EQUALITY AND SATURATED NEWTON POLYTOPES OF POSTNIKOV-STANLEY POLYNOMIALS

SERENA AN, KATHERINE TUNG, AND YUCHONG ZHANG

ABSTRACT. Dual Schubert polynomials have garnered significant interest since their introduction by Bernstein, Gelfand, and Gelfand in 1973. We prove that dual Schubert polynomials have saturated Newton polytope (SNP) and give two descriptions of their Newton polytopes, continuing a recent progression of SNP results in algebraic combinatorics by Fink-Mészáros-St. Dizier (2018), Monical-Tokcan-Yong (2019), and Castillo-Cid Ruiz-Mohammadi-Montaño (2023), among others. We also consider a generalized family of dual Schubert polynomials, called skew dual Schubert polynomials or Postnikov-Stanley polynomials. We prove nontrivial equalities between these polynomials in intervals of small rank, in analogy to results about skew Schur polynomials by Reiner-Shaw-van Willigenburg (2007).

CONTENTS

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1. Introduction

Schubert polynomials, which represent the cohomology classes of Schubert cycles in flag varieties, were introduced by Lascoux and Schützenberger [LS78]. Bernstein, Gelfand, and Gelfand defined a dual to the Schubert polynomials with respect to the D-pairing on the polynomial ring [BGG73]. These *dual Schubert polynomials* are given by weighted sums over saturated chains from the identity to a permutation w in the Bruhat order of the symmetric group S_n. Postnikov and Stanley [PS09] introduced a generalization of the dual Schubert polynomials such that these saturated chains from the identity to w can be replaced by any interval $[u, w]$ in the Bruhat order. These generalized polynomials are often called *skew dual Schubert polynomials* or *Postnikov-Stanley polynomials* in literature. We use the latter terminology.

For the last several decades, Schubert polynomials $\mathfrak{S}_w$ have captivated algebraic geometers, representation theorists, and combinatorialists alike. Highlights include the result that the Schubert polynomial $\mathfrak{S}_w$ equals the dual character of the flagged Weyl module associated to the Rothe diagram of w [KP04], and the result that Schubert polynomials form a basis for the polynomial ring $\mathbb{C}[x_1, x_2, \dots]$ which allows us to expand Schubert products uniquely into a linear combination of Schubert polynomials:

$$\mathfrak{S}_u \cdot \mathfrak{S}_w = \sum_w c_{uv}^w \mathfrak{S}_w.$$

The coefficients c_{uv}^w are called *generalized Littlewood-Richardson coefficients*, and Kleiman transversality from algebraic geometry guarantees that they are nonnegative [Kle74]. Determining a combinatorial formula for these coefficients is one of the most significant open problems in combinatorics and representation theory today. Since the dual Schubert polynomials form a D-dual basis to the Schubert polynomials, understanding dual Schubert polynomials may give more insight into Schubert polynomials.

In this report, we study two main questions.

Question 1.1. When are two Postnikov-Stanley polynomials equal?

Since Postnikov-Stanley polynomials are skew dual Schubert polynomials, we are motivated to study them analogous to results about Schur functions have extended to skew Schur functions. One interesting topic is when two skew Schur functions are nontrivially equal, which means roughly that they are not obtained from the same shape of tableaux. This topic was explored by Reiner, Shaw, and van Willigenburg [RSW07], who gave sufficient and necessary conditions are developed for two skew diagrams to give rise to the same skew Schur function, and later by McNamara and van Willigenburg [MW09], who provided an operation for constructing skew diagrams whose corresponding skew Schur functions are equal. In the same spirit, we investigate nontrivial equalities between Postnikov-Stanley polynomials by comparing their chain multisets. We provide a classification for equalities between all rank 2 Postnikov-Stanley polynomials and a large number of rank 3 Postnikov-Stanley polynomials.

Our second question is the following.

Question 1.2. Do Postnikov-Stanley polynomials have SNP?

The saturated Newton polytope property was first defined by Monical, Tokcan and Yong [MTY19]. They described polynomials with algebraic combinatorial significance known to be SNP, such as Schur polynomials [Rad52] and resultants [GKZ90], and proved SNP for additional families of polynomials, including cycle index polynomials, Reutenauer's symmetric polynomials linked to the free Lie algebra and to Witt vectors, Stembridge's symmetric polynomials associated to totally nonnegative matrices, and symmetric Macdonald polynomials. Subsequent work of Fink, Mézśaros, and St. Dizier proved SNP for key polynomials and Schubert polynomials [FMD18], and work of Castillo, Cid Ruiz, Mohammadi, and Montaño proves SNP for double Schubert polynomials [Cas+23b]. Proving SNP can be difficult given that many polynomial operations, such as multiplication, do not preserve SNP. However, a wide range of techniques have been harnessed to prove SNP. For instance, Rado uses elementary combinatorial techniques [Rad52], Fink, Mézśaros, and St. Dizier rely on representation theory [FMD18], and Castillo, Cid Ruiz, Mohammadi, and Montaño use results from algebraic geometry and commutative algebra [Cas+23b]. Continuing this program of proving SNP for various algebraic combinatorial models, we prove that all dual Schubert polynomials have SNP. Furthermore, we give descriptions of their Newton polytopes, including the vertices. Our proof technique relies on elementary combinatorial techniques and a few results about generalized permutahedra.

This paper is organized as follows. In Section 2, we give the necessary relevant background and definitions. In Section 3, we present our equality results for Postnikov-Stanley polynomials of ranks 2 and 3. In Section 4, we present our results on SNP, including the theorem that dual Schubert polynomials have SNP. In Section 5, we characterize the Newton polytope of dual Schubert polynomials.

2. Preliminaries

2.1. **Bruhat Order.** Let S_n denote the set of permutations on $[n] = \{1, \ldots, n\}$. We write permutations $w \in S_n$ in one-line notation as $w = w(1)w(2)\cdots w(n)$. We may also consider each permutation $w \in S_n$ as a product of *simple transpositions* $\{s_i = (i, i+1) : 1 \le i \le n-1\}$ which satisfy the relations

$$\begin{aligned} s_i^2 &= \mathrm{id} \\ s_i s_j &= s_j s_i \text{ for } |i-j| \ge 2 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}. \end{aligned}$$

We adopt the convention of multiplying permutations from right to left.

Definition 2.1. Let $w \in S_n$. If $w = s_{i_1} \cdots s_{i_\ell}$ is written as a product of simple transpositions and ℓ is minimal among all such expressions, then the string of indices $i_1 \cdots i_\ell$ is a *reduced decomposition* of w. We call $\ell = \ell(w)$ the *(Coxeter) length* of w.

Remark. Reduced decompositions are not necessarily unique. For example, in S_3, $w = 321$ can be written as $w = s_1 s_2 s_1 = s_2 s_1 s_2$.

Definition 2.2. Let $w \in S_n$. An *inversion* of w is an ordered pair $(a, b) \in [n]^2$ such that $a < b$ and $w(a) > w(b)$. We denote the set of all inversions of w by $\mathrm{Inv}(w)$.

Example 2.3. The permutation 231 has inversion set $\{(1, 3), (2, 3)\}$.

Let $|\mathrm{Inv}(w)|$ be the order of the set $\mathrm{Inv}(w)$. It is well-known that $\ell(w) = |\mathrm{Inv}(w)|$.

Definition 2.4. We define a partial order $\le$ on S_n called the *(strong) Bruhat order* as follows. Let $u, v \in S_n$ and $\ell = \ell(v)$. We have $u \le v$ if and only if for any reduced decomposition $i_1 \cdots i_\ell$ of $v = s_{i_1} \cdots s_{i_\ell}$, there exists a reduced decomposition $j_1 \cdots j_k$ of $u = s_{j_1} \cdots s_{j_k}$ such that $j_1 \cdots j_k$ is a subword of $i_1 \cdots i_\ell$. If additionally $\ell(v) = \ell(u) + 1$, we write $u \lessdot v$.

