k-TRIANGULATIONS ON SURFACES

SASKIA SOLOTKO, KATHERINE TUNG, MENGYUAN YANG, AND YUCHONG ZHANG

ABSTRACT. We study k-triangulations on the half cylinder with arbitrarily many marked points on a single boundary, generalizing the work of V. Pilaud and F. Santos and expanding upon work of M. Lepoutre. Extending work of C. Stump, we develop a theory of pipe dreams on cylindrical polyominoes, enumerate these pipe dreams, and show that the corresponding pipe complexes are pure, connected, and pseudomanifolds. We conjecture there is a bijection between these cylindrical pipe dreams and periodic triangulations of polygons. We prove the purity of the simplicial complexes corresponding to k-triangulations on the half cylinder when $k = 2$. We additionally conjecture a bijection between k-triangulations on the half cylinder and pipe dreams on cylindrical polyominoes. Assuming enumeration of k-triangulations of the half-cylinder with n marked points, we prove the existence of a cyclic sieving phenomenon.

CONTENTS

1. INTRODUCTION

A k-triangulation of an n-gon is a maximal set of diagonals that forbids any subset of $k+1$ pairwise crossing diagonals. The usual notion of triangulation is a 1-triangulation. V. Pilaud, F.Santos, J. Jonsson, and many other authors have proved various structural results about k-triangulations of polygons:

(i) Every k-triangulation is *star decomposable*, that is, it is a complex of k-stars, which are a generalization of triangles.
(ii) The simplicial complex of k-triangulations of a polygon is *pure*.

(*iii*) The simplicial complex of k-triangulations of a polygon has *flip property*, that is, there always exists a unique way to replace a diagonal in one k-triangulation and obtain another k-triangulation. Furthermore, this flip graph is connected.
(*iv*) The number of k-triangulations of a polygon is given by a Catalan determinantal formula.
(*v*) The simplicial complex for k-triangulations of a polygon is a *vertex-decomposable* sphere.

In [Lep19], multitriangulations on surfaces are introduced as projections of periodic multi-triangulations of the infinite polygons. Since k-triangulations on the half-cylinder with n marked points on a single boundary do not correspond to k-triangulations on its universal cover, however, a different approach is needed. In this report we define k-triangulations on surfaces in a more natural way as a subset of edges on a surface S that do not yield $(k+1)$ crossings when lifted to the universal cover $\overline{S}$. As noted in [Lep19], the primary obstacle in proving the flip property on general surfaces comes in proving the flip property on the half cylinder. As such, the flip property on the half cylinder is the primary focus of our work. To that end, Theorem 3.9 shows that 2-triangulations on half cylinder decompose as complexes of 2-stars. Using a bijection between 2-triangulations of the half cylinder and triangulations of a regular polygon invariant under certain rotations, Corollary 4.3 shows that the corresponding complexes are pure using the corresponding result in [PS08].

2. Background

Definition 2.1. Let $\mathcal{C}_n$ denote the annulus with n marked points $\alpha_{[0]}, \dots, \alpha_{[n-1]}$ on its outer boundary, where $[i]$ is the modulo class of i mod n. We refer to $\mathcal{C}_n$ as a half-cylinder or $(n, 0)$-annulus.

In general, we will let $\mathcal{S}$ denote a surface and $\overline{\mathcal{S}}$ its universal cover. For most of this section, we will only consider $\mathcal{C}_n$, the half cylinder as defined above. The universal cover of $\mathcal{C}_n$, denoted $\overline{\mathcal{C}}_n$, has vertices $\{\alpha_i \mid i \in \mathbb{Z}\}$. Take any vertex $v = \alpha_i \in \mathcal{C}_n$, we define $v + j \coloneqq \alpha_{i+j}$ for any integer j. An edge e on $\overline{\mathcal{C}}_n$ is given by two distinct vertices u, v and is denoted by $[u, v]$ or $[v, u]$. Similarly we define $e + n \coloneqq [u + n, v + n]$. We say an edge e is a translation of an edge f if $e = f + \ell n$ for some $\ell \in \mathbb{Z}$, equivalently $\pi(e) = \pi(f)$.

There is a natural projection map $\pi : \overline{\mathcal{C}}_n \to \mathcal{C}_n$. Note that on vertices, we have $\pi(v + n) = \pi(v)$ for any vertex $v \in \overline{\mathcal{C}}_n$.

Moving forward, we will not refer to vertices of $\mathcal{C}_n$ or $\overline{\mathcal{C}}_n$ using the subscripts in the definition.

Note that there is a natural order of edges on $\overline{\mathcal{C}}_n$ where $u < v$ if and only if $u = v + n$ for some $n \in \mathbb{Z}^+$ consistent with a counterclockwise orientation on $\overline{\mathcal{C}}_n$. When considering a subset of vertices of $\mathcal{C}_n$, we have a *cyclic order* $\prec$ given by the counterclockwise cyclic order on the corresponding finite or infinite polygon.

On $\overline{\mathcal{C}}_n$, edges e_1 and e_2 intersect if $a_1 \prec b_1 \prec a_2 \prec b_2$ for some labeling of vertices $e = [a_1, b_1], e_2 = [a_2, b_2]$. A k-crossing on $\overline{\mathcal{C}}_n$ is given by a set of edges $E = \{e_1, e_2, \cdots, e_k\}$ that pairwise intersect.

Take a subset $\{z_0, \cdots, z_{2k}\}$ of vertices on $\overline{\mathcal{C}}_n$ ordered such that

$$z_0 \prec z_1 \prec \cdots \prec z_{2k}.$$

A k-star polygon S defined on this subset contains all vertices z_i and edges $[z_i, z_{i+k}]$ on $\overline{\mathcal{C}}_n$ with subscripts reduced modulo n. Moving forward, the vertices of S will be denoted in star order as $s_0, \cdots, s_{2k}$ where $s_j = z_{kj}$. This numbering is unique given a choice of s_0. We use these definitions of k-crossings and k-stars on $\overline{\mathcal{C}}_n$ to define analogously k-crossings and k-stars on $\mathcal{C}_n$.

Definition 2.2. A *k-crossing* on $\mathcal{C}_n$ is the projection $\pi(E)$ of a k-crossing $E = \{e_1, \cdots, e_k\}$ on $\overline{\mathcal{C}}_n$.

Definition 2.3. A *k-star* on $\mathcal{C}_n$ is the projection $\pi(S)$ of a k-star S on $\overline{\mathcal{C}}_n$.

We define a *k-triangulation* $\overline{T}$ on $\overline{\mathcal{C}}_n$ to be a maximal set of n-periodic edges on $\overline{\mathcal{C}}_n$, that is $v \in \overline{T} \iff v + n \in \overline{T}$. As with k-crossings and k-stars, the definition of a k-triangulation on $\mathcal{C}_n$ follows from that on $\overline{\mathcal{C}}_n$.

Definition 2.4. A *k-triangulation* T of $\mathcal{C}_n$ is the projection $\pi(\overline{T})$ of a k-triangulation $\overline{T}$ on $\overline{\mathcal{C}}_n$.

For edges $u, v \in \overline{\mathcal{C}}_n$, we say $v - u = j$ if $u + j = v$. The length of an edge $[u, v] \in \overline{\mathcal{C}}_n$ is given by $|v - u|$. We also say $\pi^{-1}([v, u])$ has length $|v - u|$. This informs the following 3-definitions.

Definition 2.5. An edge $[u, v] \in \overline{T}$ is a *k-irrelevant* edge if $|v - u| < k$

Note that no k-irrelevant edge is contained in a k-crossing $E \subset \overline{\mathcal{C}}_n$. Thus every k-triangulation T (resp. $\overline{T}$) will contain every k-irrelevant edge.

Definition 2.6. An edge $[u, v] \in \overline{T}$ is a *k-boundary edge* if $|v - u| = k$.

As with irrelevant edges, no edge of length k is contained in a k-crossing $E \subset \overline{\mathcal{C}}_n$. Thus every k-triangulation T (resp. $\overline{T}$) will contain every edge of length k.

Definition 2.7. An edge $[u, v] \in \overline{T}$ is a *k-relevant* edge if $k < |v - u| \leq nk$.

In Lemma 3.2 we show that every k-triangulation T on $\mathcal{C}_n$ contains exactly one edge of length kn.

Definition 2.8. The *rank* of a k-triangulation T on $\mathcal{C}_n$ is the number of k-relevant edges in T.

Definition 2.9. An *angle* of a k-triangulation $\overline{T}$ on $\overline{\mathcal{C}}_n$ is given by a pair of edges $[u, v], [v, w] \in \overline{T}$ such that $u \prec v \prec w$ and there exist no angle bisectors $[v, w_0]$ in $\overline{T}$ with $w_0 \prec u \prec v \prec w$. We will denote such an angle by $\angle(u, v, w)$. An angle $\angle(u, v, w)$ of $\overline{T}$ is k-relevant if $[u, v]$ or $[v, w]$ is k-relevant.

Note that by the above, every k-relevant angle will either contain two k-relevant edges or a k-relevant edge and a k-boundary edge.

Many of the above definitions can be extended to orientable surfaces. In the future when we consider k-triangulations beyond $\mathcal{C}_n$, we will let $\mathcal{S}$ denote a surface and $\overline{\mathcal{S}}$ its universal cover. [Lep19] notes that similar the half-cylinder, the combinatorial information of k-crossings on the universal cover for general surfaces with finitely many marked points on finitely many boundaries is modeled by an infinite polygon.

3. Multitriangulations on the half cylinder

3.1. **Preliminary Lemmas.**

In this subsection, let T denote a 2-triangulation of $\mathcal{C}_n$, the half cylinder or $(n, 0)$-annulus corresponding to a k-triangulation $\pi^{-1}(T) = \overline{T}$ on the universal cover $\overline{\mathcal{C}}_n$.

Lemma 3.1. A k-triangulation T on $\mathcal{C}_n$ contains no edges of length $\ell > kn$.

Proof. Suppose otherwise, let $e = [v, v+\ell]$ be an element of the preimage of this length $\ell > kn$ edge on $\overline{\mathcal{C}}_n$. We have

$$v < v+n < \cdots < v+kn < v+\ell < v+\ell+n < v+\ell+2n < \ldots v+\ell+kn,$$

showing that $\{e, e+n, e+2n, \ldots, e+kn\}$ forms a $(k+1)$-crossing, which cannot be the case since $\overline{T}$ is $(k+1)$-crossing free. $\square$

Lemma 3.2. Every k-triangulation T on $\mathcal{C}_n$ has exactly one edge of length kn.

