INFINITE FREE RESOLUTIONS OVER CERTAIN FAMILIES OF NUMERICAL SEMIGROUP ALGEBRAS

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Abstract. A numerical semigroup (cofinite subsemigroup of $\mathbb{Z}_{\geq 0}$) has an associated finite poset, the Kunz poset, which encodes much of its additive structure. Recent work has shown that numerical semigroups with the same Kunz poset have semigroup algebras with similar infinite free resolutions of the base field. We expand on this by investigating such infinite free resolutions for certain notable families of numerical semigroups. We provide a combinatorial construction, using pattern-avoiding words, for a minimal free resolution of the base field $\mathbb{K}$ over the semigroup algebra $\mathbb{K}[S]$ when S is generated by a generalized arithmetic progression. We also conjecture minimal resolutions when S has smallest positive element 5 or 6.

Contents

1. Introduction

A numerical semigroup is a cofinite subset of the natural numbers that is closed under addition and contains 0. These semigroups have numerous connections to the study of planar curves and of singularities of algebraic varieties; they appear in both the theoretical setting of algebraic geometry and in the applied setting of cryptography. Given a numerical semigroup, we have a natural interpretation of its elements as exponents of monomials in a certain polynomial ring quotient known as the semigroup algebra. We gain a deeper understanding of the semigroup algebra by studying modules over it, and a principal tool in the study of modules is a free resolution.

Prior work defined the resolution of a certain ideal associated with a given numerical semigroup over a polynomial ring, resulting in a finite free resolution. The structure of infinite free resolutions associated with numerical semigroups, however, is less well-understood.

This project aims to understand the behavior of infinite free resolutions resolving a ground field over a numerical semigroup algebra, and we build on work developing infinite free resolutions in [Gom+24]. Free resolutions of various modules over different rings are computable but often offer no natural combinatorial interpretation. In fact, there are very few families of numerical semigroup algebras for which infinite free resolutions have been obtained from combinatorial data.

We aim to increase the number of families of numerical semigroups for which the behavior of infinite free resolutions is well-understood. In particular, we consider extra-generalized arithmetical numerical semigroups, which are generated by a generalized arithmetic sequence of the form

$$\langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta\rangle,$$

where $m, h, k \in \mathbb{Z}_{>0}$, and δ is any integer with $\gcd(m,\delta) = 1$ and $mh + k\delta > m$. For extra-generalized arithmetical numerical semigroups, we provide a combinatorial method rooted in the theory of pattern-avoiding words to construct free resolutions. Further, for numerical semigroups with small multiplicity and for numerical semigroups with a single maximal element, we provide conjectures for the words which index basis vectors in the free modules comprising free resolutions of a base field over the numerical semigroup algebra.

In Section 2, we provide much of the combinatorial background needed from semigroup theory, and we give a broad survey of the additive structure of a numerical semigroup S. In Section 3, we provide a (somewhat non-constructive) minimal free resolution of a ground field over an extra-generalized arithmetical numerical semigroup algebra (Theorem 3.8), and we give an explicit construction (Corollary 3.14) in a special case. Finally, Section 4 provides a survey of resolutions for semigroups of small multiplicities $m = 5, 6, 7$, as well as standing conjectures for such semigroups.

2. Background

2.1. **Numerical Semigroups and the Kunz Cone.** Let S be a numerical semigroup. We typically specify S by giving a list of generators: we write $S = \langle n_1, n_2, \ldots, n_k\rangle$ to mean

$$S = \{z_1n_1 + z_2n_2 + \cdots + z_kn_k : z_i \in \mathbb{Z}_{\geq 0}\}.$$

Because S has finite complement, we necessarily have $\gcd(n_1, \ldots, n_k) = 1$. It is well-known (see [RG09]) that a numerical semigroup S has a unique minimal generating set (which is finite) with respect to both size and inclusion. Elements of this minimal generating set are called *minimal generators*, and the smallest such generator, which is also the smallest positive element of S, is called the *multiplicity* of S and is denoted $m(S)$. The cardinality of the minimal generating set is the *embedding dimension* of S and is denoted $e(S)$.

Because a numerical semigroup S *eventually* contains every positive integer, it seems worthwhile to understand the elements of S that are minimal in some respect. For a numerical semigroup S with multiplicity m, we define its *Apéry set* $\mathrm{Ap}(S)$ by

$$\mathrm{Ap}(S) := \{n \in S : n - m \notin S\}.$$

Note that $\mathrm{Ap}(S)$ necessarily contains exactly one representative from each equivalence class modulo m (in fact, the smallest representative in S), so it follows that $|\mathrm{Ap}(S)| = m$; in particular, it is always true that $0 \in \mathrm{Ap}(S)$. Additionally, $\mathrm{Ap}(S)$ contains all minimal generators of S except m; therefore, S is generated by its nonzero Apéry set elements together with m, though this may or may not be the minimal generating set. This demonstrates the following:

Proposition 2.1. For any numerical semigroup S, we have $e(S) \leq m(S)$.

When $e(S) = m(S)$, the nonzero elements of $\mathrm{Ap}(S)$ together with the multiplicity m form the minimal generating set of S, and we say that S has *maximal embedding dimension (MED)*.

Example 2.2. Consider the numerical semigroup $S = \langle 6, 9, 20, 29\rangle$. This choice of generating set is obviously not minimal, though one can check that $\{6, 9, 20\}$ is indeed a minimal generating set. Hence, $m(S) = 6$ and $e(S) = 3$, so this semigroup is not MED. We compute its Apéry set, which, by convention, we list in order of equivalence class modulo $m = 6$:

$$\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}.$$

Given a semigroup S, define the relation $\preceq$ on $\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_{m-1}\}$ by $a_i \preceq a_j$ if and only if $a_j - a_i \in S$. It is readily checked that $\preceq$ is a well-defined partial order. In the example above, the *Apéry poset* has the diagram given in Figure 1. Observe that each cover relation $a_i \preceq a_j$ in the poset in that figure corresponds to adding a minimal generator to a_i to get to a_j.

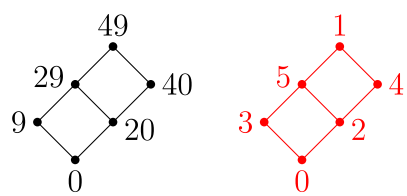


Figure 1. The Apéry poset (left) and Kunz poset (right) for $S = \langle 6, 9, 20\rangle$.

The *Kunz poset* of a numerical semigroup S is derived by reducing every element of the Apéry poset to its entry modulo the multiplicity. The power of the Kunz poset lies in the fact that they capture the additive structure of many semigroups at once; to see this, we turn to some geometry. Let $\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_{m-1}\}$, where $a_i \equiv i \pmod m$. Define the *Apéry tuple* of S to be $(a_1, a_2, \ldots, a_{m-1}) \in \mathbb{R}^{m-1}$; note that S is uniquely determined by its Apéry tuple. Observe that for any semigroup S, we must have $a_i + a_j \geq a_{i+j}$ (where the sum $i + j$ is taken modulo m) whenever $i + j \not\equiv 0 \pmod m$ by minimality of each element in $\mathrm{Ap}(S)$. Hence, all Apéry tuples must reside within the following region:

Definition 2.3. Let $m \geq 2$. We define the *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}_{\geq 0}$ as the region satisfying the inequalities

$$x_i + x_j \geq x_{i+j}$$

for all $1 \leq i \leq j \leq m-1$ with $i+j \not\equiv 0 \pmod m$, where we add the subscripts above modulo m. A *face* F of the Kunz cone is a region $F \subseteq C_m$ defined by the inequalities above, where some of the inequalites are possibly equalities.

We shall often say that a face $F \subseteq C_m$ *contains* a numerical semigroup S of multiplicity m if F (set-wise) contains the Apéry tuple of S.

Example 2.4. The Kunz cone $C_4 \subseteq \mathbb{R}^3_{\geq 0}$ is defined by the four inequalities

$$2x_1 \geq x_2, \quad x_1 + x_2 \geq x_3, \quad x_2 + x_3 \geq x_1, \quad \text{and} \quad 2x_3 \geq x_2$$

and contains (the Apéry tuples for) all numerical semigroups of multiplicity 4. We give a few examples of numerical semigroups on C_4:

- The semigroup $S = \langle 4, 5, 6\rangle$ has Apéry tuple $(5, 6, 11)$, which evidently sits on the two-dimensional face F of the Kunz cone C_4 given by $x_1 + x_2 = x_3$. Similarly, $T = \langle 4, 9, 10\rangle$ has Apéry tuple $(9, 10, 19)$, which also sits on F. Since both S and T lie on the same face, they have the same Kunz poset P — the equality $x_1 + x_2 = x_3$ on F dictates a cover relation $3 \geq 1$ and $3 \geq 2$ in P.
- The semigroup $S' = \langle 4, 5, 7\rangle$ has Apéry tuple $(5, 10, 7)$, which lies on a different two-dimensional face $F' \subseteq C_4$ given by $2x_1 = x_2$. Therefore, S' has a Kunz poset P' different from P.
- Finally, the semigroup $S'' = \langle 4, 5\rangle$ has Apéry tuple $(5, 10, 15)$ that sits on the one-dimensional face F'' of the Kunz cone given by $x_1 + x_2 = x_3$ and $2x_1 = x_2$. The facet equalities of F and F' immediately give $F'' = F \cap F'$, and so the Kunz poset P'' of S'' consists of the union of relations present in P and P'.

These observations lead to the following proposition:

Proposition 2.5. Any two semigroups on the same face of the Kunz cone have the same Kunz poset, and if a face of the Kunz cone F is the intersection of two faces F' and F'', then the Kunz poset of semigroups that lie on F consist of the union of relations present in the Kunz posets of semigroups that lie on either F' or F''.

Thus, we see that the Kunz poset is an understandable combinatorial object that encodes fundamental information about a numerical semigroup. We will attempt to use this object to glean even more information about the semigroup.

2.2. **Factorizations, Trades, and Betti Elements.**

We now give a quick survey of the additive structure of a numerical semigroup, which will motivate our constructions in Section 3. Throughout, let S be a numerical semigroup of multiplicity m, with $S = \langle m, n_1, n_2, \ldots, n_k\rangle$ where each n_i is a minimal generator.

Definition 2.6. Let $n \in S$. A *factorization* of n is an expression $n = z_0 m + z_1 n_1 + z_2 n_2 + \cdots + z_k n_k$, where $z_i \geq 0$ for all $1 \leq i \leq k$. We will often specify a factorization by giving its *factorization tuple* $(z_0, z_1, \ldots, z_k) \in \mathbb{Z}^{k+1}_{\geq 0}$. The *factorization homomorphism* $\varphi_S : \mathbb{Z}^{k+1}_{\geq 0} \to S$ is the homomorphism that maps factorization tuples to their corresponding semigroup elements, i.e. $\varphi_S(z_0, z_1, ..., z_k) = z_0 m + z_1 n_1 + z_2 n_2 + \cdots + z_k n_k$. The preimage $\varphi_S^{-1}(n) =: \mathsf{Z}_S(n)$ is the *set of factorizations* of a semigroup element $n \in S$.

Example 2.7. Observe that factorizations are not necessarily unique. Take $S = \langle 6, 9, 20\rangle$, and notice $18 = 2 \cdot 9 = 3 \cdot 6$ has two distinct factorizations. In contrast, $15 = 6 + 9$ does have a unique factorization. Observe that as n increases, the number of factorizations of n generally increases: taking $n = 63$, we have

$$63 = 7 \cdot 9 = 3 \cdot 6 + 5 \cdot 9 = 6 \cdot 6 + 3 \cdot 9 = 9 \cdot 6 + 9;$$

The existence of two distinct factorizations of 18 gives us four factorizations of 63. In terms of the factorization homomorphism φ_S, we have

$$\varphi_S(0,7,0) = \varphi_S(3,5,0) = \varphi_S(6,3,0) = \varphi(9,1,0) = 63, \text{ or}$$
$$\mathsf{Z}_S(63) = \{(0,7,0),(3,5,0),(6,3,0),(9,1,0)\}.$$

For an alternative viewpoint, take the homomorphism $\varphi : \mathbb{K}[y, x_1, x_2] \to \mathbb{K}[t]$ by $y \mapsto t^6$, $x_1 \mapsto t^9$, and $x_2 \mapsto t^{20}$. Now, the equation $2 \cdot 9 - 3 \cdot 6 = 0$ may be represented as the claim $x_1^2 - y^3 \in \ker \varphi$. Rearranging the previous statement as $x_1^2 \equiv y^3 \pmod{\ker \varphi}$, this corresponds to an *equality* $x_1^2 = y^3$ in the quotient $\mathbb{K}[y, x_1, x_2]/\ker \varphi$. Thus, the four factorizations of 63 may be succinctly denoted

$$t^{63} = x_1^7 = y^3 x_1^5 = y^6 x_1^3 = y^9 x_1.$$

Here, the first equality is an abuse of notation stemming from the first isomorphism theorem: $\operatorname{im} \varphi \cong \mathbb{K}[y, x_1, x_2]/\ker \varphi$, where we identify t^{63} with x_1^7 under the canonical isomorphism, but the other equalities are true equalities in the quotient above. Observe that φ and φ_S behave rather similarly; however, φ has the advantage of being a *ring* homomorphism, while φ_S is a semigroup homomorphism — it is this advantage (of preserving more structure) that we will exploit later.

