\right)&=\begin{ytableau}\none&1\\3&2\\4\end{ytableau}\otimes x_1x_2-\begin{ytableau}\none&1\\2&3\\4\end{ytableau}\otimes x_1x_3+\begin{ytableau}\none&1\\2&4\\3\end{ytableau}\otimes x_1x_4\\&\quad+\begin{ytableau}\none&2\\1&3\\4\end{ytableau}\otimes x_2x_3-\begin{ytableau}\none&2\\1&4\\3\end{ytableau}\otimes x_2x_4+\begin{ytableau}\none&3\\1&4\\2\end{ytableau}\otimes x_3x_4,\end{aligned}$$

and for ∂_1:

$$\begin{aligned}\partial_1\left(\begin{ytableau}\none&1\\2&3\\4\end{ytableau}\otimes 1\right)&=\begin{ytableau}\none&1&2\\4&3\end{ytableau}\otimes x_2-\begin{ytableau}\none&1&4\\2&3\end{ytableau}\otimes x_4,\\\partial_1\left(\begin{ytableau}\none&1\\2&4\\3\end{ytableau}\otimes 1\right)&=\begin{ytableau}\none&1&2\\3&4\end{ytableau}\otimes x_2-\begin{ytableau}\none&1&3\\2&4\end{ytableau}\otimes x_3,\\\partial_1\left(\begin{ytableau}\none&2\\1&3\\4\end{ytableau}\otimes 1\right)&=\begin{ytableau}\none&2&1\\4&3\end{ytableau}\otimes x_1-\begin{ytableau}\none&2&4\\1&3\end{ytableau}\otimes x_4,\\\partial_1\left(\begin{ytableau}\none&2\\1&4\\3\end{ytableau}\otimes 1\right)&=\begin{ytableau}\none&2&1\\3&4\end{ytableau}\otimes x_1-\begin{ytableau}\none&2&3\\1&4\end{ytableau}\otimes x_3,\\\partial_1\left(\begin{ytableau}\none&3\\1&4\\2\end{ytableau}\otimes 1\right)&=\begin{ytableau}\none&3&1\\2&4\end{ytableau}\otimes x_1-\begin{ytableau}\none&3&2\\1&4\end{ytableau}\otimes x_2.\end{aligned}$$

It is worth noting that since $\{e(T):T\in\mathrm{Tab}(\lambda/\mu)\}$ is linearly dependent, the well-definedness of the maps ∂_i is not trivial. However, we can prove well-definedness following a very similar argument to that in [SY23a].

Proposition 4.4. *The maps* ∂_i *are well-defined.*

Proof. We begin by proving the well-definedness of ∂_i for $1 \le i \le k-1$. We observe that because any linear relation between polytabloids is a combination of Garnir relations, it suffices to show that

$$\sum_{\pi \in G_{A,B}} \operatorname{sign}(\pi)\partial_i(e(\pi T) \otimes 1) = 0 \tag{2}$$

for any $T \in \operatorname{Tab}(\operatorname{Ribb}(k-i, \ell, 1^i))$.

Let $T \in \operatorname{Tab}(\operatorname{Ribb}(k-i, \ell, 1^i))$ be given by

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}
\cline{5-8}
\multicolumn{4}{c|}{} & b_1 & b_3 & \cdots & b_{k-i+1} \\
\hline
a_1 & c_1 & \cdots & c_{\ell-2} & b_2 & \multicolumn{3}{c}{} \\
\cline{1-5}
a_2 & \multicolumn{7}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{7}{c}{} \\
\cline{1-1}
a_{i+1} & \multicolumn{7}{c}{} \\
\cline{1-1}
\end{array} . \tag{3}$$

We observe that there are only two possible non-trivial descents: $a_1 > c_1$ or $b_1 > b_3$. If $\ell = 2$, there is a third case $a_1 > b_2$ which follows by a similar argument to what is described below. This case is shown in Example 4.5.

If $b_1 > b_3$, then $A = \{b_1, b_2\}$ and $B = \{b_3\}$. Now we consider the terms of the sum in (2) of the form $- \otimes x_{a_j}$ for some $1 \le j \le i+1$. We observe that the Garnir element associated with A and B in T is the same as the Garnir element $g_{A,B}$ associated with A and B in T^j $g_{A,B}$. Hence, the sum of terms of this form is

$$\begin{aligned}
\sum_{\pi \in G_{A,B}} (-1)^{j-1}\operatorname{sign}(\pi)e((\pi T)^j) \otimes x_{a_j} &= (-1)^{j-1} \sum_{\pi \in G_{A,B}} \operatorname{sign}(\pi)e((\pi T)^j) \otimes x_{a_j} \\
&= (-1)^{j-1} g_{A,B} e(T^j) \otimes x_{a_j} \\
&= 0.
\end{aligned}$$

Therefore, (2) holds.

If $a_1 > c_1$, then $A = \{a_1, \ldots, a_{i+1}\}$ and $B = \{c_1\}$. Similarly, we can consider the terms of the form $- \otimes x_{a_j}$ in the sum (2), as well as those of the form $- \otimes x_{c_1}$. By definition of polytabloids and since we are only concerned with Garnir relations between them, we may assume without loss of generality that $a_1 < a_2 < \cdots < a_{i+1}$.

We first consider the terms $- \otimes x_{a_j}$. Let $\pi \in G_{A,B}$ and denote $A^j = A \setminus \{a_j\}$. In this case, we must have $\pi(c_1) \neq a_j$ since otherwise we do not obtain a term with x_{a_j}. If $\pi(c_1) \ge a_{j+1}$, then $a_1 < \ldots < a_{i+1}$ implies that $\pi(a_{j+1}) = a_j$, and in fact this is both a sufficient and necessary condition. Furthermore, the permutation $t_\pi := (a_j, \pi(a_j)) \cdot \pi \in G_{A^j,B}$ in the tableau T^j. This establishes a bijective correspondence

$$\{\pi \in G_{A,B} : \pi(c_1) \ge a_{j+1}\} \longleftrightarrow \{\sigma \in G_{A^j,B} : \sigma(b_1) \ge \sigma(a_{j+1})\}$$

where $G_{A^j,B}$ is interpreted within the tableau T^j. We have $\operatorname{sign}(t_\pi) = -\operatorname{sign}(\pi)$ and it is also easy to check that $(\pi T)^{j+1} = t_\pi T^j$.

Similarly, $\pi(c_1) \le a_{j-1}$ if and only if $\pi(a_j) = a_j$ and, moreover, π is a Garnir element associated with A^j and B in the tableau T^j.

Hence, the $-\otimes x_{a_j}$ part of (2) is given by

$$\begin{aligned}
&\left(\sum_{\substack{\pi\in G_{A,B}\\ \pi(c_1)\le a_{j+1}}}(-1)^{j-1}\mathrm{sign}(\pi)e((\pi T)^j)+\sum_{\substack{\pi\in G_{A,B}\\ \pi(c_1)\ge a_{j+1}}}(-1)^{j}\mathrm{sign}(\pi)e((\pi T)^{j+1})\right)\otimes x_{a_j}\\
&=\left(\sum_{\substack{\pi\in G_{A,B}\\ \pi(c_1)\le a_{j+1}}}(-1)^{j-1}\mathrm{sign}(\pi)e(\pi T^j)+\sum_{\substack{t_\pi\in G_{A^j,B}\\ t_\pi(c_1)\ge a_{j+1}}}-(-1)^{j}\mathrm{sign}(\pi)e(t_\pi T^j)\right)\otimes x_{a_j}\\
&=(-1)^{j-1}g_{A^j,B}e(T^j)=0.
\end{aligned}$$

For the terms of the form $-\otimes x_{c_1}$, we observe that if we have such term, we must have $\pi(a_1)=c_1$. Let $T'=((a_1,c_1)T)^1$ and define $A'=A\setminus\{a_1\}$ and $B'=\{a_1\}$. If $\pi\in G_{A,B}$ and $\pi(a_1)=c_1$, then $t'_\pi := (c_1,\pi(c_1))\pi\in G_{A',B'}$ in T' and $\pi\mapsto t_\pi$ defines a bijection when $\pi(a_1)=c_1$. Hence, the x_{c_1} part of (2) is equal to $-g_{A',B'}e(T')\otimes x_{c_1}=0$.

Thus, we have shown that ∂_i is well-defined for $1\le i\le k-1$. It is straightforward to see that a very similar argument as the one applied for the $a_1>c_1$ case above shows that ∂_i is well-defined for $i\ge k+1$. Thus, it only remains to show well-definedness of ∂_k. For this, we define the maps $\varphi:\mathcal{S}_{\mathrm{Ribb}(\ell-1,1^{k+1})}\to\mathcal{S}_{\mathrm{Ribb}(\ell,1^k)}\otimes R_1$ and $\psi:\mathcal{S}_{\mathrm{Ribb}(\ell,1^k)}\to\mathcal{S}_{\mathrm{Ribb}(1,\ell,1^{k-1})}\otimes R_1$ by

$$\varphi(T)=\sum_{j=1}^{k+2}(-1)^{j-1}e(T_j)\otimes x_{a_j}$$

for $T\in\mathrm{Ribb}(\ell-1,1^{k+1})$ having first column $a_1,\dots,a_{k+2}$, and

$$\psi(T)=\sum_{j=1}^{k+1}(-1)^{j-1}e(T^j)\otimes x_{a_j}$$

for $T\in\mathrm{Ribb}(\ell,1^k)$ with first column $a_1,\dots,a_{k+1}$. Lastly, define $\sigma:R_1\otimes R_1\to R_2$ by $\sigma(x_i\otimes x_j)=\frac{1}{2}x_ix_j$.

