MIXED DIMER MODEL FOR EXCEPTIONAL TYPE QUIVER REPRESENTATIONS AND CLUSTER ALGEBRAS

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Abstract. In recent years, dimer combinatorics has presented new techniques for modeling the Laurent expansions of cluster variables in certain classes of cluster algebras. We aim to model the exceptional finite type cluster algebras, namely types E_6, E_7, and E_8 acyclic quivers, using mixed dimer configurations. This report utilizes the theory of quiver representations and focuses most specifically on type E_6 quivers. We characterize the monomials of the F-polynomial for a large class of cluster variables, using mixed dimer configurations which we show are in bijection with certain vectors associated to a given quiver representation. We conjecture a combinatorial interpretation of the coefficients in the F-polynomial, as well as ways to extend our model to type E_7 and E_8 quivers.

Contents

1. Introduction

In recent years, dimer combinatorics has proven very useful for modeling the Laurent expansion for cluster variables, the generators of cluster algebras. This line of research began when the positivity conjecture for the coefficients of Laurent polynomials was resolved in [CLS15; Gro+18]. For surface cluster algebras, single dimer configurations on planar,

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bipartite "snake graphs" were used to model the Laurent expansions for these types of cluster algebras [MSW11]. For the once-punctured polygon, it was shown that one can model these type D_n cluster algebras via not only single dimer configurations, but also double dimer configurations in a series of two papers [MW23; FMW22]. This model that mixed together single and double dimer configurations, known as mixed dimer configuration model, uses a type D_n quiver Q as well as a dimension vector for an indecomposable representation of Q and sheds more explicit light on the underlying quiver representation theory that is hinted at in [MSW11].

In parallel, dimer combinatorics has been used to model the image of Grassmannian cluster variables under a famous automorphism called the twist map [BFZ96]. These cluster algebras coming from Grassmannians are far less understood, but the simplest generators, known as Plücker coordinates, have twist images that are modeled by single dimer configurations on plabic graphs [MS16]. In a recent paper, double and triple dimer configurations on plabic graphs were used to give the twist images of the Laurent expansions for a wider family of cluster variables [EMW24].

With these inspirations in mind, this paper explores how higher dimer combinatorics can model the exceptional finite type cluster algebras coming from the E_6, E_7, and E_8 quivers with a strong focus on the underlying quiver representation theory. In particular, one inspiration for the project aims to investigate the following connection: some finite type cluster algebras, namely D_4, E_6 and E_8 arise from certain $Gr(3, n)$ Grassmannian cluster algebras. These cluster variables (or twist images of them) are modeled by higher dimer combinatorics. Our model seeks to combine these various models into a streamlined dimer model for all exceptional type cluster algebras, including E_7, which previously had no combinatorial interpretation.

In this paper, we characterize the terms that appear in the F-polynomial of type E_6 quiver representation for a given dimension vector, analogous to [Tra09] for type D_n quivers. Our proof method involves a combinatorial mixed dimer model inspired by [MW23; FMW22], as well as representation theoretic techniques. We also conjecture a correspondence between F-polynomial coefficients and cycles in mixed dimer configurations. Finally, we suggest many areas of further research, including higher dimension vectors, type E_6 quiver mutations, and quivers of types E_7 and E_8.

This paper is organized as follows. In Section 2, we give the relevant background on cluster algebras and quiver representations. In Section 3, we define our mixed dimer model for type E quivers. In Section 4, we present our main theorem: a correspondence between quiver subrepresentations and certain mixed dimer configurations. In Section 5, we use representation-theoretic techniques to formalize many observations about type E_6 quiver representations. In Section 6, we discuss extending our findings for mutations of type E_6 quivers. In Section 7, we provide a link to the code that we wrote for our research.

2. Preliminaries

We provide some background on cluster algebras and quiver representations.

2.1. **Cluster Algebras.** In this section, we define the necessary prerequisites about cluster algebras of geometric type. The ultimate goal of our project is modeling the Laurent expansion for the generators of these algebras known as cluster variables. Here, we give a streamlined and somewhat terse exposition of cluster algebras, see [FZ02] for details and our later sections for more hands-on examples. Throughout this section, let $n < m$ be positive integers, and let $\mathcal{F} := \mathbb{Q}[u_1, \ldots, u_n, u_1^{-1}, \ldots, u_n^{-1}](x_1, \ldots, x_m)$ be the field of rational functions in variables $x_1, \ldots, x_m$ with coefficients in $\mathbb{Q}[u_1, \ldots, u_n, u_1^{-1}, \ldots, u_n^{-1}]$.

Definition 2.1. A *labeled seed* in $\mathcal{F}$ is an ordered pair $(\tilde{\mathbf{x}}, \tilde{B})$ consisting of an m-tuple $\tilde{\mathbf{x}} := (x_1, \ldots, x_m)$ of elements of $\mathcal{F}$ such that $(x_1, \ldots, x_n)$ is a transcendence basis for $\mathcal{F}$ over $\mathbb{Q}(x_{n+1}, \ldots, x_m)$, and where $\tilde{B}$ is an $m \times n$ integer matrix of the form

$$\tilde{B} := \begin{bmatrix} D^{-1}\hat{B} \\ * \end{bmatrix},$$

where D is an invertible $n \times n$ diagonal matrix with nonnegative integer entries, and $\hat{B}$ is an $n \times n$ skew symmetric matrix.

For a labeled seed as above, there are several terms we will use throughout the paper:

- $\tilde{\mathbf{x}}$ is called the *extended cluster* of the labeled seed;
- $(x_1, \ldots, x_n)$ is called the *cluster* of the labeled seed;
- $\tilde{B}$ is called the *exchange matrix* of the labeled seed; and
- $B := D^{-1}\hat{B}$ is called the *principal part* of $\tilde{B}$.

To streamline future definitions, we introduce the following notation that is common in the literature; see [FZ02]:

- $[i, j] := \{k \in \mathbb{Z} : i \leq k \leq j\}$ for any $i, j \in \mathbb{Z}$.
- $\operatorname{sgn}(x) := \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}$ for any $x \in \mathbb{Q}$
- $[x]_+ := \max(x, 0)$ for any $x \in \mathbb{Q}$

Now, fix a number $k \in [1, n]$ for the following definitions. We define the following local operation on on seeds that will eventually help us define the generators for our cluster algebras.

Definition 2.2. For an integer $1 \leq k \leq n$ and an $m \times n$ matrix $\tilde{B}$, we define the *mutation of $\tilde{B}$ in direction k* to be the matrix $\tilde{B}'$ with

$$\tilde{B}'_{ij} := \begin{cases} -b_{ij} \text{ if } k \in \{i, j\} \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ \text{ otherwise} \end{cases}.$$

We now extend this definition of mutation to labeled seeds.

Definition 2.3. The *mutation of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in direction k*, denoted μ_k, is the involution on labeled seeds $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{F}$ given by

$$\mu_k((x_1, \ldots, x_m), \tilde{B}) := ((x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_m), \tilde{B}'),$$

where

$$x'_k := x_k^{-1}\Big(\prod_{1 \leq i \leq m} x_i^{[b_{ik}]_+} + \prod_{1 \leq i \leq m} x_i^{[-b_{ik}]_+}\Big)$$

and where $\tilde{B}'$ is the mutation of $\tilde{B}$ in direction k.

We now present a way to collect all labeled seeds obtainable via successive mutations. Let $\mathbb{T}_n$ be the group with group presentation given by $\langle s_1, \ldots, s_n | s_k^2 \text{ for all } 1 \leq k \leq n\rangle$. We denote the identity element of this group by e.

Definition 2.4. $\mathbb{T}_n$ acts on the set of all labeled seeds $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{F}$ by setting $s_i \cdot (\tilde{\mathbf{x}}, \tilde{B}) := \mu_k(\tilde{\mathbf{x}}, \tilde{B})$. The orbits $\mathbb{T}_n \cdot (\tilde{\mathbf{x}}, \tilde{B})$ of this action are called *cluster patterns*.

Given a cluster pattern $\mathbb{T}_n \cdot (\tilde{\mathbf{x}}, \tilde{B})$ and an element $t \in \mathbb{T}_n$, let $t \cdot (\tilde{\mathbf{x}}, \tilde{B}) = (x_{1;t}, \ldots, x_{m;t}), \tilde{B}^t)$, and let B^t be the principal part of $\tilde{B}^t$. We are now ready to state the definition of a cluster algebra.

Definition 2.5. For a given cluster seed $\mathbb{T}_n \cdot (\tilde{\mathbf{x}}, \tilde{B})$, we define the set of *cluster variables* by $\mathcal{X} := \{x_{j;t} : t \in \mathbb{T}_n, 1 \leq j \leq m\}$. The *cluster algebra* $\mathcal{A}$ associated to the cluster pattern $\mathbb{T}_n \cdot (\tilde{\mathbf{x}}, \tilde{B})$ is defined as the $\mathbb{Z}[x_{n+1}, \ldots, x_m, x_{n+1}^{-1}, \ldots, x_m^{-1}]$-subalgebra of $\mathcal{F}$ generated by the elements of $\mathcal{X}$. We say that $(\tilde{\mathbf{x}}, \tilde{B})$ is the *initial seed*.

For the purposes of this article, we will be focusing on certain classes of cluster algebras with particular nice exchange matrices. Namely, we will be focusing on cluster algebras that come from quivers, directed graphs we define in the next subsection. These cluster algebras are contained the set of cluster algebras of geometric type as in Definition 2.5. In essence, cluster algebras from quivers are a specific case of cluster algebras of geometric type when the exchange matrices are themselves skew-symmetric where the matrices are signed adjacency matrices of the associated quiver. For more details on cluster algebras from quivers, see Section 2.1 of [Wil14] for a nice exposition of this.

We are now ready to define coefficients for cluster algebras.

Definition 2.6. We say that a labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{F}$ has *principal coefficients* if $\tilde{\mathbf{x}} = (x_1, \ldots, x_{2m})$ and

$$\tilde{B} = \begin{bmatrix} B \\ I \end{bmatrix}$$

where I is the $n \times n$ identity matrix. We denote the corresponding cluster algebra by $\mathcal{A}_\bullet = \mathcal{A}_\bullet(B)$.

For $1 \leq j \leq n$, define the variables $\hat{y}_j$ by

$$\hat{y}_j := x_{n+j} \prod_{1 \leq i \leq n} x_i^{B_{ij}}.$$

One nice property of cluster algebras is the form that the generators can take. It turns out that all cluster variables must be Laurent polynomials.