Definition 2.8. The *kernel* of φ_S is a relation $\sim$ on the domain of φ_S, and relates $z \sim z'$ whenever $\varphi_S(z) = \varphi_S(z')$, i.e., z and z' are both factorizations of one $n \in S$. If $z \sim z'$, we call the pair (z, z') a *trade*.

We remark that the relation $\sim$ defined by the kernel of φ_S is an equivalence relation, and in fact a *congruence*: we have $z \sim z'$ implies $z + z'' \sim z' + z''$ for all $z, z', z'' \in \mathbb{Z}_{\geq 0}^{k+1}$.

A particularly important concept for understanding numerical semigroups and their posets is *outer Betti elements*, which help us discuss the additive structure of S that lies slightly beyond the Apéry set $\operatorname{Ap}(S)$. Before we can define outer Betti elements, we need some background definitions.

A *nilsemigroup* is a commutative semigroup $(N, +)$ with an element $\infty \in N$ (called the *nil* of N) such that $a + \infty = \infty$ for all $a \in N$. Let S be a numerical semigroup, and define the congruence $\approx$ on S by $a \approx b$ whenever $a = b$ or $a, b \notin \operatorname{Ap}(S)$. Observe that $\approx$ partitions S into the subset $S \setminus \operatorname{Ap}(S)$ and singletons $\{a\}$ for each $a \in \operatorname{Ap}(S)$, i.e., the quotient $S/{\approx}$ has a natural nilsemigroup structure with $\infty = S \setminus \operatorname{Ap}(S)$, and one non-nil element for each element of $\operatorname{Ap}(S)$.

Definition 2.9. The *Kunz nilsemigroup* N of S is obtained from $S/{\approx}$ by replacing each non-nil element with its equivalence class modulo m.

Observe that $N = \mathbb{Z}/m\mathbb{Z} \cup \{\infty\}$ as sets, and that N is *partially cancellative*, meaning $a + b = a + c \neq \infty$ implies $b = c$. The *divisibility poset* of the non-nil elements of N, the poset in which $a \preccurlyeq b$ when $b = a + c$ for some $c \in N$, is exactly the Kunz poset of S.

We may also define factorization within the Kunz nilsemigroup N, which is analogous to the situation within the semigroup S itself.

Definition 2.10. Let S be a semigroup with associated Kunz nilsemigroup N. Suppose $n_1, \dots, n_k$ are the atoms of N, so that $\{n_1, \dots, n_k\}$ is the minimal generating set for N. We define the *factorization homomorphism* on N by

$$\varphi_N : \mathbb{Z}_{\geq 0}^k \to N \text{ with } (z_1, z_2, \dots, z_k) \mapsto \sum_{i=1}^{k} z_i n_i.$$

Similarly, define the *set of factorizations* of some Kunz nilsemigroup element $n \in N$ by $\mathsf{Z}_N(n) := \varphi_N^{-1}(n)$.

Partial cancellativity of N ensures $|\mathsf{Z}_N(n)| < \infty$ for $n \neq \infty$. If $a_i \in \operatorname{Ap}(S)$, omitting the first coordinate of each factorization of $\mathsf{Z}_S(a_i)$ (which is always 0) gets us $\mathsf{Z}_N(i)$, and $\mathsf{Z}_N(\infty)$ contains all other elements of $\mathbb{Z}_{\geq 0}^k$.

We also make the following useful definition.

Definition 2.11. The *support* of a factorization $z \in \mathbb{Z}_{\geq 0}^k$ is $\operatorname{supp}(z) := \{i : z_i > 0\}$, and the support of a subset $Z \subseteq \mathbb{Z}_{\geq 0}^k$ is $\operatorname{supp}(Z) := \{i : z_i > 0 \text{ for some } z \in Z\}$.

For such a set $Z \subseteq \mathbb{Z}_{\geq 0}^k$, let ∇_Z be the graph with vertex set Z where z, z' are connected by an edge when $z \neq z'$ and $\operatorname{supp}(z) \cap \operatorname{supp}(z') \neq \emptyset$. For each $i \in \operatorname{supp}(Z)$, define $Z - e_i = \{z - e_i : z \in Z \text{ with } i \in \operatorname{supp}(z)\}$, where $e_i = (z_1, z_2, \dots, z_k)$ is 1 in the ith component and 0 everywhere else. Now we are ready to define outer Betti elements:

Definition 2.12. Let S be a numerical semigroup and let N be its Kunz nilsemigroup. An *outer Betti element* B of S is a subset $B \subseteq \mathsf{Z}_N(\infty)$ such that

(i) for every $i \in \operatorname{supp}(B)$, $B - e_i = \mathsf{Z}_N(n)$ for some $n \in N \setminus \{\infty\}$, and

(ii) the graph ∇_B is connected.

The outer Betti elements of S are essentially the things that live just outside of the Kunz poset: *very* loosely speaking, Betti elements may be identified with sums of the elements of the Kunz poset that do not occur within the Kunz poset, but for which removing one atom from *any factorization* of the sum gives something within the Kunz poset. They necessarily encode trades in S, and their sense of minimality indicates fundamental importance to the structure of the semigroup S.

Example 2.13. Let $S = \langle 13, 14, 15, 16, 17 \rangle$, with Apéry set

$$\operatorname{Ap}(S) = \{0, 14, 15, 16, 17, 31, 32, 33, 34, 48, 49, 50, 51\}.$$

We give the Kunz poset for S in Figure 2. Then as sets, the Kunz nilsemigroup N of S satisfies $N = \mathbb{Z}_{13} \cup \{\infty\}$ with the following addition law: $a + \infty = \infty$ for all $a \in N$, and if $b, c \in N \setminus \{\infty\}$, we read off the sum $b + c$ using the Kunz poset as follows:

- If $a_b + a_c = a_{b+c} \in \operatorname{Ap}(S)$, then $b + c$ is just the sum in $\mathbb{Z}/13\mathbb{Z}$.
- If $a_b + a_c \notin \operatorname{Ap}(S)$, then $b + c = \infty$.

The atoms of N are evidently $\bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$, so the factorization homomorphism $\varphi_N : \mathbb{Z}_{\geq 0}^k \to N$ is given by $(z_1, z_2, z_3, z_4) \mapsto \bar{1}z_1 + \bar{2}z_2 + \bar{3}z_3 + \bar{4}z_4$. For example, we may compute the set of factorizatioins for $10 \in N$:

$$\mathsf{Z}_N(10) = \varphi_N^{-1}(10) = \{(0, 1, 0, 2), (0, 0, 2, 1)\}.$$

Now, each sum of any two atoms among $\bar{1}, \bar{2}, \bar{3}$ gives an outer Betti element, and we have four outer Betti elements which "fill in" a new row of the poset in Figure 2 — these points are marked in red. Within this particular numerical semigroup S, the outer Betti elements correspond to the following factorizations of semigroup elements in S:

$$\begin{aligned}
28 &= a_1 + a_1,\\
29 &= a_1 + a_2,\\
30 &= a_1 + a_3 = a_2 + a_2,\\
65 &= a_{12} + a_1 = a_{11} + a_2 = a_{10} + a_3 = a_9 + a_4,\\
66 &= a_{12} + a_2 = a_{11} + a_3 = a_{10} + a_4,\\
67 &= a_{12} + a_3 = a_{11} + a_4,\\
68 &= a_{12} + a_4.
\end{aligned}$$

We notice the trades occurring at several of these outer Betti elements — it is these trades that will dictate our constructions in Section 3.

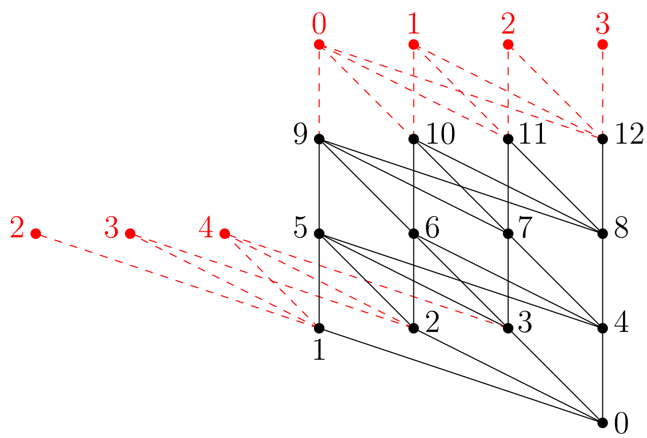


Figure 2. The Kunz poset for $S = \langle 13, 14, 15, 16, 17\rangle$. Outer Betti elements, connected by dashed lines, are in red, and we identify them with their equivalence class modulo 13.

2.3. **Toric Ideals and Semigroup Algebras.** Numerical semigroups, by their very nomenclature, have less access to properties that are present in other algebraic structures. Luckily, however, we have a clever homomorphism that allows us to examine the properties of a numerical semigroup by looking at monomials within a polynomial ring quotient. In particular, this allows us to utilize tools from ring theory and commutative algebra to study numerical semigroups. Let $S = \langle m, n_1, n_2, \ldots, n_k\rangle$, where S has multiplicity m and $\{m, n_1, \ldots, n_k\}$ is a minimal generating set. Fix a field $\mathbb{K}$ and define the ring homomorphism

$$\varphi : \mathbb{K}[y, x_1, x_2, \ldots, x_k] \to \mathbb{K}[t] \text{ by } y \mapsto t^m \text{ and } x_i \mapsto t^{n_i}.$$

We also grade the polynomial ring $\mathbb{K}[y, x_1, \ldots, x_k]$ by setting $\deg(y) = m$ and $\deg(x_i) = n_i$. Notice that $t^n \in \operatorname{im}\varphi$ if and only if $n \in S$, so the image $\operatorname{im}\varphi$ captures precisely the elements of S, endowed now with the richer structure of a ring whose multiplicative behavior is exactly the additive structure of S. Hence, we make the following definition.

Definition 2.14. The *semigroup algebra* $R = \mathbb{K}[S]$ is defined to be the image $\operatorname{im}\varphi$ of the homomorphism given above, and the *toric ideal* I_S is defined as the kernel $\ker\varphi$.

By the first isomorphism theorem, we have $\mathbb{K}[y, x_1, x_2, \ldots, x_k]/I_S \cong R$. Thus, understanding the toric ideal I_S is crucial to understanding the semigroup S. Clearly, no monomials in $\mathbb{K}[y, x_1, \ldots, x_k]$ are in $\ker \varphi$, so what elements are in the toric ideal I_S? It turns out that I_S is generated by binomials — differences of monomials that correspond precisely to distinct factorizations of the same element of S. Thus, the polynomials generating I_S are exactly those encoded by trades of elements in S. It is here we see the combinatorial power of the Kunz poset: from the Kunz poset of a numerical semigroup S, we can trace distinct paths in the poset to find trades of Apéry set elements, and we can quite easily "jump off" the poset to find outer Betti elements, thereby finding the minimal trades that seem to be needed to generate the important ideal I_S. The following proposition legitimizes this intuition.

Proposition 2.15 ([GOD23])**.** The toric ideal I_S is minimally generated by differences of monomials corresponding to distinct factorizations of Apéry set elements and differences of monomials corresponding to members of a shared outer Betti element, and in particular, one such binomial generator from each outer Betti element is contained in a minimal generating set for I_S.

The flavor of this intuition provides the motivation for our work. What more information about a semigroup does the Kunz poset provide? How does it encode larger trades or even trades between trades?