Then, it is easy to see that $((\mathrm{Id}\otimes\sigma)\circ(\psi\otimes\mathrm{Id})\circ\varphi)(e(T))$ agrees with the $\ell+k+1$-degree part of ∂_k. Identifying $F_k=\mathcal{S}_{\mathrm{Ribb}(\ell-1,1^{k+1})}\otimes R(-\ell-k-1)$ with its $\ell+k+1$ degree part $\mathcal{S}_{\mathrm{Ribb}(\ell-1,1^{k+1})}$, we then see that ∂_k is uniquely determined by its $\ell+k+1$. Lastly, we observe that proving the well-definedness of φ and ψ is equivalent to that of ∂_i for $i\ge k_1$ and $i\le k-1$, respectively. Hence, ∂_k is well-defined. $\square$

We now give an example to illustrate the proof of the proposition above.

Example 4.5. We continue with our running example $\mathrm{Ribb}(2,2)=\ydiagram{1+2,2}$. In the complex shown in Example 4.3, the only map for which well-definedness is not immediate is ∂_1. We consider the tableau

$$T=\begin{ytableau}\none & 1\\ 3 & 2\\ 4\end{ytableau}.$$

T has a descent $3>2$, and so we have $A=\{3,4\}$ and $B=\{1,2\}$. We thus have the linear relation

$$\begin{ytableau}\none & 1\\ 3 & 2\\ 4\end{ytableau}-\begin{ytableau}\none & 1\\ 2 & 3\\ 4\end{ytableau}+\begin{ytableau}\none & 1\\ 2 & 4\\ 3\end{ytableau}+\begin{ytableau}\none & 3\\ 1 & 4\\ 2\end{ytableau}-\begin{ytableau}\none & 2\\ 1 & 4\\ 3\end{ytableau}+\begin{ytableau}\none & 2\\ 1 & 3\\ 4\end{ytableau}=0.$$

Note that for the ease of the reader, we have written tableaux instead of polytabloids. We then see that

$$
\begin{aligned}
&\partial_1\left(\left(\ytableaushort{\none 1,32,4}-\ytableaushort{\none 1,23,4}+\ytableaushort{\none 1,24,3}+\ytableaushort{\none 3,14,2}-\ytableaushort{\none 2,14,3}+\ytableaushort{\none 2,13,4}\right)\otimes 1\right)\\
&=\ytableaushort{\none 13,42}\otimes x_3-\ytableaushort{\none 14,32}\otimes x_4-\ytableaushort{\none 12,43}\otimes x_2+\ytableaushort{\none 14,23}\otimes x_4\\
&\quad+\ytableaushort{\none 12,34}\otimes x_2-\ytableaushort{\none 13,24}\otimes x_3+\ytableaushort{\none 31,24}\otimes x_1-\ytableaushort{\none 32,14}\otimes x_2\\
&\quad-\ytableaushort{\none 21,34}\otimes x_1+\ytableaushort{\none 23,14}\otimes x_3+\ytableaushort{\none 21,43}\otimes x_1-\ytableaushort{\none 24,13}\otimes x_4\\
&=\left(\ytableaushort{\none 21,43}-\ytableaushort{\none 21,34}+\ytableaushort{\none 31,24}\right)\otimes x_1-\left(\ytableaushort{\none 12,43}-\ytableaushort{\none 12,34}+\ytableaushort{\none 32,14}\right)\otimes x_2\\
&\quad+\left(\ytableaushort{\none 13,42}-\ytableaushort{\none 13,24}+\ytableaushort{\none 23,14}\right)\otimes x_3-\left(\ytableaushort{\none 14,32}-\ytableaushort{\none 14,23}+\ytableaushort{\none 24,13}\right)\otimes x_4\\
&=0,
\end{aligned}
$$

where the third equality follows from the Garnir relations present in each term.

We now prove that $\mathcal{F}_\bullet^{\mathrm{Ribb}(k,l)}$ is actually a chain complex.

Proposition 4.6. $\mathcal{F}_\bullet^{\mathrm{Ribb}(k,l)}$ *is a chain complex.*

Proof. It suffices to show that $\partial_{i-1}\partial_i = 0$ for all $2 \leq i \leq k+\ell-2$.

We begin by considering $i < k$. Consider a tableau T as in (3). Every term in $\partial_{i-1}\partial_i(e(T)\otimes 1)$ will correspond to a tableau T' whose first row has entries $b_1, b_3, \ldots, b_{k-i+1}, a_{j_1}, a_{j_2}$ (in this order) for some j_1, j_2. Observe that $e(T') = e(\tau(T'))$, where $\tau = (a_{j_1}, a_{j_2}) \in \mathfrak{S}_n$. As $\mathrm{sign}(\tau) = -1$, it follows that $e(T')$ and $e(\tau(T'))$ appear with opposite signs in $\partial_{i-1}\partial_i(e(T)\otimes 1)$, so $\partial_{i-1}\partial_i = 0$. A similar argument applies when $i > k+1$, so it remains to check the cases $i = k, k+1$.

If $i = k$, let

$$
T = \begin{ytableau} a_1 & b_2 & \cdots & b_{\ell-1} \\ a_2 \\ \vdots \\ a_{k+2} \end{ytableau} \in \mathrm{Ribb}(\ell-1, 1^{k+1}).
$$

Then

$$
\partial_{k-1}\partial_k(e(T)\otimes 1) = \sum_{a_{j_1}<a_{j_2}<a_{j_3}} (-1)^{j_1+j_2+j_3}\left(e(T^{j_3}_{j_1,j_2}) - e(T^{j_2}_{j_1,j_3}) + e(T^{j_1}_{j_2,j_3})\right)\otimes x_{a_{j_1}}x_{a_{j_2}}x_{a_{j_2}},
$$

where

$$
T^{j_3}_{j_1,j_2} = \begin{ytableau} \none & \none & \none & \none & a_{j_1} & a_{j_3} \\ a_1 & b_2 & \cdots & b_{\ell-1} & a_{j_2} \\ \vdots \\ a_{k+2} \end{ytableau}, \quad T^{j_2}_{j_1,j_3} = \begin{ytableau} \none & \none & \none & \none & a_{j_1} & a_{j_2} \\ a_1 & b_2 & \cdots & b_{\ell-1} & a_{j_3} \\ \vdots \\ a_{k+2} \end{ytableau},
$$

and $T^{j_1}_{j_2,j_3}$ is obtained analogously. We remark that the first column for each of these tableaux omits $a_{j_1}, a_{j_2}, a_{j_3}$ but we have not emphasized this in the diagram to avoid convoluted notation. It is then easy to see that $e(T^{j_3}_{j_1,j_2}) - e(T^{j_2}_{j_1,j_3}) + e(T^{j_1}_{j_2,j_3}) = 0$ by a Garnir relation, so $\partial_{k-1}\partial_k = 0$.

Lastly, we consider $i = k+1$. Let

$$T = \begin{array}{|c|c|c|c|} \hline a_1 & b_2 & \cdots & b_{\ell-2} \\ \hline a_2 & \multicolumn{3}{c}{} \\ \cline{1-1} \vdots & \multicolumn{3}{c}{} \\ \cline{1-1} a_{k+3} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \in \mathrm{Ribb}(\ell-2, 1^{k+2}).$$

Then, we have

$$\partial_k\partial_{k+1}(e(T)\otimes 1) = \sum_{a_{j_1}<a_{j_2}<a_{j_2}} (-1)^{j_1+j_2+j_3}\left(e(T_{j_1,j_2,j_3}) - e(T_{j_2,j_1,j_3}) + e(T_{j_3,j_1,j_2})\right)\otimes x_{a_{j_1}}x_{a_{j_2}}x_{a_{j_2}},$$

where

$$T_{j_1,j_2,j_3} = \begin{array}{|c|c|c|c|c|c|} \cline{6-6} \multicolumn{5}{c|}{} & a_{j_2} \\ \hline a_1 & b_1 & \cdots & b_{\ell-2} & a_{j_1} & a_{j_3} \\ \hline a_2 & \multicolumn{5}{c}{} \\ \cline{1-1} \vdots & \multicolumn{5}{c}{} \\ \cline{1-1} a_{k+3} & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array}, \quad T_{j_2,j_1,j_3} = \begin{array}{|c|c|c|c|c|c|} \cline{6-6} \multicolumn{5}{c|}{} & a_{j_1} \\ \hline a_1 & b_1 & \cdots & b_{\ell-2} & a_{j_2} & a_{j_3} \\ \hline a_2 & \multicolumn{5}{c}{} \\ \cline{1-1} \vdots & \multicolumn{5}{c}{} \\ \cline{1-1} a_{k+3} & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array},$$

and T_{j_3,j_1,j_2} is defined analogously. As mentioned earlier, the first column does not contain either $a_{j_1}, a_{j_2}, a_{j_3}$.