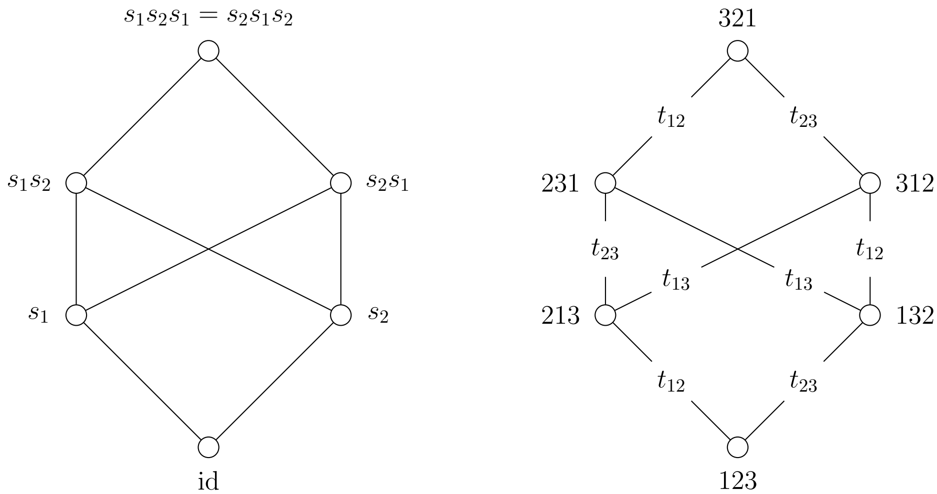


FIGURE 1. The Bruhat order of S_3.

Before giving an example, we introduce a conceptually easier definition of the Bruhat order that will be relevant to the remainder of the paper.

Definition 2.5. For $1 \le a < b \le n$, let t_{ab} act on permutations $w \in S_n$ such that wt_{ab} is the permutation given by transposing the two numbers in positions a and b in w.

Example 2.6. We have $312 = 213t_{13}$ in S_3.

In the remainder of the paper, whenever we write t_{ab}, we assume $a < b$ unless specified.

Definition 2.7. An equivalent definition of the Bruhat order is as follows: for $u, v \in S_n$, we have $u \lessdot v$ in the Bruhat order if and only if $v = ut_{ab}$ and $\ell(v) = \ell(u) + 1$.

Example 2.8. The Bruhat order of S_3 is shown in Figure 1, as described in both equivalent definitions Definition 2.4 and Definition 2.7.

Definition 2.9. For $u \le w$ in the Bruhat order of S_n, the *interval* $[u, w]$ is the subposet containing all $v \in S_n$ such that $u \le v \le w$. The *rank* of the interval $[u, w]$ is $\ell(w) - \ell(u)$.

2.2. **Postnikov-Stanley Polynomials.** In this section, we define Postnikov-Stanley Polynomials.

Definition 2.10. Given $u \lessdot v$, there exists a unique integer pair (a, b) with $1 \le a < b \le n$ such that $v = ut_{ab}$. We define the *weight* $m(u \lessdot v)$ of the edge between u and v to be the polynomial

$$m(u \lessdot v) = x_a + x_{a+1} + \cdots + x_{b-1}.$$

Example 2.11. Since $312 = 213t_{13}$, we have $m(213 \lessdot 312) = x_1 + x_2$.

Figure 2 displays all edge weights in the Bruhat order of S_3.

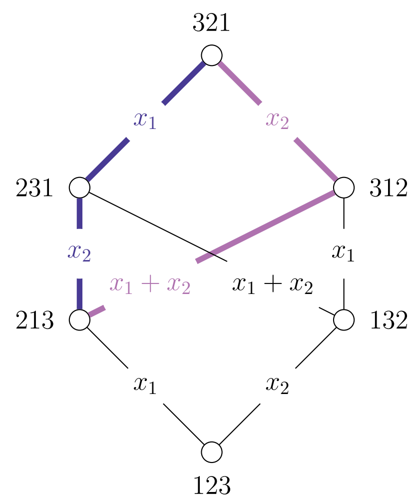


FIGURE 2. Edge weights in the Bruhat order of S_3.

Definition 2.12. For a saturated chain $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$ in the interval $[u_0, u_\ell]$, we define its *weight* $m_C(x)$ to be the product of the weights of the edges in C, that is

$$m_C(x) = \prod_{i=1}^{\ell} m(u_{i-1} \lessdot u_i).$$

Example 2.13. From Figure 2, the weight of the saturated chain $213 \lessdot 312 \lessdot 321$ in $[213, 321]$ is $(x_1 + x_2) \cdot x_2$.

Definition 2.14. For $u \leq w$ in S_n, the *Postnikov-Stanley polynomial* D_u^w is defined by

$$D_u^w = \frac{1}{(\ell(w) - \ell(u))!} \sum_C m_C(x),$$

where the sum is over all saturated chains $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$ with $u_0 = u$ and $u_\ell = w$. We call u the *bottom permutation* and w the *top permutation*.

Observe that D_u^w is homogeneous and of degree $\ell(w) - \ell(u)$.

Example 2.15. We compute D_{213}^{321} by summing the weights of the blue and purple chains in Figure 2 to obtain $D_{213}^{321} = \frac{1}{2!}(x_1 x_2 + (x_1 + x_2) \cdot x_2)$.

Remark. For the polynomial properties we will introduce later, namely nontrivial equality and saturated Newton polytope, multiplying a polynomial by a nonzero constant will not change these properties. Thus , we omit the $\frac{1}{(\ell(w)-\ell(v))!}$ coefficient in the formula for the Postnikov-Stanley polynomial for the remainder of the paper.

2.3. **Dual Schubert Polynomials.** A special Postnikov-Stanley polynomial of interest arises when the bottom permutation is the identity.

Definition 2.16. The *dual Schubert polynomial* D^w is defined as $D^w = D_{\mathrm{id}}^w$.

The "dual" terminology comes from the following definition.

Definition 2.17. [Gao20] The *D-pairing* on each graded component of the polynomial ring $\mathbb{C}[x_1, x_2, \dots]$ is defined by
$$\langle f, g\rangle \coloneqq \mathrm{CT}(f(\partial/\partial x) \cdot g(x)),$$
where CT stands for the constant term.

Theorem 2.18. [BG82, Theorem 3.13][PS09, Corollary 12.3] The collection of Schubert polynomials $\{\mathfrak{S}_w\}_{w\in S_\infty}$ and the collection of dual Schubert polynomials $\{D^w\}_{w\in S_\infty}$ form a dual basis of $\mathbb{C}[x_1, x_2, \dots]$ with respect to the D-pairing.

The coefficients of Postnikov-Stanley polynomials on the basis of dual Schubert polynomials are determined by the *generalized Littlewood-Richardson coefficients*.

Definition 2.19. The *generalized Littlewood-Richardson coefficients* $c^w_{u,v}$, parameterized by permutations $u, v, w \in S_n$ are the coefficients that satisfy
$$\mathfrak{S}_u\mathfrak{S}_v = \sum_w c^w_{u,v}\mathfrak{S}_w.$$

Theorem 2.20. [PS09, Corollary 6.9] For any $u \le w$ in S_n, we have
$$D^w_u = \sum_{v\in S_n} c^w_{u,v} D^v.$$

As a result, equalities between Postnikov-Stanley polynomials give information about generalized Littlewood-Richardson coefficients. A formula for the generalized Littlewood-Richardson coefficients is currently one of the biggest open problems in combinatorics. We discuss our equality results in Section 3.

2.4. **Saturated Newton Polytope.** The saturated Newton polytope property of polynomials is a multidimensional generalization of the log concavity property of sequences. This property has been proven for many families of polynomials, including Schur polynomials [MTY19], Schubert polynomials [FMS18], and double Schubert polynomials [Cas+23a], so it is natural to consider Postnikov-Stanley polynomials as well.

Definition 2.21. For a tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$, let x^α denote the monomial
$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}[x_1, \dots, x_n].$$

Definition 2.22. The *Newton polytope* $\mathrm{Newton}(f)$ of a polynomial $f = \sum_{\alpha\in\mathbb{Z}^n_{\ge 0}} c_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ is the convex hull of its exponent vectors; that is,
$$\mathrm{Newton}(f) = \mathrm{conv}\left(\{\alpha \in \mathbb{Z}^n_{\ge 0} \mid c_\alpha \neq 0\}\right) \subseteq \mathbb{R}^n.$$

We introduce the following fact about Newton polytopes.

Definition 2.23. For two sets of points P and Q, the *Minkowski sum* $P + Q$ is the set
$$P + Q = \{p + q \mid p \in P,\ q \in Q\}.$$

Proposition 2.24. [MTY19] For two polynomials $f, g \in \mathbb{C}[x_1, \dots, x_n]$, we have
$$\mathrm{Newton}(fg) = \mathrm{Newton}(f) + \mathrm{Newton}(g)$$
and
$$\mathrm{Newton}(f + g) = \mathrm{conv}(\mathrm{Newton}(f) \cup \mathrm{Newton}(g)).$$

We now give the main definition of this section.