Proof. Existence: Suppose otherwise, we pick an edge in T with maximal length and take $e+tn, t \in \mathbb{Z}$ to be the lifting of this edge in $\overline{T}$. Denote $e = [v, v+\ell]$. We claim: $\overline{T} \cup \{f + tn \mid t \in \mathbb{Z}\}$ is $(k+1)$-crossing free for the edge $f := [v, v+2kn]$.

Suppose there exists a $(k+1)$-crossings in $\overline{T} \cup \{f + tn \mid t \in \mathbb{Z}\}$. Note that if a $(k+1)$-crossing E involves $f + t_1 n$ and $f + t_2 n$, then for every $r \in [t_1 n, t_2 n]$ such that $f + rn \notin E$, we have that $f + rn$ intersects with every edges in E. Let $j \geq 1$ be minimal such that there exists a $(k+1)$-crossing involving only j translations of $f + tn$. Without loss of generality, let them be $f, f+n, \ldots, f+(j-1)n$. Note that $f+(j-1)n$ and $e+(j-1)n$ do not intersect. So we can replace $f + (j-1)n$ by $e + (j-1)n$, and then we get a $(k+1)$-crossing involving $j-1$ translations of $f + tn$, contradicting with the minimality of j. So we have proved the claim. By the claim, T is not maximal, and we get a contradiction. Hence there exists an edge of length kn in T.

Uniqueness: Suppose there are two distinct edges of length kn in T, let $e = [v, v+kn]$ and $f = [v+r, v+r+kn]$ be a copy of these two edges respectively on the universal cover $\overline{\mathcal{S}}$ such that $0 \leq r \leq n-1$. Then the edges $e, f, e+n, e+2n, \ldots e+(k-1)n$ form a $(k+1)$-crossing, which cannot be the case since $\overline{T}$ is $(k+1)$-crossing free. $\square$

Lemma 3.3. Let $[u, v]$ and $[w, z]$ be k-boundary or k-relevant diagonals with $v > u$ and $z > w$. If $u < z$ and $w + kn < v$, then $[u, v]$ and $[w + kn, z + kn]$ cross.

Proof. We will show that $u < w + kn < v < z + kn$. Since the middle inequality is given by the assumption, if suffices to show $u - w < kn$ and $v - z < kn$. Because $[u, v]$ and $[w, z]$ are k-boundary or k-relevant diagonals, we have $v - u \leq kn$ and $z - w \leq kn$, which implies that

$$u - w < z - w \leq kn,$$

and

$$v - z < v - u \leq kn.$$

$\square$

Lemma 3.4. Let $\overline{T}$ be the lift of a k-triangulation T of $\mathcal{C}_n$. Let $\angle(u, v, w)$ be an angle of $\overline{T}$ such one of $[u, v]$ and $[v, w]$ is not of length kn. Let $E = \{[a_1, b_1], \ldots, [a_{k-1}, b_{k-1}]\} \subseteq \overline{T}$ be the v-maximal $(k-1)$-crossing intersecting $\angle(u, v, w)$ with order $u \prec a_1 \prec a_2 \prec \cdots \prec a_{k-1} \prec v \prec b_1 \prec b_2 \prec \cdots \prec b_{k-1} \prec w$. Then $|w - a_{k-1}| \leq kn$, and similarly, $|b_1 - u| \leq kn$.

Proof. **Case 1**: Suppose $u < v < w$. Then $u < a_1 < \cdots < a_{k-1} < v < b_1 < \cdots < b_{k-1} < w$. For the sake of contradiction, suppose $w - a_{k-1} > kn$. We claim that $[u + kn, v + kn]$, $[a_1 + kn, b_1 + kn]$, $\ldots$, $[a_{k-1} + kn, b_{k-1} + kn]$, and $[v, w]$ form a $(k+1)$-crossing. Since the first k diagonals form a shift of k-crossing by kn, it suffices to show that they also cross $[v, w]$.

We first show that $[u+kn, v+kn]$ and $[v,w]$ cross. Notice that in this case, if one of $[u,v]$ and $[v,w]$ is of length kn, then the other one is necessarily of length kn. Hence, both of them have length strictly less than kn. In other words, $v < u+kn$ and $w < v+kn$. Combining with the fact that $u+kn < a_{k-1}+kn < w$, we have $v < u+kn < w < v+kn$ and thus $[u+kn, v+kn]$ and $[v,w]$ cross. Second, by Lemma 3.3, for each $1 \le i \le k-1$, we know $[a_i+kn, b_i+kn]$ and $[v,w]$ cross because $v < b_i$ and $a_i+kn < a_{k-1}+kn < w$.

Case 2: Suppose $w < u < v$. Then $w < u < a_{k-1} < v$. Since $[v,w]$ is k-relevant or k-boundary, we have $v-w \le kn$. Because $a_{k-1} < v$, we have $a_{k-1}-w < kn$ as desired.

Case 3: Suppose $v < w < u$. Then $v < b_1 < \cdots < b_{k-1} < w$. If $u < a_{k-1}$, given that $[a_{k-1}, b_{k-1}]$ is k-relevant or k-boundary, we obtain $a_{k-1}-w < a_{k-1}-b_{k-1} \le kn$. Hence, we further assume that $a_{k-1} < v$. Then there exists some $1 \le i \le k-1$ such that $a_i < \cdots < a_{k-1} < v < b_1 < \cdots < b_{k-1} < w < u < a_1 < \cdots < a_{i-1}$. For the sake of contradiction, suppose $w-a_{k-1} > kn$. We claim that $[a_1,b_1], \ldots, [a_{i-1},b_{i-1}], [a_i+kn, b_i+kn], \ldots$, and $[a_{k-1}+kn, b_{k-1}+kn]$ form a $(k-1)$-crossing intersecting $\angle(u,v,w)$, which violates the v-maximality of E. Since the first $i-1$ and the last $k-i$ diagonals are already mutually crossing, it suffices to show the following three things:

(i) $[b_j, a_j]$ and $[a_\ell+kn, b_\ell+kn]$ cross for $1 \le j \le i-1$ and $i \le \ell \le k-1$.
(ii) $[a_\ell+kn, b_\ell+kn]$ and $[v,u]$ cross for $i \le \ell \le k-1$.
(iii) $[a_\ell+kn, b_\ell+kn]$ and $[v,w]$ cross for $i \le \ell \le k-1$.

By Lemma 3.3, (i) is true because $b_j < b_\ell$ and $a_\ell+kn \le a_{k-1}+kn < w < a_j$; ($ii$) is true because $v < b_\ell$ and $a_\ell+kn \le a_{k-1}+kn < w < u$; and ($iii$) is true because $v < b_\ell$ and $a_\ell+kn \le a_{k-1}+kn < w$. □

3.2. **2-triangulations on the half-cylinder.** The following lemmas and theorems only relate to 2-triangulations on the half cylinder C_n. We begin with some preliminary lemmas about 3-crossings on C_n before proving the star decomposition and a key maximality lemma.

Lemma 3.5. Let $e=[u_1,u_2]$, $f=[w_1,w_2]$ be edges in $\overline{T}$ annulus positioned such that $u_1 < w_1 < u_2 < w_2$. Then $u_2-w_1 \le n$.

Proof. Assuming that $u_2-w_2 > n$ gives $w_1-u_1, w_2-u_2 < n$. Then we have $u_1 < w_1 < u_1+n < u_2 < w_2 < u_2+n$, yielding a 3-crossing $[u_1,u_2],[w_1,w_2],[u_1+n,u_2+n]$. Thus since $\overline{T}$ is 3-crossing free we must have $u_2-w_1 \le n$. □

Lemma 3.6. Let $e=[u_1,u_2], f=[w_1,w_2] \in \overline{T}$ be edges on the $n+0$ annulus of length $> n$ positioned such that $u_1 < w_1 < u_2 < w_2$. Then $w_2-u_1 \ge 2n$.

Proof. Assume for contradiction that $w_2-u_1 < 2n$, we will produce a 3-crossing in $\overline{T}$. Since $u_2-u_1 > n$ and $w_2-u_1 < 2n$, we must have $w_2-u_2 < n$. Similarly, $w_1-u_1 < n$, giving

$$w_1-n < u_1 < w_1 < w_2-n < u_2 < w_2$$

and yielding a 3-crossing $[u_1,u_2],[w_1,w_2],[w_1-n,w_2-n]$. □

Lemma 3.7. Let e,f be edges of length $> n$ on the $(n,0)$-annulus such that $e,f,f+n$ gives a 3-crossing. Then there exists a 3-crossing containing two elements of $\pi^{-1}(\pi(e))$ and one copy of $\pi^{-1}(\pi(f))$

Proof. Let $e=[a,b]$ with $a<b$ and $f=[c,d]$ with $c<d$. We have three cases:

Case 1: $a < c < c+n < b < d < d+n$: We have $b-c > n$, as such we also have $a-n < a < c < b-n < b < d$, giving a 3-crossing $e, e-n, f$.

Case 2: $c < c+n < a < d < d+n < b$: Again we have $b - c > n$, then also $c < a < a+n < d+n < b < b+n$, giving a 3-crossing $e, e+n, f+n$.

Case 3: $c < a < d < c+n < b < d+n$: If $a+n < d$, we get $c < a < a+n < d < b < b+n$, yielding a 3-crossing $e, e+n, f$. If $a+n \geq d$, note that $b+n > d+n$ since $b > d$, giving a 3-crossing $e, e+n, f+n$. □

In the following let $\angle(u, v, w)$ denote an angle of $\overline{T}$ such that $[u, v]$ or $[v, w]$ are not of length $2n$. Let $e = [a, b] \in \overline{T}$ be the unique v-maximal edge intersecting $\angle(u, v, w)$ labeled such that $u \prec a \prec v \prec b \prec w$.

Lemma 3.8. Assume there exists a 2-crossing f_1, f_2 intersecting $[u, b]$ with $f_1 \in \overline{T}$ and $f_2 \in \pi^{-1}(\pi([u, b]))$. Then there also exists a 3-crossing $[u, b], f_1', f_2'$ for some $f_1', f_2' \in \overline{T}$

Proof. By Lemma 3.4 we know that either f_1 or f_2 is not contained in $\pi^{-1}(\pi([u, b]))$. Without loss of generality we can then assume that $f_1 \in \overline{T}$. By the conditions laid out in the claim, we can then determine that $w \preceq c_1 \prec u \prec d_1 \preceq a$ or $v \preceq d_1 \prec b \prec c_1 \prec w$ where $f_1 = [c_1, d_1]$. Note that if $f_2 \in \overline{T}$ as well the claim is satisfied under (iii), so we will proceed assuming $f_2 \in \pi^{-1}(\pi([u, b]))$. If there exists a 3-crossing amongst the edges $f_2, [a, b], [v, w], [v, u]$ then (iii) is satisfied. Otherwise we have four possibilities:

(i) $w \preceq c_i \prec u \prec d_i \preceq a$ for $i \in \{1, 2\}$
(ii) $v \preceq d_i \prec b \prec c_i \prec w$ for $i \in \{1, 2\}$
(iii) f_2 bisects $\angle(u, v, w)$ and $w \preceq c_1 \prec u \prec d_1 \preceq a$
(iv) f_2 intersects $\angle(u, v, w)$ positioned v-farther than $[a, b]$.