It is then easy to see that $e(T_{j_1,j_2,j_3}) - e(T_{j_2,j_1,j_3}) + e(T_{j_3,j_1,j_2}) = 0$ via a Garnir relation, so $\partial_k\partial_{k+1} = 0$. This finishes the proof. □

We now give an example to illustrate the proof of Proposition 4.6.

Example 4.7. We consider Ribb(2, 2) as in Example 4.3. The composition $\partial_1\partial_2$ on the basis element of $\mathcal{S}_{(1^4)} \otimes R$ is given by:

$$\begin{aligned}
\partial_1\partial_2\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\otimes 1\right) &= \left(\begin{smallmatrix} & 1 & 3 \\ 4 & 2 & \end{smallmatrix} - \begin{smallmatrix} & 1 & 2 \\ 4 & 3 & \end{smallmatrix} + \begin{smallmatrix} & 2 & 1 \\ 4 & 3 & \end{smallmatrix}\right)\otimes x_1x_2x_3 \\
&+ \left(-\begin{smallmatrix} & 1 & 4 \\ 3 & 2 & \end{smallmatrix} - \begin{smallmatrix} & 2 & 1 \\ 3 & 4 & \end{smallmatrix} + \begin{smallmatrix} & 1 & 2 \\ 3 & 4 & \end{smallmatrix}\right)\otimes x_1x_2x_4 \\
&+ \left(\begin{smallmatrix} & 1 & 4 \\ 2 & 3 & \end{smallmatrix} - \begin{smallmatrix} & 1 & 3 \\ 2 & 4 & \end{smallmatrix} + \begin{smallmatrix} & 3 & 1 \\ 2 & 4 & \end{smallmatrix}\right)\otimes x_1x_3x_4 \\
&+ \left(\begin{smallmatrix} & 2 & 3 \\ 1 & 4 & \end{smallmatrix} - \begin{smallmatrix} & 2 & 4 \\ 1 & 3 & \end{smallmatrix} - \begin{smallmatrix} & 3 & 2 \\ 1 & 4 & \end{smallmatrix}\right)\otimes x_2x_3x_4.
\end{aligned}$$

As in previous examples, we have omitted the notation $e(T)$ for a polytabloid and instead we have written its corresponding tableau. Observe that we have arranged terms as outlined in the proof of Proposition 4.6, and it is straightforward to see that the Garnir relations imply $\partial_1\partial_2 = 0$.

4.2. **Exactness of the complex.** In [SY23b], the authors use results on the Hilbert series of $R/I_{(n-d,d)}$ to prove that the chain complex they construct is exact and minimal. These results rely on a description of the minimal primes of $I_{(n-d,d)}$ and the fact that $R/I_{(n-d,d)}$ is Cohen-Macaulay. In this section, we outline some analogous results in this direction for two-row ribbon Specht ideals.

First, we identify a generating set of $I_{\mathrm{Ribb}(b,d)}$ having a nice combinatorial description. Given a subset of $d+1$ indices $J = \{j_1 < \cdots < j_{d+1}\} \subseteq \{1, \ldots, b+d\}$ and some $1 \le i \le d$, let

$$f_{J,i} = x_{j_1} \cdots \widehat{x_{j_i}} \cdots x_{j_d}(x_{j_i} - x_{j_{d+1}}) \in R. \tag{4}$$

Then $f_{J,i}$ is the Specht polynomial of the skew tableau

<table>
<tr><td></td><td></td><td></td><td></td><td></td><td>j_i</td><td>ℓ_1</td><td>$\cdots$</td><td>ℓ_{b-1}</td></tr>
<tr><td>j_1</td><td>$\cdots$</td><td>j_{i-1}</td><td>j_{i+1}</td><td>$\cdots$</td><td>j_{d+1}</td><td></td><td></td><td></td></tr>
</table>

,

where the $\ell_1, \ldots, \ell_{b-1}$ are the remaining indices; their order is immaterial. Among these tableaux are the standard Young tableaux of $\mathrm{Ribb}(b,d)$, which implies that the $f_{J,i}$ generate $I_{\mathrm{Ribb}(b,d)}$.

Let $P_F = \langle x_i \mid i \notin F \rangle$ for $F \subseteq \{1, \ldots, b+d\}$. The P_F are precisely the *monomial prime ideals* of R; cf. [MS05].

Proposition 4.8. *A two-row ribbon Specht ideal has the following prime decomposition:*

$$I_{\mathrm{Ribb}(b,d)} = \langle x_i - x_j \mid 1 \le i < j \le b+d \rangle \cap \left(\bigcap_{\substack{\#F = d-1 \\ F \subseteq [b+d]}} P_F \right).$$

Proof. Note that all two-row ribbon Specht polynomials are binomials. By [ES96], in characteristic 0, every binomial ideal is radical.

We determine the prime decomposition of $I_{\mathrm{Ribb}(b,d)}$ by analyzing the affine variety $\mathbb{V}(I_{\mathrm{Ribb}(b,d)}) \subseteq \mathbb{A}^{b+d}_{\Bbbk}$. Toward this end, suppose $\mathbf{a} = (a_1, \ldots, a_{b+d}) \in \mathbb{V}(I_{\mathrm{Ribb}(b,d)})$. Then $f_{J,i}(\mathbf{a}) = 0$ for all subsets $J = \{j_1 < \cdots < j_{d+1}\} \subseteq [b+d]$ and $1 \le i \le d$, so that

$$a_{j_1} \cdots \widehat{a_{j_i}} \cdots a_{j_d}(a_{j_{d+1}} - a_{j_i}) = 0,$$

meaning that either $a_{j_\ell} = 0$ for some $1 \le \ell \le d$ with $\ell \neq j$, or $a_{j_i} = a_{j_{d+1}}$. This must be true for *any* choice J of $d+1$ and any $1 \le i \le d$.

Fix a set of indices $J \subseteq [b+d]$ of size $d+1$. We analyze the entries $a_{j_1}, \ldots, a_{j_{d+1}}$ in cases:

1. If $a_{j_\ell} = 0$ for only *one* $1 \le \ell \le d$, then $f_{J,i}(\mathbf{a}) = 0$ is immediate if $i \neq \ell$; when $i = \ell$, the equation reads
$$a_{j_1} \cdots \widehat{a_{j_\ell}} \cdots a_{j_d}(a_{j_{d+1}} - 0) = 0,$$
which implies that $a_{j_{d+1}} = 0$.
2. If $a_{j_\ell} = a_{j_r} = 0$ for any two $1 \le \ell, r \le d$, then there is always a zero in the product $a_{j_1} \cdots \widehat{a_{j_i}} \cdots a_{j_d}$, so $f_{J,i}(\mathbf{a}) = 0$ is clear.
3. If $a_{j_i} \neq 0$ for all $1 \le i \le d$, then $f_{J,i}(\mathbf{a}) = 0$ implies that $a_{j_{d+1}} = a_{j_i}$ for any i, so that $a_{j_1} = \cdots = a_{j_{d+1}} \neq 0$.

Thus, if $\mathbf{a} \in \mathbb{V}(I_{\mathrm{Ribb}(b,d)})$, then among any $d+1$ entries of $\mathbf{a}$, either at least two are 0, or all entries are equal and nonzero. The reverse implication is clear from the definition of the polynomials $f_{J,i}$ and the fact that they generate $I_{\mathrm{Ribb}(b,d)}$, so the implication goes both ways.

Now, we argue on the entries of $\mathbf{a} \in \mathbb{A}_{\Bbbk}^{b+d}$ at large. First, note that if $\mathbf{a}$ has fewer than d nonzero entries, then any choice of $d+1$ entries must contain at least two 0s, so $\mathbf{a} \in \mathbb{V}(I_{\mathrm{Ribb}(b,d)})$. This implies that for each $F \subseteq [b+d]$ of size $d-1$, $\mathbb{V}(I_{\mathrm{Ribb}(b,d)})$ contains the irreducible variety $V_F = \{\mathbf{a} \in \mathbb{A}_{\Bbbk}^{b+d} \mid a_i = 0 \text{ if } i \notin F\}$, which corresponds to the prime ideal $P_F \subseteq \Bbbk[x_1, \ldots, x_{b+d}]$ defined above.

Now, suppose $\mathbf{a}$ has at least d nonzero entries, corresponding to some subset $A \subseteq [b+d]$ of size d. Then, for any $\ell \in [b+d] \setminus A$, consider the set $J = A \cup \{\ell\}$. If $\mathbf{a} \in \mathbb{V}(I_{\mathrm{Ribb}(b,d)})$, there either exist distinct indices $j, j' \in J$ for which $a_j = a_{j'} = 0$, or $a_j = a_{j'} \neq 0$ for all $j, j' \in J$. But since J has at least d nonzero entries, the first condition cannot hold; thus the entries of $\mathbf{a}$ indexed in J are all nonzero and equal. As $\ell \in [b+d] \setminus A$ is arbitrary, this implies that *all* entries of $\mathbf{a}$ are equal and nonzero, that is, $\mathbf{a}$ lies in the irreducible variety $E = \{(a, \ldots, a) \mid a \in \Bbbk\} \subseteq \mathbb{V}(I_{\mathrm{Ribb}(b,d)})$, which corresponds to the prime ideal $B = \langle x_i - x_j \mid 1 \leq i < j \leq b+d \rangle \subseteq \Bbbk[x_1, \ldots, x_{b+d}]$. Conversely, $\mathbf{a} \in E$ clearly implies $\mathbf{a} \in \mathbb{V}(I_{\mathrm{Ribb}(b,d)})$. Therefore, we have the following expression of $\mathbb{V}(I_{\mathrm{Ribb}(b,d)})$ in terms of irreducible components:

$$\mathbb{V}(I_{\mathrm{Ribb}(b,d)}) = E \cup \left(\bigcup_{\substack{\#F = d-1 \\ F \subseteq [b+d]}} V_F \right).$$

As $I_{\mathrm{Ribb}(b,d)}$ is radical, this implies that

$$I_{\mathrm{Ribb}(b,d)} = B \cap \left(\bigcap_{\substack{\#F = d-1 \\ F \subseteq [b+d]}} P_F \right),$$

which is the desired result. □

We have observed that certain ribbons exhibit a particularly nice Betti table. In particular, we have the following.