Example 2.16. Let $S = \langle 5, 6, 7\rangle$. Clearly, $\{5, 6, 7\}$ is a minimal generating set, so the homomorphism that defines the toric ideal is $\varphi : \mathbb{K}[y, x_1, x_2] \to \mathbb{K}[t]$ by $y \mapsto t^5$, $x_1 \mapsto t^6$, and $x_2 \mapsto t^7$. Notice that elements in semigroups may have different factorizations; for example, $12 = 5 + 7 = 6 + 6$. This is reflected in the fact that $\varphi(yx_2) = t^{12} = \varphi(x_1^2)$; in particular, $yx_2 - x_1^2 \in \ker \varphi = I_S$. Here, $yx_2 - x_1^2$ encodes a trade between the factorizations of 12, and as such, the defining toric ideal is generated by trades of factorizations of elements in S.

It will also be useful for us to consider a generalized type of toric ideal, defined as follows. Fix a numerical semigroup S with multiplicity m, and as usual, we denote its Apéry set by listing its elements in equivalence class order: $\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_{m-1}\}$, with $a_i \equiv i \pmod m$, and we set $a_0 := m$. Consider the polynomial ring $\mathbb{K}[y, x_1, \ldots, x_{m-1}]$ together with the grading $\deg(y) := m$ and $\deg(x_i) := a_i$. Setting $x_0 := y$, define the homomorphism

$$\varphi : \mathbb{K}[y, x_1, \ldots, x_{m-1}] \to \mathbb{K}[t] \text{ by } x_i \mapsto t^{a_i}.$$

From here, we get two analogous definitions.

Definition 2.17. The *Apéry toric ideal* is $J_S := \ker \varphi$, and the *semigroup algebra* R is:

$$R := \mathbb{K}[y, x_1, \ldots, x_{m-1}]/J_S \cong \operatorname{im} \varphi.$$

The use of the term *semigroup algebra* for two seemingly different quotients is quite disconcerting, but it need not be, for the two semigroup algebras R are isomorphic. This is a result of the following relationship between I_S and J_S:

Proposition 2.18. We have $I_S = J_S \cap \mathbb{K}[y, x_i : a_i \text{ is a minimal generator}]$.

In particular, $I_S \subseteq J_S$, and from this, also note that $I_S = J_S$ if and only if S has maximal embedding dimension. This is because when S is MED, all nonzero Apéry set elements are minimal generators, as we have mentioned before.

The following fact about the Apéry toric ideal is well-known, and is useful as a computation aid as well as for understanding outer Betti elements.

Proposition 2.19. Let S be a numerical semigroup with multiplicity m. Then

$$J_S = \langle x_i x_j - y^{b_{i,j}} x_{i+j} : 1 \leq i \leq j \leq m-1 \rangle,$$

where $b_{i,j} := \frac{1}{m}(a_i + a_j - a_{i+j})$.

Observe that each generator $x_i x_j - y^{b_{i,j}} x_{i+j}$ represents a trade corresponding to an outer Betti element whenever $b_{i+j} > 0$.

Example 2.20. Let $S = \langle 5, 6, 7 \rangle$. Then $\mathrm{Ap}(S) = \{0, 6, 7, 13, 14\}$, and we compute the Apéry toric ideal using the proposition above:

$$\begin{aligned} J_S = \langle & x_1^2 - yx_2, x_1x_2 - x_3, x_1x_3 - yx_4, x_1x_4 - y^4, x_2^2 - x_4, x_2x_3 - y^4, \\ & x_2x_4 - y^3x_1, x_3^2 - y^4x_1, x_3x_4 - y^4x_3, x_4^2 - y^3x_3 \rangle. \end{aligned}$$

Here, the binomial $x_1^2 - yx_2$ corresponds to a trade of 10, which corresponds to an outer Betti element. We may also compute the defining toric ideal by noting y, x_1, x_2 correspond to minimal generators: under the canonical isomorphism $\mathbb{K}[y, x_1, x_2, x_3, x_4]/J_S \cong \mathbb{K}[y, x_1, x_2]/I_S$ we may (abusively) identify $x_3 = x_1x_2$ and $x_4 = x_2^2$. Making these replacements in J_S above and eliminating redundant generators yields

$$I_S = \langle x_1^2 - yx_2, y^3x_1 - x_2^3, y^4 - x_1x_2^2 \rangle.$$

To study the semigroup algebra R, we will examine modules over R; to do this, we introduce some terminology.

Definition 2.21. Let M be an R-module. A *free resolution* of M over R is an exact sequence of R-linear maps

$$\mathcal{F}_\bullet : 0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots,$$

where each F_d is a free R-module. If M is a graded module, then $\mathcal{F}_\bullet$ is a *graded free resolution* if each *boundary map* $\partial_d : F_d \to F_{d-1}$ is homogeneous. If the number of free modules that occur in the free resolution is finite, we say $\mathcal{F}_\bullet$ is a *finite free resolution*; otherwise, it is an *infinite free resolution*. Finally, $\mathcal{F}_\bullet$ is *minimal* if $\partial_d(F_d) \subseteq (y, x_1, \ldots, x_k)F_{d-1}$ for all $d \geq 1$.

This fact follows immediately from the definition of minimality:

Proposition 2.22. Let M be an R-module. Then a free resolution $\mathcal{F}_\bullet$ of M is minimal if and only if the matrices representing the boundary maps ∂_d have no nonzero constant entries.

In particular, we will study the infinite free resolution of the ground field $\mathbb{K}$, over the semigroup algebra R. Results about such resolutions may be found in [Gom+24] in the case that the numerical semigroup S is MED; in particular, the results construct the *infinite Apéry resolution* $\mathcal{F}_\bullet$, which is minimal if and only if S is MED. When S is not MED, minimal resolutions may be found by row reducing the matrices in $\mathcal{F}_\bullet$, and this row reduction depends only on the Kunz poset of S. However, the proof of this fact is nonconstructive, but the hope is that the "niceness" of the Kunz poset in certain cases will allow us to construct an explicit resolution for certain non-MED semigroups S.

2.4. **Extra-Generalized Arithmetical Numerical Semigroups.** One important family of numerical semigroups is of interest to us, due to the symmetry and predictability of their Kunz poset structure.

Definition 2.23. An *extra-generalized arithmetical numerical semigroup* S takes the form

$$S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta\rangle,$$

where m, h, k, δ are integers such that $0 < k < m$, $h \geq 1$, $\gcd(m, \delta) = 1$, and $mh + k\delta > m$. If $\delta < 0$, we say that S is *pessimistic*.

Remark 2.24. The semigroup S above is *generalized arithmetical* because of the "scaling" term h; if $h = 1$, then S is generated by an arithmetic progression and is called simply *arithmetical*. S is "extra-generalized" because δ is allowed to be negative.

Example 2.25. The semigroup $S = \langle 13, 14, 15, 16, 17\rangle$ is an arithmetical semigroup, as we have $h = 1$. Here, $m = 13$, $k = 4$, and $d = 1$. Recall that

$$\operatorname{Ap}(S) = \{0, 14, 15, 16, 17, 31, 32, 33, 34, 48, 49, 50, 51\}.$$

As usual, we list the elements in order of their equivalence classes, but notice that this is the same as listing the elements in increasing order. Examining the Kunz poset of S (Figure 2) explains why: we observe a nice, graded "staircase" structure, which essentially "counts up" the equivalence classes in order. This nice structure holds for the Kunz poset of any extra-generalized arithmetical numerical semigroup, which follows from an explicit description of the elements of $\operatorname{Ap}(S)$, together with its poset relations:

Theorem 2.26 (Theorem 3.4 from [Aut+21])**.** Let $S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta\rangle$ be an extra-generalized arithmetical numerical semigroup, and write $a - 1 = qk + r$ for $q, r \in \mathbb{Z}_{\geq 0}$ with $r < k$, and write $\operatorname{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$.

- (i) Each nonzero element $a_i \in \operatorname{Ap}(S)$ takes the form $a_i = x_i mh + \big((x_i - 1)k + y_i\big)\delta$, for either $x_i \in \{1, 2, \ldots, q\}$ and $y_i \in \{1, 2, \ldots, k\}$, or $x_i = q + 1$ and $y_i \in \{1, r\}$.
- (ii) In $\operatorname{Ap}(S)$, $a_i \preceq a_j$ if and only if $x_i < x_j$ and $y_i \geq y_j$.
- (iii) In $\operatorname{Ap}(S)$, a_j covers a_i if and only if $x_j = x_i + 1$ and $y_i \geq y_j$.

Reading the theorem sufficiently carefully reveals that we are stating that the Apéry (or Kunz) poset of S has the "tower" structure demonstrated in Figure 2, with q full rows, each full row having a width of k, with the top (possibly incomplete) row containing r elements. Note that the top row is a complete row with k elements if and only if $m \equiv 1 \pmod{k}$.

In particular, the structure of the poset only depends on m and k, and the labeling of each vertex depends only on the equivalence class of δ modulo m. The relations between Apéry set elements in the theorem simply tell us how the elements are placed within each row.

Proposition 2.27 (Lemma 4.1 from [Aut+21])**.** Let $S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta\rangle$ be an extra-generalized arithmetical numerical semigroup, whose Apéry tuple is contained in a face $F \subseteq C_m$. Applying an automorphism on C_m induced by multiplication of a unit $u \in \mathbb{Z}_m^\times$ to F yields a face $F' \subseteq C_m$, and F' contains the Apéry tuple corresponding to

$$S' = \langle m, mh+u'\delta, mh+2u'\delta, \ldots, mh+ku'\delta\rangle,$$

where u' is an integer in the equivalence class u.

This proposition greatly simplifies the work we must do to understand an extra-generalized arithmetical numerical semigroup — hence, without loss of generality, we fix $\delta \equiv 1 \pmod{m}$; this is reflected in our construction in Section 3.

3. Minimal Free Resolutions for Extra-Generalized Arithmetical Numerical Semigroups

3.1. **The Infinite Apéry Resolution.** In this section, we prove a result about the Betti numbers (Corollary 3.10) for the minimal free resolution for an extra-generalized arithmetical numerical semigroup $\langle m, mh+\delta, mh+2\delta, \dots, mh+k\delta\rangle$, where $k \nmid m$. Our proof is somewhat nonconstructive, but it utilizes the chain-mapping technique employed in Section 4 of [Gom+24]. We also borrow the strategy of indexing our free modules by permissible words, which correspond to a specific grading on said modules, used to define the *infinite Apéry resolution* in the same source.

First, we define the infinite Apéry resolution, as this will form the basis of our proof. Throughout, let S be any numerical semigroup with multiplicity m, and as per convention write

$$\mathrm{Ap}(S) = \{0, a_1, a_2, \dots, a_{m-1}\} \text{ with } a_i \equiv i \bmod m,$$

and also set $a_0 := m$. Let $R = \mathbb{K}[S]$ be the semigroup algebra of S, setting $y := x_0$. We define the grading on R by $\deg(x_i) := a_i$.

Definition 3.1 ([Gom+24] Definition 3.1)**.** The *infinite Apéry resolution* of the ground field $\mathbb{K}$ over R is the free resolution

$$\mathcal{F}_\bullet : 0 \longleftarrow \mathbb{K} \longleftarrow F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow F_3 \longleftarrow \cdots,$$

where each F_d is a free module with basis

$$\{e_{\mathbf{w}} : \mathbf{w} = (w_1, w_2, \dots, w_d) \in \mathbb{Z}_m^d, w_i \neq 0 \text{ for all } i \geq 2\} \text{ with } \deg(e_{\mathbf{w}}) := \sum_{i=1}^{d} \deg(x_{w_i}).$$

The boundary maps $\partial_d : F_d \to F_{d-1}$ for $d \geq 2$ are given by

$$\partial_d : e_{\mathbf{w}} \mapsto x_{w_d} e_{\widehat{\mathbf{w}}} + \sum_{i=1}^{d-1} (-1)^{d-i} y^{b_{w_i, w_{i+1}}} e_{\tau_i w},$$

where for $\mathbf{w} = (w_1, w_2, \dots, w_n)$, we define $\widehat{\mathbf{w}} := (w_1, w_2, \dots, w_{n-1})$ and

$$\tau_i \mathbf{w} := (w_1, \dots, w_{i-1}, w_i + w_{i+1}, w_{i+2}, \dots, w_n),$$

both of which shorten the length of $\mathbf{w}$ by one letter. The exponents $b_{w_i, w_{i+1}}$ are given by

$$b_{w_i, w_{i+1}} := \frac{1}{m}(a_{w_i} + a_{w_{i+1}} - a_{w_i + w_{i+1}}).$$

The map $\partial_0 : F_0 \to \mathbb{K}$ is the quotient map taking $F_0 = R \cong \mathbb{K}[y, x_1, x_2, \dots, x_{m-1}]/J_S$ to $\mathbb{K}$, and $\partial_1 : F_1 \to R$ is simply the map sending $e_i \mapsto x_i$.