In case (i), a translation of the 3-crossing $[v, u], f_1, f_2$ will satisfy the claim. Likewise in case (ii), a translation of the 3-crossing $[b, a], f_1, f_2$ will satisfy the claim.

In case (iii), let $f_2 = [v, c_2]$. If $c_2 \in \pi(\pi^{-1}(u))$ then $c_1 < c_2 < u < d_1$ or $d_1 < v < b < c_1$, giving $|c_1 - d_1| > n$ and satisfying the claim by Lemma 3.7. If $v \in \pi(\pi^{-1}(u))$, we have two possibilities: $u < c_1 \leq a < v < b < w \leq d_1$ and $b < w \leq c_1 < u < d_1 < a < v < b$, if $|c_1 - d_1| > n$ we again are done by Lemma 3.7. Thus we proceed assuming $|c_1 - d_1| \leq n$ and $v = u \pm n$. We have two possibilities: $u < d_1 \leq a < v < b < w \leq c_1$ and $b < w \leq c_1 < u < d_1 < a < v < b$. If $u < d_1 \leq a < v < b < w \leq c_1$, we additionally have $u < d_1 < v = u+n < b < c_1 < v+n$, giving a 3-crossing $[u+n, v+n], [u, b], [c_1, d_1]$, satisfying the claim. If $b < w \leq c_1 < u < d_1 < a < v < b$, since $a - b > u - b > n$ we get a 3-crossing $[b, a], [b+n, a+n], [c_1, d_1]$, which cannot be the case since $\overline{T}$ is 3-crossing free. This concludes case (iii).

For case (iv), we need only consider the position of f_2. Set $f_2 = [c_2, d_2]$ with $u \prec d_2 \preceq a \prec v \prec b \preceq c_2 \prec w \prec u$. Again if $|d_2 - c_2| > n$ we are done by Lemma 3.7, so we proceed assuming $|d_2 - c_2| \leq n$. If $c_2 = b \pm n$, then we must have $|v - w| > n$. Then since $[v, w], [b, u], [c_2, d_2]$ gives a 3-crossing, Lemma 3.7 produces a 3-crossing containing one translate of $[b, u]$ and two translates of $[v, w]$, in which case we are done If $d_2 = b \pm n$ and $|v - w| > n$, again we can use Lemma 3.7 to produce a 3-crossing satisfying the claim. Thus we proceed assuming $|v - w| \leq n$. We have two cases: $c_2 \leq w < u < d_2 \leq a < v < b$ and $d_2 \leq a < v < b < c_2 \leq w < u$. Assuming $c_2 \leq w < u < d_2 \leq a < v < b$, we have that $b - a < b - u$ and $b - n = d_1$. This gives $a - n < w < u < b - n = d_2 < v$, showing that $[a - n, b - n]$ intersects $\angle(u, v, w)$ is v-farther than $[a, b]$ (which is positioned $a < v < b$) and contradicting the conditions of the claim. In the case that $d_1 \leq a < v < b < c_2 \leq w < u$, since $u - v > u - b > n$ and $[v, u], f_2, [u, b]$ yields a 3-crossing, Lemma 3.7 gives a 3-crossing which translated will satisfy the claim. This completes case (iv). □

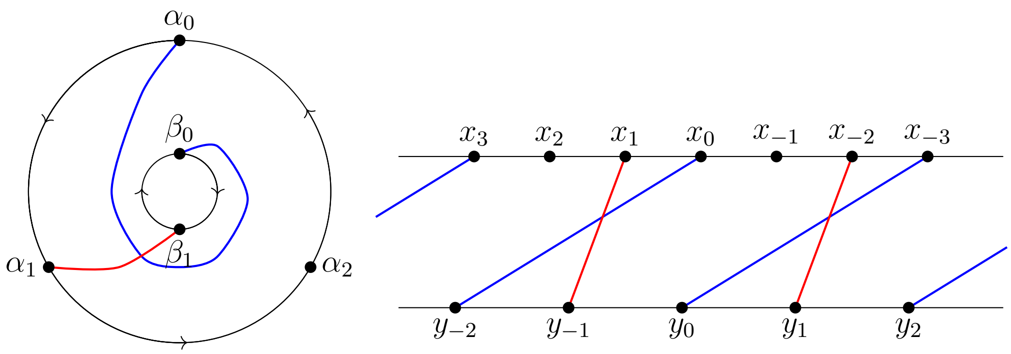


FIGURE 1. A $(3, 2)$-annulus with two edges (left) and the lifts of the vertices and edges on the universal cover (right)

Theorem 3.9. Every angle $\angle(u, v, w) \subset$ of $\overline{T}$ with $|u - v| < 2n$ or $|w - v| < 2n$ belongs to a unique 2-star contained in $\overline{T}$.

Proof. Let $e = [a, b]$ denote the unique v-maximal edge intersecting $\angle(u, v, w)$ labeled such that $u \prec a \prec b \prec v \prec w$. Suppose that $[u, b] \notin \overline{T}$. Then there exists a 2-crossing $\{f_1, f_2\}$ that prevents the edge $[u, b]$. By Lemma 3.4 we know that $f_1\overline{T}$ or $f_2 \in \overline{T}$, then by Lemma 3.8 we may additionally assume that $f_1, f_2 \in \overline{T}$. However every possible positioning of f_1, f_2 will yield a 3-crossing $\{f_1, f_2, [u, v]\}$, $\{f_1, f_2, [w, v]\}$, or $\{f_1, f_2, [a, b]\}$, which cannot be the case since $\overline{T}$ is 3-crossing free. Thus we conclude that $[u, b]$ (and likewise $[a, w]$) are contained in $\overline{T}$. This shows that $\overline{T}$ contains the 2-star polygon about vertices $\{u, a, v, b, w\}$. Similarly to the polygon case, any angle of a 2-star polygon must be an angle of $\overline{T}$, showing uniqueness. □

Theorem 3.10. Let e be an edge on the universal cover of the $(n, 0)$-annulus of length $< 2n$ not contained in the preimage $\overline{T}$ of a 2-triangulation T. Then $T \cup \{e\}$ yields a 3-crossing.

Proof. Set $e = [v_1, v_2]$. Since $|v_2 - v_1| < 2n$, e bisects an angle $\angle(u, v_1, w)$ of $\overline{T}$ such that $[u, v_1]$ or $[v_1, w]$ has length $< 2n$. Then by Theorem 3.9, $\angle(u, v_1, w)$ is contained in a 2-star about vertices $v \prec b \prec w \prec u \prec a$. Since $[v_1, v_2]$ bisects $\angle(u, v_1, w)$, $[v_1, v_2]$, $[a, w]$, $[b, u]$ gives our desired 3-crossing. □

4. BIJECTION: k-TRIANGULATIONS OF ANNULUS AND PERIODIC k-TRIANGULATIONS OF POLYGON

Definition 4.1. An n-periodic k-triangulation of a $2kn$-gon is a k-triangulation of the polygon that is invariant under rotation by $\frac{2\pi}{2k} = \frac{\pi}{k}$ radians and all its integer multiples.

As in the definition of k-triangulation on a polygon, in the definition of n-periodic k-triangulation, we assume that it is a *maximal* collection of diagonals that avoids a $(k + 1)$-crossing. The following theorem will shows n-periodic k-triangulations of the $2kn$-gon exist. The remainder of theorems in this section depend on Theorem 3.9 and Theorem 3.10. As such, for the remainder of this section we will set $k = 2$. In the future we hope to generalize Theorem 3.9 and Theorem 3.10, allowing the work in this section to apply for general k.

Theorem 4.2. For $k = 2$, There is a bijection ϕ between k-triangulations of $\mathcal{C}_n$ and n-periodic triangulations of the $2kn$-gon.

The bijection is as follows: Let E be a k-triangulation of $\mathcal{C}_n$ corresponding to $\overline{E}$ on $\overline{\mathcal{C}}_n$. Label all the vertices of the $2kn$-gon counter-clockwise by $\alpha_{[i]}$ where $[i]$ is the congruence class of i modulo $2kn$. We define $\phi(E) = \{[\alpha_{[i]}, \alpha_{[j]}] \mid [\alpha_i, \alpha_j] \in \overline{E}\}$. When restricted to k-triangulations T on $\overline{\mathcal{C}}_n$, ϕ gives our desired bijection

Proof. Observe that ϕ yields a bijection between subsets of edges of $\mathcal{C}_n$ and n-periodic subsets of edges on the $2kn$-gon. Thus it will suffice to show ϕ and ϕ^{-1} preserve $(k+1)$-crossings and maximality.

First note that a $(k+1)$-crossing on $\overline{\mathcal{C}}_n$ is given by edges $[u_1, v_1], [u_2, v_2], \cdots, [u_{k+1}, v_{k+1}]$ with vertices $\{u_1 \cdots u_{k+1}, v_1, \cdots, v_{k+1}\}$ in some cyclic order, for example $u_1 \prec v_1 \prec u_2 \prec v_2$ when $k = 2$. Since the span of these vertices is less than $2kn$ on $\overline{\mathcal{C}}_n$, the given cyclic orientation is preserved by projection to $\mathcal{C}_n$ and ϕ. Since information about crossing is uniquely given by cyclic order on the endpoints, or $(k+1)$-crossing $\overline{E}$ on $\overline{\mathcal{C}}_n$ will yield a $(k+1)$-crossing on the $2kn$-gon. Similarly, a $(k+1)$-crossing F on the $2kn$-gon corresponds to a unique $(k+1)$ crossing $\phi^{-1}(E) = F$ on $\mathcal{C}_n$. From this we determine that a collection of edges $E \subset \mathcal{C}_n$ is $(k+1)$-crossing free if and only if $\phi(E)$ is $(k+1)$ crossing free.