Proposition 4.9. *Assuming that $\mathcal{F}_\bullet^{\mathrm{Ribb}(k,l)}$ is a free resolution for $I_{\mathrm{Ribb}(k,l)}$, then $I_{\mathrm{Ribb}(n-2,2)}$ has a self-dual resolution.*

Proof. We begin by noting that in $\mathcal{F}_\bullet^{\mathrm{Ribb}(n-2,2)}$, we have $F_{n-2} = \mathcal{S}_{(1^n)} \otimes R$ and so it has rank 1 as an R-module. It now suffices to show that for $0 \leq i \leq \frac{n-3}{2}$, the rank of F_i equals the rank of F_{n-3-i} as R-modules. We have $F_{n-3-i} = \mathcal{S}_{\mathrm{Ribb}(1+i,2,1^{n-3-i})} \otimes R$ and $F_i = \mathcal{S}_{\mathrm{Ribb}(n-2-i,2,1^i)} \otimes R$. Observe that $\mathrm{Ribb}(1+i, 2, 1^{n-3-i})$ is the unique ribbon obtained by conjugating the Young diagram $\mathrm{Ribb}(n-2-i, 2, 1^i)$, so $\#\mathrm{SYT}(\mathrm{Ribb}(1+i,2,1^{n-3-i})) = \#\mathrm{SYT}(\mathrm{Ribb}(n-2-i,2,1^i))$. Thus, the claim follows. □

4.3. **$\mathfrak{S}_n$-equivariant Hilbert series.** We are interested in investigating the $\mathfrak{S}_n$-equivariant Hilbert series of $R/I_{\mathrm{Ribb}(k,l)}$ in order to extract more information about the structure of the Specht ideals for these skew shapes. Below, we give an example of the computation of this Hilbert series.

Example 4.10. We consider the ribbon $\mathrm{Ribb}(2,2) = \ydiagram{1+2,2}$. If Conjecture 4 holds, the free resolution for $R/I_{\ydiagram{1+2,2}}$ is given by

$$0 \longrightarrow \mathcal{S}_{\ydiagram{1,1,1,1}} \otimes R(-5) \xrightarrow{\partial_2} \mathcal{S}_{\ydiagram{1+1,2,1}} \otimes R(-3) \xrightarrow{\partial_1} \mathcal{S}_{\ydiagram{1+2,2}} \otimes R(-2) \longrightarrow \mathcal{S}_{\ydiagram{4}} \otimes R \longrightarrow R/I_{\ydiagram{1+2,2}} \longrightarrow 0.$$

We examine the degree i strands of the minimal free resolution for $R/I_{\mathrm{Ribb}(2,2)}$.

- Degree 0: It is easy to verify that $(R/I_{\mathrm{Ribb}(2,2)})_0 \cong \mathcal{S}_{(4)}$, as expected.
- Degree 1: We obtain

$$0 \longrightarrow \mathcal{S}_{(4)} \otimes R_1 \longrightarrow (R/I_{\mathrm{Ribb}(2,2)})_1 \longrightarrow 0 \ .$$

  Therefore, $(R/I_{\mathrm{Ribb}(2,2)})_1 \cong R_1 \otimes \mathcal{S}_{(4)} \cong \mathcal{S}_{(3,1)}$.
- Degree 2: We have

$$0 \longrightarrow R_0(-2) \otimes \mathcal{S}_{\mathrm{Ribb}(2,2)} \longrightarrow \mathcal{S}_{(4)} \otimes R_2 \longrightarrow (R/I_{\mathrm{Ribb}(2,2)})_2 \longrightarrow 0 \ .$$

  Thus, $(R/I_{\mathrm{Ribb}(2,2)})_2 \cong R_2 - R_0 \otimes \mathcal{S}_{\mathrm{Ribb}(2,2)} \cong \mathcal{S}_{(3,1)} + 2\mathcal{S}_{(4)}$, where the Specht-module decomposition of the $\mathfrak{S}_4$-module $R_2 \cong \mathcal{S}_{(2,2)} + 2\mathcal{S}_{(3,1)} + 2\mathcal{S}_{(4)}$ was obtained via Sage.

Continuing, one observes that the representations stabilize at $(R/I_{\mathrm{Ribb}(2,2)})_i \cong \mathcal{S}_{(3,1)} + 2\mathcal{S}_{(4)}$ for any $i \geq 3$.

From the example above and from further computation, we conjecture the following:

Conjecture 4.11. *For a two-row ribbon* $\mathrm{Ribb}(n-2,2)$ *let* $R = \Bbbk[x_1, \dots, x_n]$. *Then:*

$$(R/I_{\mathrm{Ribb}(n-2,2)})_i = \begin{cases} \mathcal{S}_{(n)} & \text{if } i = 0, \\ \mathcal{S}_{(n-1,1)} + \mathcal{S}_{(n)} & \text{if } i = 1, \\ \mathcal{S}_{(n-1,1)} + 2\mathcal{S}_{(n)} & \text{if } i \geq 2. \end{cases}$$

We have a more general conjecture about this phenomena, involving the notion of *$\mathfrak{S}_n$-representation stability* introduced by Church and Farb in [CF13] and further explored in their joint work with Ellenberg in [CEF15].

Definition 4.12. Let $\{V_n\}$ be a sequence of $\mathfrak{S}_n$-representations equipped with linear maps $\phi_n : V_n \to V_{n+1}$. The sequence $\{V_n\}$ is said to be *consistent* is for all $g \in \mathfrak{S}_n$, the following diagram commutes:

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ \downarrow g & & \downarrow g \\ V_n & \xrightarrow{\phi_n} & V_{n+1} \end{array}$$

In the definition below, we will use the following notation. If $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash k$, then for any $n \geq k + \lambda_1$, we let $\lambda[n] := (n-k, \lambda_1, \dots, \lambda_\ell)$.

Definition 4.13. [CF13, Definition 2.3] Let $\{V_n\}$ be a consistent sequence of $\mathfrak{S}_n$-representations. Then, $\{V_n\}$ is *representation stable* if for sufficiently large n, the following hold:

(i) The natural map $\phi_n : V_n \to V_{n+1}$ is injective.

(ii) The span of the $\mathfrak{S}_{n+1}$-orbit of $\phi_n(V_n)$ is V_{n+1}.

(iii) Let $V_n = \bigoplus_\lambda c_{\lambda,n} \mathcal{S}_{\lambda[n]}$ be the decomposition of V_n into irreducibles. For each λ, $c_{\lambda,n}$ is eventually independent of n.

As remarked by Church and Farb, sometimes there is no natural choice for the maps ϕ_n. In some cases, we might still be interested in studying some notion of stability that does not involve choosing these maps. In particular, if only condition (iii) above holds, we say that $\{V_n\}$ is *multiplicity stable*.

We have observed that the free modules in the $\mathfrak{S}_n$-equivariant Hilbert series for $R/I_{\text{Ribb}(n-k,k)}$ seem to be representation stable. In particular, we have the following conjecture:

Conjecture 4.14. *Let k be a fixed positive integer. For each $n \geq 1$, let $V_n^i = (R/I_{\text{Ribb}(n-k,k)})_i$, considered as a $\mathfrak{S}_n$-representation. Then, $\{V_n^i\}$ is representation stable for all $i \geq 0$. Moreover, the coefficients $c_{\lambda,n}$ stabilize for $n \geq 2(k-1)$.*

5. Hooks and their generalizations

Watanabe–Yanagawa [WY19] showed that in the case where λ is a *hook*, i.e. $\lambda = (n-k, 1^k)$, the Specht ideal I_λ is minimally resolved by the *Eagon-Northcott complex*. In this section, we investigate the $\mathfrak{S}_n$-action on the free modules of this resolution, and seek to generalize this free resolution to certain (not necessarily skew) diagrams corresponding to permuting rows of a hook diagram.

The Eagon-Northcott complex gives a minimal free resolution for ideals generated by maximal minors of a matrix with sufficiently large depth. First, we recall how to view I_λ for λ a hook as an ideal of maximal minors; this was first observed by Watanabe–Yanagawa.