It was proven in [Gom+24] that the infinite Apéry resolution is indeed a resolution, and that it is minimal precisely when S has maximal embedding dimension.

Given a numerical semigroup $S = \langle m, mh+\delta, mh+2\delta, \dots, mh+k\delta\rangle$ with $m \not\equiv 0 \pmod{k}$, we construct a minimal infinite free resolution of the base field $\mathbb{K}$ over the semigroup algebra $R = \mathbb{K}[y, x_1, x_2, \dots, x_k]/I_S$ via a chain map applied to the Apéry resolution of that field. In other words, we construct the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & \mathbb{K} & \xleftarrow{\partial_0} & R & \xleftarrow{\partial_1} & F_1 & \xleftarrow{\partial_2} & F_2 & \longleftarrow & \cdots \\
 & & & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \\
0 & \longleftarrow & \mathbb{K} & \xleftarrow{\partial'_0} & R & \xleftarrow{\partial'_1} & F'_1 & \xleftarrow{\partial'_2} & F'_2 & \longleftarrow & \cdots
\end{array}$$

wherein the top row is the Apéry resolution of $\mathbb{K}$ over R and the bottom row is our desired resolution. The power of our method is attained in virtue of the fact that the words which will index basis elements in the free modules F'_d also index basis elements in the original Apéry modules F_d.

Throughout the remainder of this section, fix $S = \langle m, mh + \delta, mh + 2\delta, \ldots, mh + k\delta\rangle$ to be an extra-generalized arithmetical numerical semigroup with $k \nmid m$ and $\gcd(m, \delta) = 1$, and write $m = qk + \alpha$, for $0 < \alpha < k$ and $q \in \mathbb{Z}$. Without loss of generality, take $h = \delta = 1$ — this has no effect on the Betti numbers of the free resolutions which we will study, by virtue of Proposition 2.27.

3.2. **Valid Words.** The construction of our minimal free resolutions relies on pattern-avoiding words of a given length, which gives our study a combinatorial flavor.

Definition 3.2. A *valid word* of length $d \geq 0$ is an element $\mathbf{w} = (w_1, w_2, \ldots, w_d) \in \mathbb{Z}_m^d$ such that

(*i*) $w_i \in \{0, 1, \ldots, k, m - \alpha\}$ for all i.
(*ii*) $w_i \neq 0$ for all $i > 1$ and $w_1 \neq m - \alpha$,
(*iii*) $\mathbf{w}$ does not contain the subword (j, k) for any $j \in \{1, 2, \ldots, k\}$, and
(*iv*) $\mathbf{w}$ does not contain the subword $(j, m - \alpha)$ for any $j \in \{0, 1, \ldots, \alpha - 1, m - \alpha\}$.

The set of all valid words of length d is denoted W_d, and we take the empty word $\varnothing$ to be the only valid word of length 0. We will often omit the tuple notation and write $\mathbf{w} = (w_1, w_2, \ldots, w_d) = w_1 w_2 \cdots w_d$, where the letter $m - \alpha$ is abbreviated with "$-$".

Detailed inspection of these words reveals a sharp connection between concatenations of letters with paths in the Kunz poset of S.

Example 3.3. Let $S = \langle 13, 14, 15, 16, 17\rangle$, so $m = 13$ and $k = 4$. In this case, $m = 3k + 1$, so $q = 3$ and $\alpha = 1$. Thus, our valid words $\mathbf{w} = (w_1, w_2, \ldots, w_d)$ of length d are elements of $\mathbb{Z}_{13}^d$, with the following forbidden subwords:

- Any subword containing a letter among $\{5, 6, 7, \ldots, 11\}$.
- 0 occurring after w_1,
- $-$ occurring at w_1,
- $14, 24, 34, 44$ anywhere in $\mathbf{w}$, and
- $0-$ anywhere in $\mathbf{w}$.

For example, there are 20 valid words of length 2, and we may compute the 80 valid words of length 3:

011	012	013	01−	021	022	023	02−	031	032	033	03−	041	042	043	04−
111	112	113	11−	121	122	123	12−	131	132	133	13−	1−1	1−2	1−3	1−4
211	212	213	21−	221	222	223	22−	231	232	233	23−	2−1	2−2	2−3	2−4
311	312	313	31−	321	322	323	32−	331	332	333	33−	3−1	3−2	3−3	3−4
411	412	413	41−	421	422	423	42−	431	432	433	43−	4−1	4−2	4−3	4−4

Now, consider a free module F_3', generated by basis elements $F_3' = \langle e_{\mathbf{w}} : \mathbf{w} \in W_3 \rangle$. In the resolutions that follow, we will grade each basis element by setting

$$\deg(e_{\mathbf{w}}) = a_{w_1} + a_{w_2} + a_{w_3},$$

similar to the situation for the infinite Apéry resolution. Hence, each valid word corresponds to a graded degree; in particular, *different* valid words of each graded degree will reflect the additive structure of the semigroup. As an example,

$$\deg(e_{21-}) = a_2 + a_1 + a_{-1} = 80 = a_0 + a_3 + a_{-1} = \deg(e_{03-}),$$

which, after canceling the a_{-1}, gives the relation $a_2 + a_1 = 29 = a_0 + a_3$, which by Example 2.13, represents something pertaining to the outer Betti element corresponding to trades of $29 \in S$.

Example 3.4. For larger values of α, the number of valid words decreases, with the effect being much larger on the number of longer words. Take $S = \langle 11, 12, 13, 14, 15 \rangle$, so $m = 11$ and $k = 4$. Here, $\alpha = 3$, and thus our valid words $\mathbf{w} = (w_1, w_2, \ldots, w_d)$ of length d are elements of $\mathbb{Z}_{11}^d$, with the following forbidden subwords:

- Any subword containing a letter among $\{5, 6, 7, 8, 9\}$.
- 0 occurring after w_1,
- $-$ occurring at w_1,
- $14, 24, 34, 44$ anywhere in $\mathbf{w}$, and
- $0-, 1-, 2-$ anywhere in $\mathbf{w}$.

Notice that when compared to the previous example, the list of forbidden rules is nearly the same, except that we now eliminate the two substrings $1-$ and $2-$, both of length 2. We have 18 valid words of length 2, which is a small decrease from the $k = 4$, $\alpha = 1$ case in the previous example, but this non-existence of certain length 2 (otherwise) valid words (in the case $\alpha = 1$) has a propagating effect on the length 3 valid words:

011	012	013	021	022	023	031	032	033	03−	041	042	043	04−
111	112	113	121	122	123	131	132	133	13−				
211	212	213	221	222	223	231	232	233	23−				
311	312	313	321	322	323	331	332	333	33−	3−1	3−2	3−3	3−4
411	412	413	421	422	423	431	432	433	43−	4−1	4−2	4−3	4−4

In general, counting valid words is a routine combinatorial task:

Proposition 3.5. Fix $m, k \geq 2$ and a corresponding $1 \leq \alpha < k$, and let $\beta_d = |W_d|$. Then $\beta_0 = 1$, $\beta_1 = k + 1$, and $\beta_d = k\beta_{d-1} - (\alpha - 1)\beta_{d-2}$ for all $d \geq 2$.

Proof. The empty word $\varnothing$ is the unique valid word of length 0, and $W_1 = \{0, 1, \ldots, k\}$. Now, we proceed by induction on d, and it is not too hard to check $\beta_2 = k(k+1) - (\alpha - 1)$. Suppose $\beta_d = k\beta_{d-1} - (\alpha - 1)\beta_{d-2}$ holds for some $d \geq 2$. Let $w = (w_1, \ldots, w_d)$ be a valid word of length d, and consider the concatenation $wr = (w_1, \ldots, w_d, r)$ for some $r \in \{0, 1, 2, \ldots, k, -\alpha\}$. If $w_d = -\alpha$, observe that any choice of $r \neq -\alpha$ gives a valid word. Otherwise, if $w_d \neq -\alpha$, observe that any choice of $r \neq k$ gives a word satisfying all conditions of being a valid word, except possibly condition (iv) given in Definition 3.2 — which is failed if and only if $w_d \in \{0, 1, 2, \ldots, \alpha - 1\}$. Now, careful counting gives the recurrence relation by induction. ☐

In particular, we stress that when $m \equiv 1 \pmod{k}$, the second-order linear recurrence given in the proposition above becomes first-order, (as $\alpha - 1 = 0$), from which the number of non-empty valid words grows strictly exponentially as a function of length.

3.3. **Modules and Maps.** In what follows, for all $d \geq 0$, the free R-module F'_d has the basis $\{e_{\mathbf{w}} : \mathbf{w} \in W_d\}$, again with the grading $\deg(e_{\mathbf{w}}) = \sum_{i=1}^{d} \deg(x_{w_i}) = \sum_{i=1}^{d} a_{w_i}$.

Definition 3.6. Given a term $p_{\mathbf{w}} e_{\mathbf{w}} \in F'_d$ with $p_{\mathbf{w}} \in R$, the *class* of $p_{\mathbf{w}} e_{\mathbf{w}}$, denoted $C(p_{\mathbf{w}} e_{\mathbf{w}})$, is the equivalence class of $\deg(p_{\mathbf{w}})$ modulo m. We call a term $p_{\mathbf{w}} e_{\mathbf{w}}$ *reduced* if

- (i) $C(p_{\mathbf{w}} e_{\mathbf{w}}) \in \{0, k, 2k, 3k, \ldots, (q-1)k, qk\}$;
- (ii) if $C(p_{\mathbf{w}} e_{\mathbf{w}}) \neq 0$, then $w_d \notin \{0, m-\alpha\}$, and
- (iii) if $C(p_{\mathbf{w}} e_{\mathbf{w}}) = m - \alpha$, then $w_d \in \{1, 2, \ldots, \alpha - 1\}$.

An element $f = \sum_{\mathbf{w} \in W_d} p_{\mathbf{w}} e_{\mathbf{w}} \in F'_d$ is *reduced* if all of its terms $p_{\mathbf{w}} e_{\mathbf{w}}$ are reduced.

Lemma 3.7. Suppose $\partial' : F'_d \to F'_{d-1}$ is an R-linear map such that for all $\mathbf{w} \in W_d$,

- (i) the term $x_{w_d} e_{\widehat{\mathbf{w}}}$ appears in $\partial' e_{\mathbf{w}}$, and
- (ii) each term of $\partial' e_{\mathbf{w}}$ not involving $e_{\widehat{\mathbf{w}}}$ is reduced.

Then for each $f \in F'_{d-1}$, there exists some reduced $g \in F'_d$ such that $f - \partial' g$ is reduced, and if $h \in \ker \partial'$ is reduced, then $h = 0$.

Proof. Consider a term $p_{\mathbf{w}} e_{\mathbf{w}}$ of f with class not among $\{0, k, 2k, 3k, \ldots, m-\alpha\}$, so that $p_{\mathbf{w}} e_{\mathbf{w}} = \ell y^b x_k^c x_r e_{\mathbf{w}}$ for $\ell \in \mathbb{K}$, $b, c \geq 0$, and $r \in \{1, 2, 3, \ldots, k-1\}$. Then $\mathbf{w}r$ is a valid word, so by hypothesis on ∂', replacing f by $f - \partial'(\ell y^b x_k^c e_{\mathbf{w}r})$ results in an expression with one less term of this type. Repeating this process finitely many times, we may assume that f only has terms with classes among $\{0, k, 2k, 3k, \ldots, qk\}$.

Let $p_{\mathbf{w}} e_{\mathbf{w}}$ be a term of f with class among $\{k, 2k, \ldots, (q-1)k\}$ where $w_d \in \{0, m-\alpha\}$, so $p_{\mathbf{w}} = \ell y^b x_k^c$ for $1 \leq c \leq q-1$. Replacing f by $f - \partial'(\ell y^b x_k^{c-1} e_{\mathbf{w}k})$ results in an expression with one less term of this type. Repeating this process finitely often, we may assume that f does not have any terms of this type.