Definition 2.25. A polynomial f has *Saturated Newton Polytope (SNP)* if $c_\alpha \neq 0$ for every integer point $\alpha \in \mathrm{Newton}(f)$.

Example 2.26. As computed in Example 2.15, $D_{213}^{321} = x_1x_2 + \frac{1}{2}x_2^2$, so its Newton polytope $\mathrm{Newton}(D_{213}^{321})$ is the line segment from $(1,1)$ to $(0,2)$ in $\mathbb{R}^2$. There are no integer points on this line segment besides the endpoints, so D_{213}^{321} has SNP.

Example 2.27. A nonexample for SNP is the polynomial

$$f = x^{(0,1,3)} + x^{(0,3,1)} + x^{(1,0,3)} + x^{(1,3,0)} + x^{(3,0,1)} + x^{(3,1,0)},$$

because $\mathrm{Newton}(f)$ contains the integer point $(0,2,2)$, but there is no $cx^{(0,2,2)}$ monomial in f for c nonzero.

2.5. **Matroid Polytopes.** Matroid polytopes, a special type of Newton polytope, were heavily used in [Cas+23a] to prove that double Schubert polynomials have SNP and to characterize their Newton polytopes. We also leverage the following results about matroid polytopes to discuss products of linear polynomials in Section 4.1 and characterize the Newton polytope of dual Schubert polynomials in Section 5.1.

Definition 2.28. A *matroid* $M = (E, \mathcal{B})$ consists of a finite set E and a nonempty collection of subsets $\mathcal{B}$ of E, called the *bases* of M, which satisfy the *basis exchange axiom*: if $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 \setminus B_2$, then there exists $b_2 \in B_2 \setminus B_1$ such that $B_1 \setminus \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

In the remainder of this section, we let $E = [n]$.

Definition 2.29. The *matroid polytope* $P(M)$ of a matroid $M = ([n], \mathcal{B})$ is

$$P(M) = \mathrm{conv}(\{\zeta^B : B \in \mathcal{B}\}),$$

where $\zeta^B = (\mathbb{1}_{i\in B})_{i=1}^n$ denotes the indicator vector of B.

Generalized permutahedra were first studied in [Pos05], and many nice connections to matroid polytopes have been found since. A generalized permutahedron is a deformation of the standard permutahedron–the convex hull in $\mathbb{R}^n$ of the vector $(0,1,\dots,n-1)$ and all permutations of its entries–and has the following explicit characterization.

Definition 2.30. A *generalized permutahedron* $P_n^z(\{z_I\})$, parameterized by collections of real numbers $\{z_I\}$ for $I \subseteq [n]$, is given by

$$P_n^z(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i\in I} t_i \geq z_I \text{ for } I \neq [n], \sum_{i=1}^n t_i = z_{[n]} \right\}.$$

Proposition 2.31. [ABD08] Generalized permutahedra are closed under Minkowski sums:

$$P_n^z(\{z_I\}) + P_n^z(\{z_I'\}) = P_n^z(\{z_I + z_I'\}).$$

Definition 2.32. For a matroid $M = ([n], \mathcal{B})$, the *rank function* $r_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ on subsets of $[n]$ is given by

$$r_M(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\}.$$

Proposition 2.33. [ABD08] Matroid polytopes are generalized permutahedra with

$$\begin{aligned} P(M) &= P_n^z(\{r_M([n]) - r_M([n] \setminus I)\}_{I\subseteq[n]}) \\ &= \left\{ t \in \mathbb{R}^n : \sum_{i\in I} t_i \leq r_M(I) \text{ for } I \neq [n], \sum_{i=1}^n t_i = r_M([n]) \right\}. \end{aligned}$$

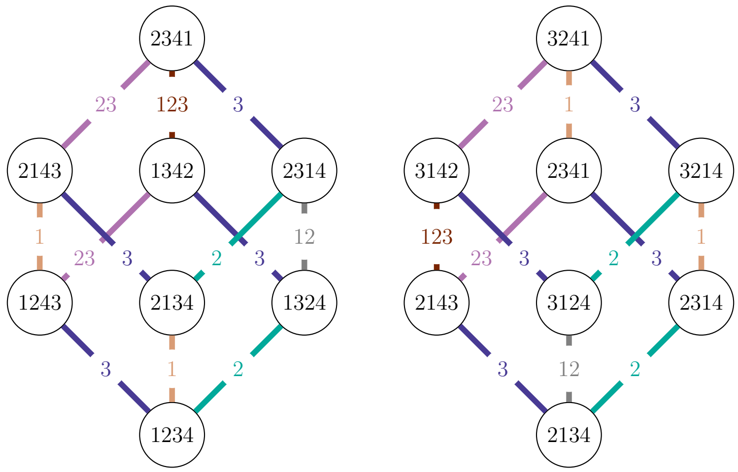


Figure 3. The Bruhat intervals for the equality $D_{1234}^{2341} = D_{2134}^{3241}$. The edge label 123 stands for $x_1 + x_2 + x_3$, etc.

Proposition 2.34. [Sch03, Corollary 46.2c] Let $M_1, \ldots, M_k$ be matroids, and let $Q = P(M_1) + \cdots + P(M_k)$. Every integer point $q \in Q$ can be written as $q = p_1 + \cdots + p_k$ for integer points $p_i \in P(M_i)$.

3. Equality Results

3.1. **Defining Nontrivial Equalities.** While the same edge labels and interval structure appearing in two places in the Bruhat order yield equal Postnikov-Stanley polynomials, we care about equalities involving Bruhat intervals that are nonobvious. This notion is formalized as follows.

Definition 3.1. For a saturated chain $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$, define its *weight multiset* M_C to be the multiset containing all weights $m(u_{i-1} \lessdot u_i)$ for $1 \le i \le \ell$. For an interval $[u, w]$, define its *chain multiset* to be the multiset containing all weight multisets M_C for C a saturated chain in $[u, w]$.

Example 3.2. From Figure 2, the chain multiset of $[123, 321]$ is

$$\{\{x_1, x_2, x_1\}, \{x_1, x_1 + x_2, x_2\}, \{x_2, x_1 + x_2, x_1\}, \{x_2, x_1, x_2\}\}.$$

Definition 3.3. An equality between two Postnikov-Stanley polynomials is *nontrivial* if their chain multisets are distinct, and *trivial* otherwise.

We wish to consider nontrivial equalities between Postnikov-Stanley polynomials.

Example 3.4. The equality $D_{1234}^{2341} = D_{2134}^{3241}$, displayed in Figure 3, is nontrivial: both polynomials evaluate to $6x_1x_2x_3 + 3x_1x_3^2 + 3x_2^2x_3 + 3x_2x_3^2 + x_3^3$, and the chain multisets are different for the two intervals. For example, the weight multiset $\{x_1 + x_2 + x_3, x_3, x_2\}$ appears in $[1234, 2341]$, but not $[2134, 3241]$.

One class of trivial equalities is described as follows.

Proposition 3.5. [BB05] For $w \in S_n$, let $w' \in S_n$ denote the permutation with $w'(i) = n + 1 - w(i)$, for all $1 \le i \le n$. Then for $u, w \in S_n$, we have $u \lessdot w$ if and only if $w' \lessdot u'$.

Example 3.6. If $w = 2341 \in S_4$, then $w' = 3214$.

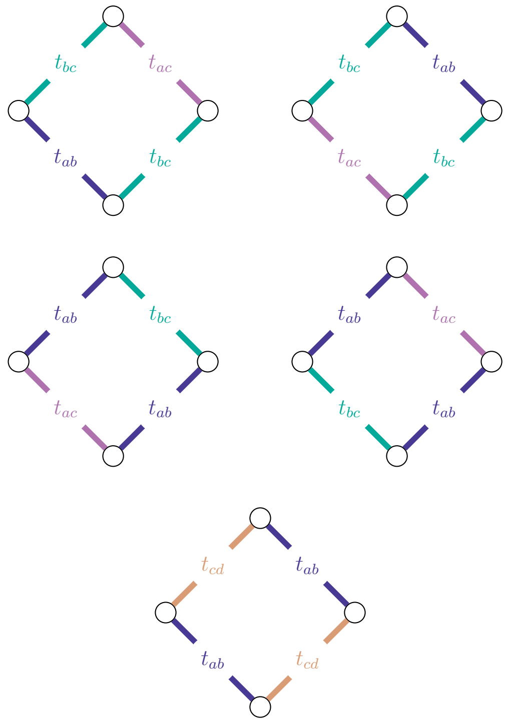


FIGURE 4. The five types of rank 2 intervals.