Proposition 5.1. *For hooks $\lambda := (n-k, 1^k)$, the Specht ideal I_λ is equal to the ideal generated by the maximal $k \times k$ minors of the $k \times n-1$ matrix $A^{(\lambda)}$ having entries given by*

$$A^{(\lambda)}_{i,j} := x^i_{j+1} - x^i_1 \text{ for all } 1 \leq i \leq k, 1 \leq j \leq n-1.$$

Proof. The standard young tableaux of shape $(n-k, 1^k)$ are in bijection with the set of size $k+1$ subsets of $\{1, \ldots, n\}$ containing 1 by sending $T \in \text{SYT}((n-k, 1^k))$ to the sequence of numbers $1 = i_1 < i_2 < \cdots < i_{k+1}$ appearing in its leftmost column, listed from top to bottom. The Specht polynomial of T is then

$$f_T = \prod_{1 \leq j_1 < j_2 \leq k+1} (x_{i_{j_1}} - x_{i_{j_2}}) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_{i_2} & \cdots & x_{i_{k+1}} \\ x_1^2 & x_{i_2}^2 & \cdots & x_{i_{k+1}}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_{i_2}^k & \cdots & x_{i_{k+1}}^k \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ x_1 & x_{i_2} - x_1 & \cdots & x_{i_{k+1}} - x_1 \\ x_1^2 & x_{i_2}^2 - x_1^2 & \cdots & x_{i_{k+1}}^2 - x_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_{i_2}^k - x_1^k & \cdots & x_{i_{k+1}}^k - x_1^k \end{vmatrix}.$$

By cofactor expansion, this is equal to

$$f_T = \begin{vmatrix} x_{i_2} - x_1 & \cdots & x_{i_{k+1}} - x_1 \\ x_{i_2}^2 - x_1^2 & \cdots & x_{i_{k+1}}^2 - x_1^2 \\ \vdots & \ddots & \vdots \\ x_{i_2}^k - x_1^k & \cdots & x_{i_{k+1}}^k - x_1^k \end{vmatrix},$$

the $k \times k$ minor of $A^{(\lambda)}$ for the column indices $i_2 - 1 < \cdots < i_{k+1} - 1$. The claim then follows from the fact that the polynomials f_T for $T \in \text{SYT}((n-k, 1^k))$ generate $I_{(n-k,1^k)}$. ☐

Whenever we have an ideal I with sufficiently large depth in a commutative graded $\Bbbk$-algebra R that is generated by the $s \times s$ minors of an $s \times r$ matrix A whose entries are each degree ≥ 1 elements of R, the Eagon-Northcott complex described in page 191 of [EN62] gives an explicit minimal free resolution of I. The free modules in this resolution are of the form $F_q = \wedge^{s+q}(R^r) \otimes S^q(R^s)$, and the differentials $d_q : F_q \to F_{q-1}$ (with $d_0 : F_0 \twoheadrightarrow I$) in this resolution depend linearly on the matrix A (aside from d_0, which is alternating multilinear in the rows of A).

Proposition 5.2. *Let $\lambda = (n-k, 1^k)$ be a hook partition. Then the free modules in the minimal free resolution of the Specht ideal I_λ have $\mathfrak{S}_n$-equivariant structure given by*

$$F_q \cong \mathcal{S}_{(n-k-q,1^{k+q})} \otimes_{\Bbbk} \bigoplus_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\ldots+i_k=q}} R\left(-\binom{k+1}{2} - \sum_{1\leq j \leq k} j i_j\right)$$

for all $0 \leq q \leq n-k$ and $F_{n-k} = 0$.

Example 5.3. We consider the hook $(4,1^2)$. By Proposition 5.2, the modules in the free resolution for $I_{(4,1^2)}$ are:

$$\begin{array}{ccccc}
0 \longrightarrow \mathcal{S}_{(1^6)} \otimes (R(-6) \oplus R(-7) \oplus R(-8) \oplus R(-9)) & & & & \\
\downarrow & & & & \\
\mathcal{S}_{(2,1^4)} \otimes (R(-5) \oplus R(-6) \oplus R(-7)) & & & & \\
\downarrow & & & & \\
\mathcal{S}_{(3,1^3)} \otimes (R(-4) \oplus R(-5)) & \longrightarrow & \mathcal{S}_{(4,1^2)} \otimes R(-3) & \longrightarrow & 0
\end{array}$$

This agrees with the character decomposition table given by `BettiCharacters` in Macaulay2 below.

```
o15 = Decomposition table

                |          2      3      4     6
                | (6)  (4,1 )  (3,1 )  (2,1 )  (1 )
      ----------+----------------------------------
      (0, {0})  |  1      0      0      0     0
      (1, {3})  |  0      1      0      0     0
      (2, {4})  |  0      0      1      0     0
      (2, {5})  |  0      0      1      0     0
      (3, {5})  |  0      0      0      1     0
      (3, {6})  |  0      0      0      1     0
      (3, {7})  |  0      0      0      1     0
      (4, {6})  |  0      0      0      0     1
      (4, {7})  |  0      0      0      0     1
      (4, {8})  |  0      0      0      0     1
      (4, {9})  |  0      0      0      0     1

o15 : CharacterDecomposition
```

Proof. Consider the action of $\mathrm{GL}_r(R)$ on the $s \times r$ matrices A over R by $T \cdot A := AT^t$. This induces a $\mathrm{GL}_r(R)$-action on I, as well as on the differentials d_q. $\mathrm{GL}_r(R)$ also acts on F_q by having $\mathrm{GL}_r(R)$ act on $\wedge^{q+s}(R^r)$ in the usual way by linear change of variables. It follows from [EN62] that these actions are compatible with each other, i.e., the Eagon-Northcott complex for $T \cdot A$ is the same as the complex you get by acting by T on the F_q's and d_q's in the Eagon-Northcott complex for A.

Consequently, for the case of I_λ, by restricting the action of $\mathrm{GL}_{n-1}(\Bbbk)$ to $\rho_{(n-1,1)}(\mathfrak{S}_n)$, where $\rho_{(n-1,1)} : \mathfrak{S}_n \to \mathrm{GL}_{n-1}(\Bbbk)$ is the standard representation of $\mathfrak{S}_n$, we get that this resolution is also $\mathfrak{S}_n$-equivariant.

Regarding the Betti table for $I_{(n-k,1^k)}$, this implies that for all $q \geq 0$,

$$F_q \cong \wedge^{k+q}(\mathcal{S}_{(n-1,1)}) \otimes_{\Bbbk} \bigoplus_{\substack{i_1,\dots,i_k \geq 0 \\ i_1+\dots+i_k=q}} R\left(-\binom{k+1}{2} - \sum_{1\leq j\leq k} j i_j\right).$$

Now, observe that while on the one hand,

$$\wedge^s(\mathcal{S}_{(n,1)/(1)}) \cong \wedge^s(\Bbbk_{\mathrm{triv}} \oplus \mathcal{S}_{(n-1,1)}) \cong \wedge^s(\mathcal{S}_{(n-1,1)}) \oplus \wedge^{s-1}(\mathcal{S}_{(n-1,1)}),$$

we also have

$$\wedge^s(\mathcal{S}_{(n,1)/(1)}) \cong \wedge^s(\Bbbk[\mathfrak{S}_n/\mathfrak{S}_{n-1}]) \cong \mathrm{Ind}_{\mathfrak{S}_s\times\mathfrak{S}_{n-s}}^{\mathfrak{S}_n}(\mathrm{sgn} \boxtimes \Bbbk_{\mathrm{triv}}) \cong \mathrm{Ind}_{\mathfrak{S}_s\times\mathfrak{S}_{n-s}}^{\mathfrak{S}_n}(\mathcal{S}_{(1^s)} \boxtimes \mathcal{S}_{(n-s)}),$$

which from the Littlewood-Richardson rule for inducing irreducible representations $\mathcal{S}_\lambda \boxtimes \mathcal{S}_\mu$ of $\mathfrak{S}_s \times \mathfrak{S}_{n-s}$ to $\mathfrak{S}_n$, is just $\mathcal{S}_{(n-s,1^s)} \oplus \mathcal{S}_{(n-(s-1),1^{s-1})}$. So then by inducting on s, we find that $\wedge^s(\mathcal{S}_{(n-1,1)}) \cong \mathcal{S}_{(n-s,1^s)}$. Therefore, for all $0 \leq q \leq n-k$, we have

$$F_q \cong \mathcal{S}_{(n-k-q,1^{k+q})} \otimes_{\Bbbk} \bigoplus_{\substack{i_1,\dots,i_k \geq 0 \\ i_1+\dots+i_k=q}} R\left(-\binom{k+1}{2} - \sum_{1\leq j\leq k} j i_j\right).$$

□

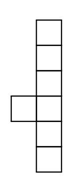
Unlike the hook case where a full $\mathfrak{S}_n$-equivariant resolution is obtained easily through the use of the Eagon-Northcott complex, $\mathfrak{S}_n$-equivariant free resolutions for Specht ideals of "upside down hooks" $\lambda/\mu := ((n-k)^{k+1})/((n-k-1)^k)$ are not well understood. This is especially true for the free modules in the resolution past the first two steps (generators and relations) of the resolution. Though even the relations between the generators of $I_{\lambda/\mu}$ are not fully understood, there is a way of generating a large family of relations between them using Jacobi's bialternant formula. Since these relations involve lots of Vandermonde determinants, we introduce some shorthand notation:

Definition 5.4. For any n variables $z_1, \dots, z_n$, we denote by $\mathrm{VD}(z_1, \dots, z_n)$ the determinant of the $n \times n$ Vandermonde matrix in the variables $z_1, \dots, z_n$. i.e.,

$$\mathrm{VD}(z_1,\dots,z_n) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ z_1^2 & z_2^2 & \cdots & z_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{vmatrix} = \prod_{1\leq i<j\leq n} (z_j - z_i).$$

Definition 5.5. For a fixed $n \in \mathbb{N}$ and some $1 \leq r \leq n-1$, define $H_{n,k,r}$ to be the left-justified (not necessarily skew) diagram with n boxes and row length $n-k$ in the r-th row and 1 in all other rows, with this picture.