Next, consider a term $p_{\mathbf{w}} e_{\mathbf{w}}$ of f with class $qk = m - \alpha$, so that $p_{\mathbf{w}} e_{\mathbf{w}} = y^b x_k^q e_{\mathbf{w}}$. If $w_d \in \{1, 2, \ldots, \alpha - 1\}$, then we are done as this term is already reduced; otherwise, replace f with $f - \partial' y^b e_{\mathbf{w}(m-\alpha)}$. The result is now that f is reduced, and we only applied ∂' to reduced elements, which proves the first claim in the lemma.

For the second claim in the lemma, suppose $h \in \ker \partial'$ is reduced, and write $h = h_1 + h_2 + \cdots + h_n \in \ker \partial'$, where each h_i is a term. We first claim that h only has terms of class 0. Say that h has some term $p_{\mathbf{w}} e_{\mathbf{w}}$ of nonzero class. Since h is reduced, write $p_{\mathbf{w}} e_{\mathbf{w}} = \ell y^b x_k^c e_{\mathbf{w}}$ for $b \geq 0$, $0 < c < q$, and $w_d \notin \{0, m-\alpha\}$. Now, $\partial'(p_{\mathbf{w}} e_{\mathbf{w}})$ must contain the term $\ell y^b x_k^c x_{w_d} e_{\widehat{\mathbf{w}}}$; since $h \in \ker \partial'$, there must exist some other term $-p_{\mathbf{w}'} e_{\mathbf{w}'} = -\ell' y^{b'} x_k^{c'} e_{\mathbf{w}'}$ of h such that either $\ell y^b x_k^c x_{w_d} e_{\widehat{\mathbf{w}}}$ cancels with the "leading term" $-\ell' y^{b'} x_k^{c'} x_{w'_d} e_{\widehat{\mathbf{w}'}}$ of $\partial'(-p_{\mathbf{w}'} e_{\mathbf{w}'})$, or some reduced term of $\partial'(-p_{\mathbf{w}'} e_{\mathbf{w}'})$ multiplied by $-\ell' y^{b'} x_k^{c'}$; we consider these cases separately (without regard to ℓ).

Case I: $y^b x_k^c x_{w_d} e_{\widehat{\mathbf{w}}} = y^{b'} x_k^{c'} x_{w'_d} e_{\widehat{\mathbf{w}'}}$. When $w_d \notin \{0, k, m-\alpha\}$, the equality $\mathbf{w} = \mathbf{w}'$ follows immediately, so $p_{\mathbf{w}} e_{\mathbf{w}} = p_{\mathbf{w}'} e_{\mathbf{w}'}$. When $w_d = 0$, then $d = 1$ and $\mathbf{w} = 0$, from which the same claim follows easily. Now, if $w_d = m - \alpha$, then this forces $w'_d = k$ or $w'_d = m - \alpha$. In the latter case, the claim follows; in the former, note $w_{d-1} \in \{\alpha, \alpha+1, \ldots, k\} = w'_{d-1}$, which contradicts the assumption that $\mathbf{w}'$ is a valid word. A completely symmetric argument follows for the case where $w_d = k$.

Case II: $y^b x_k^c x_{w_d} e_{\widehat{\mathbf{w}}}$ cancels with a reduced term g of $\partial'(-e_{\mathbf{w}'})$ multiplied by $y^{b'} x_k^{c'}$. Write $g = y^s x_k^j e_{\mathbf{v}}$, where $0 \le j \le q$, so that

$$y^b x_k^c x_{w_d} e_{\widehat{\mathbf{w}}} = y^{b'+s} x_k^{c'+j} e_{\mathbf{v}}$$

Notice that when $w_d \notin \{0, k, m-\alpha\}$, this case cannot happen by comparing the classes of the terms, and when $w_d = 0$, a straightforward verification shows that this also cannot happen. When $w_d = m-\alpha$, notice that $y^b x_k^c x_{w_d} e_{\widehat{\mathbf{w}}}$ has class $ck+m-\alpha = ck-\alpha$; since $c > 0$ (by class nonzero) and $\alpha \neq 0$ we have $ck - \alpha \notin \{k, 2k, \ldots, qk\}$, contradicting reducedness.

Finally, when $w_d = k$, we are forced to have $c \le q-1$ by reducedness. Because the term involving $e_{\mathbf{v}}$ is reduced, we have $v_{d-1} \notin \{0, -\alpha\}$. But this forces $v_{d-1} = w_{d-1} \in \{1, 2, \ldots, k\}$, contradicting the validity of $\mathbf{w}$, so Case II is impossible.

Hence, f consists only of terms of class zero. Take a term $y^b e_{\mathbf{w}}$ of f, so that the "leading" term of $\partial' y^b e_{\mathbf{w}}$ is $y^b x_{w_d} e_{\widehat{\mathbf{w}}}$. Again, the term $-y^b x_{w_d} e_{\widehat{\mathbf{w}}}$ must appear in $\partial(f)$ to cause cancellation, which can happen in two ways: given a term $y^{b'} e_{\widehat{\mathbf{w}'}}$ of f, we again consider two cases:

Case I′: $y^b x_{w_d} e_{\widehat{\mathbf{w}}} = y^{b'} x_{w'_d} e_{\widehat{w}'}$. Here, the same argument as Case I from before shows that $y^b e_{\mathbf{w}} = y^{b'} e_{\widehat{\mathbf{w}'}}$.

Case II′: $y^b x_{w_d} e_{\widehat{\mathbf{w}}}$ cancels with a reduced term g of $\partial'(-e_{\widehat{\mathbf{w}'}})$, multiplied by $y^{b'}$. In this case, write $g = y^s x_k^j e_{\mathbf{v}}$, where $0 \le j \le q$, so that $y^b x_{w_d} e_{\widehat{\mathbf{w}}} = y^{b'+s} x_k^j e_{\mathbf{v}}$. When $w_d \notin \{0, k, m-\alpha\}$, this case cannot happen by comparing classes of the terms; similarly, when $w_d = 0$, it is easy to show that this case cannot happen. When $w_d = m - \alpha$, we have $j = q$; by reducedness of g we have a similar contradiction to Case II above. Finally, when $w_d = k$, we have $j = 1$, but the same contradiction as Case II follows. This establishes the second claim of the lemma. □

Theorem 3.8. Given a numerical semigroup $S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta \rangle$ with $k \nmid m$, under the above definitions, applying a chain map to the infinite Apéry resolution of $\mathbb{K}$ over $\mathbb{K}[y, x_1, x_2, \ldots, x_{m-1}]/J_S \cong \mathbb{K}[y, x_1, x_2, \ldots, x_k]/I_S = R$, the numerical semigroup algebra of S, yields a free resolution

$$\mathcal{F}'_\bullet : 0 \longleftarrow \mathbb{K} \longleftarrow R \longleftarrow F'_1 \longleftarrow F'_2 \longleftarrow F'_3 \longleftarrow \cdots$$

of $\mathbb{K}$ over R.

Proof. Recall that we aim to construct the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & \mathbb{K} & \xleftarrow{\partial_0} & R & \xleftarrow{\partial_1} & F_1 & \xleftarrow{\partial_2} & F_2 & \longleftarrow & \cdots \\
 & & & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \\
0 & \longleftarrow & \mathbb{K} & \xleftarrow{\partial'_0} & R & \xleftarrow{\partial'_1} & F'_1 & \xleftarrow{\partial'_2} & F'_2 & \longleftarrow & \cdots
\end{array}$$

First, define ∂'_0 to again be the usual quotient map, surjecting $\mathbb{K}[y, x_1, \ldots, x_k]/I_S$ onto $\mathbb{K}$, and define p_0 to be the canonical isomorphism of $\mathbb{K}[y, x_1, \ldots, x_{m-1}]/J_S \cong R$ with $R = \mathbb{K}[y, x_1, \ldots, x_k]/I_S$. Similarly to the Apéry resolution, define the R-linear map ∂'_1 by $\partial'_1(e_{\mathbf{w}}) = x_{w_1}$ for all $\mathbf{w} = (w_1) \in W_1$. It is clear that the image of ∂'_1 in R is the ideal $(y, x_1, x_2, \ldots, x_k)$; thus, the bottom row is exact at $R = F'_0$. Further, we define $p_1 : F_1 \to F'_1$ by

$$p_1(e_i) = \begin{cases} e_i, & i \in \{0, 1, 2, \ldots, k\} \\ x_k^c e_r & \text{if } i > k, \text{ where } i = qc + r \text{ with } 0 \le r < k, \end{cases}$$

wherein we might allow $r = k$ in the case that $i = m - \alpha$ (implying an R-coefficient of x_k^{q-1}) to ensure that p_1 outputs strictly reduced terms. Note that our definition of p_1 implies $p_0(\partial_1(e_i)) = \partial_1'(p_1(e_i))$ for all $e_i \in F_1$; indeed, if $i \in \{0, 1, \ldots, k\}$, then both $p_0\partial_1$ and $\partial_1' p_1$ map e_i to x_i, and if otherwise, writing $i = kq' + r$ for $q' \in \{1, 2, \ldots, q\}$ and $r \in \{0, 1, 2, \ldots, k\}$, then $\partial_1'(p_1(e_i)) = x_k^{q'} x_r$, the isomorphic copy of $\partial_1(e_i) = x_i$ in $R = \mathbb{K}[y, x_1, \ldots, x_k]/I_S$. This shows that the leftmost square in our diagram commutes.

Next, define $\partial_2' : F_2' \to F_1'$ by, for all $\mathbf{w} = (w_1, w_2) \in W_2$,

$$\partial_2'(e_{\mathbf{w}}) = x_{w_2} e_{w_1} - y^{b_{w_1, w_2}} p_1 e_{w_1 + w_2}.$$

Note that by our construction and the definition of ∂_2 in the Apéry resolution, $p_1\partial_2(e_{\mathbf{w}}) = \partial_2'(e_{\mathbf{w}})$ for all $\mathbf{w} \in W_2$. Thus, commutativity of the leftmost square and the exactness of the top row show clearly that $\text{im}(\partial_2') \subseteq \ker(\partial_1')$. To see that $\ker(\partial_1') \subseteq \ker(\partial_2')$, note that both ∂_1' and ∂_2' satisfy the conditions of Lemma 3.7, so for all $f \in \ker(\partial_1')$, $f - \partial_2' g$ is reduced and in $\ker(\partial_1')$ by our earlier claim that $\partial_2' \subseteq \ker(\partial_1')$ and R-linearity of ∂_1'. Thus, we have $f - \partial_2' g = 0$ and hence $f \in \text{im}(\partial_2')$. Thus, $\text{im}(\partial_2') = \ker(\partial_1')$ and the bottom row is exact at F_1'. Careful inspection reveals that the images of basis vectors in F_2' under ∂_2' precisely encode minimal trades between monomials that generate the toric ideal I_S and ones of basic commutativity of the variables $y, x_1, \ldots, x_k$.

We define the rest of the chain map and resolution inductively: let $n \geq 2$, and assume that

- (i) ∂_d' is defined and degree-preserving (homogeneous) for all $d \in \{0, 1, \ldots, n\}$ and $\text{im}(\partial_d') = \ker(\partial_{d-1}')$ for all $d \in \{1, \ldots, n\}$,
- (ii) the term $x_{w_d} e_{\widehat{\mathbf{w}}}$ appears in $\partial_d'(e_{\mathbf{w}})$, and each term of $\partial_d'(e_{\mathbf{w}})$ not involving $e_{\mathbf{w}}$ is reduced for all $e_{\mathbf{w}} \in F_d'$, for all $d \in \{1, 2, \ldots, n\}$,
- (iii) p_d is defined for all $d \in \{0, 1, \ldots, n-1\}$ and $p_{d-1}\partial_d = \partial_d' p_d$ for all $d \in \{1, \ldots, n-1\}$,
- (iv) $p_d \partial_{d+1}(e_{\mathbf{w}}) = \partial_{d+1}'(e_{\mathbf{w}})$ for all $\mathbf{w} \in W_{d+1}$, for all $d \in \{0, 1, \ldots, n-1\}$, and
- (v) the maps p_d are homogeneous and $p_d(e_{\mathbf{w}})$ is reduced for all $e_{\mathbf{w}} \in F_d$, for all $d \in \{0, 1, \ldots, n-1\}$.