Now consider a k-triangulation T of $\mathcal{C}_n$. If $e \cup \phi(T)$ is $(k+1)$-crossing free, we can use Theorem 3.10 and the above to show that $\phi^{-1}(e) \cup T$ is $(k+1)$-crossing free, showing $\phi^{-1}(e) \in T$ and thus $e \in \phi(T)$. Thus ϕ preserves k-triangulations, specifically the maximality property. Similarly we can show ϕ^{-1} preserves k-triangulations. □

As illustrated in the proof of Theorem 4.2, ϕ sends edges of $\mathcal{C}_n$ of length $\leq 2n$ to edges of the $2kn$-gon: In particular, when applied to a k-triangulation T of $\mathcal{C}_n$, ϕ will yield a k-triangulation of the $2kn$-gon that is also n-periodic. For the below proofs, note that the size of the orbit of an edge e of the $2kn$-gon under our action of $\mathbb{Z}/2k\mathbb{Z}$ via rotation is $2k$ if e has length $< nk$ and k if e has length $= nk$. Additionally, the size of the orbit of any k-star is $2k$. Using the bijection outlined in Theorem 4.2 and this information about orbits, we can determine the number of k-stars, k-relevant edges, and edges in a given k-triangulation of $\overline{\mathcal{C}}_n$.

Corollary 4.3. For $k = 2$, any k-triangulation of $\mathcal{C}_n$ contains exactly $n-1$ k-stars, $k(n-1)$ k-relevant edges, and $k(2n-1)$ edges.

Proof. Corollary 4.4 of [PS08] states that every k-triangulation of the $2kn$-gon has exactly $2nk - 2k$ k-stars, $k(2nk-2k-1)$ k-relevant edges, and $k(4kn-2k-1)$ edges. This gives that every k-triangulation of $\mathcal{C}_n$ has $(2nk-2k)/2k = n-1$ k-stars, $((2nk^2 - 2k^2 - k) + k)/2k = k(n-1)$ k-relevant edges, and $((4nk^2 - 2k^2 - k) + k)/2k = k(2n-1)$ total edges. □

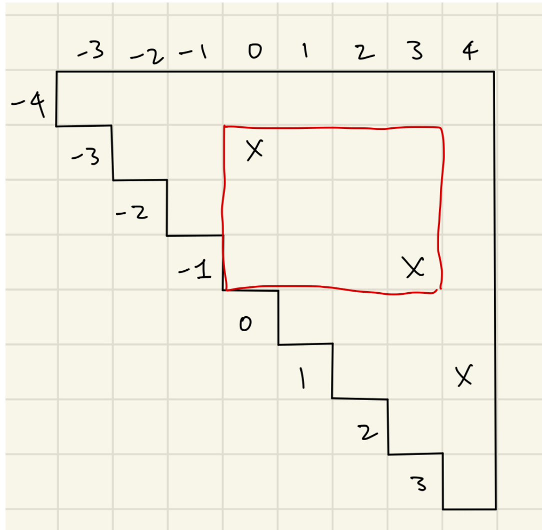
Corollary 4.4. For $k = 2$, the k-triangulations of $\mathcal{C}_n$ are exactly the $(k+1)$-crossing free subsets of edges on $\mathcal{C}_n$ of cardinality $k(2n-1)$.

Proof. This follows directly from Corollary 4.3, Theorem 4.2, and [PS08] corollary 4.4. □

5. Cylindrical Polyominoes

In this section, we transform the problem of k-triangulations of $\mathcal{C}_n$ into an equivalent problem framed in terms of subsets of a specified polyomino, as introduced by [Jon05]. This language may help us visualize some technical proofs.

We start with a special polyomino $P = \{(i,j) \in \mathbb{Z}^2 \mid i < j\} \subseteq \mathbb{Z}^2$, in which every pair (i,j) represents a diagonal $[\alpha_i, \alpha_j]$ on the universal cover $\overline{\mathcal{C}}_n$. To be consistent with the next section of pipe dreams, we think of the y-axis in the opposite direction, as shown by the figure below. Visually, P contains the lattice points in a half plane above the line $\{(-i,i) \mid i \in \mathbb{R}\}$.



On the universal cover, a *k-crossing* is a set of diagonals $(\alpha_{i_1},\alpha_{j_1}),\dots,(\alpha_{i_k},\alpha_{j_k})$ such that $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. The inequalities of the form $i_1 < i_2$ and $j_1 < j_2$ hold if and only if (i_1,j_1) lies in the northwest section of (i_2,j_2) in P. The middle inequality $i_k < j_1$ holds if and only if the smallest rectangle containing (i_1,j_1) and (i_k,j_k) is contained in P. For example, in the figure above, $(-3,0)$ and $(-1,3)$ forms a 2-crossing as the red box is contained in P; however, they do not form a 3-crossing with $(1,4)$.

Definition 5.1. A *k-crossing* in P is a set of lattice points $(i_1,j_1),\dots,(i_k,j_k) \in P$ satisfying

$$i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k.$$

For a k-triangulation of $\mathcal{C}_n$, we know that every diagonal on the universal cover repeats itself after shifting its endpoints by n. Similarly, we make the following definition for triangulations of P.

Definition 5.2. An *n-cylindrical k-triangulation* of P is a maximal subset $T \subseteq P$ that satisfies:

(i) T is free of $(k+1)$-crossing, and
(ii) for each $(i,j) \in T$, we have $(i \pm n, j \pm n) \in T$.

Here are some observations on n-cylindrical $(k+1)$-crossing-free subsets of P.

First, for $(i,j) \in \mathbb{Z}^2$ with $j-i > kn$, it always forms a $(k+1)$-crossing with k translational copies of itself, so it is never contained in an n-cylindrical k-triangulation. In the previous section, such a diagonal is called *k-irrelevant*. Second, for $(i,j) \in \mathbb{Z}^2$ with $0 < j-i < k$, it never forms a $(k+1)$-crossing with other diagonals, so it is contained in every n-cylindrical k-triangulation, and it is also called *k-irrelevant* in the previous section. Then the *k-boundary* diagonals are of the form $(i,i+k) \in \mathbb{Z}$, and the *k-relevant* diagonals are of the form $(i,j) \in \mathbb{Z}^2$ with $k+1 \le j-i \le kn$. Visually, all possible k-relevant diagonals form a bi-infinite staircase, and each level has length $kn-k-1$.

Because P contains all the translational copies of each of its element, one can easily show the following proposition.

Proposition 5.3. There is a bijection between k-triangulations of $\mathcal{C}_n$ and n-cylindrical k-triangulations of P.

Proof. This bijection is induced by the fact that every (i, j) corresponds to the diagonal $[\alpha_i, \alpha_j]$ on the universal cover $\overline{\mathcal{C}}_n$. □

Next, we will define k-relevant angles and k-stars on P. Let $T \subseteq P$ be an n-cylindrical k-triangulation. Then we define a *k-relevant angle* to be a pair of lattice points $\{(i, j), (r, s)\} \subseteq T$ if (i, j) and (r, s) are k-relevant or k-boundary, and if they satisfy one of the following condition:

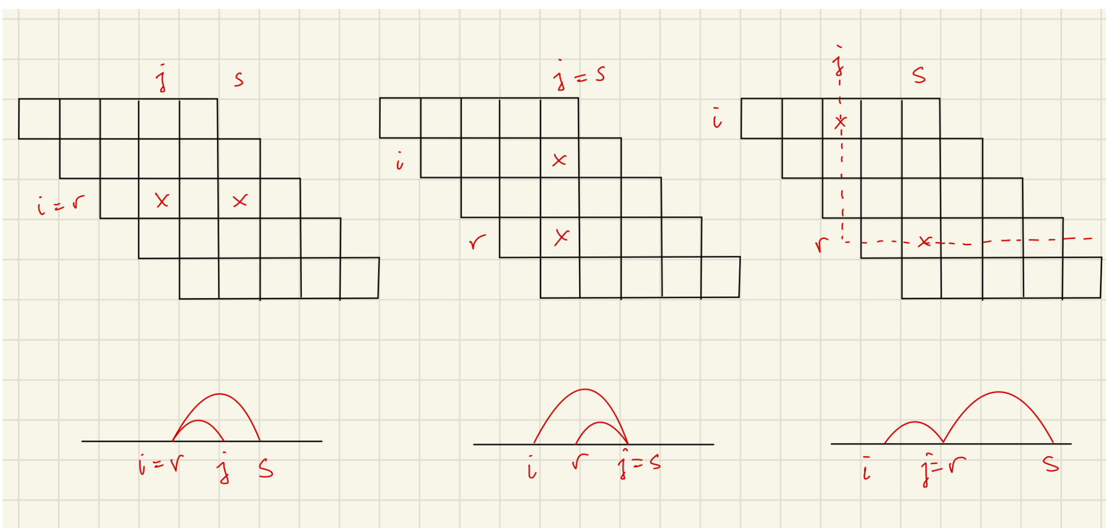
1. $i = r$, $j < s$, and $(i, \ell) \notin T$ for all $j < \ell < s$;
2. $j = s$, $i < r$, and $(\ell, j) \notin T$ for all $i < \ell < r$;
3. $j = r$, and $(\ell, j), (r, \ell) \notin T$ for all $\ell < i$ or $\ell > s$, respectively.

This definition is tiresome, but visually, a k-relevant angle is:

1. a pair of lattice points on the same row with no elements of T in between;
2. a pair of lattice points on the same column with no elements of T in between;
3. the column of one point and the row of the other intersects at some $(i, i) \in \mathbb{Z}^2$, and no elements of T are beyond these two points.

In the universal cover, these three cases are:

1. two diagonals with common end on the left;
2. two diagonals with common end on the right;
3. two diagonals with common end in the middle.



Now we walk zigzag on the k-relevant angles according to the following rule:

1. Choose some element in T.
2. Increase the second entry until reaching an element in T. If it terminates, go to step (3); if not, go to step (4).
3. Increase the first entry until reaching an element in T. Go to step (2).
4. Move the first entry to the second. Find the element in T with this fixed second entry and the smallest first entry. Go to step (2).

Intuitively, we travel rightwards on P, hit some element in T, travel downwards, hit another element, and repeat. In a special case, we sometimes need to "loop" to the top. The equivalent process on the universal cover is that we travel along diagonals, and at each endpoint, we rotate counter-clockwise until hitting another diagonal in the k-triangulation and travel again. Because of the existence of k-boundary diagonals at each vertex, we know step (3) and step (4) always terminate. For better visualization of this idea, see the next section of pipe dreams.

Definition 5.4. If the process above traverses exactly $2k+1$ elements in T, and if in each period, it uses step (4) exactly once, then we say the set of such $2k+1$ elements form a *k-star*.

Using the polyomino model, we can visualize all the possible diagonals that cross a specific one.