Theorem 2.7. [FZ02, Theorem 3.1] For $j \in [n]$ and $t \in \mathbb{T}_n$, any cluster variable $x_{j;t} \in \mathcal{A}$ can be expressed in the following form:

$$x_{j;t} = \frac{N(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$$

where $N(x_1, \ldots, x_n) \in \mathbb{Z}[x_{n+1}^{\pm 1}, \ldots, x_m^{\pm 1}][x_1, \ldots, x_n]$ and is not divisible by any x_i. We call the underlying vector $\underline{d} = (d_1, \ldots, d_n)$ the **denominator vector** of $x_{j;t}$. It notably only depends on B^0, the principal part of the exchange matrix, rather than on $\tilde{B}^0$, the extended exchange matrix.

We are now able to define the main object of our model: the F-polynomial and also its associated g-vector.

Definition 2.8. [FZ07] Let B be an $n \times n$ integer matrix which is the principal part of the exchange matrix $\tilde{B}$ of a labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{F}$ having principal coefficients. For any $1 \leq l \leq n$ and $t \in \mathbb{T}_n$, there exists a unique polynomial $F^B_{l;t}(u_0, \ldots, u_{n-1}) \in \mathbb{Z}[u_0, \ldots, u_{n-1}]$ with nonzero constant coefficient and a unique vector $\mathbf{g}^B_{l;t} = (g_1, \ldots, g_n) \in \mathbb{Z}^n$ such that the cluster variable $x_{l;t} \in \mathcal{A}_\bullet(B)$ is given by

$$x_{l;t} = F^B_{l;t}(\hat{y}_1, \ldots, \hat{y}_n) \prod_{1 \leq j \leq n} x_i^{g_i}.$$

The polynomials $F^B_{l;t}$ are called *F-polynomials* and the vectors $\mathbf{g}^B_{l;t}$ are called *g-vectors*.

Remark. The notion of F-polynomial and g-vector allow us to understand the Laurent expansion of any cluster variable in terms of just initial data, see Theorem 3.7/Corollary 6.7 of [FZ07].

2.2. **Quiver Representations.** We now discuss some representation theory that we will connect back to cluster algebras in the next subsection.

Definition 2.9. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a directed graph consisting of vertices Q_0, and arrows Q_1, and two maps $s, t : Q_1 \to Q_0$ which associates to each arrow its source and target respectively.

Example 2.10. Let Q be the following quiver.

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & & \uparrow{\scriptstyle \gamma} & & & & \\ 0 & \xleftarrow{\alpha} & 1 & \xrightarrow{\beta} & 2 & \xleftarrow{\delta} & 4 & \xleftarrow{\phi} & 5 \end{array}$$

Then $Q_0 = \{0, 1, 2, 3, 4, 5\}$ and $Q_1 = \{\alpha, \beta, \gamma, \delta, \phi\}$. Additionally, we have $s(\beta) = 1$ and $t(\beta) = 2$.

Definition 2.11. Let Q be a quiver. A *quiver representation* $M = (V_i, \varphi_V)$ of Q consists of the following data:

- For each vertex, $a \in Q_0$, there is an associated k-vector space, V_a.
- For each arrow, $\alpha \in Q_1$, there is an associated linear transformation $\varphi_{V_\alpha} : V_{s(\alpha)} \to V_{t(\alpha)}$.

Example 2.12. Consider the quiver from Example 2.10. One representation of Q over a field k is given by

$$\begin{array}{ccccccccc} & & & & k & & & & \\ & & & & {\scriptstyle [1\ \ 1]}\uparrow & & & & \\ k & \xleftarrow[{[1]}]{} & k & \xrightarrow{[1\ \ 0]^T} & k^2 & \xleftarrow[{[0\ \ 1]}]{} & k & \xleftarrow[{[0]}]{} & 0 \end{array}.$$

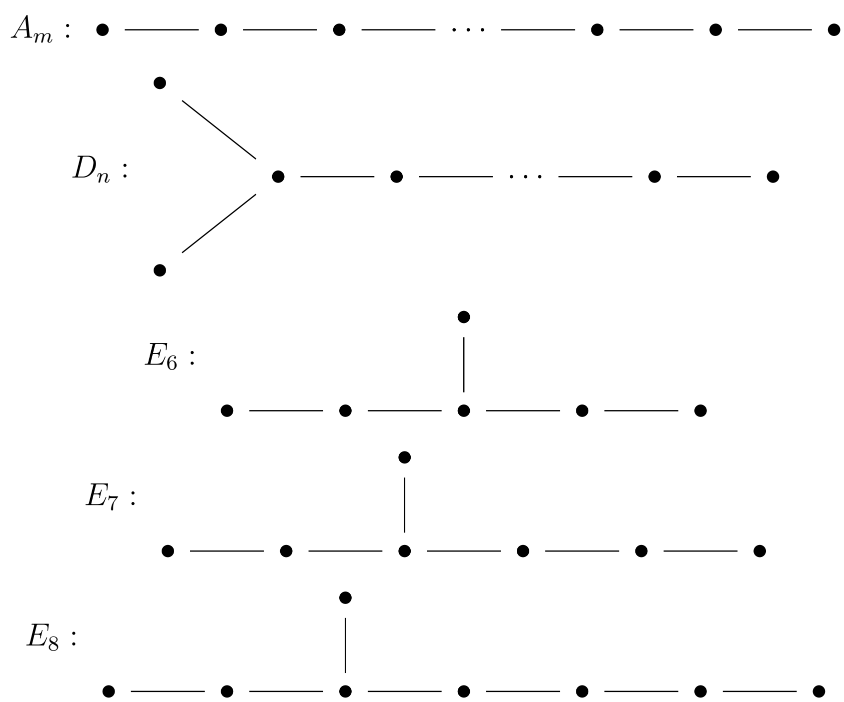
Definition 2.13. The *direct sum* of two representations, $M = (V_i, \varphi_V)$ and $N = (W_i, \varphi_W)$, of a quiver Q is $(V_i \oplus W_i, \varphi_V \oplus \varphi_W)$.

Example 2.14.

$$\begin{array}{ccc} V_1 & \xleftarrow{\varphi_{V_{4,1}}} & V_4 \\ {\scriptstyle \varphi_{V_{2,1}}}\uparrow & & \uparrow{\scriptstyle \varphi_{V_{3,4}}} \\ V_2 & \xrightarrow{\varphi_{V_{2,3}}} & V_3 \end{array} \oplus \begin{array}{ccc} W_1 & \xleftarrow{\varphi_{W_{4,1}}} & W_4 \\ {\scriptstyle \varphi_{W_{2,1}}}\uparrow & & \uparrow{\scriptstyle \varphi_{W_{3,4}}} \\ W_2 & \xrightarrow{\varphi_{W_{2,3}}} & W_3 \end{array} = \begin{array}{ccc} V_1 \oplus W_1 & \xleftarrow{(\varphi_V \oplus \varphi_W)_{4,1}} & V_4 \oplus W_4 \\ {\scriptstyle (\varphi_V \oplus \varphi_W)_{2,1}}\uparrow & & \uparrow{\scriptstyle (\varphi_V \oplus \varphi_W)_{3,4}} \\ V_2 \oplus W_2 & \xrightarrow[(\varphi_V \oplus \varphi_W)_{2,3}]{} & V_3 \oplus W_3 \end{array}$$

Definition 2.15. A quiver representation is *indecomposable* if it cannot be written as the direct sum of two nonzero quiver representations.

Definition 2.16. The *Dynkin diagrams* are A_m ($m \geq 1$), D_n ($n \geq 4$), E_6, E_7, and E_8, which are displayed below. The subscript indicates the number of vertices in the graph.



Theorem 2.17 (Gabriel's Theorem)**.** [Gab72] A quiver has finitely many isomorphism classes of indecomposable representations if and only if its underlying graph is a Dynkin graph.

Definition 2.18. A *subrepresentation* M' of a quiver representation M consists of the following data:

- For each $\alpha \in Q_1$, we associate the same linear transformation as that in M.
- For each $a \in Q_0$, an associated vector space M'_a such that $M'_a \subseteq M_a$, and $\varphi_{V_\alpha}(M'_{s(\alpha)}) \subseteq M'_{t(\alpha)}$.

Example 2.19. Let M be the following quiver representation.

$$\begin{array}{ccccccc} & & k & & & & \\ & & \downarrow{\scriptstyle [1\ \ 0]^T} & & & & \\ k & \xleftarrow[{[1\ \ 1]}]{} & k^2 & \xleftarrow[{[0\ \ 1]^T}]{} & k & \xleftarrow[{[0\ \ 0]}]{} & 0 \end{array} .$$

A subrepresentation of M is

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \Big\downarrow{\scriptstyle [0\ \ 0]^T} & & & & \\ k & \xleftarrow[{[1\ \ 1]}]{} & k^2 & \xleftarrow[{[0\ \ 1]^T}]{} & k & \xleftarrow[{[0\ \ 0]}]{} & 0 \end{array}$$

While the following is not a subrepresentation of M as $\begin{bmatrix}1\\0\end{bmatrix} k \not\subseteq 0$.

$$\begin{array}{ccccccc} & & k & & & & \\ & & \Big\downarrow{\scriptstyle [1\ \ 0]^T} & & & & \\ k & \xleftarrow[{[0\ \ 0]}]{} & 0 & \xleftarrow[{[0\ \ 1]^T}]{} & k & \xleftarrow[{[0\ \ 0]}]{} & 0 \end{array}$$

Theorem 2.20 (Krull-Schmidt Theorem)**.** [ASS06] A decomposition of a quiver representation into a direct sum of indecomposable representations is unique.

2.3. **Connection between Cluster Algebras and Quiver Representations.** Now that we have defined cluster algebras and quiver representations, make explicit the connection between the two. For an acyclic Dynkin quiver Q of type A, D, or E having n vertices, we saw that Gabriel's theorem shows that there are only finitely many isomorphism classes of indecomposable representations of Q. Similarly, cluster algebras have a finite type classification using Dynkin quivers.

Definition 2.21. We say a cluster algebra is of *finite type* if it has finitely many seeds.

Definition 2.22. Let B be an $n \times n$ exchange matrix. Let $\Gamma(B)$ be the *diagram of* B given by a directed graph on nodes $v_1, \ldots, v_n$ with v_i directed towards v_j if and only if $b_{ij} > 0$ and weighted by $|b_{ij}b_{ji}|$.