For example, the diagram $H(7,5,4)$ looks like so:



.

Before we can discuss how the Jacobi bialternant formula comes into play when finding relations between the generators of the Specht ideals $I_{H(n,k,r)}$, we must first describe the generating set of $I_{H(n,k,r)}$ we will be using.

Remark 5.6. Let $X_1, ..., X_n$ be a basis of $\Bbbk^n$. Then the map $g_{n,k,r} : \bigwedge^{k+1}(\Bbbk^n) \otimes R \to I_{H(n,k,r)}$ given by

$$g_{n,k,r}(X_{i_0} \wedge \cdots \wedge X_{i_k}) := \left(\prod_{j \in [n] \setminus \{i_0, \ldots, i_k\}} x_j \right)^{r-1} \mathrm{VD}(x_{i_0}, \ldots, x_{i_k})$$

for any $1 \leq i_0 < \cdots < i_k \leq n$ is a surjective R-module homomorphism.

Now, for any $1 \leq i_0 < \cdots < i_k \leq n$, denote the elements of the complement $[n] \setminus \{i_0, \ldots, i_k\}$ by $1 \leq i_1^c < \cdots < i_{n-k-1}^c \leq n$. For any sequence of $n+k-1$ distinct nonnegative integers $0 \leq d_1 < \cdots < d_{n-k-1}$, let $\lambda^{(d_1,\ldots,d_{n-k-1})}$ be the partition with parts given by $\lambda_j^{(d_1,\ldots,d_{n-k-1})} := d_{n-k-j} + (r-1) - (n-j)$, and let $r^{(d_1,\ldots,d_{n-k-1})} \in \bigwedge^{k+1}(\Bbbk^n) \otimes R$ be given by

$$r^{(d_1,\ldots,d_{n-k-1})} := \sum_{1 \leq i_0 < \cdots < i_k \leq n} (-1)^{\sum_{0 \leq j \leq k} i_j - j} \begin{vmatrix} x_{i_1^c}^{d_1} & x_{i_2^c}^{d_1} & \cdots & x_{i_{n-k-1}^c}^{d_1} \\ x_{i_1^c}^{d_2} & x_{i_2^c}^{d_2} & \cdots & x_{i_{n-k-1}^c}^{d_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_1^c}^{d_{n-k-1}} & x_{i_2^c}^{d_{n-k-1}} & \cdots & x_{i_{n-k-1}^c}^{d_{n-k-1}} \end{vmatrix} \cdot X_{i_0} \wedge \cdots \wedge X_{i_k}.$$

Proposition 5.7 (Jacobi Bialternant Relations)**.**

$$g_{n,k,r}\left(r^{(d_1,\ldots,d_{n-k-1})}\right) = \mathrm{VD}(x_1, \ldots, x_n) \cdot s_{\lambda^{(d_1,\ldots,d_{n-k-1})}}(x_1, \ldots, x_n),$$

where $s_\lambda(x_1, \ldots, x_n)$ is the Schur polynomial in n variables for any partition λ with at most n parts. We take the convention that if $\lambda_1^{(d_1,\ldots,d_{n-k-1})} < 0$, then $s_{\lambda^{(d_1,\ldots,d_{n-k-1})}}(x_1, \ldots, x_n) := 0$. In particular, $g_{n,k,r}(r^{(d_1,\ldots,d_{n-k-1})} - s_{\lambda^{(d_1,\ldots,d_{n-k-1})}}(x_1, \ldots, x_n) r^{(k+1,\ldots,n-1)}) = 0$.

Proof. By iterating cofactor expansion $k+1$ times on the top $k+1$ rows of the following matrix M, we have

$$|M| := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & \cdots & x_n^k \\ x_1^{(r-1)+d_1} & x_2^{(r-1)+d_1} & \cdots & x_n^{(r-1)+d_1} \\ x_1^{(r-1)+d_2} & x_2^{(r-1)+d_2} & \cdots & x_n^{(r-1)+d_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(r-1)+d_{n-k-1}} & x_2^{(r-1)+d_{n-k-1}} & \cdots & x_n^{(r-1)+d_{n-k-1}} \end{vmatrix}$$

$$= \sum_{1 \leq i_0 < \cdots < i_k \leq n} (-1)^{\sum_{0 \leq j \leq k} i_j - j} \cdot \begin{vmatrix} x_{i_1^c}^{(r-1)+d_1} & x_{i_2^c}^{(r-1)+d_1} & \cdots & x_{i_{n-k-1}^c}^{(r-1)+d_1} \\ x_{i_1^c}^{(r-1)+d_2} & x_{i_2^c}^{(r-1)+d_2} & \cdots & x_{i_{n-k-1}^c}^{(r-1)+d_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_1^c}^{(r-1)+d_{n-k-1}} & x_{i_2^c}^{(r-1)+d_{n-k-1}} & \cdots & x_{i_{n-k-1}^c}^{(r-1)+d_{n-k-1}} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_{i_0} & x_{i_1} & \cdots & x_{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_0}^k & x_{i_1}^k & \cdots & x_{i_k}^k \end{vmatrix}$$

$$= \sum_{1\le i_0<\cdots<i_k\le n} (-1)^{\sum_{0\le j\le k} i_j - j} \cdot \left(\prod_{j\in[n]\setminus\{i_0,\dots,i_k\}} x_j\right)^{r-1} \cdot \begin{vmatrix} x_{i_1^c}^{d_1} & x_{i_2^c}^{d_1} & \cdots & x_{i_{n-k-1}^c}^{d_1} \\ x_{i_1^c}^{d_2} & x_{i_2^c}^{d_2} & \cdots & x_{i_{n-k-1}^c}^{d_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_1^c}^{d_{n-k-1}} & x_{i_2^c}^{d_{n-k-1}} & \cdots & x_{i_{n-k-1}^c}^{d_{n-k-1}} \end{vmatrix} \cdot \mathrm{VD}(x_{i_0},\dots,x_{i_k})$$

$$= \sum_{1\le i_0<\cdots<i_k\le n} (-1)^{\sum_{0\le j\le k} i_j - j} \cdot \begin{vmatrix} x_{i_1^c}^{d_1} & x_{i_2^c}^{d_1} & \cdots & x_{i_{n-k-1}^c}^{d_1} \\ x_{i_1^c}^{d_2} & x_{i_2^c}^{d_2} & \cdots & x_{i_{n-k-1}^c}^{d_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_1^c}^{d_{n-k-1}} & x_{i_2^c}^{d_{n-k-1}} & \cdots & x_{i_{n-k-1}^c}^{d_{n-k-1}} \end{vmatrix} \cdot g_{n,k,r}(X_{i_0}\wedge\cdots\wedge X_{i_k})$$

$$= g_{n,k,r}\left(r^{(d_1,\dots,d_{n-k-1})}\right).$$

Observe that if $\lambda_1^{(d_1,\dots,d_{n-k-1})} < 0$, then $0 \le (r-1)+d_1 \le k$, so M has a repeated row and thus $|M| = 0$. Otherwise, note that we must have $\lambda_1^{(d_1,\dots,d_{n-k-1})} \ge \cdots \ge \lambda_{n-k-1}^{(d_1,\dots,d_{n-k-1})} \ge 0$, and so letting $\lambda_j^{(d_1,\dots,d_{n-k-1})} = 0$ for any $j \ge n-k$, by the Jacobi bialternant formula, we have

$$|M| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & \cdots & x_n^k \\ x_1^{(r-1)+d_1} & x_2^{(r-1)+d_1} & \cdots & x_n^{(r-1)+d_1} \\ x_1^{(r-1)+d_2} & x_2^{(r-1)+d_2} & \cdots & x_n^{(r-1)+d_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(r-1)+d_{n-k-1}} & x_2^{(r-1)+d_{n-k-1}} & \cdots & x_n^{(r-1)+d_{n-k-1}} \end{vmatrix}$$

$$= \begin{vmatrix} x_1^{\lambda_n^{(d_1,\dots,d_{n-k-1})}} & x_2^{\lambda_n^{(d_1,\dots,d_{n-k-1})}} & \cdots & x_n^{\lambda_n^{(d_1,\dots,d_{n-k-1})}} \\ x_1^{\lambda_{n-1}^{(d_1,\dots,d_{n-k-1})}+1} & x_2^{\lambda_{n-1}^{(d_1,\dots,d_{n-k-1})}+1} & \cdots & x_n^{\lambda_{n-1}^{(d_1,\dots,d_{n-k-1})}+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_1^{(d_1,\dots,d_{n-k-1})}+(n-1)} & x_2^{\lambda_1^{(d_1,\dots,d_{n-k-1})}+(n-1)} & \cdots & x_n^{\lambda_1^{(d_1,\dots,d_{n-k-1})}+(n-1)} \end{vmatrix}$$

$$= \mathrm{VD}(x_1,\dots,x_n)\cdot s_{\lambda^{(d_1,\dots,d_{n-k-1})}}(x_1,\dots,x_n).$$

□

Fix $0 \le \ell \le k$, and $1 \le i_1 < \cdots < i_\ell \le n$.