Observe that if $u \lessdot w$ and $w = ut_{ab}$, then $u' = w't_{ab}$, i.e. $m(u \lessdot w) = m(w' \lessdot u')$.

Example 3.7. For $u \leq w$ in S_n, we have $D_u^w = D_{w'}^{u'}$ because there is a correspondence of saturated chains with the same weight multisets. Namely, there exists a bijection by Proposition 3.5 of saturated chains

$$(u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = w) \leftrightarrow (w' = u'_\ell \lessdot \cdots \lessdot u'_1 \lessdot u'_0 = u')$$

with the same weights in opposite order. As a concrete example, the equality $D_{1234}^{2341} = D_{3214}^{4321}$ in S_4 is trivial.

3.2. **The Rank 2 Case.** In this section, we classify all Postnikov-Stanley polynomials of rank 2.

Lemma 3.8. [BW82] Up to a reordering of chains, there are five types of rank 2 Bruhat intervals, shown in Figure 4.

The corresponding Postnikov-Stanley polynomials are

$$(x_b + \cdots + x_{c-1})(2x_a + \cdots + 2x_{b-1} + x_b + \cdots + x_{c-1})$$
$$(x_a + \cdots + x_{b-1})(x_a + \cdots + x_{b-1} + 2x_b + \cdots + 2x_{c-1})$$

for $a < b < c$, or

$$2(x_a + \cdots + x_{b-1})(x_c + \cdots + x_{d-1})$$

for $a < b$ and $c < d$.

Proposition 3.9. There are no nontrivial equalities between Postnikov-Stanley polynomials of length 2 in any S_n.

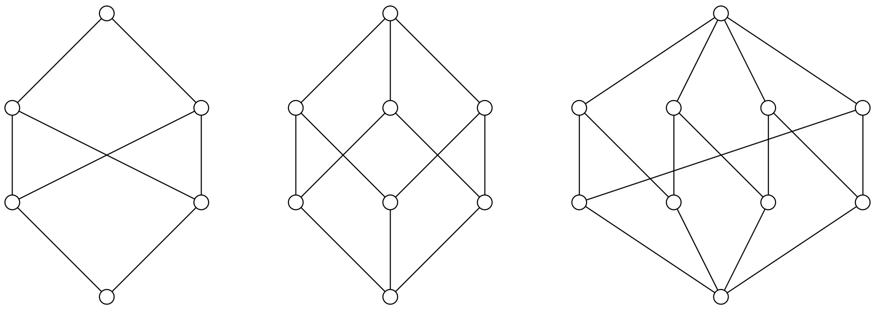


FIGURE 5. From left to right, a 2-crown, a 3-crown, and a 4-crown.

Proof. Casework and unique factorization shows that equality in the parameters $a < b < c$, or $a < b$ and $c < d$ is forced. □

By inspection of all rank 2 intervals in Figure 4, we have the following results.

Lemma 3.10. All rank 2 intervals contain two nonadjacent edges with the same weight u_1. Moreover, the weights on the other two edges u_2, u_3 are either equal or satisfy $u_2 - u_3 = \pm u_1$.

In later sections, we write the latter condition more concisely as $|u_2 - u_3| = u_1$.

Corollary 3.11. Given three edge weights of a rank 2 interval, the fourth edge weight is uniquely determined.

Definition 3.12. We call rank 2 intervals in the former case of Lemma 3.10 as *type I* and rank 2 intervals in the latter case as *type II*.

3.3. **The Rank 3 Case.** There is an explicit characterization of the structure of rank 3 Bruhat intervals that we use to consider rank 3 Postnikov-Stanley polynomial equalities.

Theorem 3.13. [BB05] The only poset structures of rank 3 Bruhat intervals are 2-crowns, 3-crowns, and 4-crowns, displayed in Figure 5.

The posets in this section have natural geometric interpretations. We refer to 3-crowns as *cubes* and the rank 2 Bruhat intervals as *squares*. We refer to edges of a cube or square poset as *parallel* if the corresponding edges in a cube or square, respectively, are parallel.

Definition 3.14. Two posets with labeled edges are *rotations* of each other if there is a graph isomorphism between them, i.e. an edge-preserving bijection between their vertices.

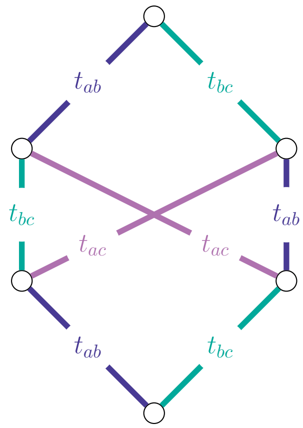
Example 3.15. The two cubes in Figure 3 are rotations of each other.

Lemma 3.16. There is only one type of 2-crown in the Bruhat order, and the Postnikov-Stanley polynomial of a 2-crown is of the form

$$3(x_a + \cdots + x_{b-1})(x_b + \cdots + x_{c-1})(x_a + \cdots + x_{c-1})$$

for some $a < b < c$.

Proof. Up to a relabeling of chains, there is only one type of 2-crown, which has the desired Postnikov-Stanley polynomial.



□

By inspection, an equality between Postnikov-Stanley polynomials of 2-crowns must come from identical parameters $a < b < c$ and be trivial.

Corollary 3.17. There are no nontrivial equalities between Postnikov-Stanley polynomials of 2-crowns in any S_n.

The following is a sufficient condition for two rank 3 Postnikov-Stanley polynomials to be equal, motivated by the observation that all nontrivial rank 3 equalities in S_4 come from two cubes that are rotations of each other. In the theorem statement, we extend the definition of a Postnikov-Stanley polynomial to any poset interval with edges labeled by polynomials.

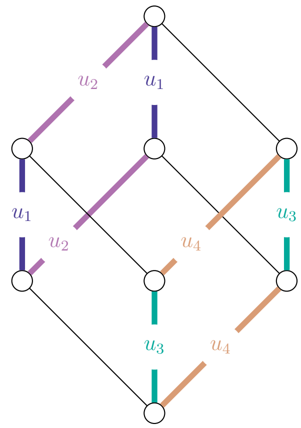
Theorem 3.18. All rotations of a given cube in the Bruhat order have equal Postnikov-Stanley polynomials.

Note that not all possible rotations of a given cube necessarily appear in the Bruhat order. Furthermore, some rotations of the cube may yield trivial equalities, while others may yield nontrivial equalities.

We first prove two lemmas that help characterize the structure of cubes in the Bruhat order.

Lemma 3.19. Every cube in the Bruhat order has four parallel edges with the same label.

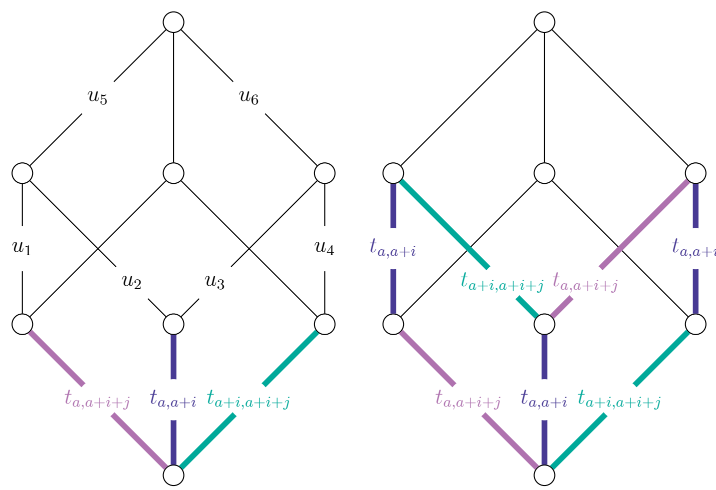
Proof. Each vertex of the cube is a vertex of exactly three squares. First suppose that every vertex is contained in at most two type II squares. Then there must exist two squares on opposite faces of the cube that are not type II: otherwise, we could choose one type II square from each of the three pairs of opposite faces, which meet at a vertex of the cube. Therefore in the following diagram, either $u_1 = u_3$, $u_2 = u_4$, or the remaining four edges have the same label.