Theorem 2.23 (Finite Type Classification of Cluster Algebras)**.** [FZ03] A cluster algebra $\mathcal{A}$ is of finite type if and only if it contains a seed (x, B) such that $\Gamma(B)$ is an orientation of an ADE Dynkin diagram.

This suggests a deeper connection between quiver representations with finitely many isomorphism classes of indecomposables and finite type cluster algebras. On the surface of this connection, we have that these sets are indexed by the same set of vectors i.e. the dimension vectors and denominator vectors.

Theorem 2.24. [FZ03, Theorem 1.9] Let $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$ be a finite type cluster algebra and let D denote the set of all denominator vectors of cluster variables in $\mathcal{A}$ which do not occur in the initial cluster $\tilde{\mathbf{x}}$. There is a one-to-one correspondence between these cluster variables in $\mathcal{A}$ and the dimension vectors of the indecomposable quiver representations of $\Gamma(\tilde{B})$.

Definition 2.25. Using the correspondence above, for $\underline{\mathbf{d}}$ the dimension vector of an indecomposable representation M of an acyclic Dynkin quiver Q, we define $F_{\underline{\mathbf{d}}} := F_{l;t}^{B^Q}$ to be the F-polynomial of the corresponding cluster variable, and $\mathbf{g}_{\underline{\mathbf{d}}} := \mathbf{g}_{l;t}^{B^Q}$ to be the $\mathbf{g}$-vector of the corresponding cluster variable. We say that $F_{\underline{\mathbf{d}}}$ and $\mathbf{g}_{\underline{\mathbf{d}}}$ are the F-polynomial and $\mathbf{g}$-vector respectively for the denominator (or dimension) vector $\underline{\mathbf{d}}$.

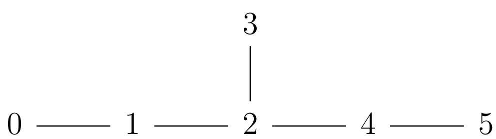
3. Dimers

In this section, we define the graph theoretic objects we will use to model the F-polynomials of our exceptional type cluster algebras. Many of the definitions in this section are inspired by the type D_n case in [MW23; FMW22]. We extend the notions developed in these papers to fit the quivers of type E_6, E_7, and E_8.

3.1. Defining the Base Graph.

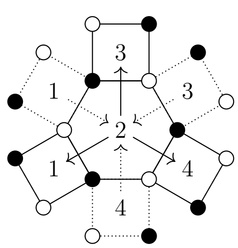
We begin by defining a graph associated to a given type E quiver that we will be endowing with more combinatorics to model the F-polynomials in the associated cluster algebra. We focus on the construction for type E_6 quivers, since the construction for type E_7 and E_8 quivers will easily follow.

We label the vertices of the Dynkin diagram for E_6 with the numbers 0, 1, 2, 3, 4, and 5 as shown.



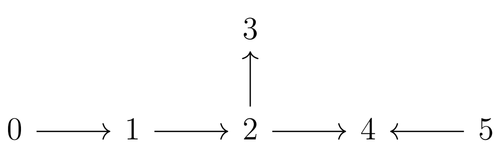
Assigning an orientation to each edge of the diagram yields a directed acyclic quiver of type E_6. We now construct the *base graph* associated to a given E_6 quiver, in an analogous manner to the type D_n case discussed in [MW23].

We begin with a hexagon tile that represents the trivalent vertex labeled 2 and fix a bipartite black and white coloring of its vertices. All other vertices of the quiver are represented by square tiles. The tiles for the vertices labeled 1, 3, and 4 are attached along one of two potential edges of the hexagon as shown according to the convention that each arrow $i \to j$ in the quiver "sees white on the right." An image of the possibilities superimposed is given below.

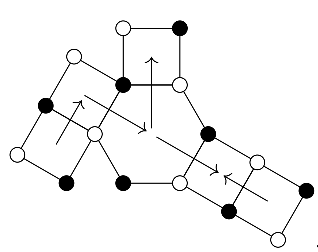


Finally, we attach square tiles for vertices 0 and 5 adjacent to the square tiles for vertices 1 and 4, respectively, following the "white on the right" convention.

Example 3.1. The base graph associated to the quiver



is



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This definition can be extended for type E_7 quivers by attaching a square tile for the vertex 6 adjacent to the square tile for vertex 5, and furthermore extended for type E_8 quivers by attaching a square tile for the vertex 7 adjacent to the square tile for vertex 6.

3.2. **Mixed Dimer Configurations.** Now that we have described how to obtain a base graph from the quiver, we will now discuss a generalization of a perfect matching of this graph known as a mixed dimer configuration. The set of mixed dimer configurations will eventually be in bijection with terms of an F-polynomial of the associated cluster algebra.

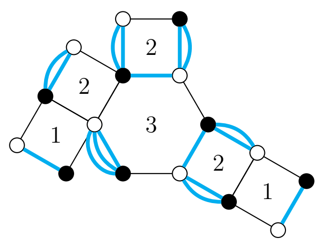
Definition 3.2. A *mixed dimer configuration* D of a base graph G is a multiset of the edges of G. We refer to these edges as *dimers*.

We represent a mixed dimer configuration by drawing the dimers in cyan on a base graph.

Definition 3.3. Let $\underline{d} \in \{0,1,2,3\}^6$. A mixed dimer configuration D satisfies the *valence condition* with respect to $\underline{d}$ if for each vertex v incident to a tile labeled i,

- If $d_i = 3$, then v is contained in 3 edges in D.
- If $d_i = 2$, then v is contained in at least 2 edges in D.
- If $d_i = 1$, then v is contained in at least 1 edge in D.

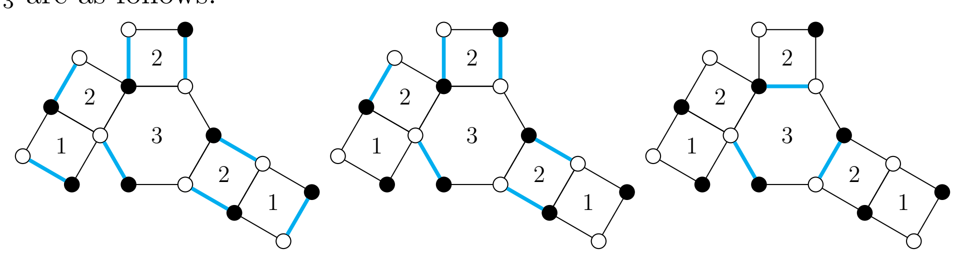
Example 3.4. The following mixed dimer configuration on the base graph from Example 3.1 satisfies the valence condition for $d = (1,2,3,2,2,1)$.



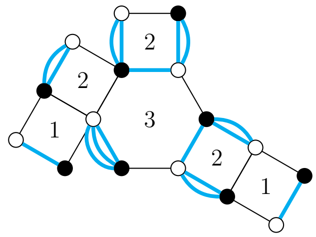
Definition 3.5. Fix a base graph G and a dimension vector $\underline{d}$. For $1 \le r \le 3$, let G_r be the union of the tiles in G with $d_i \ge r$, and let D_r contain the edges along the boundary of G_r that go from black to white clockwise. The *minimal matching* D_- is the multiset sum of D_1, D_2, and D_3.

Using the structure of the dimension vectors for type E quiver representations, one can observe that each G_r for $1 \le r \le 3$ is connected. One may then verify that the minimal matching D_- satisfies the valence condition with respect to $\underline{d}$ by construction.

Example 3.6. Let the base graph be as in Example 3.1 and $d = (1,2,3,2,2,1)$. Then D_1, D_2, and D_3 are as follows.



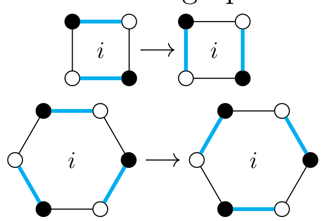
Taking the multiset sum of the edges in D_1, D_2, and D_3 yields the following minimal matching.



3.3. **Poset of Mixed Dimer Configurations.** We may define a local "flip" operation on mixed dimer configurations which is originally inspired by [KW10; MS10] and used in the type D_n case in [MW23; FMW22].

Definition 3.7. A *flip* on a square (resp. hexagonal) tile exchanges two (resp. three) dimers on distinct black-to-white-clockwise edges for two (resp. three) dimers on distinct white-to-black-clockwise edges.

The flip operation on a square and on a hexagon are depicted below. Observe that flips preserve the degree of every vertex of the base graph.



Definition 3.8. Fix a quiver Q and dimension vector $\underline{d}$, and let D_- be the associated minimal matching. The *full poset of mixed dimer configurations* $(\bar{P}, \leq)$ contains as elements the mixed dimer configurations attainable by a sequence of flips from D_-. The covering relation $\lessdot$ is given by $D \lessdot D'$ if and only if D' is attainable from D by one flip.

Every mixed dimer configuration D in $\bar{P}$ corresponds to a vector $\underline{e} = (e_0, \ldots, e_5)$, where e_i is the number of times that tile i is flipped to obtain D from D_-. We may then associate D with the monomial $u_0^{e_0} \cdots u_5^{e_5}$.

Definition 3.9. The *refined poset of mixed dimer configurations* $(P, \leq)$ is a refinement of $(\bar{P}, \leq)$ that only contains mixed dimer configurations D such that the associated monomial $u_0^{e_0} \cdots u_5^{e_5}$ has nonzero coefficient in the F-polynomial for $\underline{d}$.

Example 3.10. Consider the same minimal matching as in Example 3.6 with $\underline{d} = (1, 2, 3, 2, 2, 1)$. The terms of degree at most 2 the F-polynomial for $\underline{d}$ are

$$F_{\underline{d}} = 1 + 2u_3 + 2u_4 + u_2u_3 + u_2u_4 + u_3^2 + 4u_3u_4 + u_4^2 + u_4u_5 + \cdots .$$

The bottom three layers in the refined poset $(P, \leq)$ are shown in Figure 1.

3.4. **Correspondence Between Dimers and Subrepresentations.** We first present the correspondence between dimers and subrepresentations in the type D_n case as described in [Tra09] and [MW23], which motivates our construction in the E_6 case.

Define a partial order $\leq$ on $\mathbb{Z}^n$ by $\underline{a} \leq \underline{b}$ if and only if $\underline{b} - \underline{a} \in \mathbb{Z}^n_{\geq 0}$.