Definition 5.5. Let $(u, v) \in P$. We call

$$W^L_{(u,v)} = \{(a, b) \in \overline{T} \mid a < u < b < v\}$$

the *left wing*, and

$$W^R_{(u,v)} = \{(a, b) \in \overline{T} \mid u < a < v < b\}$$

the *right wing*.

By a careful drawing, one can see that the tips of the wings have the same shape, as shown by the blue shaded regions in the figure below. In fact, these two tips are exactly the translational copies of each other by kn. Formally, we can prove

Lemma 5.6. Let $(i, j), (r, s) \in P$ be k-relevant or k-boundary diagonals. If $(r, s) \in W^L_{(i,j)}$ with $r + kn < j$, then $(r + kn, s + kn) \in W^R_{(i,j)}$.

Proof. We need to show that $i < r + kn < j < s + kn$. Since the middle inequality is given by the assumption, it suffices to show $i - r < kn$ and $j - s < kn$. Since (i, j) and (r, s) are k-relevant or k-boundary, we have $j - i \leq kn$ and $s - r \leq kn$. Because $(r, s) \in W^L_{(i,j)}$, we have $i < s$, which implies that

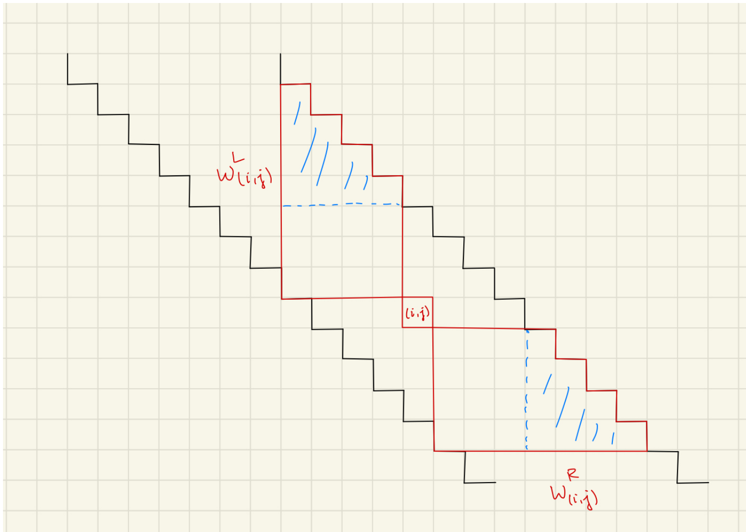
$$i - r < s - r \leq kn,$$

and

$$j - s < j - i \leq kn.$$

□

This is the visualization of Lemma 3.3.



6. Cylindrical Pipe Dreams

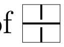
6.1. **Definitions and Fundamental Properties.** In this section, we define the model of *cylindrical pipe dreams* and utilize this model to prove the purity, flip property, and connectedness of the flip graph. This model also enables us to complete the enumeration of k-triangulations of $\mathcal{C}_n$.

The *RC-graphs*, or *(reduced) pipe dreams*, are originally introduced by [FK96] and [BB93] as a combinatorial description of Schubert polynomials. It is a tiling of 1/4 plane $\{(x,y) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0}\}$ by two kinds of pieces and (finitely many) such that no pair of pipes crossing twice. For any pipe dream α, it determines a permutation $w(\alpha)$, say, $w(i) = j$ if the leftmost i-th pipe is connected to the upper j-th pipe. In [Stu11], Stump established the connection between pipe dreams and multi-triangulations. We extend this work and introduce the concept of cylindrical pipe dreams.

Definition 6.1. Given integers n and k, a *cylindrical Young diagram* $\mathbb{Y}$ of type (n,k) is an infinite skew Young diagram (reflected along the y-axis) with a box centered at every point in $\{(i,j) \in \mathbb{Z}^2 \mid k \leq i < j \leq kn\} \subseteq \mathbb{Z}^2$.

Definition 6.2. Given a cylindrical Young diagram $\mathbb{Y}$ of type (n,k), a *cylindrical pipe dream* $\mathbb{P}$ of $\mathbb{Y}$ is a tiling of $\mathbb{Y}$ by four kinds of pieces , , , and such that

- (i) The pipe dream is n-*cylindrical*, that is, all the piles at the position $(i+rn, j+rn)$ for arbitrary $r \in \mathbb{Z}$ is the same as a pile at the position (i,j);
- (ii) There is a tiled at the position $(i, k-i)$ for all $i \in \mathbb{Z}$;
- (iii) For every pipe, the total number of , , it passes through is $2k+1$ (equivalently, the number of it passes through is k);
- (iv) Each pipe connects $(i, kn-i)$ and $(i+kn, -i)$ for some $i \in \mathbb{Z}$, or equivalently, if there is a pipe passing through the pile at the position $(i, kn-i)$, and it corresponds to the part of the pile , then this pipe also passes through the pile at the position $(i+kn, -i)$, which corresponds to the part of the pile . The *position* of the pipe connecting $(i, kn-i)$ and $(i+kn, -i)$ is defined to be i. Given two pipes at the position i and j, the *distance* of these two pipes is defined to be $|i-j|$;

(v) For every pair of pipes, they do not cross twice, that is, the number of  piles both pipes pass through is no more than 1.

(vi) There is exactly one  in each successive n rows, tiled at the position $(i, kn-i)$ for some $i \in \mathbb{Z}$.

Lemma 6.3. Given a cylindrical pipe dream of type (n, k), two pipes cross once if and only if they have distance no more than kn.

Proof. Denote the two pipes are at the position i and j respectively, with $i < j$. By the definition of cylindrical pipe dreams, the pipe at the position i connects

$$A \coloneqq (i, kn-i)$$

and

$$B \coloneqq (i+kn, -i),$$

while the pipe at the position j connects

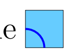
$$C \coloneqq (j, kn-j)$$

and

$$D \coloneqq (j+kn, -j).$$

Note that all four points lie on the line $x+y=kn$. Suppose $|i-j| > kn$, then the relative position of A, B, C, D is A, B, C, D, and hence the two pipes do not cross. Conversely, suppose $|i-j| \leq kn$, then the relative position of A, B, C, D is A, C, B, D, and hence the two pipes must have an intersection. Moreover, by the definition of cylindrical pipe dreams, these two pipes cannot cross more than once, so they cross exactly once. □

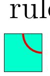
Lemma 6.4. For any sequence of n successive integers $l, l+1, \ldots, l+n-1$, there are precisely $n-1$ pipes at these positions.

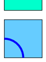
Proof. According to the definition of cylindrical pipe dreams, there is exactly one  in these n successive rows, positioned at $(i, kn-i)$ for some $i \in \mathbb{Z}$. Consequently, there is a pipe at every position except for i. □

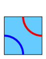
Lemma 6.5. Given a pipe, there are exactly $2k \cdot (n-1)$ pipes intersects with it.

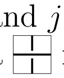
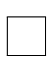
Proof. Denote the position of this pipe by l. Then, by Lemma 6.3 and Lemma 6.4, there are $k \cdot (n-1)$ pipes intersecting with the pipe at position l whose positions are greater than l, and another $k \cdot (n-1)$ pipes intersecting with the pipe at position l whose positions are smaller than l. □

Proposition 6.6. Let $k=2$. Given a cylindrical Young diagram $\mathbb{Y}$ of type (n, k) and a k-triangulation $\overline{T}$ of $\overline{\mathcal{C}}_n$, a cylindrical pipe dream $\mathbb{P}$ of $\mathbb{Y}$ can be obtained by tiling $\mathbb{Y}$ by the following rule:

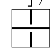
(i) a  is tiled at (i, j) if there is a k-boundary edge in $\overline{T}$ connecting i and j;

(ii) a  is tiled at (i, j) if there is a k-relevant edge of length kn in $\overline{T}$ connecting i and j;

(iii) a  is tiled at (i, j) if there is a k-relevant edge of length $< kn$ in $\overline{T}$ connecting i and j;

(iv) a  is tiled in every other  of $\mathbb{Y}$.

Moreover, each pipe in $\mathbb{P}$ corresponds to a k-star on $\overline{T}$.

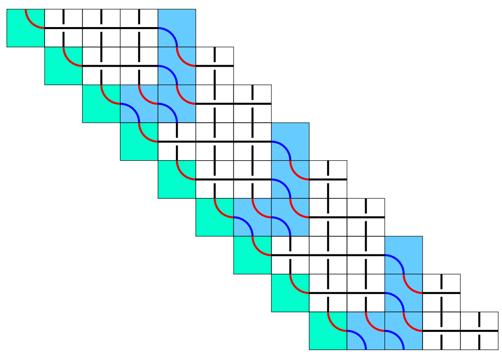
Sketch of Proof. The requirement (i) is satisfied because of the n-periodicity of k-triangulations of $\overline{T}$. Since k-boundary edges are exactly edges of length k, the requirement (ii) is satisfied. Every angle made of edges of the same direction correspond to a "longest line segments" in a pipe dream. By Theorem 3.9, every pipe then corresponds to a k-star on $\overline{T}$, and hence the requirement (iii) and (iv) are satisfied. Analogous to [PS08, Corollary 4.3], two k-stars on $\overline{T}$ can have at most one common angle bisector, which corresponds to a  intersected by the two pipes. So the requirement (v) is satisfied. By Lemma 3.2, there is exactly one edge of length kn in the k-triangulation of $\mathcal{C}_n$, hence the requirement (vi) is satisfied.

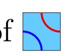
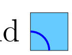
Remark. Proposition 6.6 for arbitrary k remains to be a conjecture, and it can be proved if we have proved Theorem 3.9 for arbitrary k.

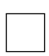
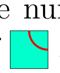
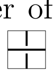
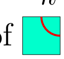
□

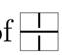
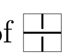
Conjecture 6.7. Let $k \in \mathbb{Z}$, then Proposition 6.6 gives a canonical bijection between k-triangulations of $\mathcal{C}_n$ and cylindrical pipe dreams of (n, k).

Example 6.8. A cylindrical pipe dream $\mathbb{P}$ of type $(3, 2)$, which corresponds to a 2-triangulation of $\mathcal{C}_3$, is shown in the figure below.

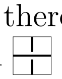


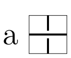
Proposition 6.9. Given a cylindrical pipe dream of type (n, k), the number of  and  in each successive n rows is $k \cdot (n - 1)$.

Proof. The number of  in each successive n rows is $n \cdot (kn - k + 1)$. We compute the number of  and  in each successive n rows. The number of  in these n rows is n.

Then we compute the number of  in n rows. We denote this number as S_1. Furthermore, given an arbitrary integer $r \geq 2k + 1$, we denote the number of  in these rn rows by S_r. Since the pipe dream is n-cylindrical, $S_1 = S_r/r$ is a constant integer.