Now suppose that some vertex is adjacent to three type II squares. By Lemma 3.10, the labels of the three edges adjacent to that vertex pairwise share an element. There are two possibilities:

Case 1: The labels are $\{t_{a,a+i}, t_{a,a+i+j}, t_{a+i,a+i+j}\}$ for some integers $a, i, j \geq 1$.

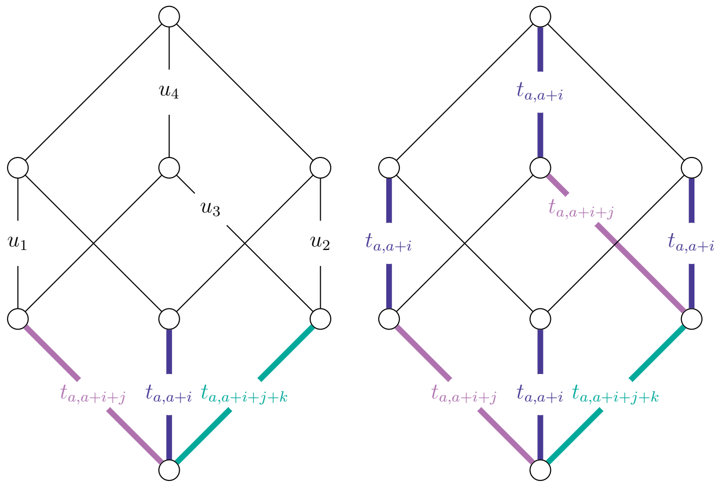
We claim that this case yields a contradiction. In the following diagram, we draw the bottom vertex adjacent to those labels but the same reasoning works for any other vertex.



From the square with unknown labels u_1 and u_2, we may deduce by Lemma 3.10 that $u_1 = t_{a,a+i}$ and $u_2 = t_{a+i,a+i+j}$. Since u_3 is adjacent to $u_2 = t_{a+i,a+i+j}$ and thus does not equal $t_{a+i,a+i+j}$, we must have that $u_4 = t_{a,a+i}$ and $u_3 = t_{a,a+i+j}$ in order for the square with unknown labels u_3 and u_4 to have two equal parallel sides. Considering u_5 and u_6, the possibilities are $u_5 = t_{a,a+i+j}$ and $u_6 = t_{a,a+i}$), or $u_5 = t_{a,a+i}$ and $u_6 = t_{a+i,a+i+j}$). Both of these are contradictions because u_5 and u_6 are already adjacent to edges labeled $t_{a,a+i}$, so neither can equal $t_{a,a+i}$.

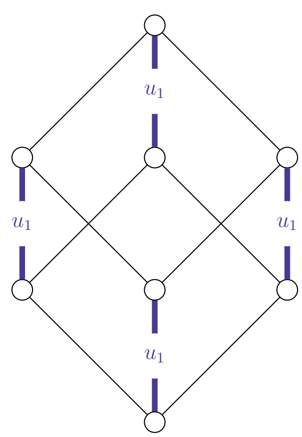
Case 2: The labels are $\{t_{a,a+i}, t_{a,a+i+j}, t_{a,a+i+j+k}\}$ for some integers $a, i, j, k \geq 1$.

In the following diagram, we draw the bottom vertex adjacent to those labels but the same reasoning works for any other vertex. We claim that this case yields four parallel edges with the label $t_{i,i+a}$. If two adjacent edges of a square have labels t_{ab} and t_{ac} for some $a < b < c$, then by Lemma 3.10, the edge opposite to the edge labeled t_{ab} is also labeled t_{ab}. As a result, from the three squares adjacent to the bottom vertex, we deduce $u_1 = t_{a,a+i}$, $u_2 = t_{a,a+i}$, and $u_3 = t_{a,a+i+j}$. Finally from u_2 and u_3, we deduce that $u_4 = t_{a,a+i}$.

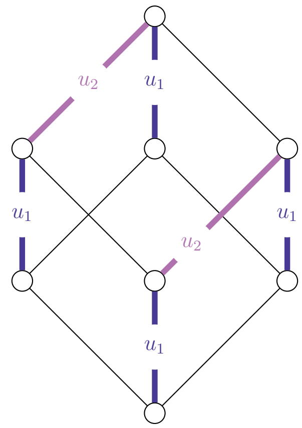


The case of $\{t_{a,a-i}, t_{a,a-i-j}, t_{a,a-i-j-k}\}$ is analogous. $\square$

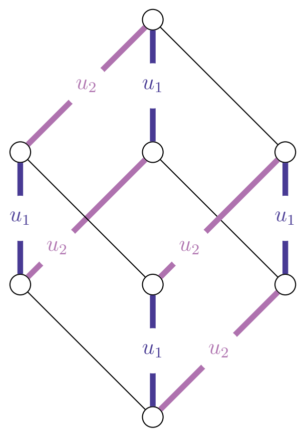
Proof of Theorem 3.18. First by Lemma 3.19, there exist four parallel edges with the same label, say u_1. Observe that rotations of the cube fixing its top and bottom vertices preserve the chains of the poset, so it does not matter which four parallel edges are chosen.

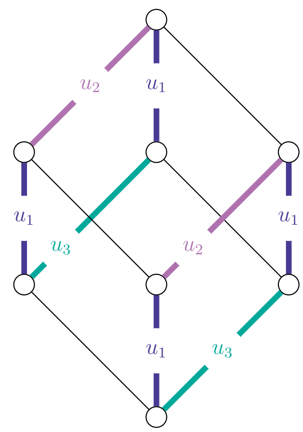


Now by Lemma 3.10, there exist two parallel edges on the top face with the same label, say u_2. It does not matter which two parallel edges on the top face are chosen, since we can reflect the poset.



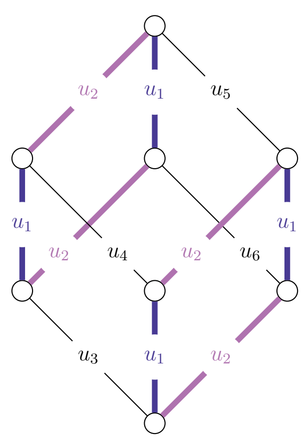
The bottom face also has two parallel edges with the same label. For a square with two u_1 edges and one u_2 edge, the label u_3 on the remaining edge is determined by u_1 and u_2: by Lemma 3.10, either $u_3 = u_2$, or $|u_3 - u_2| = u_1$, which yield the following two cubes.





Case 1: $u_3 = u_2$.

We label the remaining edges of the cube with u_3, u_4, u_5, and u_6 as shown.



The possible Postnikov-Stanley polynomials of a rotation of the cube are

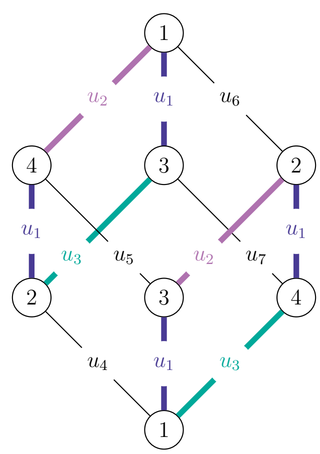
$$u_1 u_2 (2u_3 + u_4 + 2u_5 + u_6)$$

and
$$u_1u_2(u_3 + 2u_4 + u_5 + 2u_6),$$
which are equal if and only if
$$u_3 + u_5 = u_4 + u_6.$$

- If $u_4 = u_5$, then we must have $u_3 = u_6$ by Corollary 3.11. Similarly, if $u_3 = u_6$, if $u_4 = u_5$, or if $u_5 = u_6$, the other two edge labels must be equal. In these cases, we obtain $u_3 + u_5 = u_4 + u_6$.
- Now we are in the case that $u_3 \neq u_4 \neq u_5 \neq u_6 \neq u_3$, which yields the relations
$$|u_3 - u_4| = |u_5 - u_6| = u_1$$
$$|u_3 - u_6| = |u_4 - u_5| = u_2.$$
If $u_3 + u_5 \neq u_4 + u_6$, then the above absolute value equalities become
$$u_3 - u_4 = u_5 - u_6$$
$$u_3 - u_6 = u_5 - u_4.$$
Solving yields $u_3 = u_5$ and $u_4 = u_6$, which then implies
$$u_1 = |u_3 - u_4| = |u_3 - u_6| = u_2,$$
a contradiction.