Definition 3.11. Let Q be a type D_n quiver. Fix $\underline{d} \in \Phi_+$ and $\underline{e} = (e_0, \ldots, e_{n-1}) \in \mathbb{Z}^n$ with $0 \leq \underline{e} \leq \underline{d}$. We say that the arrow $i \to j$ in Q is a *type D_n critical arrow* if $(d_i, e_i) \to (d_j, e_j)$ equals $(1, 1) \to (2, 1)$ or $(2, 1) \to (1, 0)$.

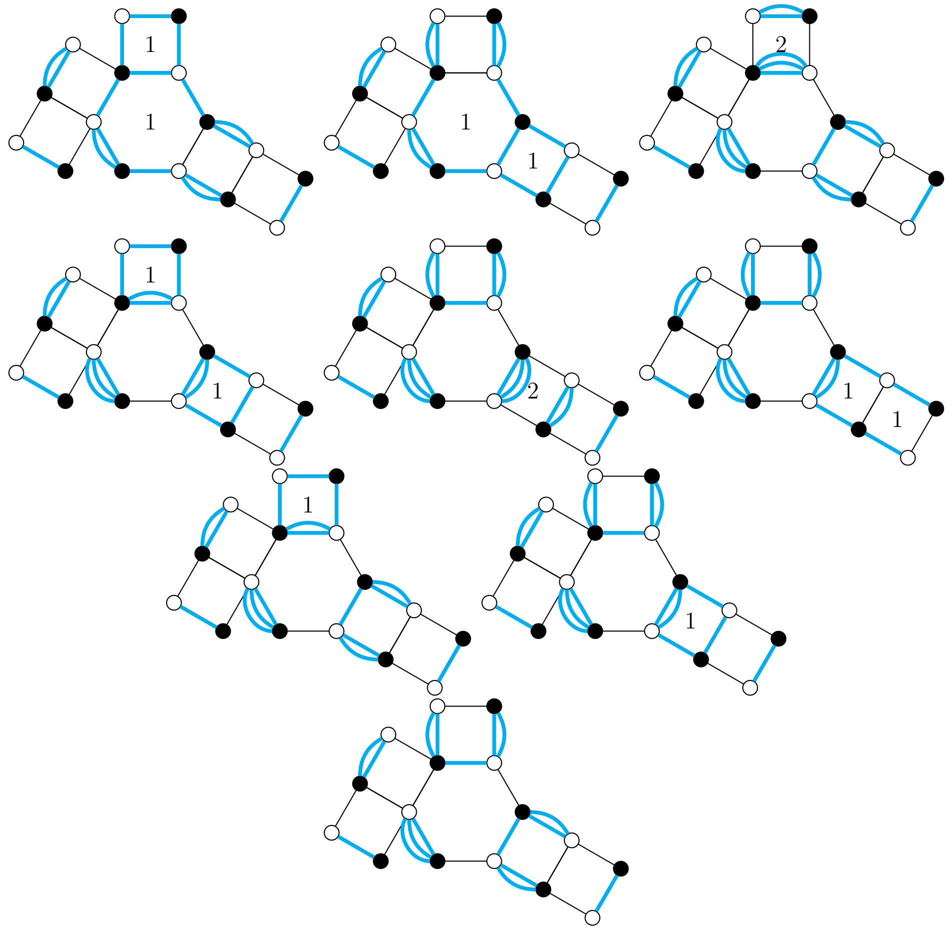


FIGURE 1. A number in a tile is the number of times that tile has been flipped, with 0s omitted.

Theorem 3.12. [Tra09] Let Q be a type D_n quiver. Fix $\underline{d} \in \Phi_+$ and $\underline{e} = (e_0, \ldots, e_{n-1}) \in \mathbb{Z}^n$. In the F-polynomial for $\underline{d}$, the coefficient of the monomial $u_0^{e_0} \cdots u_{n-1}^{e_{n-1}}$ is nonzero if and only if all of the following conditions (referred to as *Tran's conditions*) are satisfied.

(*i*) $0 \leq \underline{e} \leq \underline{d}$.
(*ii*) All arrows $i \to j$ in Q satisfy $e_i - e_j \leq \max(d_i - d_j, 0)$.
(*iii*) For each connected component C of tiles i with $(d_i, e_i) = (2, 1)$, the number of type D_n critical arrows with an endpoint in C is at most 1.

We wish to find an analog of Theorem 3.12 for type E_6 quivers and first give intuition for why Tran's conditions (i) and (ii) are reasonable to be directly applied to the E_6 case.

Theorem 3.13. [MW23] For $\underline{e} \in \mathbb{Z}^n$ satisfying (i) and (ii) of Tran's conditions, there exists a sequence of flips from D_- such that in the end, tile i is flipped e_i times, for all $1 \leq i \leq n$. Call the resulting mixed dimer configuration $D_{\underline{e}}$. In particular,

(*i*) The condition $0 \leq \underline{e} \leq \underline{d}$ guarantees that all edges on the boundary of $D_{\underline{e}}$ have a nonnegative number of dimers.
(*ii*) The condition $e_i - e_j \leq \max(d_i - d_j, 0)$ for all arrows $i \to j$ guarantees that all edges on the interior of $D_{\underline{e}}$ have a nonnegative number of dimers.

Corollary 3.14. As a consequence of Theorem 3.12 and Theorem 3.13, there is a bijection

$$\{\text{vectors } \underline{e} \text{ satisfying Tran's conditions}\} \leftrightarrow \{\text{mixed dimer configurations } D_{\underline{e}} \text{ in } (P, \leq)\}$$

given by $\underline{e} \mapsto D_{\underline{e}}$.

Remark. Representation theoretically, $\underline{e}$ is thought of as the dimension vector of a subrepresentation of an indecomposable representation of Q with dimension vector $\underline{d}$. Tran's conditions (i) and (ii) have the following representation-theoretic interpretations.

(*i*) The condition $0 \leq \underline{e} \leq \underline{d}$ implies that for $\underline{e}$ to be a subrepresentation dimension vector, each entry must necessarily be nonnegative and at most the corresponding entry in $\underline{d}$.

(*ii*) The condition $e_i - e_j \leq \max(d_i - d_j, 0)$ implies that every map in the indecomposable representation of Q with dimension vector $\underline{d}$ must be full rank. It is not obvious in general that the maps in any indecomposable representation of Q must be full rank, but maps which are not full rank often lead to a way of decomposing the representation as a nontrivial direct sum of its subrepresentations.

4. Critical Arrows for Type E Quivers

We find an analog of Tran's condition (iii) for type E_6 quivers. We color-code the arrows in the remainder of the section for ease of reading.

Definition 4.1. Let Q be a quiver of Dynkin type E_6. Fix a dimension vector $\underline{d} \in \Phi_+$ and let $\underline{e} = (e_0, \ldots, e_5) \in \mathbb{Z}^6$. The arrow $i \to j$ in Q is a *type* E_6 *critical arrow* if $(d_i, e_i) \to (d_j, e_j)$ is of one of the following forms.

- $(1, 1) \to (2, 1)$
- $(2, 1) \to (1, 0)$
- $(2, 1) \to (3, 1)$
- $(3, 1) \to (2, 0)$
- $(2, 2) \to (3, 2)$
- $(3, 2) \to (2, 1)$

We classify the type E_6 critical arrows into types I, II, and III as follows.

- Type I: $(1, 1) \to (2, 1)$ and $(2, 1) \to (1, 0)$
- Type II: $(2, 1) \to (3, 1)$ and $(3, 2) \to (2, 1)$
- Type III: $(2, 2) \to (3, 2)$ and $(3, 1) \to (2, 0)$

Theorem 4.2. Let Q be a quiver of Dynkin type E_6. Let $\underline{d} = (1, 2, 3, 2, 2, 1)$ and $\underline{e} = (e_0, \ldots, e_5) \in \mathbb{Z}^6$. In the F-polynomial corresponding to $\underline{d}$, the coefficient of the monomial $u_0^{e_0} \cdots u_5^{e_5}$ is nonzero if and only if all of the following conditions are satisfied:

(*i*) $0 \leq \underline{e} \leq \underline{d}$.

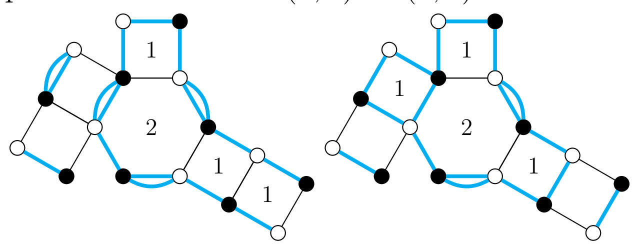
(*ii*) All arrows $i \to j$ in Q satisfy $e_i - e_j \leq \max(d_i - d_j, 0)$.

(*iii*) None of the following combinations of critical arrows occur.

Type I	Type II	Type III
		2
	1	1
	3	
2	1	
1		1
1	2	

Theorem 4.2 has been verified for all acyclic type E_6 quivers by the code in Section 7.

Example 4.3. The mixed dimer configuration on the left with $\underline{e} = (0, 0, 2, 1, 1, 1)$ does not appear in the F-polynomial, because it contains one type I critical arrow $(1, 1) \to (2, 1)$ and two type II critical arrows $(3, 2) \to (2, 1)$. The mixed diagram configuration on the right with $\underline{e} = (0, 1, 2, 1, 1, 0)$ does appear in the F-polynomial, because the only critical arrows that appear are two type II critical arrows $(3, 2) \to (2, 1)$.



Lemma 4.4. [MW23] Let $\underline{d} \in \Phi_+$ and let $0 \leq \underline{e} \leq \underline{d}$. If $i \to j$ in Q, then the number of dimers in $D_{\underline{e}}$ on the edge between tiles i and j is

$$\max(d_i - d_j, 0) + (e_j - e_i).$$

Lemma 4.5. [MW23] Let $\underline{d} \in \Phi_+$ and let $0 \leq \underline{e} \leq \underline{d}$. For any $0 \leq i \leq n - 1$, let D be the mixed dimer configuration obtained by flipping tile i e_i number of times from M. Let the outer face of our graph G be indexed by ∞. We assign an arrow to each of the boundary edges of G where the orientation of this arrow depends on the bipartite coloring of G following the convention that we "see white on the right."
Let $m_{i,\infty}(\alpha)$ (respectively $m_{\infty,i}(\alpha)$) be the number of edges distinguished on α on tile i in D where $i \to \infty$ about α (respectively $\infty \to i$ about α.) Then:

$$m_{i,\infty}(\alpha) = \max(d_i, 0) - e_i \;\text{ if } i \to \infty \text{ about } \alpha$$
$$m_{\infty,i}(\alpha) = \max(d_i, 0) + e_i \;\text{ if } \infty \to i \text{ about } \alpha$$

Corollary 4.6. The number of dimers on each white-to-black clockwise exterior edge of tile i is e_i, and the number of dimers on each black-to-white-clockwise exterior edge of tile i is $d_i - e_i$.