We define p_n and ∂_{n+1}' in such a way that the diagram again commutes and the bottom row is exact at F_n'. We first define p_n. Given $e_{\mathbf{w}} \in F_n$, we have $\partial_{n-1}\partial_n(e_{\mathbf{w}}) = 0$ by exactness of the top row; thus, $p_{n-2}\partial_{n-1}\partial_n(e_{\mathbf{w}}) = 0$, so by part (iii) of the inductive hypothesis, $\partial_{n-1}' p_{n-1}\partial_n(e_{\mathbf{w}}) = 0$. Thus, $p_{n-1}\partial_n(e_{\mathbf{w}}) \in \ker(\partial_{n-1}')$. By exactness of the bottom row at F_{n-1}' (part (i) of the inductive hypothesis), there exists $a_{\mathbf{w}'} \in F_n'$ such that $\partial_n'(a_{\mathbf{w}'}) = p_{n-1}\partial_n(e_{\mathbf{w}})$. Thus, set $p_n(e_{\mathbf{w}}) = a_{\mathbf{w}'}$. If $\mathbf{w} \in W_n$, let $a_{\mathbf{w}'} = e_{\mathbf{w}}$, which we may do by part (iv) of the inductive hypothesis, and if $\mathbf{w} \notin W_n$, let $p_n(e_{\mathbf{w}}) = a_{\mathbf{w}'}$ be reduced, which we may do by part (ii) of the inductive hypothesis and Lemma 3.7. Thus, we have defined p_n, and note that our construction forces $p_{n-1}\partial_n = \partial_n' p_n$; thus, p_n is homogeneous. We now define ∂_{n+1}' in the natural way: for all $e_{\mathbf{w}} \in F_{n+1}'$, define

$$\partial_{n+1}'(e_{\mathbf{w}}) = x_{w_{n+1}} p_n(e_{\widehat{\mathbf{w}}}) + \sum_{i=1}^{n} (-1)^{n+1-i} y^{b_{w_i, w_{i+1}}} p_n(e_{\tau_i \mathbf{w}}).$$

Thus, by definition of ∂_{n+1} in the Apéry resolution and since $\widehat{\mathbf{w}} \in W_n$ if $\mathbf{w} \in W_{n+1}$, we again have $\partial_{n+1}'(e_{\mathbf{w}}) = p_n\partial_{n+1}(e_{\mathbf{w}})$ for all $\mathbf{w} \in W_{n+1}$ (implying homogeneity again), and we leverage this fact to show that $\text{im}(\partial_{n+1}') \subseteq \ker(\partial_n')$.

Let $e_{\mathbf{w}} \in F'_{n+1}$. Then $\partial'_n \partial'_{n+1}(e_{\mathbf{w}}) = \partial'_n p_n \partial_{n+1}(e_{\mathbf{w}})$ by the above claim, and since we have shown that $p_{n-1}\partial_n = \partial'_n p_n$, we have $\partial'_n \partial'_{n+1}(e_{\mathbf{w}}) = p_{n-1}\partial_n \partial_{n+1}(e_{\mathbf{w}}) = p_{n-1}(0) = 0$ by exactness of the Apéry resolution in the top row. Thus, $\partial'_n \partial'_{n+1}(e_{\mathbf{w}}) = 0$ for all $e_{\mathbf{w}} \in F'_{n+1}$, which shows that $\operatorname{im}(\partial'_{n+1}) \subseteq \ker(\partial'_n)$. This can be seen by chasing the diagram below.

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & F_{n-1} & \xleftarrow{\partial_n} & F_n & \xleftarrow{\partial_{n+1}} & F_{n+1} & \longleftarrow & \cdots \\
 & & \big\downarrow{\scriptstyle p_{n-1}} & & \big\downarrow{\scriptstyle p_n} & & & & \\
\cdots & \longleftarrow & F'_{n-1} & \xleftarrow[\partial'_n]{} & F'_n & \xleftarrow[\partial'_{n+1}]{} & F_{n+1} & \longleftarrow & \cdots
\end{array}$$

Now, we show $\ker(\partial'_n) \subseteq \operatorname{im}(\partial'_{n+1})$. Note that if $\mathbf{w} \in W_{n+1}$, then $\widehat{\mathbf{w}} \in W_n$, so by our construction of both p_n and ∂'_{n+1}, the term $x_{w_{n+1}} e_{\widehat{\mathbf{w}}}$ appears in $\partial'_{n+1}(e_{\mathbf{w}})$, and each term of $\partial'_{n+1}(e_{\mathbf{w}})$ not involving $e_{\widehat{\mathbf{w}}}$ is reduced. Thus, by Lemma (3.7), for all $f \in \ker(\partial'_n)$, we have $f - \partial'_{n+1} g$ is reduced for some $g \in F'_{n+1}$. Thus, $\partial'_n(f - \partial'_{n+1}g) = \partial'_n f - \partial'_n \partial'_{n+1} g = 0$ by our earlier claim that $\operatorname{im}(\partial'_{n+1}) \subseteq \ker(\partial'_n)$. Thus, $f - \partial'_{n+1}g \in \ker(\partial'_n)$ and $f - \partial'_{n+1}g$ is reduced, implying $f = \partial'_{n+1}g$ by Lemma 3.7, which shows that $f \in \operatorname{im}(\partial'_{n+1})$. Thus, $\operatorname{im}(\partial'_{n+1}) = \ker(\partial'_n)$. This completes the inductive step. Thus, the bottom row of our desired commutative diagram is a free resolution of $\mathbb{K}$ over R. □

Conjecture 3.9. The above resolution of $\mathbb{K}$ over R is minimal.

Recall that the *Poincaré series* of R is the formal power series

$$P^R_{\mathbb{K}}(z) = \sum_{d=0}^{\infty} \beta_d z^d,$$

where β_d are the Betti numbers of the minimal free resolution of $\mathbb{K}$ over R. Assuming the conjecture above, we record our Betti numbers for the resolution given in Theorem 3.8 in a clean way:

Corollary 3.10. Let $S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta \rangle$ be a numerical semigroup with $k \nmid m$. If Conjecture 3.9 holds, then the Poincaré series of the semigroup algebra R is

$$P^R_{\mathbb{K}}(z) = \frac{1+z}{1-kz+(\alpha-1)z^2}.$$

Proof. We have already established a minimal free resolution when $\delta \equiv 1 \pmod{m}$. Now, for a general δ with $\gcd(\delta, m) = 1$, Proposition 2.27 tells us that the equivalence class of δ modulo m induces an $\mathbb{Z}_m^\times$ automorphism on the Kunz poset of S. Applying this automorphism to the minimal free resolution obviously does not change the degrees of the free modules, so the result follows after noticing that $|W_d| = \beta_d$, where β_d are the Betti numbers for $\mathbb{K}$ over R. Finally, apply Proposition 3.5 to finish the proof, noting that the generating function of the recursion given in that proposition is exactly the Poincaré series claimed here. □

The careful reader will notice that the assumption $\alpha \neq 0$ (equivalently, $k \nmid m$) is only used **twice** throughout this entire proof — namely, in Case II of the proof of Lemma 3.7, where we needed to conclude $ck - \alpha \notin \{k, 2k, \ldots, qk\}$ to derive a contradiction and in small considerations of the minimality of our constructed resolution. The usage of this condition at these specific stages in the proof is quite strange, and seemingly arbitrary. In fact, when $k \mid m$, we have the following conjecture:

Conjecture 3.11. Let $S = \langle m, mh+\delta, mh+2\delta, \dots, mh+k\delta\rangle$ be a numerical semigroup with $k \mid m$. Then the Poincaré series of the semigroup algebra R is

$$P^R_{\mathbb{K}}(z) = \frac{1+z}{1-kz+(k-1)z^2}.$$

In actuality, this is an extension of the corollary, as in this case $m \equiv 0 \equiv k \pmod{k}$, so we simply take the "shifted" remainder $\alpha = k$.

Example 3.12. Let $S = \langle 10, 11, 12, 13, 14\rangle$, so $m = 10$, $k = 4$, and $\alpha = 2$. Though Theorem 2.26 does not give an explicit expression for the maps $\partial'_d : F'_d \to F'_{d-1}$, but we can still construct the first two matrices from scratch. Theorem 3.8 gives the free resolution

$$\mathcal{F}'_\bullet : 0 \longleftarrow \mathbb{K} \longleftarrow R \xleftarrow{\begin{bmatrix} y & x_1 & x_2 & x_3 & x_4\end{bmatrix}} R^5 \xleftarrow{\partial'_2} R^{19} \xleftarrow{\partial'_3} R^{71} \longleftarrow \cdots,$$

where the matrix for $\partial'_2 : R^{19} \to R^5$ is given by

$$\begin{array}{c|ccccccccccccccccccc}
 & 01 & 02 & 03 & 04 & 11 & 12 & 13 & 21 & 22 & 23 & 2- & 31 & 32 & 33 & 3- & 41 & 42 & 43 & 4- \\
\hline
0 & x_1 & x_2 & x_3 & x_4 & & & & & & & -y^4 & & & & & & & & \\
1 & -y & & & & x_1 & x_2 & x_3 & & & -x_4 & & & -x_4 & & -y^3 & -x_4 & & & \\
2 & & -y & & & -y & & & x_1 & x_2 & x_3 & x_4^2 & & & -x_4 & & & -x_4 & & -y^3 \\
3 & & & -y & & & -y & & -y & & & & x_1 & x_2 & x_3 & x_4^2 & & & -x_4 & \\
4 & & & & -y & & & -y & & -y & & & -y & & & & x_1 & x_2 & x_3 & x_4^2
\end{array}$$

Observe that our 19 columns are indexed by precisely the elements of W_2; in particular, the $1*$ "block" (with the asterisk denoting a letter among $\{0, 1, 2, 3, 4, -1\}$) is three columns wide, instead of four (as with the $0*, 2*, 3*$, and $4*$ blocks), as the word $1-$ is forbidden.

Remark 3.13. Let $S = \langle m, mh+\delta, mh+2\delta, \dots, mh+k\delta\rangle$ be a numerical semigroup with $k \nmid m$. As usual, let $\alpha \in \{1, \dots, k-1\}$ denote the remainder of m when divided by k. Then generalizing the previous example, the existence of the term $x_{w_d} e_{\widehat{\mathbf{w}}}$ in the map definition for $\partial'_d : F'_d \to F'_{d-1}$ guarantees a block form specified as follows: only the three following row matrices can fill the $(\mathbf{w}, \mathbf{w}*)$-entry of the block matrix (here, the columns are blocked, as represented by the asterisk):

(*i*) If $w_{d-1} = -\alpha$, then $\widetilde{X} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k\end{bmatrix}$ appears.

(*ii*) Else, if $w_{d-1} \in \{1, 2, \dots, \alpha-1\}$, then $\widehat{X} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{k-1}\end{bmatrix}$ appears. Note that if $\alpha = 1$, then $\widehat{X}$ never appears down a block diagonal.

(*iii*) Otherwise, $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k^q\end{bmatrix}$ appears, where $q = \frac{1}{k}(m-\alpha)$.

Again, the behavior of the other entries varies, based on the maps $p_d : F_d \to F'_d$.

3.4. **One Constructive Case: A Complete Top Row.** We remark that the proof of Theorem 3.8 is somewhat nonconstructive, in that the "replacement" maps $p_d : F_d \to F'_d$ are not explicitly defined. However, in the case where $S = \langle m, mh+\delta, mh+2\delta, \dots, mh+k\delta\rangle$ with $m \equiv 1 \pmod{k}$, we have explicit definitions of the maps p_d. Notice that this is precisely the case where the Kunz poset of S has a complete top row, as per our comment after Theorem 2.26.

Throughout (except where specified), take $\alpha = 1$, W_d as the set of valid words of length d corresponding to $\alpha = 1$, F_d to be the free modules in the Apéry resolution of S, and F'_d as defined in the previous subsection.