Without loss of generality, suppose the rn rows we take is from $y = 0$ to $y = 1 - rn$.

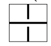
By Lemma 6.4, there are $(r-2k)(n-1)$ pipes in the positions between kn and $(r-k)n-1$. And moreover, all  they pass through lie in the rn rows we choose.

By Lemma 6.3, every pairs of pipes with distance no more than kn correspond to a . We consider the number of pairs of pipes that use at least one of these $(r-2k)(n-1)$ pipes. By Lemma 6.5, this number is greater than or equal to

$$\frac{1}{2}(r - 2k)(n - 1) \cdot 2k \cdot (n - 1) = (r - 2k) \cdot k(n - 1)^2.$$

Hence we have

$$(r - 2k) \cdot k(n - 1)^2 \leq S_r.$$

On the other hand, by Lemma 6.4, there are $(r+2k)(n-1)$ pipes in the position between $-kn$ and $(r+k)n-1$. And moreover, every  between $y=0$ and $y=1-rn$ are passed by at least one of these pipes.

Again, by Lemma 6.3, every pairs of pipes with distance no more than kn correspond to a . We consider the number of pairs of pipes that use both pipes among these $(r+2k)(n-1)$ pipes. By Lemma 6.5, this number is less than or equal to

$$\frac{1}{2}(r+2k)(n-1)\cdot 2k\cdot(n-1)=(r+2k)\cdot k(n-1)^2.$$

Hence we have

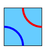
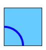
$$S_r\leq(r+2k)\cdot k(n-1)^2.$$

Conclusively, we have

$$(r-2k)\cdot k(n-1)^2\leq S_r\leq(r-2k)\cdot k(n-1)^2.$$

Now let $r\to\infty$, then we have

$$S_1=\lim_{r\to\infty}S_r/r=k\cdot(n-1)^2.$$

That is to say, the number of  in each successive n rows is $k\cdot(n-1)^2$. So the number of  and  in each successive n rows is

$$n\cdot(kn-k+1)-n-k\cdot(n-1)^2=k\cdot(n-1),$$

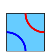
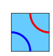
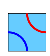
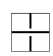
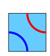
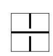
as desired. □

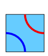
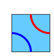
Proposition 6.10. Let $k=2$. Every k-triangulation of $\mathcal{C}_n$ has the rank $k\cdot(n-1)$.

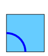
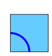
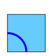
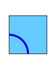
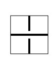
Proof. By Proposition 6.6 and Proposition 6.9. □

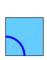
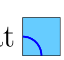
Remark. Again, Proposition 6.10 for arbitrary k remains to be a conjecture, and it can be proved if we have proved Theorem 3.9 for arbitrary k.

Definition 6.11. An *n-cylindrical mutation* is a mutation of piles such that if the piles at the position (i,j) and (t,l) are mutated, then the piles at the position $(i+rn,j+rn)$ and $(t+rn,l+rn)$ are mutated for every $r\in\mathbb{Z}$.

Definition 6.12. Given a tile , the *regular pipe flip* of that  is defined as follows: select the two pipes passing through the tile , identify the intersection  of these two pipes, and apply an n-cylindrical mutation from the  to the  for every translation of these two pipes.

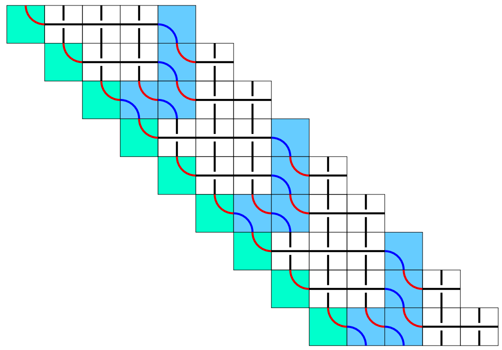
Remark. Given a tile , by Lemma 6.3, the *regular pipe flip* of that  is uniquely determined.

Definition 6.13. Given a tile , the *exceptional pipe flip* of that  is defined as follows: select the pipe passing through the , choose a translation of this pipe with a distance of kn, and apply an n-cylindrical mutation from the  to the  where these two pipes intersect for every translation of these two pipes.

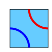
Remark. Given a tile , by Lemma 6.3, the *exceptional pipe flip* of that  is also uniquely determined.

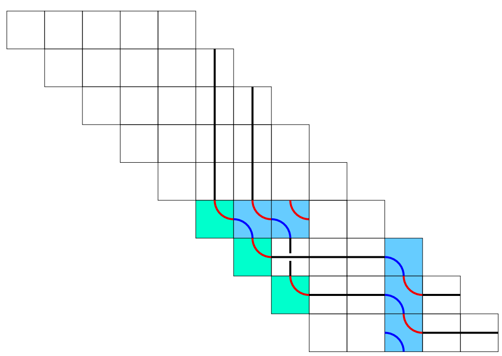
Definition 6.14. Both regular pipe flips and exceptional pipe flips are called *pipe flips*.

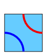
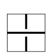
Example 6.15. Consider the cylindrical pipe dream $\mathbb{P}$

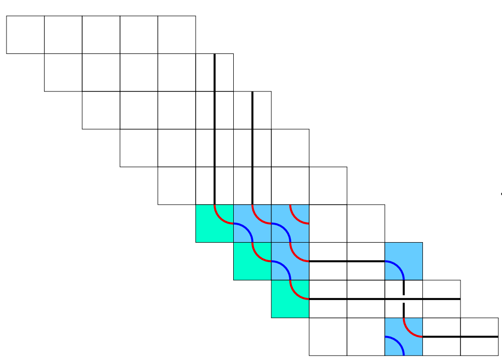


in Example 6.8. A regular pipe flip applying to the second pile from the right in the penultimate row is given by the following procedure:

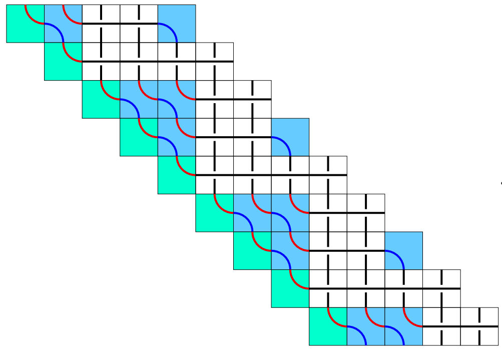
First, choose the two pipes passing through that  :



Next, mutate this  with the  intersected by the two pipes:

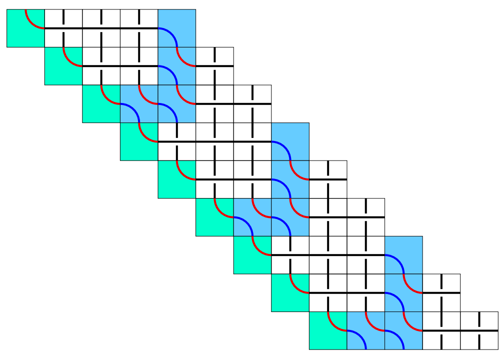
 .

Finally, do the same thing for every translation of these two pipes to make it an n-cylindrical mutation:

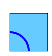


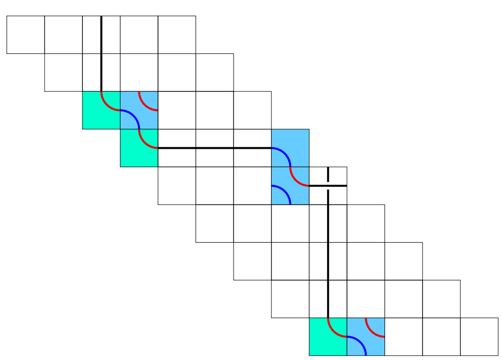
.

Example 6.16. Consider the cylindrical pipe dream $\mathbb{P}$

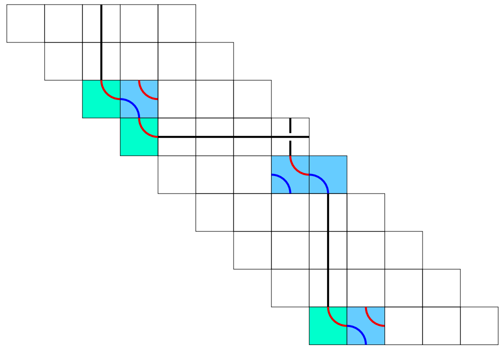


in Example 6.8. A regular pipe flip applying to the first pile to the right of the fourth row is given by the following procedure:

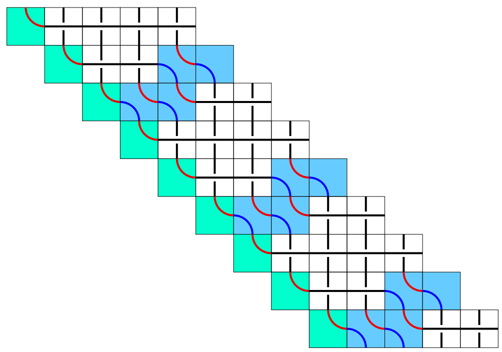
First, choose the pipe passing through that  , and a translation of this pipe with a distance of kn:



Next, mutate this  with the  intersected by the two pipes:

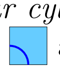


Finally, do the same thing for every translation of these two pipes to make it an n-cylindrical mutation:

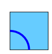
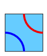
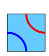
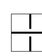


Proposition 6.17. Cylindrical pipe dreams of type (n, k) are closed under pipe flips.

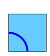
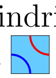
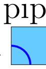
Proof. By checking the definition of both regular pipe flips and exceptional pipe flips. □

Definition 6.18. A *regular cylindrical pipe dream* is a cylindrical pipe dream $\mathbb{P}$ such that for every , there exists a  at the same row.

Proposition 6.19. Every cylindrical pipe dream can be flipped to a regular cylindrical pipe dream with a sequence of regular pipe flips.

Proof. We choose n successive rows with the first row has a  . Suppose the pipe dream is not regular, we find the bottommost row that contains a  , and apply a regular pipe flip to the rightmost  in that row. Then, this  will mutate with another  located in an upper row. Repeat this process, this process ends in finitely many step, and then we get a regular cylindrical pipe dream. □

Corollary 6.20. The pipe flip graph of cylindrical pipe dreams of type (n, k) is connected.