Theorem 4.7. The arrow $i \to j$ is a type E_6 critical arrow or of the form $(2, 1) \to (2, 1)$ or $(1, 1) \to (3, 1)$, if and only if there is no dimer on the edge between tiles i and j. And there is at least one dimer on all boundary edges adjacent to the two vertices on both tiles i and j, or at least one dimer on an edge adjacent to the edge straddling i and j rather than a dimer on an adjacent boundary edge.

Proof. For each type E_6 critical arrow $i \to j$ listed in Definition 4.1, direct computation shows that

$$\max(d_i - d_j, 0) + (e_j - e_i) = 0.$$

Then by Lemma 4.4, there is no dimer on the edge straddling i and j.

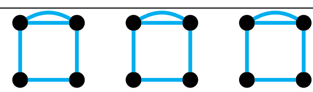
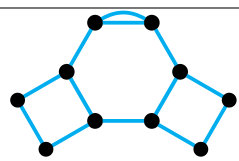
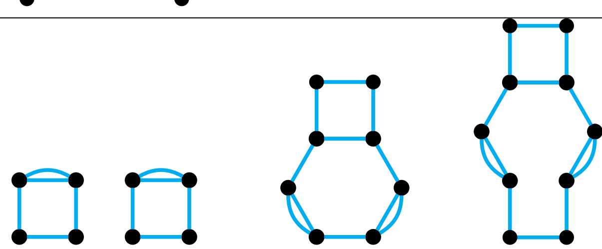
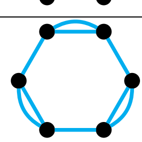
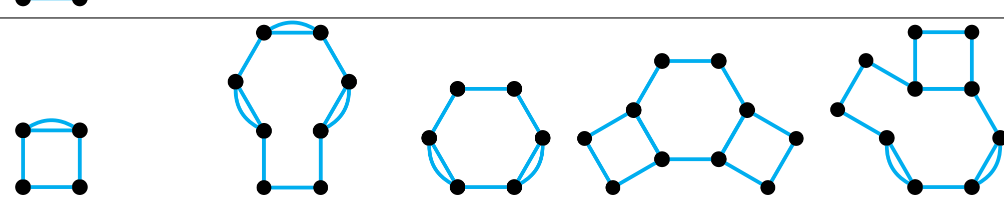
If we consider the tiles i and j locally, as well, it similarly follows from Corollary 4.6 and direct computation that its boundary edges have at least one dimer each. In the case that i or j is a hexagon, say j is a hexagon without loss of generality, and a vertex incident to the edge straddling i and j has degree 4, let the square tile adjacent to i and j be denoted by k. Then, since flips preserve the degree of the vertices and the presence of tile k does not change the number of dimers on the boundary of tiles i or j or on the edge straddling i and j, it is clear that either the edge straddling j and k must have at least one dimer, or the boundary edge on tile k incident to the edge straddling i and j must have at least one dimer.

In order to have no dimer on the edge straddling i and j, by Lemma 4.4 we must have that $d_i - d_j + e_j - e_i = 0$ and $d_i > d_j$, or $e_j - e_i = 0$ and $d_j \geq d_i$. In the case that $d_i - d_j + e_j - e_i = 0$ and $d_i > d_j$, when $d_i = e_i$ and $d_j = e_j$, all dimers will be on white-to-black clockwise edges, so we can disregard such cases. All other choices of d_i, d_j, e_i, e_j meeting these conditions correspond to critical arrows.

In the case that $e_i = e_j$ and $d_j \geq d_i$, by the preceding reasoning we can the cases where $d_i = e_i$ and $d_j = e_j$. All other choices of d_i, d_j, e_i, e_j meeting these conditions correspond to critical arrows or are $(2, 1) \to (2, 1)$, or $(1, 1) \to (3, 1)$. □

4.1. **Coefficients.** We focus on the $\underline{d} = (1, 2, 3, 2, 2, 1)$ case.

Conjecture 4.8. Coefficients greater than 1 in the F-polynomial for $\underline{d} = (1, 2, 3, 2, 2, 1)$ correspond to the following cycles present in the mixed dimer configuration.

Coefficient	Cycles
8	
5	
4	
3	
2	

5. Quiver Representation Reduction Operations

In this section, we introduce tools for a representation-theoretic proof of Theorem 4.2. Although the theorem is a result about (acyclic) type E_6 quivers for the specific dimension vector $\underline{d} = (1, 2, 3, 2, 2, 1)$, many of the methods presented in this section can be generalized to other dimension vectors $\underline{d}$.

For (acyclic) type E_6 indecomposable quiver representations, the only dimension vectors $\underline{d}$ containing a 3 are $(1, 2, 3, 1, 2, 1)$ and $(1, 2, 3, 2, 2, 1)$. In all other cases, we wish to show that the critical arrow conditions from Theorem 4.2 for subrepresentation dimension vectors $0 \leq \underline{e} \leq \underline{d}$ still hold. In these cases, our conditions reduce to Tran's conditions in Theorem 3.12 characterizing subrepresentation dimension vectors in type D_n.

One case in which the relationship between subrepresentations of indecomposable representations for types E_6 and D_n (or sometimes A_n) is clear is when some entry of the dimension vector $\underline{d}$ of the indecomposable representation M of the type E_6-quiver Q is equal to 0. In these cases, the reduction from type E_6 quiver representations to type A_n and D_n quiver representations is made possible by the following result.

Definition 5.1. Let Q be a quiver and $i \in Q_0$ a vertex of Q. The *vertex deletion* $Q \setminus i$ of Q at i is the quiver with vertex set

$$(Q \setminus i)_0 \coloneqq Q_0 \setminus \{i\}$$

and edge set

$$(Q \setminus i)_1 \coloneqq Q_1 \setminus \{r \in Q_1 : t(r) = i \text{ or } s(r) = i\}.$$

Proposition 5.2. Let $\mathrm{Rep}_{Q\setminus i}$ be the category of quiver representations of $Q\setminus i$. Let $\mathcal{C}$ be the full subcategory of the category Rep_Q of representations of Q consisting of the representations M of Q satisfying $M_i = 0$. There is an equivalence of categories

$$Z_i^- : \mathcal{C} \to \mathrm{Rep}_{Q\setminus i}$$
$$Z_i^+ : \mathrm{Rep}_{Q\setminus i} \to \mathcal{C}$$

between $\mathrm{Rep}_{Q\setminus i}$ and $\mathcal{C}$.

Proof. We give explicit definitions for the functors Z_i^- and Z_i^+. For any morphism $\phi : M \to N$ between representations $M, N \in \mathcal{C}$, we let Z_i^- be the functor given by

- $Z_i^-(M)_j \coloneqq M_j$ for all $j \in (Q \setminus i)_0$
- $Z_i^-(M)(r) \coloneqq M(r)$ for all $r \in (Q \setminus i)_1$
- $Z_i^-[\phi]_j \coloneqq \phi_j$ for all $j \in (Q \setminus i)_0$.

Similarly, for any morphism $\phi' : M' \to N'$ between representations $M', N' \in \mathrm{Rep}_{Q\setminus i}$, we let Z_i^+ be the functor given by

- $Z_i^+(M')_j \coloneqq \begin{cases} 0 \text{ if } j = i \\ M'_j \text{ otherwise} \end{cases}$ for all $j \in Q_0$
- $Z_i^+(M')(r) \coloneqq \begin{cases} M'(r) \text{ if } r \in (Q \setminus i)_0 \\ 0 \text{ otherwise} \end{cases}$ for all $r \in Q_1$
- $Z_i^+[\phi']_j \coloneqq \begin{cases} 0 \text{ if } j = i \\ \phi'_j \text{ otherwise} \end{cases}$ for all $j \in Q_0$

It can then be checked that $Z_i^- \circ Z_i^+ : \mathrm{Rep}_{Q\setminus i} \to \mathrm{Rep}_{Q\setminus i}$ and $Z_i^+ \circ Z_i^- : \mathcal{C} \to \mathcal{C}$ are equal to the identity functors $\mathrm{id}_{\mathrm{Rep}_{Q\setminus i}}$ and $\mathrm{id}_{\mathcal{C}}$ respectively. □

Definition 5.3. We call Z_i^- and Z_i^+ from Proposition 5.2 the *zero deletion* and *zero extension* functors at i, respectively.

Remark. The functors Z_i^- and Z_i^+ above respect the k-Vect enriched category structures of $\mathcal{C}$ and $\mathrm{Rep}_{Q\setminus i}$, so the equivalence of categories above is an equivalence of k-Vect enriched categories.

Observe that any subrepresentation of a representation of Q lying in $\mathcal{C}$ is also in $\mathcal{C}$.

The following data can be defined entirely in terms of the k-linear category structures of $\mathcal{C}$ and $\mathrm{Rep}_{Q\setminus i}$.

- Indecomposable representations, since this correspondence preserves all limits and colimits–in particular the 0 representation, direct sums and isomorphisms
- Objects isomorphic to subrepresentations of another object, since those are equivalent to monomorphisms into the latter object
- Dimension vectors of representations, since those can be defined alternatively in terms of composition series of representations, as well as the simple quotients at each step of the composition series (and simple modules are preserved since they can be characterized by a condition on their subrepresentations).
- The intersection, i.e. meet, of two subrepresentations N, N' of a representation M (up to isomorphism), since this amounts to forming the pullback of their monomorphisms to M
- The sum, i.e. join, of two subrepresentations N, N' of M (up to isomorphism), since this can be characterized as the subrepresentation obtained by forming the cokernels M/N and M/N' of the monomorphisms of N and N' into M, then forming the pushout of those cokernels, $M/(N+N')$, and then forming the kernel of the diagonal $M \twoheadrightarrow M/(N+N')$ of that pushout square

We may deduce that for an indecomposable representation M in $\mathcal{C}$ having dimension vector $\underline{d}$, there exists a subrepresentation of M having dimension vector $\underline{e}$ if and only if the indecomposable representation $Z_i^-(M)$ has a subrepresentation with dimension vector

$$Z_i^-(\underline{e}) := Z_i^-(\text{The multiset of simple representations which correspond to } \underline{e})$$

Theorem 5.4. The combinatorial characterization of allowed subrepresentation dimension vectors of indecomposable representations M of type E_6 quivers Q whose dimension vector $\underline{d}$ has $d_i = 0$ for some $i \in Q_0$ is word for word the same as that given for type D_n quivers in [Tra09]

Remark. Among the 36 possible dimension vectors $\underline{d}$ of indecomposable representations of type E_6 quivers, only seven of them have $d_i \neq 0$ for all $0 \leq i \leq 5$. Only two of these seven dimension vectors have $d_i = 3$ for some $0 \leq i \leq 5$, namely $(1,2,3,2,2,1)$ and $(1,2,3,1,2,1)$.