Corollary 3.14. Let $S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta\rangle$ be an extra-generalized arithmetical numerical semigroup with $\delta \equiv 1 \pmod{m}$ and $m \equiv 1 \pmod{k}$. Then a minimal infinite free resolution of the ground field $\mathbb{K}$ over the semigroup algebra R is

$$\mathcal{F}'_\bullet : 0 \longleftarrow \mathbb{K} \longleftarrow F'_0 \longleftarrow F'_1 \longleftarrow F'_2 \longleftarrow F'_3 \longleftarrow \cdots,$$

where the free modules F'_d are defined by $F'_d = \langle e_{\mathbf{w}} : \mathbf{w} \in W_d\rangle$, and each boundary map $\partial'_d : F'_d \to F_{d-1}$ is given by

$$\partial'_d(e_{\mathbf{w}}) = x_{w_{d+1}} p_{d-1}(e_{\widehat{\mathbf{w}}}) + \sum_{i=1}^{d-1} (-1)^{d-i} y^{b_{w_i,w_{i+1}}} p_{d-1}(e_{\tau_i \mathbf{w}}),$$

with $\widehat{\mathbf{w}}$, $\tau_i\mathbf{w}$, and $b_{w_i,w_{i+1}}$ are as defined in the Apéry resolution. The maps $p_d : F_d \to F'_d$ are given by $p_d(e_{\mathbf{w}}) := e_{\mathbf{w}}$ for all $\mathbf{w} \in W_d$. In the case that $\mathbf{w}$ contains exactly one letter w_i with $w_i \in \{k+1, k+2, \ldots, 2k-1\}$, with $w_j \in \{0, 1, 2, \ldots, k, -1\}$ for all $j \neq i$, we define

$$\widetilde{\mathbf{w}} := \begin{cases} (w_1, w_2, \ldots, w_i - k, w_{i+1}, w_d) & \text{if } w_j \neq -1 \text{ for all } j > i \\ (w_1, w_2, \ldots, w_i - k, \ldots, w_{j_0} + k, \ldots, w_d) & \text{if } j_0 \text{ is the minimal } j > i \text{ with } w_j = -1, \end{cases}$$

so that

$$p_d(e_{\mathbf{w}}) := \begin{cases} x_k e_{\widetilde{\mathbf{w}}} & \text{if } w_j \neq -1 \text{ for all } j > i \\ y^{qh+\delta} e_{\widetilde{\mathbf{w}}} & \text{otherwise,} \end{cases}$$

where $q = \frac{1}{k}(m-1)$.

Proof. Simply observe that this choice of p_d satisfies $p_{d-1}\partial_d = \partial'_d p_d$ and $p_d\partial_{d+1}(e_{\mathbf{w}}) = \partial'_{d+1}(e_{\mathbf{w}})$ for all $w \in W_{d+1}$. Finally, our choice of p_d restricts to the identity on F'_d, and for $\mathbf{w} \notin W_d$, $p_d(e_{\mathbf{w}})$ is clearly reduced. $\square$

Notice that we have only specified the linear maps $p_d : F_d \to F'_d$ for certain words $\mathbf{w}$; however, even with p_d partially constructed, this still gives a complex explicit description of the minimal free resolution for S, as the words $\mathbf{w}$ for which we specified p_d are exactly the ones that appear in the calculation of the maps ∂'_d.

Example 3.15. Let $S = \langle 13, 14, 15, 16, 17\rangle$, so $m = 13$ and $k = 4$. Then Corollary 3.14 gives the free resolution

$$\mathcal{F}'_\bullet : 0 \longleftarrow \mathbb{K} \longleftarrow R \xleftarrow{\begin{bmatrix} y & x_1 & x_2 & x_3 & x_4 \end{bmatrix}} R^5 \xleftarrow{\partial'_2} R^{20} \xleftarrow{\partial'_3} R^{80} \longleftarrow \cdots,$$

where the matrix for $\partial'_2 : R^{20} \to R^5$ is given by

$$\begin{array}{c|cccccccccccccccccccc}
 & 01 & 02 & 03 & 04 & 11 & 12 & 13 & 1- & 21 & 22 & 23 & 2- & 31 & 32 & 33 & 3- & 41 & 42 & 43 & 4- \\
\hline
0 & x_1 & x_2 & x_3 & x_4 & & & & -y^5 & & & & & & & & & & & & \\
1 & -y & & & & x_1 & x_2 & x_3 & x_4^3 & & & -x_4 & -y^4 & & -x_4 & & & -x_4 & & & \\
2 & & -y & & & -y & & & & x_1 & x_2 & x_3 & x_4^3 & & & -x_4 & -y^4 & & -x_4 & & \\
3 & & & -y & & & -y & & & -y & & & & x_1 & x_2 & x_3 & x_4^3 & & & -x_4 & -y^4 \\
4 & & & & -y & & & -y & & & -y & & & -y & & & & x_1 & x_2 & x_3 & x_4^3
\end{array}$$

Notice that the graded degrees corresponding to each $e_{\mathbf{w}}$, where $\mathbf{w} \in W_2$, relates to either a basic commutativity relation within the poset, a trade of factorizations within the Kunz poset, or a trade corresponding to an outer Betti element in the Kunz poset of S (Figure 2, also cf. the discussion of Example 3.3). This is listed in Table 1:

Degree n	Words $\mathbf{w} \in W_2$ with $\deg(e_{\mathbf{w}}) = n$	Type
27	01	Commutativity: $a_0 + a_1 = a_1 + a_0$
28	02	Commutativity: $a_0 + a_2 = a_2 + a_0$
29	03	Commutativity: $a_0 + a_3 = a_3 + a_0$
30	04	Commutativity: $a_0 + a_4 = a_4 + a_0$
28	11	Outer Betti: $a_1 + a_1 = 28$
29	12, 21	Outer Betti: $a_1 + a_2$
30	13, 22, 31	Outer Betti: $a_1 + a_3$ and $a_2 + a_2$
31	23, 32	Inner Trade: $a_2 + a_3 = a_1 + a_4 = 31$
32	33, 42	Inner Trade: $a_3 + a_3 = a_2 + a_4 = 32$
33	43	Commutativity: $a_3 + a_4 = a_4 + a_3$
65	1−	Outer Betti: $a_1 + a_{-1}$
66	2−	Outer Betti: $a_2 + a_{-1}$
67	3−	Outer Betti: $a_3 + a_{-1}$
68	4−	Outer Betti: $a_4 + a_{-1}$

TABLE 1. Words indexing the columns of ∂_2', $S = \langle 13, 14, 15, 16, 17\rangle$.

To demonstrate the replacement maps p_d, we compute

$$\begin{aligned}\partial_3'(e_{23-}) &= x_{-1}p_2e_{23} - y^4p_2e_{22} + y^0p_2e_{5-}\\ &= x_{-1}e_{23} - y^4e_{22} + y^0(y^4e_{13})\\ &= x_{-1}e_{23} - y^4e_{22} + y^4e_{13},\end{aligned}$$

with each term above having graded degree 82, and

$$\begin{aligned}\partial_5'(e_{42212}) &= x_2p_4e_{4221} - y^1p_4e_{4223} + y^1p_4e_{4232} - y^1p_4e_{4412} + y^0p_4e_{6212}\\ &= x_2e_{4221} - ye_{4223} + ye_{4232} - y\cdot 0 + y^0(y^1e_{6212})\\ &= x_2e_{4221} - ye_{4223} + ye_{4232} + ye_{6212},\end{aligned}$$

with each term above (including the zero term) having graded degree 76.

4. Numerical Semigroups with Small Multiplicity

The infinite resolutions of $\mathbb{K}$ over all numerical semigroup algebras with $m \leq 4$ have already been categorized in previous papers (*e.g.,* [Gom+24]) In this section we aim to categorize the infinite resolutions of all numerical semigroups with $m \in \{5, 6, 7\}$. Since semigroups with the same Kunz poset have the same resolution up to changes in the exponent of y, and Kunz posets that are permutations of each other have identical resolutions up to an automorphism of $\mathbb{Z}_m^\times$ given in Proposition 2.27, we can limit our work to one Kunz poset in each permutation orbit within the Kunz cone. Emily O'Sullivan provides the $m = 5$ Kunz cone in [OSu23], and other poset data may be readily computed. As demonstrated in the extra-generalized arithmetical case, pattern-avoiding words provide a useful combinatorial framework for understanding the maps that define free resolutions of the base field over the semigroup algebra. In this section, we provide conjectures for the valid words that define free resolutions of semigroups with small multiplicity. In particular, we consider semigroups with 5, 6, and 7 generators. For each number of generators we consider each possible Kunz poset and provide a conjecture for the set of patterns that must be avoided by its valid words. These conjectures are supported by computational data.

In the following section, the alphabet of letters in a word is $\mathbb{Z}_m$, where m is the multiplicity of the poset, and in each case, the valid words of length 1 are $\{0, 1, 2, \ldots, m-1\}$. For all conjectures, the valid words of length d correspond exactly to the graded Betti numbers of F_d via $\deg(w_1, \ldots, w_d) = \sum_{i=1}^{d} a_{w_i}$.

4.1. **Conjectured Resolutions for Multiplicity 5.** In this section, we give a short survey of all Kunz posets corresponding to faces of the Kunz cone C_5. Note that we have categorizations for four of the posets on C_5: three are the posets of extra generalized arithmetical numerical semigroups, so their resolutions are categorized earlier in this paper, and the total order Kunz poset resolution is also already known. These four posets are illustrated in Figure 3.

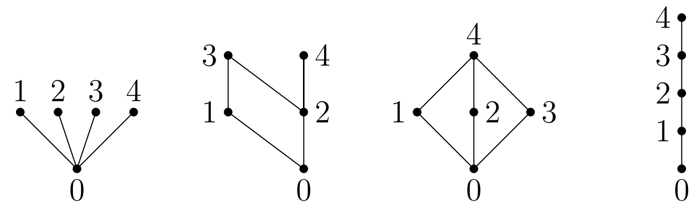


FIGURE 3. Kunz posets for $m = 5$ for which infinite free resolutions are known.

Figure 4 provides the five Kunz posets, up to an $\mathbb{Z}_5^\times$ automorphism, that do not correspond to extra-generalized arithmetical numerical semigroups.

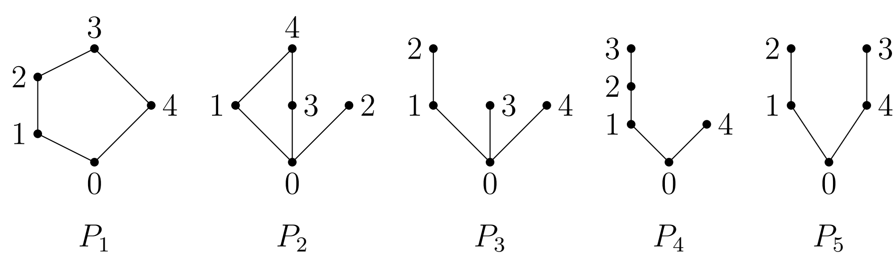


FIGURE 4. Kunz posets for $m = 5$ for which infinite free resolutions are not known.

Poset	Betti numbers	Forbidden subwords
P_1	$\beta_d = 2d + 1$	$\Phi = \{2, 3, 00, 10, 11, 40, 144\}$
P_2	$\beta_1 = 4,\ \beta_d = 3\beta_{d-1} - \beta_{d-2}$	$\Phi = \{4, 00, 10, 13, 20, 30\}$
P_3	$\beta_1 = 4,\ \beta_d = 3\beta_{d-1}$	$\Phi = \{00, 02, 10, 11, 20, 22, 30, 32, 40, 42\}$
P_4	$\beta_1 = 3,\ \beta_d = 2\beta_{d-1}$	$\Phi = \{2, 00, 03, 10, 11, 30, 33, 40, 43\}$
P_5	$\beta_1 = 3,\ \beta_d = 2\beta_{d-1}$	$\Phi = \{00, 02, 03, 10, 11, 13, 20, 22, 23, 30, 32, 33, 40, 42, 44\}$

TABLE 2. $m = 5$ Betti Number Conjectures

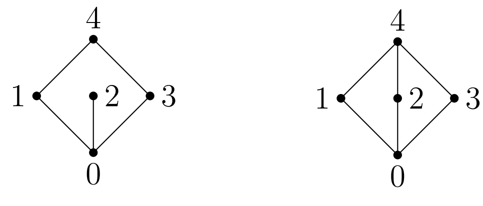
We now give a conjecture for infinite free resolutions of $\mathbb{K}$ over R, where S corresponds to the posets P_i above.

Conjecture 4.1. The valid words of the above posets are all possible words except those containing the following forbidden subwords, given in Table 2, with the beginning of conjectured minimal resolutions in Table 3.