Conjecture 5.8. *Let $D = H(n, n-3, k)$ and consider the Specht ideal $I_D \subset R = \Bbbk[x_1,\dots,x_n]$. Then the following is a minimal free resolution of I_D:*

$$\mathcal{F}_\bullet^D : 0 \longrightarrow \mathcal{S}_{(1^n)} \otimes \bigoplus_{\substack{1\le j\le n-1 \\ j\ne n-r}} R(-\tbinom{n}{2} - r - j) \xrightarrow{M} R^{n-1}(-\tbinom{n}{2} - r) \xrightarrow{g} I_D.$$

Here, $g : R^{n-1} \to I_D$ *and* $M : R^{n-2} \to R^{n-1}$ *are as follows in matrix form:*

$$g_i \coloneqq x_i^{n-k-2} \cdot \prod_{\substack{1 \le j_1 < j_2 \le n \\ j_1, j_2 \neq i}} (x_{j_2} - x_{j_1}),$$

$$M_{i,j} \coloneqq \begin{cases} (-1)^i \cdot (x_i^j - x_1^j) & \text{for } 1 \le j \le n-r-1, \\ (-1)^i \cdot \left(\prod_{n-j+1 \le s \le n} (x_i - x_s) - \prod_{r+1 \le s \le n} (x_i - x_s) h_{j-n+r}(x_1, ..., x_{n-j}) \right) & \text{for } n-r+1 \le j \le n-1, \end{cases}$$

where g *is a* $1 \times (n-1)$ *matrix with column indices* $i = 2, ..., n$*,* M *is a* $(n-1) \times (n-2)$ *matrix with row indices* $i = 2, ..., n$ *and column indices* $j = 1, ..., n-r-1, n-r+1, ..., n-1$*, and* h_t *are the homogeneous symmetric polynomials (*$h_t = 0$ *if* $t < 0$*, and* $h_0 = 1$*).*

Below, we give evidence supporting Conjecture 5.8. In particular, we show the character decomposition tables obtained for shapes obtained by moving a box through the rows of the hook $(2, 1^4)$.

```
Decomposition table (2,1^4)

            |           4     6
            |  (6)  (2,1 )  (1 )
------------+-------------------
  (0, {0})  |   1       0     0
 (1, {10})  |   0       1     0
 (2, {11})  |   0       0     1
 (2, {12})  |   0       0     1
 (2, {13})  |   0       0     1
 (2, {14})  |   0       0     1
```

```
Decomposition table (1,2,1^3)

            |           4     6
            |  (6)  (2,1 )  (1 )
------------+-------------------
  (0, {0})  |   1       0     0
 (1, {11})  |   0       1     0
 (2, {12})  |   0       0     1
 (2, {13})  |   0       0     1
 (2, {14})  |   0       0     1
 (2, {16})  |   0       0     1
```

```
Decomposition table (1,1,2,1,1)

            |           4     6
            |  (6)  (2,1 )  (1 )
------------+-------------------
  (0, {0})  |   1       0     0
 (1, {12})  |   0       1     0
 (2, {13})  |   0       0     1
 (2, {14})  |   0       0     1
 (2, {16})  |   0       0     1
 (2, {17})  |   0       0     1
```

```
Decomposition table (1^3,2,1)

            |           4     6
            |  (6)  (2,1 )  (1 )
------------+-------------------
  (0, {0})  |   1       0     0
 (1, {13})  |   0       1     0
 (2, {14})  |   0       0     1
 (2, {16})  |   0       0     1
 (2, {17})  |   0       0     1
 (2, {18})  |   0       0     1
```

```
Decomposition table (1^4,2)

            |           4     6
            |  (6)  (2,1 )  (1 )
------------+-------------------
  (0, {0})  |   1       0     0
 (1, {14})  |   0       1     0
 (2, {16})  |   0       0     1
 (2, {17})  |   0       0     1
 (2, {18})  |   0       0     1
 (2, {19})  |   0       0     1
```

6. Skew shapes from $(d, d, 1)$

In [SY23a], Shibata and Yanagawa described and proved the minimal free resolution for $R/I_{(d,d,1)}$. They defined the chain complex

$$\mathcal{F}_{\bullet}^{(d,d,1)} : 0 \longrightarrow F_d \xrightarrow{\partial_d} F_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0 \ , \tag{5}$$

where $F_0 = R$ and

$$F_i = \mathcal{S}_{(d,d-i+1,1^i)} \otimes R(-d-i-1)$$

for $1 \leq i \leq d$. In other words, the free modules are obtained by moving the rightmost box from the second row to the bottom of the first column.

The maps admit a nice combinatorial description that bear a resemblance to those described in Section 4. Let $T \in \mathrm{Tab}(d, d-i+1, 1^i)$ be given by

$$T = \begin{array}{|c|c|c|c|c|c|c|}
\hline
a_1 & b_2 & \cdots & b_{d-i+1} & b_{d-i+2} & \cdots & b_d \\
\hline
a_2 & c_2 & \cdots & c_{d-i+1} & \multicolumn{3}{c}{} \\
\cline{1-4}
\vdots & \multicolumn{6}{c}{} \\
\cline{1-1}
a_{i+2} & \multicolumn{6}{c}{} \\
\cline{1-1}
\end{array} \ .$$

For $3 \leq i \leq d$, Shibata and Yanagawa defined

$$\partial_i(e(T) \otimes 1) = \sum_{j=1}^{i+2} \sum_{\sigma \in H} (-1)^{j-1} e(\sigma(T_j)) \otimes x_{a_j} \in F_{i-1},$$

where H is the set of permutations of $\{b_{d-i+2}, \ldots, b_d\}$ such that $\sigma(b_{d-i+2}) < \cdots < \sigma(b_d)$. For $i = 2$, we have

$$\partial_2(e(T) \otimes 1) = e(T_1) \otimes x_{a_1} - e(T_2) \otimes x_{a_2} + e(T_3) \otimes x_{a_3} \in F_1.$$

Throughout this section, we study some possible generalizations of Shibata and Yanagawa's work on Specht ideals of partitions $(d, d, 1)$.

6.1. $(d, d, 1)/\mu$.

In this subsection, we explore the free resolutions for skew shapes of the form $(d, d, 1)/\mu$. In the remainder of this section, we let R be the polynomial ring in $|(d, d, 1)/\mu|$ variables over a field $\Bbbk$ of characteristic zero.

Example 6.1. The first example that we have computed in this case is $(2, 2, 1)/(1) = \ydiagram{1+1,2,1}$. The Betti table for $R/I_{\ydiagram{1+1,2,1}}$ is

	0	1	2
total:	1	5	4
0 :	1	.	.
1 :	.	.	.
2 :	.	.	.
3 :	.	5	4

This agrees with the description of the Specht modules at each step of the free resolution described in (5). At each step, we remove a box from the second row and move it to the bottom of the first column. The modules in the free resolution are given by

$$0 \longrightarrow \mathcal{S}_{(2,1,1,1)/(1)} \otimes R(-5) \longrightarrow \mathcal{S}_{(2,2,1)/(1)} \otimes R(-4) \longrightarrow R \longrightarrow 0.$$

Further computation suggests that this pattern holds for any skew shape $(d,d,1)/(a)$ where $a < d$, such as the following example.

Example 6.2. Consider the shape $(3,3,1)/(2) = (3,3,1)/(2)$. The Betti table for $R/I_{(3,3,1)/(2)}$ is

	0	1	2	3
total:	1	11	15	5
0 :	1	.	.	.
1 :	.	.	.	.
2 :	.	.	.	.
3 :	.	.	.	.
4 :	.	11	15	5

Yet again, this matches the pattern described above. In particular, the modules in the free resolution are given by:

$$0 \longrightarrow \mathcal{S}_{(3,1,1,1,1)/(2)} \otimes R(-7) \longrightarrow \mathcal{S}_{(3,2,1,1)/(2)} \otimes R(-6) \longrightarrow \mathcal{S}_{(3,3,1)/(2)} \otimes R(-5) \longrightarrow R \longrightarrow 0.$$

These computations lead us to pose the following conjecture:

Conjecture 6.3. *The minimal free resolution of $R/I_{(d,d,1)/(a)}$ for $a < d$ is*

$$\mathcal{F}_{\bullet}^{(d,d,1)/(a)} : 0 \longrightarrow F_d \xrightarrow{\partial_d} F_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0 \ ,$$

where $F_0 = R$ and $F_i = \mathcal{S}_{(d,d-i+1,1^i)/(a)} \otimes R(-d-i-1)$ for $1 \leq i \leq d$. Furthermore, the maps ∂_i are defined the same way as in (5).