For the other five dimension vectors for type E_6, the characterization also appears to be the same as the one given in [Tra09] discussed above. In some cases, we can explain this coincidence nicely by a procedure that reduces dimension vectors $0 \leq \underline{e} \leq \underline{d}$ in type E_6 to related dimension vectors $0 \leq \underline{e}' \leq \underline{d}'$ in type D_n.

Definition 5.5. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. If $e \in Q_1$ has source vertex $s(e) = a \in Q_0$ and target vertex $t(e) = b \in Q_0$, then the *contraction of Q at e*, denoted Q/e is the quiver $Q/e = ((Q/e)_0, (Q/e)_1, s/e, t/e)$ with

- $(Q/e)_0 \coloneqq Q_0/\sim_e$, where $\sim_e$ is the equivalence relation on Q_0 defined by $x \sim_e y \iff x = y$ or $\{x, y\} \subseteq \{a, b\}$
- $(Q/e)_1 \coloneqq Q_1 \setminus \{e\}$
- $(s/e) = \pi_e \circ s$ and $(t/e) = \pi_e \circ t$, where $\pi_e : Q_0 \to Q_0/\sim_e$ is the quotient map sending each vertex of Q_0 to its $\sim_e$-equivalence class

This is a purely combinatorial operation on quivers which effectively contracts a and b along e. We show that this operation lifts to an operation which takes certain quiver representations of Q to quiver representations of Q/e.

In the following, let Q be a quiver, and let M be a representation of Q over a field k. We denote by M_x the vector space M assigns to each vertex x of Q, and denote by $M(e)$ the k-linear map M assigns to each directed edge e of Q.

Definition 5.6. Given an arrow $e \in Q_1$, M is said to be *contractible at e* if $M(e)$ is an isomorphism. In this case, the *contraction of M at e*, denoted M/e, is the representation of Q/e given by

- $M_x \coloneqq \begin{cases} M_{\pi_e^{-1}(x)} \text{ if } x \neq \pi_e(s(e)) \\ M_{\pi_e(t(e))} \text{ if } x = \pi_e(s(e)) \end{cases}$ for any $x \in (Q/e)_0$
- $M(e') \coloneqq \begin{cases} M(e') \text{ if } t(e') \neq s(e) \\ M(e) \circ V(e') \text{ if } t(e') = s(e) \end{cases}$ for any $e' \in (Q/e)_1 = Q_1$.

We also require "expansion" operations that are inverse to contractions to go in between representations of Q and Q/e.

Definition 5.7. Let Q be a quiver, x a vertex of Q, and $(A, B) = (\{f_1, \dots, f_q\}, \{g_1, \dots, g_p\})$ be a partition of the set of edges in Q which have either source or target x into two disjoint subsets. Then the *expansion* of Q at x with respect to $(\{f_1, \dots, f_q\}, \{g_1, \dots, g_p\})$ is the quiver $Q^{(A|B)} \coloneqq (Q'_0, Q'_1, s', t')$ with

- $Q'_0 \coloneqq Q_0 \setminus \{x\} \cup \{x_s, x_t\}$
- $Q'_1 \coloneqq Q_1 \cup \{e_x\}$
- $t' : Q'_1 \to Q'_0$ given by $t'(e) \coloneqq \begin{cases} x_t \text{ if } e = e_x \\ x_t \text{ if } t(e) = x \text{ and } e \in B \\ x_s \text{ if } t(e) = x \text{ and } e \in A \\ t(e) \text{ otherwise} \end{cases}$
- $s' : Q'_1 \to Q'_0$ given by $s'(e) \coloneqq \begin{cases} x_s \text{ if } e = e_x \\ x_t \text{ if } s(e) = x \text{ and } e \in B \\ x_s \text{ if } s(e) = x \text{ and } e \in A \\ s(e) \text{ otherwise} \end{cases}$

Remark. If Q is a quiver, e is an edge of Q, $x = \pi_e(s(e)) = \pi_e(t(e))$, and $(A, B) = (\{r \in (Q/e)_1 \subset Q_1 : r \in s^{-1}(\{s(e)\}) \cup t^{-1}(\{s(e)\}), \{r \in (Q/e)_1 \subset Q_1 : r \in s^{-1}(\{t(e)\}) \cup t^{-1}(\{t(e)\})\})$, then the expansion of Q/e at x with respect to (A, B) is isomorphic to Q.

This is the motivation behind the previous definition. We would now like to extend this "expansion" to an operation on quiver representations.

Definition 5.8. Let Q be a quiver, x a vertex of Q, and $(A, B) = (\{f_1, ..., f_p\}, \{g_1, ..., g_q\})$ a partition of the set of edges in Q incident with x as in the definition above. Let M be a representation of Q. Then the *expansion* at vertex x of M is the representation $M^{(A|B)}$ of $Q^{(A|B)}$ given by

- $M^{(A|B)}(i) := \begin{cases} M(x) \text{ if } i = x_s, x_t \\ M(i) \text{ otherwise} \end{cases}$ for $i \in Q_0^{(A|B)}$
- $M^{(A|B)}(e) := \begin{cases} \mathrm{id}_{M(x)} \text{ if } e = e_x \\ M(e) \text{ otherwise} \end{cases}$ for $e \in Q_1^{(A|B)}$.

Proposition 5.9. Let Q be a quiver, M a representation of Q contractible at $r := i \to j$ with dimension vector $\underline{d}$, and $0 \leq \underline{e} \leq \underline{d}$ a nonnegative integer vector. If $e_i = e_j$, there exists a subrepresentation N of M with dimension vector $\underline{e}$ if and only if there exists a subrepresentation of M/r with dimension vector $\underline{e}'$, where $e'_x := e_{\pi_r^{-1}(x)}$.

Proof. Note that e'_x is well defined because $e_i = e_j$. For the backwards direction, if N is a subrepresentation of M with dimension vector $\underline{e}$, then N/r is a subrepresentation of M/r with dimension vector $\underline{e}'$. For the fowards direction, if N' is a subrepresentation of M/r with dimension vector $\underline{e}'$, then $N'^{(A|B)}$ with (A, B) chosen as in the previous remark is a subrepresentation of M with dimension vector $\underline{e}$. □

To prove the validity of these reductions, we use a result which allows us to check that our reductions induce a nice map on the subrepresentation lattices of the corresponding representations. We describe the lattice structure on the subrepresentations of M as follows.

Definition 5.10. The *subrepresentation lattice* of a representation M of a quiver Q is the bounded lattice $\mathcal{L}(M)$ whose

- elements are the subrepresentations N of M
- partial order $\leq$ is given by $N \leq N'$ if and only if N is a subrepresentation of N'
- minimal and maximal elements are trivial subrepresntations 0 and M respectively
- meet $\cap$ is defined by $(N \cap N')_i := N_i \cap N'_i$ for all $i \in Q$, and
- join $+$ is defined by $(N + N')_i := N_i + N'_i$ for all $i \in Q$.

Proposition 5.11. Let M be a representation of a quiver Q and M' a representation of a quiver Q'. Suppose there exists a map of subrepresentation lattices $\phi : \mathcal{L}(M') \hookrightarrow \mathcal{L}(M)$ for which $\phi(N') = 0 \implies N' = 0$. In other words, $\phi : \mathcal{L}(M') \to \mathcal{L}(M)$ satisfies the following.

- For subrepresentations $N'_1 \subset N'_2 \subset M'$, we have $\phi(N'_1) \subset \phi(N'_2)$ is a subrepresentation
- $\phi(0) = 0$ and $\phi(M') = M$
- $\phi(N') = 0 \implies N' = 0$
- $\phi(N'_1) \cap \phi(N'_2) = \phi(N'_1 \cap N'_2)$ for any $N'_1, N'_2 \in \mathcal{L}(M')$
- $\phi(N'_1) + \phi(N'_2) = \phi(N'_1 + N'_2)$ for any $N'_1, N'_2 \in \mathcal{L}(M')$

If M is an indecomposable representation of Q, then M' is an indecomposable representation of Q'.

Proof. Suppose that M' decomposes as a direct sum $M' = N'_1 \oplus N'_2$, i.e. there exist subrepresentations N'_1, N'_2 of M' such that $N'_1 \cap N'_2 = 0$ and $N'_1 + N'_2 = M'$. Then $\phi(N'_1), \phi(N'_2)$

are subrepresentations of M such that $\phi(N_1') \cap \phi(N_2') = \phi(N_1' \cap N_2') = \phi(0) = 0$ and $\phi(N_1') + \phi(N_2') = \phi(N_1' + N_2') = \phi(M') = M$, so $M = \phi(N_1') \oplus \phi(N_2')$. Furthermore, since ϕ doesn't take any nonzero subrepresentation of M' to 0, we see that one of $\phi(N_1'), \phi(N_2')$ equals 0 if and only if one of N_1', N_2' equals. Thus, whenever M' is decomposable, so is M. □

Remark. The same argument works verbatim if ϕ is instead a map of $\mathcal{L}(M')$ into the opposite lattice $\mathcal{L}(M)^{op}$ of $\mathcal{L}(M)$ such that only the minimal element of $\mathcal{L}(M')$ is sent to the minimal element of $\mathcal{L}(M)^{op}$. In other words, $\phi : \mathcal{L}(M') \to \mathcal{L}(M)$ satisfies for all $N_1', N_2' \in \mathcal{L}(M')$

- $N_1' \leq N_2' \implies \phi(N_2') \leq \phi(N_1')$
- $\phi(N_1' + N_2') = \phi(N_1') \cap \phi(N_2')$
- $\phi(N_1' \cap N_2') = \phi(N_1') + \phi(N_2')$
- $\phi(0) = M, \phi(M') = 0$
- $\phi(N_1') = M \implies N_1' = 0$.