Case 2: We may now assume that besides the four parallel u_1 edges, there are no other four parallel edges with the same label.

We label the remaining edges of the cube with u_4, u_5, u_6, and u_7 as shown. We also label opposite vertices of the cube with the same number out of 1, 2, 3, and 4 as shown; a rotation of the cube is determined by its top and bottom vertex.



The Postnikov-Stanley polynomials for rotations of the cube with top and bottom vertices labeled 1 and rotations with top and bottom vertices labeled 2 are equal by symmetry, and similarly for the rotations labeled 3 and 4. The polynomials for the two cases, after omitting the factor of u_1 common to all chains in the cube, are
$$u_2(u_4 + u_5 + u_6) + u_3(u_4 + u_6 + u_7)$$
$$u_2(u_5 + u_6 + u_7) + u_3(u_4 + u_5 + u_7).$$

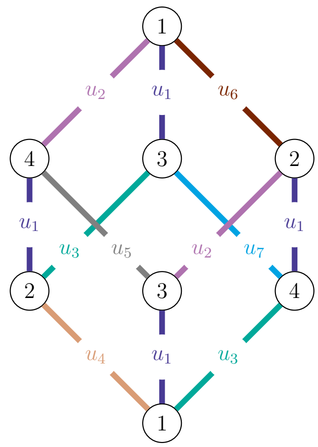
To show that these polynomials are equal, we prove that

$$u_2(u_4 - u_7) - u_3(u_5 - u_6) = 0.$$

If $u_4 = u_7$, then we must have $u_5 = u_6$ by Corollary 3.11; similarly, if $u_5 = u_6$, then $u_4 = u_7$. In both of these cases, we are done.

Now assume $u_4 \neq u_7$ and $u_5 \neq u_6$, so $|u_4 - u_7| = u_3$ and $|u_5 - u_6| = u_2$. We split into two subcases.

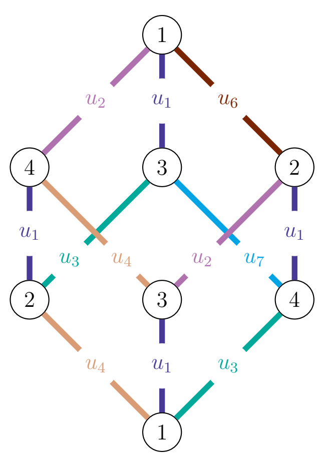
- We first show that the case $u_4 \neq u_5$ and $u_6 \neq u_7$ can not occur in the Bruhat order.



  This configuration would yield the equations

$$\begin{aligned} u_1 &= |u_2 - u_3| = |u_4 - u_5| = |u_6 - u_7| \\ u_2 &= |u_5 - u_6| \\ u_3 &= |u_4 - u_7| \end{aligned}$$

  From the first line, we obtain that $(u_4 - u_5) + (u_6 - u_7)$ equals 0, $2u_1$, or $-2u_1$. However, the possibilities for $(u_4 - u_5) + (u_6 - u_7) = (u_4 - u_7) - (u_5 - u_6)$ from the second and third lines are $u_2 + u_3$, $u_2 - u_3$, $-u_2 + u_3$, and $-u_2 - u_3$, none of which can equal 0, $2u_1$, or $-2u_1$.
- Now we may without loss of generality suppose that $u_4 = u_5$ (instead of $u_6 = u_7$).



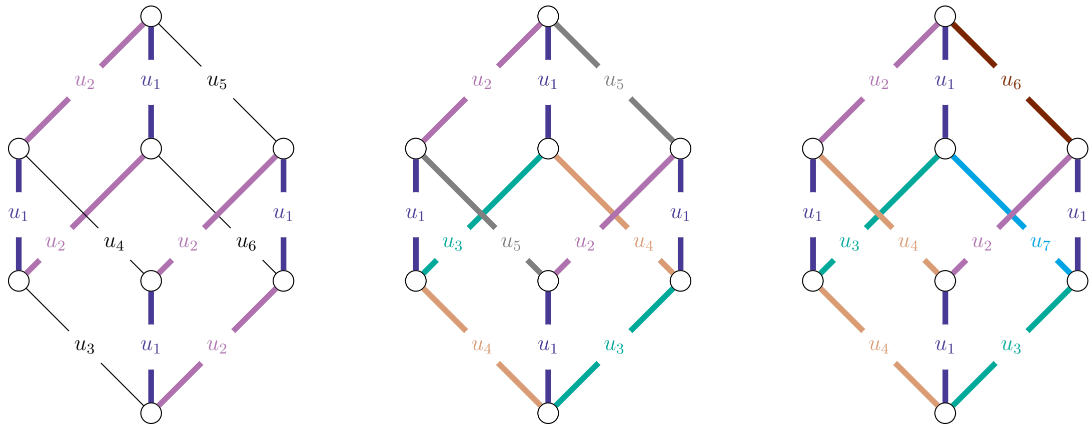


FIGURE 6. Types of cubes in the Bruhat order.

Note that $u_6 \neq u_7$, as otherwise we would obtain $u_2 = |u_5 - u_6| = |u_4 - u_7| = u_3$, a contradiction. Now $|u_4 - u_6| = u_2$ and $|u_4 - u_7| = u_3$, so

$$\begin{aligned} |u_6 - u_7| &= |u_2 - u_3| \\ &= ||u_4 - u_6| - |u_4 - u_7||. \end{aligned}$$

This implies that $(u_4 - u_6, u_4 - u_7)$ are the same sign, and equal to (u_2, u_3) or $(-u_2, -u_3)$. Then in this case, we have

$$\begin{aligned} u_2(u_4 - u_7) - u_3(u_5 - u_6) &= u_2(u_4 - u_7) - u_3(u_4 - u_6) \\ &= 0, \end{aligned}$$

as desired.

Note that the equality in Figure 3 is an example of this final case. □

From the casework in the proof of Theorem 3.18, we have the following corollary.

Corollary 3.20. Cubes in the Bruhat order can be classified into the three types shown in Figure 6.

This classification of cubes in the Bruhat order may be used to prove the following.

Theorem 3.21. There is never an equality between the Postnikov-Stanley polynomials of a 2-crown and a cube.

Proof. We denote the Postnikov-Stanley polynomials of the 2-crown and the cube by f and g, respectively. For the sake of contradiction, suppose that $f = g$. By Lemma 3.16, we may write

$$f = 3(x_i + \cdots + x_{j-1})(x_j + \cdots + x_{k-1})(x_i + \cdots + x_{k-1})$$

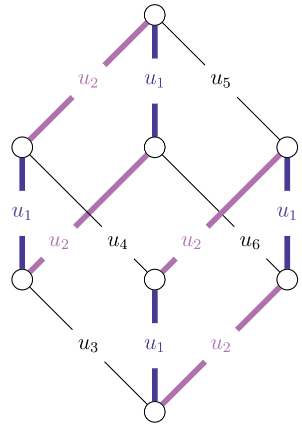
for positive integers $i < j < k$. By Lemma 3.19, there exist four parallel edges of the cube with the same label t_{ab}, so we may write

$$g = (x_a + \cdots + x_{b-1}) \cdot g'$$

for positive integers $a < b$ and a polynomial g'. This forces $\{a, b\} \subset \{i, j, k\}$.

We now split into three cases based the type of cube from Corollary 3.20. Let the four parallel equal edges of the cube be labeled with t_{ab}.

Case 1: We have $u_1 = x_a + \cdots + x_{b-1}$ and $u_2 = u_c + \cdots + x_{d-1}$ for $a \neq c$ and $b \neq d$ in the following diagram.



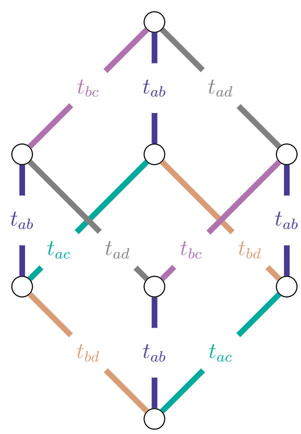
Then

$$g = u_1 u_2 (2u_3 + u_4 + 2u_5 + u_6).$$

As a result, the factors u_1 and u_2 of g must both be factors of f, but no two factors $x_a + \cdots + x_{b-1}$ and $u_c + \cdots + x_{d-1}$ of f satisfy $a \neq c$ and $b \neq d$, a contradiction.