P_1	$\overset{\cdots}{\longleftarrow} R^3 \xleftarrow{\begin{array}{c|ccccc} & 01 & 04 & 14 & 41 & 44 \\ \hline 0 & x_1 & x_4 & -y^{\bullet} & -y^{\bullet} & \\ 1 & -y & & x_4 & & -x_1^2 \\ 4 & & -y & & x_1 & x_4 \end{array}} R^5 \xleftarrow{\begin{array}{c|ccccccc} & 014 & 041 & 044 & 141 & 414 & 441 & 444 \\ \hline 01 & x_4 & & -x_1^2 & y^{\bullet} & & & -y^{\bullet}x_1 \\ 04 & & x_1 & x_4 & & y^{\bullet} & y^{\bullet} & \\ 14 & y & & & x_1 & & x_4 & \\ 41 & & y & & & x_4 & & -x_1^2 \\ 44 & & & y & & & x_1 & x_4 \end{array}} R^7 \overset{\cdots}{\longleftarrow}$
P_2	$\overset{\cdots}{\longleftarrow} R^4 \xleftarrow{\begin{array}{c|ccccccccccc} & 01 & 02 & 03 & 11 & 12 & 21 & 22 & 23 & 31 & 32 & 33 \\ \hline 0 & x_1 & x_2 & x_3 & & & & & -y^{\bullet} & & -y^{\bullet} & \\ 1 & -y & & & x_1 & x_2 & & & & x_3 & & -y^{\bullet} \\ 2 & & -y & & -y^{\bullet} & & x_1 & x_2 & x_3 & & & \\ 3 & & & -y & & -y^{\bullet} & -y^{\bullet} & -y^{\bullet}x_1 & & x_1 & x_2 & x_3 \end{array}} R^{11} \overset{\cdots}{\longleftarrow}$
P_3	$\overset{\cdots}{\longleftarrow} R^4 \xleftarrow{\begin{array}{c|cccccccccccc} & 01 & 03 & 04 & 12 & 13 & 14 & 31 & 33 & 34 & 41 & 43 & 44 \\ \hline 0 & x_1 & x_3 & x_4 & & & -y^{\bullet} & & & & -y^{\bullet} & & \\ 1 & -y & & & x_1^2 & x_3 & x_4 & & -y^{\bullet} & -y^{\bullet}x_1 & & -y^{\bullet}x_1 & \\ 3 & & -y & & -y^{\bullet} & & & x_1 & x_3 & x_4 & & & -y^{\bullet} \\ 4 & & & -y & & -y^{\bullet} & & -y^{\bullet} & & & x_1 & x_3 & x_4 \end{array}} R^{12} \overset{\cdots}{\longleftarrow}$
P_4	$0 \longleftarrow R \xleftarrow{\begin{array}{ccc} 0 & 1 & 4 \\ \hline y & x_1 & x_4 \end{array}} R^3 \xleftarrow{\begin{array}{c|cccccc} & 01 & 04 & 13 & 14 & 41 & 44 \\ \hline 0 & x_1 & x_4 & & -y^{\bullet} & -y^{\bullet} & \\ 1 & -y & & x_1^3 & x_4 & & -y^{\bullet}x_1^2 \\ 4 & & -y & -y^{\bullet} & & x_1 & x_4 \end{array}} R^6 \overset{\cdots}{\longleftarrow}$
P_5	$0 \longleftarrow R \xleftarrow{\begin{array}{ccc} 0 & 1 & 4 \\ \hline y & x_1 & x_4 \end{array}} R^3 \xleftarrow{\begin{array}{c|cccccc} & 01 & 04 & 12 & 14 & 41 & 43 \\ \hline 0 & x_1 & x_4 & & -y^{\bullet} & -y^{\bullet} & \\ 1 & -y & & x_1^2 & x_4 & & -y^{\bullet}x_4 \\ 4 & & -y & -y^{\bullet}x_1 & & x_1 & x_4^2 \end{array}} R^6 \overset{\cdots}{\longleftarrow}$

TABLE 3. $m = 5$ Resolution Conjectures

4.1.1. *Notes on Connections to Known Cases.* Consider the poset P_2 and the following poset E that corresponds to an extra-generalized arithmetical numerical semigroup.

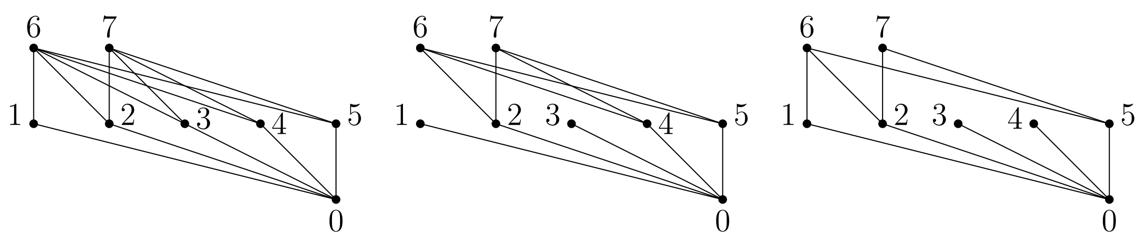


P_2 E

The conjectured resolution of $\mathbb{K}$ over the semigroup algebra described by P_2 is very similar to the resolution of $\mathbb{K}$ over the semigroup algebra described by E. In particular, the two resolutions appear to have the same ungraded Betti numbers. We suspect that they also have the same graded Betti numbers because E is on a lower dimensional subface of the face containing P_2, meaning the resolution of $\mathbb{K}$ over the semigroup algebra associated with P_2 must be a resolution of the semigroup algebra of E as well, albeit not necessarily a minimal one.

An algebraic argument for why resolutions are likely to be similar is because the inner trade $x_1x_3 = x_2^2$ in the EGANS case becomes an outer Betti element $x_2^2 = y^\bullet x_1x_3$ in P_2 case. Since outer Betti elements and inner trades play essentially the same role in the second matrix of the resolution, the second matrices of the two resolutions differ only by the power of y appearing in the corresponding column, which is 0 in the EGANS case and must be non-zero in the P_2 case.

Similar patterns seem to occur in other situations where one starts with an EGANS poset and moves to a higher dimensional face by removing inner trades. For example, the corresponding minimal resolutions of these three posets have the same Betti numbers, suggesting they have the same structure:



These observations suggest that the minimal resolutions we have constructed here corresponding to EGANS posets may also give us the corresponding resolutions of a wider set of posets. Of course, one cannot remove inner trades indefinitely without impacting the minimal resolution. In future work, we hope to examine this pattern further and determine exactly when it occurs.

4.2. **Future Work for $m = 6$.** Figure 6 shows the five posets (up to the unique non-trivial automorphism of $\mathbb{Z}_6^\times$) for which we know minimal free resolutions over their corresponding semigroup algebras — the chain has an easy resolution with $\beta_d = 2$ for all d, and the rest correspond to posets for extra-generalized arithmetical numerical semigroups.

Figure 5 contains all remaining multiplicity 6 Kunz posets corresponding (up to a $\mathbb{Z}_6^\times$ automorphism). The infinite free resolutions over their corresponding semigroup algebras are known. In future work, we aim to find subword avoidance rules that govern the infinite

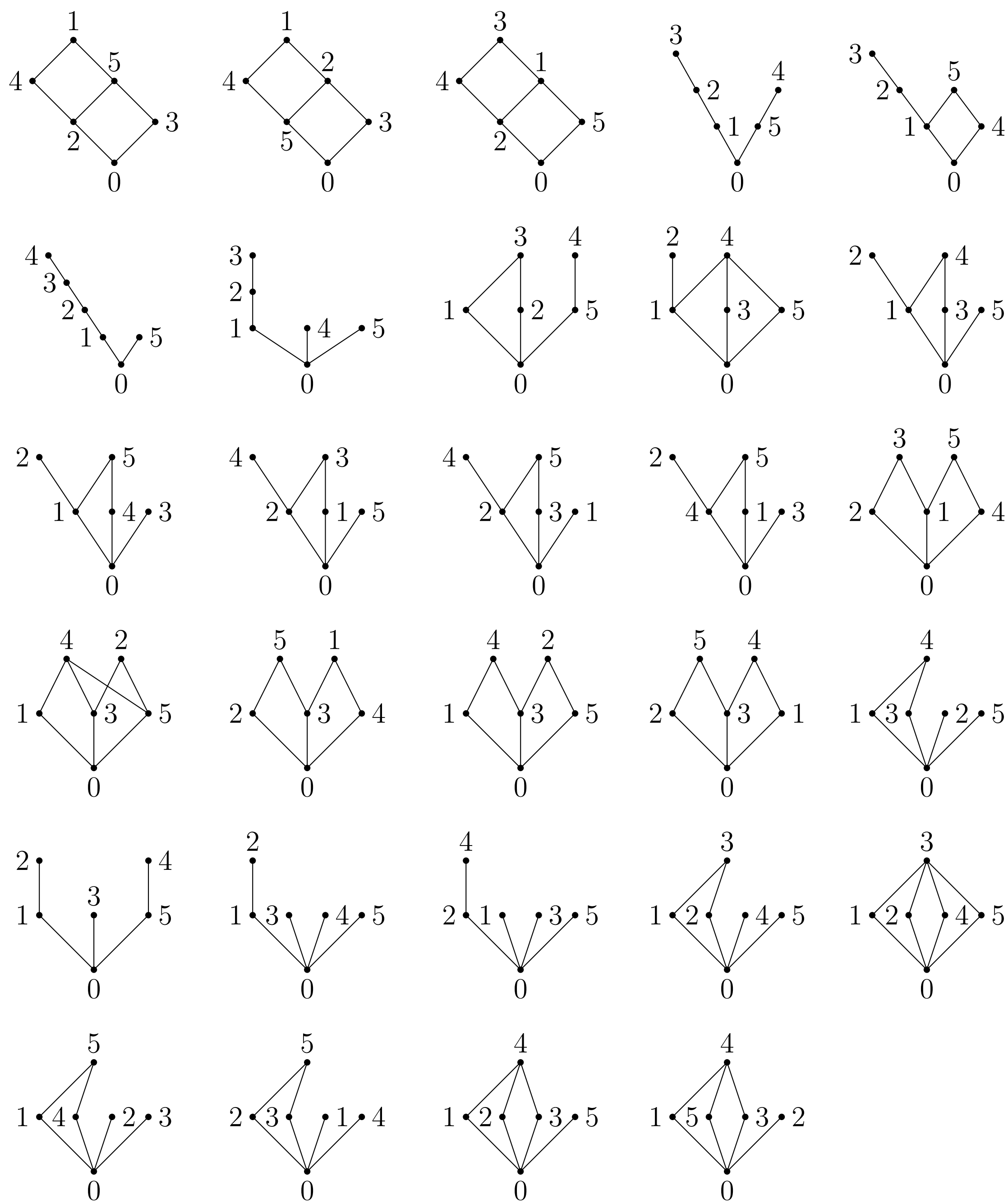


FIGURE 5. Kunz posets for $m = 6$, up to a $\mathbb{Z}_6^\times$ automorphism, for which infinite free resolutions are not known.

free resolutions of $\mathbb{K}$ over the semigroup algebras associated with the above posets. We suspect that many of the above sets of forbidden words are predictable extensions of the sets of forbidden words in corresponding $m = 4$ cases. We hope to uncover similar resolutions for embedding dimension 7.

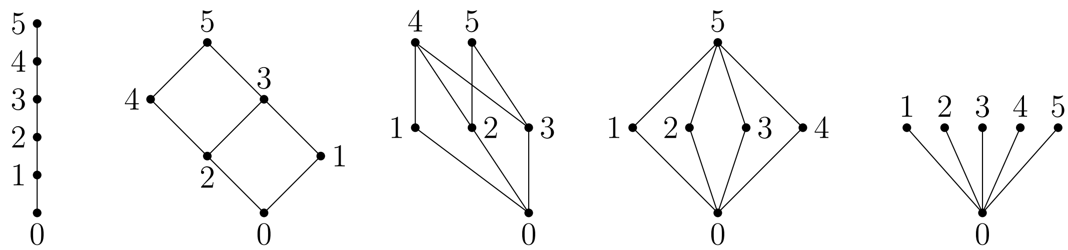


FIGURE 6. Kunz posets for $m = 6$ for which infinite free resolutions are known, up to a $\mathbb{Z}_6^\times$ automorphism.

CODE

We used Macaulay2 to generate minimal free resolutions and Betti numbers in order to formulate conjectures. The reader may find the code used at this GitHub repository:

`https://github.com/aljones3/NSFreeResolutions/tree/main`

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