We now introduce a few reduction operations and prove their validity. In the following, let Q be a quiver, and let M be a representation of Q.

Definition 5.12. Given an arrow $r \in Q_1$, we define the *deletion of Q at r* to be the quiver $Q \setminus r := (Q_0, Q_1 \setminus \{r\}, s, t)$. We define the *deletion of M at r* to be the quiver representation $M \setminus r$ of $Q \setminus r$ defined by $(M \setminus r)_i := M_i$ for all $i \in Q_0$ and $(M \setminus r)(r') = M(r')$ for all $r' \in Q_1 \setminus \{r\}$. We define the *deletion functor* $d_r : \mathrm{Rep}_Q \to \mathrm{Rep}_{Q \setminus r}$ to be the k-linear functor given by

- $d_r(M) := M \setminus r$ for all $M \in \mathrm{Rep}_Q$
- $d_r[\phi]_i := \phi_i : (M \setminus r)_i \to (N \setminus r)_i$ for any $i \in Q_0$ and any homomorphism $\phi : M \to N$ of quiver representations $M, N \in \mathrm{Rep}_Q$.

Proposition 5.13. For any $r \in Q_1$, d_r is an exact faithful functor.

Proof. Observe that for any map $\phi : M \to N$ of quiver representations, its kernel and cokernel in Rep_Q are given by $\ker(\phi)_i = \ker(\phi_i)$ for all $i \in Q_0$ and $\mathrm{coker}(\phi)_i = \mathrm{coker}(\phi_i)$ for all $i \in Q_0$, with $\ker(\phi)(r') = M(r')|_{\ker(\phi_{s(r')})}$ and $\mathrm{coker}(\phi)(r') = \pi_{\mathrm{coker}(\phi_{t(r')})} \circ N(r') \circ \pi^{-1}_{\mathrm{coker}(\phi_{s(r')})}$, where $\pi_{\mathrm{coker}(\phi_i)} : M_i \twoheadrightarrow \mathrm{coker}(\phi_i)$ denotes the cokernel projection map which has the image of ϕ_i as its kernel. Note that although the inverse $\pi^{-1}_{\mathrm{coker}(\phi_{s(r')})}$ is not technically well defined because the value of the inverse is not unique, the composition above involving $\pi^{-1}_{\mathrm{coker}(\phi_{s(r')})}$ is still well defined because it doesn't depend on which inverse you choose. Furthermore, even the kernel inclusion map $\iota_\phi : \ker(\phi) \hookrightarrow M$ and cokernel projection map $\pi_\phi : N \twoheadrightarrow \mathrm{coker}(\phi)$ are given vertex-wise: $(\iota_\phi)_i = \iota_{\phi_i}$ and $(\pi_\phi)_i = \pi_{\phi_i}$ for all $i \in Q_0$. Since each $\ker(\phi)_i$ and $\mathrm{coker}(\phi)_i$ only depends on $M(i), N(i), \phi_i$, and similarly $\ker(\phi)(r')$ and $\mathrm{coker}(\phi)(r')$ only depend on $M(r'), N(r'), \phi_{t(r')}, \phi_{s(r')}$, from the way d_r is defined by just forgetting a single arrow r, it is clear that we have $\ker(d_r(\phi)) = d_r(\ker(\phi))$ and $\mathrm{coker}(d_r(\phi)) = d_r(\mathrm{coker}(\phi))$. It is also immediate from the fact that $d_r(M)_i = M_i$ for all $i \in Q_0$ that d_r preserves finite direct sums, and so d_r is exact. Furthermore, the map on morphisms $\phi \to d_r[\phi]$ just amounts to the inclusion map obtained from the fact that morphisms of representations $\phi : M \to N$ in Rep_Q are the same as morphisms of their deletions at r in $\mathrm{Rep}_{Q \setminus r}$ which satisfy the extra condition that $\phi_{t(r)} \circ M(r) = N(r) \circ \phi_{s(r)}$. It then follows that $\phi \to d_r[\phi]$ is injective, and so d_r is a faithful functor. □

In particular, d_r preserves all finite limits and colimits, and does not send any nonzero representation $M \in \mathrm{Rep}_Q$ to 0, since otherwise, d_r would have to send both the zero map $0 : M \to M$ and the identity map $\mathrm{id}_M : M \to M$ to the zero map $0 : 0 \to 0$, which contradicts faithfulness of d_r.

It follows from the same argument as Proposition 5.11 that since d_r preserves direct sums and subrepresentations, and it takes nonzero representations to nonzero representations, if $d_r(M)$ is indecomposable, then M is indecomposable.

We now introduce an operation that allows us to add arrows back to the quiver.

Definition 5.14. Let $a, b \in Q_1$ be arrows satisfying $t(b) = s(a)$. The (a, b)-*filling* of Q is the quiver $\Delta_{a,b}(Q) := (Q_0, Q_1 \cup \{a \circ b\}, s', t')$ where

$$s'(r) := \begin{cases} s(b) \text{ if } r = a \circ b \\ s(r) \text{ otherwise} \end{cases}$$

$$t'(r) := \begin{cases} t(a) \text{ if } r = a \circ b \\ t(r) \text{ otherwise} \end{cases}$$

Similarly, we define the (a, b)-*filling* of a representation M of Q to be the representation $\Delta_{a,b}(M)$ of $\Delta_{a,b}(Q)$ given by $\Delta_{a,b}(M)_i := M_i$ for all $i \in Q_0$, $\Delta_{a,b}(M)(r) := M(r)$ for all $r \in Q_1$, and $\Delta_{a,b}(M)(a \circ b) := M(a) \circ M(b)$.

We denote by $\Delta_{a,b} : \mathrm{Rep}_Q \to \mathrm{Rep}_{\Delta_{a,b}(Q)}$ the k-linear functor given by $\Delta_{a,b}(M) := \Delta_{a,b}(M)$ for all $M \in \mathrm{Rep}_Q$ and $\Delta_{a,b}[\phi]_i := \phi_i$ for all morphisms $\phi : M \to N$ in Rep_Q and all $i \in Q_0$

Proposition 5.15. $\Delta_{a,b} : \mathrm{Rep}_Q \to \mathrm{Rep}_{\Delta_{a,b}(Q)}$ is an exact functor which is fully faithful and injective on objects.

Proof. Observe that since $d_{a\circ b} \circ \Delta_{a,b} = \mathrm{id}_{\mathrm{Rep}_Q}$, the functor $\Delta_{a,b} : \mathrm{Rep}_Q \to \mathrm{Rep}_{\Delta_{a,b}(Q)}$ is fully faithful and injective on objects. For exactness, the fact that $\Delta_{a,b}$ is fully faithful implies that, it suffices to show that the image of $\Delta_{a,b}$ contains the kernel and cokernel of any morphism of the form $\Delta_{a,b}(\phi) : \Delta_{a,b}(M) \to \Delta_{a,b}(N)$, as well as the direct sum $\Delta_{a,b}(M) \oplus \Delta_{a,b}(N)$. This is because if the image of $\Delta_{a,b}$ contains the (co)limit of a (co)cone over some diagram contained in the image of $\Delta_{a,b}$, then the image of $\Delta_{a,b}$ must automatically also contain the (co)limiting (co)cone over the diagram, as well as all (co)cones over the diagram for which the vertex of the (co)cone lies in the image of $\Delta_{a,b}$, as well as any map between the vertices of those (co)cones. This includes all of the morphisms guaranteed to exist by the universal property of (co)limits which factor (co)cones through the limiting (co)cone.
Now, observe that

- $\Delta_{a,b}(M)_i = M_i$ for all $i \in Q_0$
- $\Delta_{a,b}(M)(r) = M(r)$ for all $r \in Q_1$
- Each $\ker(\phi)_i$ and $\mathrm{coker}(\phi)_i$ only depends on $M(i), N(i), \phi_i$
- Each $\ker(\phi)(r)$ and $\mathrm{coker}(\phi)(r)$ only depends on $M(r), N(r), \phi_{t(r)}, \phi_{s(r)}$
- $(M \oplus N)_i$ only depends on M_i, N_i, and
- $(M \oplus N)(r)$ only depends on $M(r), N(r)$.

From this, it follows that the only thing that remains to be checked to show that $\Delta_{a,b}$ is exact are that $\Delta_{a,b}(\ker(\phi))(a\circ b) := \ker(\phi)(a)\circ\ker(\phi)(b) = \ker(\Delta_{a,b}[\phi])(a\circ b)$, $\Delta_{a,b}(\mathrm{coker}(\phi))(a\circ b) := \mathrm{coker}(\phi)(a) \circ \mathrm{coker}(\phi)(b) = \mathrm{coker}(\Delta_{a,b}[\phi])(a \circ b)$, and $(M(a) \oplus N(a)) \circ (M(b) \oplus N(b)) = (M(a) \circ M(b)) \oplus (N(a) \circ M(b))$. The last of these three criteria is clear. The other two criteria can be verified in a straightforward way using the expressions for $\ker(\phi)(r)$ and $\operatorname{coker}(\phi)(r)$ present in the proof of the exactness of d_r above. □

We now introduce an operation that "reverses" or "dualizes" the relationships between the quiver subrepresentations it is applied to.

Definition 5.16. Let Q^{op} denote the quiver obtained from Q by reversing all the arrows, i.e. $Q^{op} := (Q_0, Q_1, t, s)$. Fix a representation M of Q, and let M^* denote the representation of Q^{op} obtained from M by taking the duals of all the vector spaces and the transposes of all of the maps, i.e. $M_i^* := (M_i)^*$ for all $i \in Q_0$ and $M^*(r) := (M(r))^* = M(r)^\intercal$. We define *subrepresentation inversion* to be the function $\operatorname{Inv}_M : \mathcal{L}(M) \to \mathcal{L}(M^*)$ defined by

$$\operatorname{Inv}_M(N)_i := \{l : M_i \to k : l(N_i) = 0\}$$
$$\operatorname{Inv}_M(N)(r) := N(r)^\intercal.$$

In the following, we use the identification of finite dimensional vector spaces with their double duals. We state the following elementary properties of Inv_M without proof, as they reduce to simple exercises in linear algebra.