That the maps are well-defined is again nontrivial, since there are linear relations among polytabloids. However, if $a = d-1$, then $(d,d,1)/(d-1) = \mathrm{Ribb}(1,d,1)$, and we can relate these maps to those in (1). In particular, showing that they are well-defined and that $\partial_{i-1}\partial_i = 0$ can be done in the same way as for the maps in the second linear strand of $\mathcal{F}_{\bullet}^{\mathrm{Ribb}(k,\ell)}$.

Proposition 6.4. *The maps ∂_i in $\mathcal{F}_{\bullet}^{(d,d,1)/(d-1)}$ are well-defined and satisfy $\partial_{i-1}\partial_i = 0$.* □

7. Characteristic dependence

The free resolutions of three-row shapes seem to depend on the characteristic of the underlying field $\Bbbk$, even in the case where the diagram is a ribbon. In this section we present the smallest such example that where we have observed this phenomenon corresponding to the ribbon $(2,2,1)/(1)$. Over characteristic 0, the Betti table can be found in Section 4.3.3. However, we see a change in

characteristic 2, since the Betti table over $\mathbb{Z}/2\mathbb{Z}$ is:

	0	1	2	3
total:	1	5	5	1
0 :	1	.	.	.
1 :	.	.	.	.
2 :	.	.	.	.
3 :	.	5	4	1
4 :	.	.	1	.

For all other primes we have tested, the Betti table matches the one we observed in characteristic 0.

Acknowledgements

This project was supported in large part by a grant from the D.E. Shaw group, and also by NSF grant DMS-2053288. It was supervised as part of the University of Minnesota School of Mathematics Summer 2024 REU program. The authors would also like to thank our mentor Ayah Almousa and graduate student TA Miranda Moore for supervising and guiding the research, as well as providing valuable feedback throughout the editing process. We also thank Federico Galetto since we have greatly benefited from his Macalauy2 package `BettiCharacters` [Gal23] for experimentation.

References

[BE77] David A. Buchsbaum and David Eisenbud. "Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3". In: *Amer. J. Math.* 99.3 (1977), pp. 447–485. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2373926.

[BPS05] Anders Björner, Irena Peeva, and Jessica Sidman. "Subspace arrangements defined by products of linear forms". In: *J. London Math. Soc. (2)* 71.2 (2005), pp. 273–288. ISSN: 0024-6107,1469-7750. DOI: 10.1112/S0024610705006356.

[Bro+11] Abraham Broer et al. "Extending the coinvariant theorems of Chevalley, Shephard-Todd, Mitchell, and Springer". In: *Proc. Lond. Math. Soc. (3)* 103.5 (2011), pp. 747–785. ISSN: 0024-6115,1460-244X. DOI: 10.1112/plms/pdq027.

[Bro+16] Aaron Brookner et al. "On Cohen-Macaulayness of S_n-invariant subspace arrangements". In: *Int. Math. Res. Not. IMRN* 7 (2016), pp. 2104–2126. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnv200.

[CEF15] Thomas Church, Jordan S. Ellenberg, and Benson Farb. "FI-modules and stability for representations of symmetric groups". In: *Duke Math. J.* 164.9 (2015), pp. 1833–1910. ISSN: 0012-7094,1547-7398. DOI: 10.1215/00127094-3120274.

[CF13] Thomas Church and Benson Farb. "Representation theory and homological stability". In: *Adv. Math.* 245 (2013), pp. 250–314. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2013.06.016.

[DK24] Sebastian Debus and Andreas Kretschmer. "Symmetric Ideals and Invariant Hilbert Schemes". In: *arXiv preprint arXiv:2404.15240* (2024).

[EN62] J. A. Eagon and D. G. Northcott. "Ideals Defined by Matrices and a Certain Complex Associated with Them". In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 269.1337 (1962), pp. 188–204. ISSN: 00804630.

[ES96] David Eisenbud and Bernd Sturmfels. "Binomial ideals". In: *Duke Mathematical Journal* 84.1 (1996), pp. 1–45. DOI: 10.1215/S0012-7094-96-08401-X.

[Gal20] Federico Galetto. "On the ideal generated by all squarefree monomials of a given degree". In: *Journal of Commutative Algebra* 12.2 (2020), pp. 199–215. DOI: 10.1216/jca.2020.12.199.

[Gal23] Federico Galetto. "Setting the scene for Betti characters". In: *J. Softw. Algebra Geom.* 13.1 (2023), pp. 45–51. ISSN: 1948-7916. DOI: 10.2140/jsag.2023.13.45.

[Liu10] Ricky Ini Liu. *Specht modules and Schubert varieties for general diagrams.* Thesis (Ph.D.)–Massachusetts Institute of Technology. ProQuest LLC, Ann Arbor, MI, 2010, (no paging).

[LL81] Shuo-Yen Robert Li and Wen Ch'ing Winnie Li. "Independence numbers of graphs and generators of ideals". In: *Combinatorica* 1.1 (1981), pp. 55–61. ISSN: 0209-9683. DOI: 10.1007/BF02579177.

[Loe95] Jesus Antonio de Loera. *Triangulations of polytopes and computational algebra.* Thesis (Ph.D.)–Cornell University. ProQuest LLC, Ann Arbor, MI, 1995, p. 188.

[Lov94] L. Lovász. "Stable sets and polynomials". In: vol. 124. 1-3. Graphs and combinatorics (Qawra, 1990). 1994, pp. 137–153. DOI: 10.1016/0012-365X(92)00057-X.

[MR22] Satoshi Murai and Claudiu Raicu. "An equivariant Hochster's formula for -invariant monomial ideals". In: *Journal of the London Mathematical Society* 105.3 (2022), pp. 1974–2010. DOI: https://doi.org/10.1112/jlms.12551. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/jlms.12551.

[MRV21] Philippe Moustrou, Cordian Riener, and Hugues Verdure. "Symmetric ideals, Specht polynomials and solutions to symmetric systems of equations". In: *J. Symbolic Comput.* 107 (2021), pp. 106–121. ISSN: 0747-7171,1095-855X. DOI: 10.1016/j.jsc.2021.02.002.

[MS05] Ezra. Miller and Bernd. Sturmfels. *Combinatorial Commutative Algebra.* eng. Graduate Texts in Mathematics, 227. Springer, 2005. ISBN: 9780387271033.

[Pee75] M. H. Peel. "Specht modules and symmetric groups". In: *J. Algebra* 36.1 (1975), pp. 88–97. ISSN: 0021-8693. DOI: 10.1016/0021-8693(75)90158-1.

[RS95a] Victor Reiner and Mark Shimozono. "Plactification". In: *Journal of Algebraic Combinatorics* 4 (1995), pp. 331–351.

[RS95b] Victor Reiner and Mark Shimozono. "Specht series for column-convex diagrams". In: *Journal of Algebra* 174.2 (1995), pp. 489–522.

[RS98] Victor Reiner and Mark Shimozono. "Percentage-avoiding, northwest shapes and peelable tableaux". In: *J. Combin. Theory Ser. A* 82.1 (1998), pp. 1–73. ISSN: 0097-3165,1096-0899. DOI: 10.1006/jcta.1997.2841.

[Sag01] Bruce E. Sagan. *The symmetric group.* Second. Vol. 203. Graduate Texts in Mathematics. Representations, combinatorial algorithms, and symmetric functions. Springer-Verlag, New York, 2001, pp. xvi+238. ISBN: 0-387-95067-2. DOI: 10.1007/978-1-4757-6804-6.

[SY23a] Kosuke Shibata and Kohji Yanagawa. "Elementary construction of minimal free resolutions of the Specht ideals of shapes $(n-2, 2)$ and $(d, d, 1)$". In: *J. Algebra Appl.* 22.9 (2023), Paper No. 2350199, 26. ISSN: 0219-4988,1793-6829. DOI: 10.1142/S0219498823501992.

[SY23b] Kosuke Shibata and Kohji Yanagawa. "Elementary construction of the minimal free resolution of the Specht ideal of shape $(n-d, d)$". In: *J. Algebra* 634 (2023), pp. 563–584. ISSN: 0021-8693,1090-266X. DOI: 10.1016/j.jalgebra.2023.07.028.

[Woo05] Alexander Kar-Man Woo. *Ideals of the polynomial ring generated by irreducible symmetric group representations and Ellingsrud-Stromme cells on the Hilbert scheme.* Thesis (Ph.D.)–University of California, Berkeley. ProQuest LLC, Ann Arbor, MI, 2005, p. 66. ISBN: 978-0542-62127-7.

[WY19] Junzo Watanabe and Kohji Yanagawa. "Vandermonde determinantal ideals". In: *Math. Scand.* 125.2 (2019), pp. 179–184. ISSN: 0025-5521,1903-1807. DOI: 10.7146/math.scand.a-114906.

[ZGS14] Christine Berkesch Zamaere, Stephen Griffeth, and Steven V Sam. "Jack Polynomials as Fractional Quantum Hall States and the Betti Numbers of the (k + 1)-Equals Ideal". eng. In: *Communications in mathematical physics* 330.1 (2014), pp. 415–434. ISSN: 0010-3616.

University of Illinois Urbana-Champaign

Email address: caseyta2@illinois.edu

Mercer University

Email address: Zachary.Chance.Medlin@live.mercer.edu

Dartmouth College

Email address: mario.tomba.morillo.25@dartmouth.edu