Proof. Given a regular cylindrical pipe dream $\mathbb{P}$, the  can be flipped to the previous row by an exceptional pipe flip. After that, by Proposition 6.19, we can apply a sequence of regular pipe flips to flip it to a regular cylindrical pipe dream $\mathbb{P}'$. Now $\mathbb{P}'$ is the regular cylindrical pipe dream obtained by shifting all  and  in $\mathbb{P}$ upward by one row. Since n is finite, all regular cylindrical pipe dreams can be connected by a sequence of pipe flips, and again by Proposition 6.19, every pipe dream can be connected by a sequence of pipe flips. □

If Conjecture 6.7 can be proved, then we will get: the flip graph of k-triagulations of $\mathcal{C}_n$ is connected.

6.2. **Enumeration of Cylindrical Pipe Dreams.**

Definition 6.21. A *nonnegative integer k-tuple* is a sequence of k nonnegative integers, where $k \in \mathbb{Z}_{\geq 0}$.

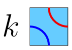
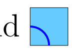
In this subsection, we use e_i to denote the k-tuple such that there is a 1 at the i-th position, and a 0 at the other positions. The main theorem of this subsection is the following bijection.

Theorem 6.22. There exists a canonical bijection between cylindrical pipe dreams of type (n, k) and nonnegative integer k-tuples valuations $v_0, v_1, \ldots v_{n-1}$ on n vertices such that

$$\sum_{i=0}^{n-1} v_i = (n-1, n-1, \ldots, n-1).$$

Sketch of proof. For a given cylindrical pipe dream $\mathbb{P}$ of type (n, k), the nonnegative integer k-tuple valuation $\phi(\mathbb{P})$ is defined as follows:

We first choose the $n-1$ pipes between positions 0 to $n-1$ (this can be done by Lemma 6.4).

For each pipe, by the definition of cylindrical pipe dreams, there are k  and  piles it passes through in total. If the i-th "" on that pipe lies in the j-th column (where $i \in [k]$, and j is considered modulo n), then we add e_i to the j-th vertex.

The valuations $v_0, v_1, \ldots, v_{n-1}$ are obtained by applying the procedure described above to all $n-1$ pipes we have chosen.

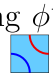
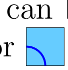
For a given cylindrical pipe dream of type (n, k), it can be verified that $\phi(\mathbb{P})$ is indeed a nonnegative integer k-tuple valuation $v_0, v_1, \ldots v_{n-1}$ on n vertices such that

$$\sum_{i=0}^{n-1} v_i = (n-1, n-1, \ldots, n-1).$$

By definition of ϕ, each pipe can add an $e_1, e_2, \ldots, e_k$ to some vertex. Therefore, each pipe contributes a $(1, 1, \ldots, 1)$ in total. Since there are $n-1$ pipes, we have

$$\sum_{i=0}^{n-1} v_i = (n-1, n-1, \ldots, n-1),$$

as desired.

To demonstrate that such a ϕ is indeed a bijection, we explicitly construct the inverse mapping ϕ^{-1}. The inverse mapping ϕ^{-1} can be constructed as follows: Given the sequence $v_0, v_1, \cdots, v_{n-1}$, we inductively fill  (or ) into the polyomino with respect to the i-th coordinate of $v_0, v_1, \cdots, v_{n-1}$ to ensure that the i-th "" on each pipe lies in the desired column. It can be shown that such a filling is uniquely determined. □

Using this bijection, we complete the enumeration of cylindrical pipe dreams of type (k, n).

Lemma 6.23. The number of nonnegative integer k-tuples valuations $v_0, v_1, \ldots v_{n-1}$ on n vertices such that

$$\sum_{i=0}^{n-1} v_i = (n-1, n-1, \ldots, n-1)$$

is $\binom{2(n-1)}{(n-1)}^k$.

Proof. The number of nonnegative integer valuations on n vertices such that the sum is $n-1$ is given by $\binom{2(n-1)}{(n-1)}$. Since nonnegative integer k-tuples valuations can be regarded as nonnegative integer valuations on n vertices componentwise, we obtain the desired result. ☐

Corollary 6.24. The number of cylindrical pipe dreams of type (n,k) is $\binom{2(n-1)}{(n-1)}^k$.

Proof. By Lemma 6.23 and Theorem 6.22. ☐

Remark. If we can prove Conjecture 6.7, then this also gives us the enumeration of k-triangulations of $\mathcal{C}_n$.

7. The cyclic sieving phenomenon

Definition 7.1. Let $M_{2nk,k}$ denote the set of k-triangulations of the $2nk$-gon.

Definition 7.2. Let $M^n_{2nk,k}$ denote the set of n-periodic k-triangulations of the $2nk$-gon.

Jonsson enumerated the k-triangulations of a $2nk$-gon using a Catalan determinantal formula that can be expressed more simply as the following:

$$M_{2nk,k}(q) := \prod_{1\le a\le b\le 2nk-2k-1} \frac{[a+b+2k]_q}{[a+b]_q}$$

where, for positive integer m, $[m]_q = 1+q^1+\cdots+q^{m-1}$.

Definition 7.3. Let $C \simeq \mathbb{Z}/(2k\mathbb{Z})$ be the group generated by the rotation c of the $2kn$-gon by $\frac{2\pi}{2k} = \frac{\pi}{k}$ radians.

Lemma 7.4. When $k=2$, the triple $(M_{2nk,k}, C, M_{2nk,k}(q))$ is CSP.

Proof. We opt for a direct combinatorial proof, as opposed to one using the representation theory paradigm/linear algebra model of [Reiner-Stanton-White]. In [CITE REFERENCE IN THIS REPORT] we proved that the number of n-periodic (i.e., preserved under c) 2-triangulations of $4n$-gon equals $\binom{2(n-1)}{n-1}^2$, and a formula in [CLS] implies that the number of 2-triangulations of $4n$-gon preserved under a 180-degree rotation is $(4n-3)\left(\frac{(4n-4)!}{(2n-1)!(2n-2)!}\right)^2$. So it suffices to show that

(a) $M_{2nk,k}(i) = \binom{2(n-1)}{n-1}^2$.

(b) $M_{2nk,k}(-1) = (4n-3)\left(\frac{(4n-4)!}{(2n-1)!(2n-2)!}\right)^2$.

(c) $M_{2nk,k}(-i) = \binom{2(n-1)}{n-1}^2$

Here are the proofs.

(a) We write

$$M_{2nk,k}(q) = \prod_{1\le a\le b\le 2nk-2k-1} \frac{1-q^{a+b+2k}}{1-q^{a+b}} \tag{1}$$

$$= \prod_{a+b+c=2nk-2k-2} \frac{1-q^{2a+b+2k+2}}{1-q^{2a+b+2}} \tag{2}$$

$$= \prod_{\substack{a+c\le 2nk-2k-2\\ a,c\ge 0}} \frac{1-q^{a-c+2nk}}{1-q^{a-c+2nk-2k.}} \tag{3}$$

We now determine the number of ways to write a given $b \in \mathbb{Z}$ as $a - c$ with $a + c \leq 2nk - 2k - 2$. Suppose $b \geq 0$. If we know c, we have $a = b + c$, so $b + 2c \leq 2nk - 2k - 2$. Thus, $0 \leq c \leq nk - k - 1 - b/2$ are all the possibilities. The number of such possibilities is $\lfloor nk - k - b/2 \rfloor$ for $b \leq 2nk - 2k$, and 0 otherwise.

If $b \in \mathbb{Z}$ is arbitrary, we get $\lfloor nk - k - |b|/2 \rfloor$ for $|b| \leq 2nk - 2k - 2$, and 0 otherwise. So we may write

$$M_{2nk,k}(q) = \prod_{|b| \leq 2nk-2k-2} \left(\frac{1 - q^{b+2nk}}{1 - q^{b+2nk-2k}} \right)^{\lfloor nk-k-|b|/2 \rfloor}.$$

Note

$$M_{2nk,k}(q) = \prod_{|b| \leq 2nk-2k-2} \Big(\frac{[b+2nk]_q}{[b+2nk-2k]_q} \Big)^{\lfloor nk-k-|b|/2 \rfloor}$$

$$= \Big(\frac{[2nk]_q}{[2nk-2k]_q} \Big)^{nk-k} \cdot \prod_{b=1}^{nk-k-1} \Big(\frac{[2b-1+2nk]_q [2b+2nk]_q [-2b+1+2nk]_q [-2b+2nk]_q}{[2b-1+2nk-2k]_q [2b+2nk-2k]_q [-2b+1+2nk-2k]_q [-2b+2nk-2k]_q} \Big)^{nk-k-b}.$$

Now, suppose $q = i$ and k is even. Then

$$\frac{[r]_q}{[r-2k]_q} = \frac{1-q^r}{1-q^{r-2k}}.$$

Evaluating at $q = i$, using L'Hospital's rule if necessary, we get

$$\frac{[r]_q}{[r-2k]_q} = \begin{cases} \frac{r}{r-2k} & \text{if } r \equiv 0 \pmod 4 \\ 1 & \text{otherwise} \end{cases}$$

So we get

$$M_{2nk,k}(i) = \Big(\frac{2nk}{2nk-2k} \Big)^{nk-k} \cdot \prod_{b=1,\ b \text{ even}}^{nk-k-1} \Big(\frac{2b+2nk}{2b+2nk-2k} \cdot \frac{-2b+2nk}{-2b+2nk-2k} \Big)^{nk-k-b} \tag{4}$$

$$= \Big(\frac{n}{n-1} \Big)^{(n-1)k} \prod_{b=1}^{(n-1)k/2} \Big(\frac{2b+nk}{2b+(n-1)k} \cdot \frac{-2b+nk}{-2b+(n-1)k} \Big)^{(n-1)k-2b}. \tag{5}$$

If $k = 2$, this becomes

$$M_{4n,2}(i) = \Big(\frac{n}{n-1} \Big)^{2(n-1)} \prod_{b=1}^{n-1} \Big(\frac{n+b}{n+b-1} \cdot \frac{n-b}{n-b-1} \Big)^{2(n-b-1)} \tag{6}$$

$$= \Big(\frac{n}{n-1} \Big)^{2(n-1)} \prod_{b=1}^{n-1} \Big(\frac{2n-b}{2n-b-1} \cdot \frac{b}{b-1} \Big)^{2(b-1)} \tag{7}$$

$$= \Big(\Big(\frac{n}{n-1} \Big)^{n-1} \prod_{b=1}^{n-1} \Big(\frac{2n-b}{2n-b-1} \cdot \frac{b}{b-1} \Big)^{b-1} \Big)^2 \tag{8}$$

Now, observe that

$$\left(\frac{n}{n-1}\right)^{n-1}\prod_{b=1}^{n-1}\left(\frac{2n-b}{2n-b-1}\cdot\frac{b}{b-1}\right)^{b-1} = \left(\frac{n}{n-1}\right)^{n-1}\frac{\prod_{b=1}^{n-1} b^{b-1}}{\prod_{b=1}^{n-2} b^{b}}\cdot\frac{\prod_{b=1}^{n-2}(2n-b-1)^{b}}{\prod_{b=1}^{n-1}(2n-b-1)^{b-1}} \tag{9}$$

$$= \left(\frac{n}{n-1}\right)^{n-1}\cdot(n-1)^{n-2}\left(\prod_{b=1}^{n-2}\frac{1}{b}\right)\cdot\frac{1}{n^{n-2}}\left(\prod_{b=1}^{n-2}(2n-b-1)\right) \tag{10}$$

$$= \left(\frac{n}{n-1}\right)^{n-1}\cdot(n-1)^{n-2}\cdot\frac{1}{(n-2)!}\cdot\frac{1}{n^{n-2}}\cdot\frac{(2n-2)!}{n!} \tag{11}$$

$$= \frac{n}{n-1}\cdot\frac{(2n-2)!}{(n-2)!n!} \tag{12}$$

$$= \frac{(2n-2)!}{(n-1)!^2} \tag{13}$$

$$= \binom{2(n-1)}{n-1}. \tag{14}$$

We have thus shown that $M_{4n,2}(i) = \binom{2(n-1)}{n-1}^2$, as desired.