For the remainder of this proof, a label t_{ij} on an edge of the cube does not imply $i < j$ unless specified.

Case 2: Besides t_{ab}, the other two bottom-most edge labels are t_{ac} and t_{bd} in some order for c and d such that a, b, c, d are pairwise distinct.



By repeatedly applying Lemma 3.10, the rest of the edge labels may be determined as shown. Each of c and d is less than a or greater than b, since there exist squares containing two t_{ab} labels and one t_{ac} or t_{bd} label. As a result, we cannot have $(a, b) = (i, k)$, because the term x_c or x_{c-1} appears in g. Now suppose $(a, b) = (i, j)$, which forces $b < c, d \leq k$. However, there is no way to obtain a $3x_{k-1}^2$ term in g', because no chain in the diagram contains an x_{d-1}^2 term if $c < d$, or an x_{c-1}^2 term if $d < c$. The case of $(a, b) = (j, k)$ leads to a similar contradiction by considering the $3x_{i-1}^2$ term.

Case 3: Besides t_{ab}, the other two bottom-most edge labels are t_{ac} and t_{cd} in some order for c and d such that a, b, c, d are pairwise distinct.

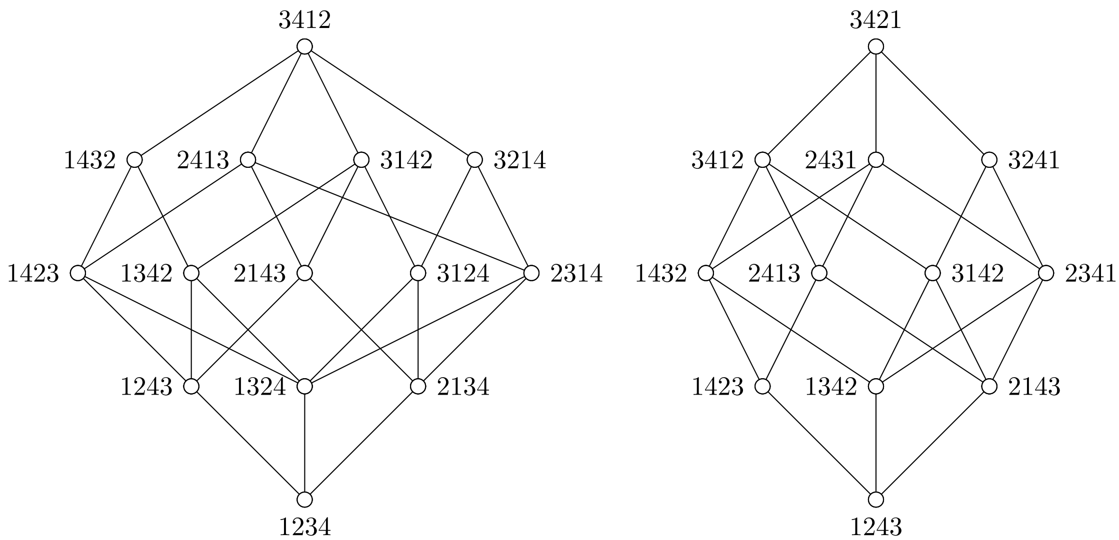
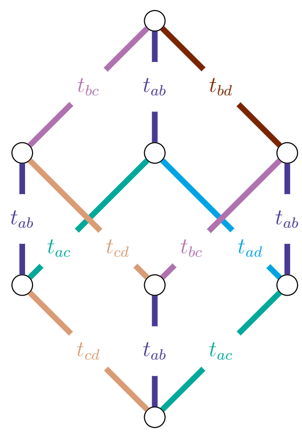


FIGURE 7. A nontrivial equality between two different poset structures.



By repeatedly applying Lemma 3.10, the rest of the edge labels may be determined as shown. We can not have $(a, b) = (i, k)$ by the same reasoning as Case 2. If $(a, b) = (i, j)$, then we have $b < c, d \leq k$. We need $c = k$ or $d = k$ to obtain a $3x_{k-1}^2$ term in g'. However, the coefficient of x_{d-1}^2 in g' is 0 if $d > c$, so $c = k$. However, we may verify that the coefficient of $x_a x_{c-1} = x_i x_{k-1}$ in g' is 2, while is supposed to be 3. The case of $(a, b) = (j, k)$ is analogous. ☐

Remark. The corresponding poset intervals being rotations of each other is not a necessary condition for equality. For example, the nontrivial equality

$$D_{1234}^{3412} = D_{1243}^{3421}$$

in S_4 occurs between two rank 4 intervals which have different poset structures, as shown in Figure 7. Both polynomials equal

$$6x_1^2x_2^2 + 12x_1^2x_2x_3 + 8x_1x_2^3 + 24x_1x_2^2x_3 + 2x_2^4 + 8x_2^3x_3 + 12x_1x_2x_3^2 + 6x_2^2x_3^2.$$

Even among poset intervals of the same structure, being rotations of each other is not a necessary condition for equality. For example, we have computed that the nontrivial equality

$D^{2431}_{1234} = D^{3421}_{2134}$ in S_4 occurs between two rank 4 intervals with the same poset structure that are not rotations of each other.

Some questions of interest are the following.

Question 3.22. Do Theorem 3.18 and Lemma 3.19 proven for cubes extend to larger boolean intervals in the Bruhat order of S_n?

Conjecture 3.23. Nontrivial equalities between two rank 3 Postnikov-Stanley polynomials only occur between two cubes that are rotations of each other.

In light of Corollary 3.17 and Theorem 3.18, this conjecture is equivalent to there being no nontrivial equalities involving a 4-crown.

4. Saturated Newton Polytope Results

4.1. **Single Chain Newton Polytope.**

We introduce a property called single chain Newton polytope which we prove is a sufficient condition for SNP.

Definition 4.1. For $u \leq w$ in S_n, the Postnikov-Stanley polynomial D^w_u has *single chain Newton polytope (SCNP)* if there exists a saturated chain C in the interval $[u, w]$ such that

$$\mathrm{Newton}(m_C) = \mathrm{Newton}(D^w_u).$$

We call such a saturated chain C a *dominant chain* of the interval $[u, w]$.

The following propositions are the motivation for defining SCNP.

Proposition 4.2. Any product of linear factors in $x_1, \ldots, x_n$ with all coefficients nonnegative has SNP.

Proof. Since all coefficients are nonnegative, we may assume without loss of generality for an SNP proof that all nonzero coefficients are 1.

Consider the product of linear factors $\prod_{i=1}^k (x_{a_{i1}} + x_{a_{i2}} + \cdots + x_{a_{im_i}})$ with all $1 \leq a_{ij} \leq n$. Let $M_i = ([n], \mathcal{B})$ be the matroid on $[n]$ with bases $\{a_{i1}\}, \{a_{i2}\}, \ldots, \{a_{im_i}\}$ and $P(M_i)$ its matroid polytope so that

$$P(M_i) = \mathrm{Newton}(x_{a_{i1}} + x_{a_{i2}} + \cdots + x_{a_{im_i}}).$$

Then each monomial $c_\alpha x^\alpha$ in the expansion $\prod_{i=1}^k (x_{a_{i1}} + x_{a_{i2}} + \cdots + x_{a_{im_i}})$ corresponds to a point α in

$$Q = \sum_{i=1}^k P(M_i) = \mathrm{Newton}\left(\prod_{i=1}^k (x_{a_{i1}} + x_{a_{i2}} + \cdots + x_{a_{im_i}})\right).$$

By Proposition 2.34, every integer point $\alpha \in Q$ can be written as $\alpha = p_1 + \cdots + p_k$, where $p_i \in P(M_i)$ is an integer point. Writing $x_{a_{ih_i}}$ as the monomial in $x_{a_{i1}} + x_{a_{i2}} + \cdots + x_{a_{im_i}}$ corresponding to $p_i \in P(M_i)$, we have $x^\alpha = \prod_{i=1}^k x_{a_{ih_i}}$. This shows that every integer point $\alpha \in Q$ appears as an exponent vector in the product $\prod_{i=1}^k (x_{a_{i1}} + x_{a_{i2}} + \cdots + x_{a_{im_i}})$, as desired. $\square$

Proposition 4.3. If D^w_u has SCNP, then D^w_u has SNP.