Proposition 5.17. $\operatorname{Inv}_M : \mathcal{L}(M) \to \mathcal{L}(M^*)$ satisfies the following for all $N_1, N_2 \in \mathcal{L}(M)$:

- $N_1 \leq N_2 \implies \operatorname{Inv}_M(N_2) \leq \operatorname{Inv}_M(N_1)$
- $\operatorname{Inv}_M(N_1 + N_2) = \operatorname{Inv}_M(N_1) \cap \operatorname{Inv}_M(N_2)$
- $\operatorname{Inv}_M(N_1 \cap N_2) = \operatorname{Inv}_M(N_1) + \operatorname{Inv}_M(N_2)$
- $\operatorname{Inv}_M(0) = M^*$
- $\operatorname{Inv}_M(M) = 0$
- $\operatorname{Inv}_{M^*} \circ \operatorname{Inv}_M = \operatorname{id}_{\mathcal{L}(M)}$
- $\operatorname{Inv}_M \circ \operatorname{Inv}_{M^*} = \operatorname{id}_{\mathcal{L}(M^*)}$

In particular, by applying the contravariant version of 5.11 to the maps $\operatorname{Inv}_M, \operatorname{Inv}_{M^*}$, we find that M is indecomposable if and only if M^* is.

5.1. **Representation Theoretic Justification.** We now give a representation theoretic justification for why a particular combination of critical arrows in Theorem 4.2 cannot occur. In the following, let M be the indecomposable representation of Q with dimension vector $\underline{d} = (1, 2, 3, 2, 2, 1)$. We work under the assumption that all maps in M are full rank.

Proposition 5.18. If $(\underline{d}, \underline{e})$ admits two type III critical arrows, then M does not have any subrepresentations N with dimension vector $\underline{e}$.

Proof. Suppose that M has a subrepresentation N with dimension vector $\underline{e}$. There are two possibilities for which critical arrows appear:

(i) two critical arrows $(3, 1) \to (2, 0)$, or
(ii) two critical arrows $(2, 2) \to (3, 2)$.

We claim that these cases are equivalent. Note that if N is a subrepresentation of M resulting in two $(3, 1) \to (2, 0)$ arrows, $\operatorname{Inv}_M(N)$ is a subrepresentation of M^* with two $(2, 2) \to (3, 2)$ arrows. Thus, without loss of generality, we may assume that we are dealing with case (ii). We call the two arrows r, r'.

Since N is a subrepresentation of M, it follows that $M(r)$ and $M(r')$ are injective maps with the same image. Thus, there exists an isomorphism $u : M(s(r)) \to M(s(r'))$ such that $M(r') \circ u = M(r)$. We now consider the auxiliary quiver Q', which appears identical to Q except with the arrow r of Q replaced with an arrow $r'^{-1} \circ r$ having source $s(r)$ and target $s(r')$. We then let M' be the representation of Q' which is identical to M on all vertices of Q' and all arrows of Q aside from r, and has $M'(r'^{-1} \circ r) := u$. Then we find that $M = d_r(\Delta_{r', r'^{-1} \circ r}(M'))$, and so by Proposition 5.13 and Proposition 5.15, since M is indecomposable, M' must also be indecomposable. Q' is either a type A_6 quiver or a type E_6 quiver. If Q' is a type A_6 quiver, then since M' has a dimension vector not consisting of exclusively 0's and 1's, by the classification of indecomposable representations of type A_n quivers, M' cannot possibly be indecomposable in this case. On the other hand, if Q' is of type E_6, then M' cannot be indecomposable because it it has 3 in its dimension vector at a degree 2 vertex of Q', which is not the case for any of the dimension vectors of the indecomposable E_6 quiver representations. □

6. Quiver Mutations

To extend our mixed dimer model for type E_6 quivers to quivers containing oriented cycles, we consider mutations of E_6 quivers to obtain base graphs for dimer covers.

We assume that quivers have no loops or 2-cycles.

Definition 6.1. A mutation of a quiver $Q = (Q_0, Q_1, s, t)$ at $i \in Q_0$, $\mu_i(Q) = (Q'_0, Q'_1, s', t')$ is given by the following procedure.

- For all arrows leaving or entering i, we reverse their direction.
- For any two vertices, $(r, q) \in Q_0$ such that there are $a > 0$ arrows from r to i and $b > 0$ arrows from i to q in Q, we add ab arrows from r to q. Then, we remove all 2-cycles.

Definition 6.2. Two quivers are *mutation equivalent* if we can obtain one by applying some sequence of mutations to the other. The mutation class of a quiver, Q, contains all quivers that are mutation equivalent to Q.

We are interested in the mutation class of E_6.

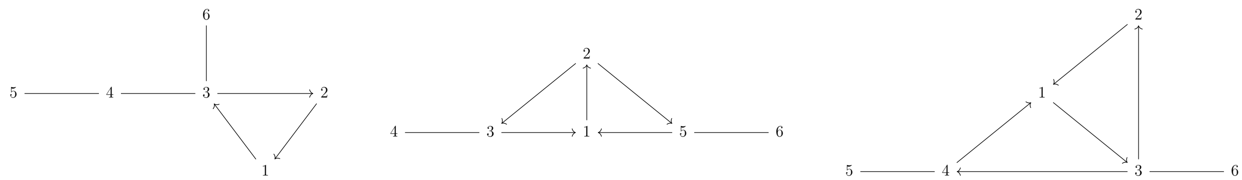
Lemma 6.3. For any of the quivers in Figure 2, we can obtain an E_6 quiver by applying at most four mutations.

Proof. Mutating any type I quiver at vertex 1 will result in a quiver isomorphic to some E_6 quiver. Mutating any type II quiver at vertices 1 and 2 will result in a quiver isomorphic to some E_6 quiver. Mutating any type III quiver at vertices 1, 2, and 3 will result in a quiver isomorphic to some E_6 quiver. Mutating any type IV quiver at vertices 1, 2, 3, and 4 will result in a quiver isomorphic to some E_6 quiver. □

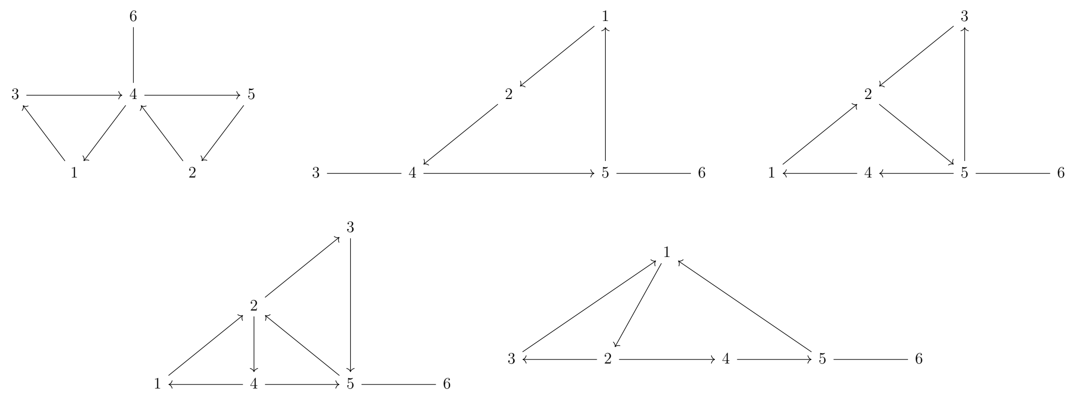
Lemma 6.4. For any quiver in the mutation class of E_6, reversing the directions of all its arrows results in a quiver that is also in the mutation class of E_6.

Proof. None of the arrows with unspecified direction are incident to the vertices at which we mutate to obtain an E_6 quiver in Lemma 6.3. In particular, it follows from the definition of a quiver mutation that if we reverse the direction of every arrow in any of these quiver, and mutate at the vertices specified in Lemma 6.3, the underlying graph obtained from each mutation will be the same as the underlying graphs obtained from mutating at those vertices in the initial quiver. In particular, the underlying graph obtained after mutating at all specified vertices will be the underlying graph of an E_6 quiver. □

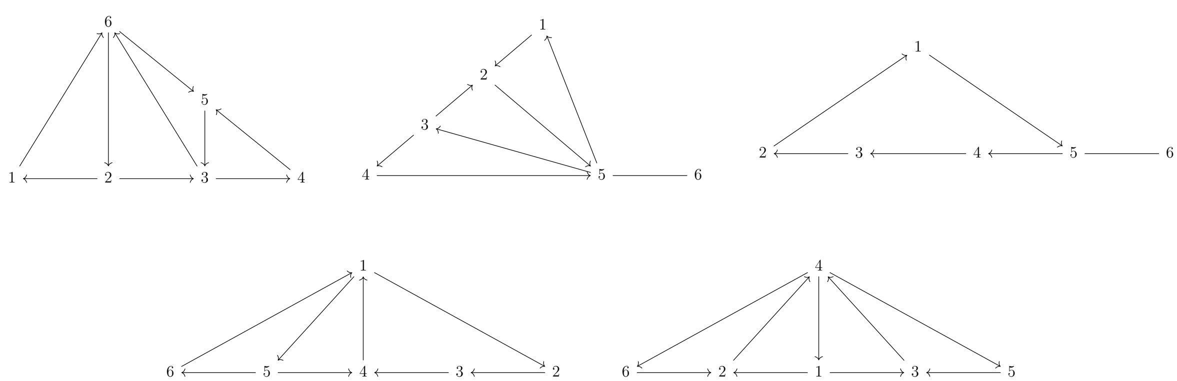
Type I E_6 mutations:



Type II E_6 mutations:



Type III E_6 mutations:



Type IV E_6 mutations:

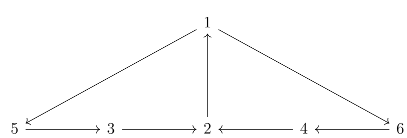


FIGURE 2. All quivers mutation-equivalent to a quiver of type E_6.

Theorem 6.5. Each quiver in the mutation class of E_6 is isomorphic to an (acyclic) E_6 quiver, or one of the quivers in Figure 2, or a quiver obtained from reversing all arrows in one of the aforementioned quivers, where the edges with no direction can be arrow of any direction.

Proof. By Lemma 6.3 and Lemma 6.4, the above quivers, as well as any quiver obtained from reversing the direction of all arrows in one of the above quivers, is in the mutation class of E_6.

When mutating at a vertex of degree 1, the resulting graph is identical except for the arrow incident to the vertex at which we mutated is reversed. In particular, it is straightforward to check that mutating at any vertex of the above quiver (or those obtained by reversing the direction of all arrows) will result in another quiver above (or one that can be obtained by reversing the direction of all arrows of a quiver above). □

7. Code

The code we wrote for our research can be found at `https://github.com/anser0/cluster-algebras`.

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