(b) Suppose $q = -1$. Then

$$\frac{[r]_q}{[r-2k]_q} = \frac{1-q^r}{1-q^{r-2k}}.$$

Evaluating at $q = -1$, using L'Hospital's rule, we get

$$\frac{[r]_q}{[r-2k]_q} = \begin{cases} \frac{r}{r-2k} & \text{if } r \equiv 0 \pmod 2 \\ 1 & \text{otherwise} \end{cases}.$$

So we obtain

$$M_{2nk,k}(-1) = \Big(\frac{2nk}{2nk-2k}\Big)^{nk-k} \prod_{b=1}^{nk-k-1} \Big(\frac{(2b+2nk)(-2b+2nk)}{(2b+2nk-2k)(-2b+2nk-2k)}\Big)^{nk-k-b} \tag{15}$$

$$= \Big(\frac{n}{n-1}\Big)^{(n-1)k} \prod_{b=1}^{(n-1)k} \Big(\frac{(b+nk)(-b+nk)}{(b+(n-1)k)(-b+(n-1)k)}\Big)^{(n-1)k-b} \tag{16}$$

$$= \Big(\frac{n}{n-1}\Big)^{2(n-1)} \prod_{b=1}^{2(n-1)} \Big(\frac{(b+2n)(-b+2n)}{(b+2(n-1))(-b+2(n-1))}\Big)^{2(n-1)-b} \quad \text{set } k=2 \tag{17}$$

$$= \Big(\frac{n}{n-1}\Big)^{2(n-1)} \frac{\prod_{b=3}^{2n}(b-2(n-1))^{2n-b}}{\prod_{b=1}^{2(n-1)}(b+2(n-1))^{2(n-1)-b}} \cdot \frac{\prod_{b=1}^{2(n-1)}(-b+2n)^{2(n-1)-b}}{\prod_{b=3}^{2n}(-b+2n)^{2n-b}} \tag{18}$$

$$= \Big(\frac{n}{n-1}\Big)^{2(n-1)} \cdot \frac{1\cdot(4n-3)}{(2n-1)^{2n-3}(2n)^{2n-4}} \Big(\prod_{b=3}^{2(n-1)} (b+2(n-1)^2\Big) \tag{19}$$

$$\cdot \frac{(2n-1)^{2n-3}(2n-2)^{2n-4}}{1\cdot 1} \Big(\prod_{b=3}^{2(n-1)} (-b+2n)^{-2}\Big) \tag{20}$$

$$= (4n-3)\Big(\frac{n}{n-1}\Big)^{2(n-1)} \cdot \Big(\frac{n-1}{n}\Big)^{2n-4} \Big(\prod_{b=3}^{2(n-1)} (2(n-1)+b)\Big)^2 \Big(\prod_{b=3}^{2(n-1)} (2n-b)\Big)^{-2} \tag{21}$$

$$= (4n-3)\Big(\frac{(4n-4)!}{(2n-1)!(2n-2)!}\Big)^2. \tag{22}$$

(c) The function $M_{2nk,k}(q)$ has real coefficients, so its evaluation at $-i$ is the complex conjugate of its evaluation at i. Since $M^n_{2nk,k}(i)$ is real, we have $M^n_{2nk,k}(i) = M^n_{2nk,k}(-i) = \binom{2(n-1)}{n-1}^2$.

□

Conjecture 7.5. The number of k-triangulations of $2kn$-gon invariant under rotation by $\frac{2\pi}{2k}\cdot j$ radians is

$$\prod_{a=1}^{k} \frac{((2n-1)d-\lceil\frac{2a}{m}\rceil)!}{((n-1)d+\lceil\frac{a}{m}\rceil-1)!} \cdot \frac{(\lceil\frac{2a}{m}\rceil-1)!}{(nd-\lceil\frac{a}{m}\rceil)!}$$

where $d=\gcd(2k,j)$ and $m=2k/d$.

Let $\zeta_m = \exp(2\pi i/m)$.

Lemma 7.6. $M_{2nk,k}(\zeta_{2k}^j) = \prod_{a=1}^k \frac{((2n-1)d-\lceil\frac{2a}{m}\rceil)!}{((n-1)d+\lceil\frac{a}{m}\rceil-1)!} \cdot \frac{(\lceil\frac{2a}{m}\rceil-1)!}{(nd-\lceil\frac{a}{m}\rceil)!}$

Proof. We seek to evaluate

$$\prod_{1\le a\le b\le 2nk-2k-1} \frac{[a+b+2k]_q}{[a+b]_q}$$

for $q = \zeta_{2k}^j$. Start by noting that

$$\frac{[a+b+2k]_q}{[a+b]_q} = \lim_{z\to q} \frac{1-z^{a+b+2k}}{1-z^{a+b}} \tag{23}$$

$$= \begin{cases} \frac{a+b+2k}{a+b} & \text{if } (a+b)j \equiv 0 \pmod{2k} \\ 1 & \text{otherwise.} \end{cases} \tag{24}$$

Equivalently, setting $d := \gcd(j, 2k)$ and $m := 2k/d$, we have

$$\frac{1-z^{a+b+2k}}{1-z^{a+b}} = \begin{cases} \frac{a+b+2k}{a+b} & \text{if } (a+b) \equiv 0 \pmod{m} \\ 1 & \text{otherwise.} \end{cases}$$

Thus, we may write

$$M_{2nk,k}(\zeta_{2k}^j) = \prod_{\substack{1\le a\le b\le 2(n-1)k-1 \\ a+b\equiv 0 \pmod{m}}} \frac{a+b+2k}{a+b} \tag{25}$$

$$= \Bigg(\prod_{\substack{k+1\le a\le b\le 2nk-k-1 \\ a+b\equiv 0 \pmod{m}}} (a+b)\Bigg)\Bigg(\prod_{\substack{1\le a\le b\le 2(n-1)k-1 \\ a+b\equiv 0 \pmod{m}}} (a+b)^{-1}\Bigg). \tag{26}$$

Note that for all integers a, b, we have $(k+1 \le a \le b \le 2nk-k-1) \wedge \neg(1 \le a \le b \le 2(n-1)k-1)$ iff $(2(n-1)k \le b \le 2nk-k-1) \wedge (k+1 \le a \le b)$.

Meanwhile, we have $(1 \le a \le b \le 2(n-1)k-1) \wedge \neg(k+1 \le a \le b \le 2nk-k-1)$ iff $(1 \le a \le k) \wedge (a \le b \le 2(n-1)k-1)$.

So we obtain

$$M_{2nk,k}(\zeta_{2k}^j) = \Big(\prod_{b=2(n-1)k}^{2nk-k-1} \prod_{\substack{k+1\le a\le b \\ a+b\equiv 0 \pmod m}} (a+b)\Big)\Big(\prod_{a=1}^{k} \prod_{\substack{a\le b\le 2(n-1)k-1 \\ a+b\equiv 0 \pmod m}} (a+b)^{-1}\Big) \tag{27}$$

$$= \Big(\prod_{b=1}^{k} \prod_{\substack{k+1\le a\le 2nk-k-b \\ a\equiv b+k \pmod m}} (a+2nk-k-b)\Big)\Big(\prod_{a=1}^{k} \prod_{\substack{a\le b\le 2(n-1)k-1 \\ a+b\equiv 0 \pmod m}} (a+b)^{-1}\Big) \tag{28}$$

$$= \Big(\prod_{b=1}^{k} \prod_{\substack{1\le a\le 2(n-1)k-b \\ a\equiv b \pmod m}} (a+2nk-b)\Big)\Big(\prod_{a=1}^{k} \prod_{\substack{a\le b\le 2(n-1)k-1 \\ a+b\equiv 0 \pmod m}} (a+b)^{-1}\Big) \tag{29}$$

$$= \prod_{a=1}^{k} \prod_{\substack{a\le b\le 2(n-1)k-1 \\ a+b\equiv 0 \pmod m}} \frac{4nk-2k-a-b}{a+b} \tag{30}$$

$$= \prod_{a=1}^{k} \prod_{\substack{1\le b\le 2(n-1)k-a \\ a\equiv b \pmod m}} \frac{2nk+b-a}{2(n-1)k+a-b} \tag{31}$$

$$= \prod_{a=1}^{k} \prod_{\substack{-a< b\le 2(n-1)k-2a \\ m\mid b}} \frac{2nk+b}{2(n-1)k-b} \tag{32}$$

$$= \prod_{a=1}^{k} \prod_{-a/m< b\le (n-1)d-2a/m} \frac{nd+b}{(n-1)d-b} \tag{33}$$

$$= \prod_{a=1}^{k} \frac{((2n-1)d-\lceil\frac{2a}{m}\rceil)!}{((n-1)d+\lceil\frac{a}{m}\rceil-1)!} \cdot \frac{(\lceil\frac{2a}{m}\rceil-1)!}{(nd-\lceil\frac{a}{m}\rceil)!} \tag{34}$$

□

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Department of Mathematics, Tufts University, Medford, MA
Email address: saskia.solotko@tufts.edu

Department of Mathematics, Harvard University, Cambridge, MA
Email address: katherinetung@college.harvard.edu

Deaprtment of Mathematics, Swarthmore College, Swarthmore, PA
Email address: myang7@swarthmore.edu

Deaprtment of Mathematics, University of Michigan, Ann Arbor, MI
Email address: zongxun@umich.edu