Proof. By Proposition 4.2, the chain weight m_C has SNP for any saturated chain C in the Bruhat order. Then if there exists a saturated chain C in the interval $[u, w]$ such that $\mathrm{Newton}(m_C) = \mathrm{Newton}(D^w_u)$ by the definition of SCNP, D^w_u also has SNP. $\square$

Example 4.4. Recall from Example 2.15 that $D_{213}^{321} = \frac{1}{2!}(x_1x_2 + (x_1+x_2)\cdot x_2)$. The saturated chain $C = (213 \lessdot 312 \lessdot 321)$ has weight $m_C = (x_1+x_2)\cdot x_2$ which satisfies

$$\mathrm{Newton}(m_C) = \mathrm{Newton}(D_{213}^{321}).$$

Thus, C is a dominant chain of $[213, 321]$ and D_{213}^{321} has SCNP.

Example 4.5. As a nonexample, we used SageMath to verify that D_{1324}^{4231} does not have SCNP: none of the 24 saturated chains in $[1324, 4231]$ are dominant.

Remark. We also verified that D_u^w has SCNP for all intervals $[u, w]$ in S_2 and S_3, and that D_{1324}^{4231} is the only exception to SCNP in S_4. However, there are 77 exceptions in S_5.

As SCNP implies SNP, we ask the following natural question.

Question 4.6. For which intervals $[u, w]$ in S_n does D_u^w has SCNP?

We have the following partial results and conjectures.

Lemma 4.7. All Postnikov-Stanley polynomials of rank 2 have SCNP.

Proof. In all types of rank 2 intervals from Lemma 3.8, we may verify that the two saturated chains C_1 and C_2 have the property that $\mathrm{Newton}(m_{C_1}) \subseteq \mathrm{Newton}(m_{C_2})$ or $\mathrm{Newton}(m_{C_2}) \subseteq \mathrm{Newton}(m_{C_1})$. □

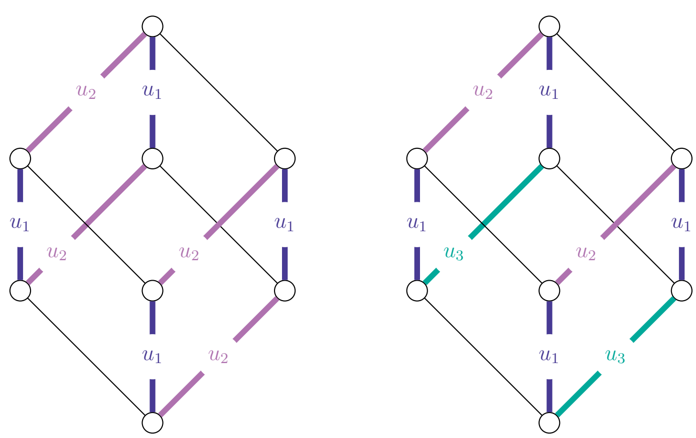
Lemma 4.8. All Postnikov-Stanley polynomials of 2-crowns have SCNP.

Proof. From the characterization of 2-crowns in Lemma 3.16, the saturated chain with edge labels t_{ab}, t_{ac}, t_{bc} is a dominant chain. □

Lemma 4.9. All Postnikov-Stanley polynomials of cubes have SCNP.

Proof. This has been verified via a brute-force computation with all cubes in S_6, but we outline the following cleaner approach.

Recall the following two cases from Theorem 3.18.



In the first case, let the the four remaining edges not labeled u_1 or u_2 be $t_{a_1b_1}, t_{a_2b_2}, t_{a_3b_3}, t_{a_4b_4}$. We select a saturated chain including $t_{a_ib_i}$ which maximizes $b_i - a_i$ for $1 \le i \le 4$. This saturated chain will be a dominant chain.

In the second case, casework yields that there exists an edge between the middle two ranks that maximizes $b_i - a_i$ over all edge labels $t_{a_ib_i}$. There is a unique saturated chain that contains this middle edge, and that saturated chain is dominant. □

Conjecture 4.10. All Postnikov-Stanley polynomials of rank 3 have SCNP.

In light of Theorem 3.13, Lemma 4.8, and Lemma 4.9, it remains to show that Postnikov-Stanley polynomials of 4-crowns have SCNP. We believe that a detailed case analysis is possible, but hope to find a cleaner approach.

Conjecture 4.11. For a Bruhat interval $[u, w]$ in S_n, if the number of reflections t_{ab} such that $u < ut_{ab} \leq w$ equals $\ell(w) - \ell(u)$ (or equivalently, the *Kazhdan-Lusztig polynomial* $P_{u,w} = 1$), then D_u^w has SCNP.

This conjecture has been checked for all $n \leq 5$.

Definition 4.12. Given $w \in S_n$ and $p \in S_k$ for $k \leq n$, we say the permutation w *contains* p if there exist $1 \leq i_1 < \cdots < i_k \leq n$ such that $w(i_1) \cdots w(i_k)$ is in the same relative order as $p(1) \cdots p(k)$. If w does not contain p, then we say w *avoids* p.

Example 4.13. Let $w = 1324 \in S_4$ and $p = 213 \in S_3$. Since the subword 324 in 1324 is in the same relative order as 213, we have that w contains p.

Conjecture 4.14. For a permutation $u \in S_n$, there exists a permutation $w \in S_n$ such that D_u^w does not have SCNP if and only if u contains 1324. Analogously, for a permutation $w \in S_n$, there exists a permutation $u \in S_n$ such that D_u^w does not have SCNP if and only if w contains 4231.

We have verified Conjecture 4.14 for all Bruhat intervals in S_3, S_4, and S_5, as well as some Bruhat intervals in S_6. A corollary of Conjecture 4.14 is that if either u avoids 1324 or w avoids 4231, then D_u^w has SCNP.

A special case of both Conjecture 4.11 and Conjecture 4.14 is the dual Schubert polynomial D^w, which we show has SCNP in Section 4.2.

4.2. **Dual Schubert Polynomials.** In this section, we prove that dual Schubert polynomials D^w have SCNP. As a corollary, dual Schubert polynomials also have SNP.

Definition 4.15. For a Bruhat interval $[u, w]$, a saturated chain

$$u = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_\ell = w$$

from u to w is a *greedy chain* if it satisfies the following condition for all $i \in [\ell]$: writing $w_i = w_{i-1}t_{ab}$ for $a < b$, there does not exist $w'_{i-1} \lessdot w_i$ with $w'_{i-1} \in [u, w]$ such that

(*i*) $w_i = w'_{i-1}t_{ab'}$ for $b' > b$, or
(*ii*) $w_i = w'_{i-1}t_{a'b}$ for $a' < a$.

Example 4.16. In the Bruhat interval $[123, 321]$, the saturated chain $123 \lessdot 132 \lessdot 231 \lessdot 321$ is greedy, while $123 \lessdot 213 \lessdot 231 \lessdot 321$ is not. This chain fails to be greedy because

$$w_2 = 231 = 213t_{23} = w_1t_{23}$$

but $132 \lessdot w_2$ with $231 = 132t_{13}$, violating condition (*ii*). In general, greedy chains are not unique. For example, $123 \lessdot 213 \lessdot 312 \lessdot 321$ is another greedy chain in $[123, 321]$.

Lemma 4.17. There exists a greedy chain in every Bruhat interval $[u, w]$.

Proof. We construct a greedy chain $u = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_\ell = w$ inductively as follows. Given a greedy chain $u = w_0 \lessdot \cdots \lessdot w_{i-1}$ for $i \in [\ell]$, set $w_i := w_{i-1}t_{ab}$ such that (a, b) cannot be replaced by some longer (a', b) or (a, b'). $\square$

Definition 4.18. For a permutation $w \in S_n$, the *global weight* $\mathrm{GW}(w)$ of w is the polynomial

$$\mathrm{GW}(w) = \prod_{(a,b)\in \mathrm{Inv}(w)} (x_a + x_{a+1} + \cdots + x_{b-1}).$$

Example 4.19. Since $\mathrm{Inv}(231) = \{(1,3),(2,3)\}$ we have that $\mathrm{GW}(231) = (x_1 + x_2)\cdot x_2$.

Lemma 4.20. Given a permutation $w \in S_n$, the weight of every greedy chain in $[\mathrm{id}, w]$ is $\mathrm{GW}(w)$.

Proof. We now prove lemma 4.20. We induct on $\ell(w)$, with the base case of $\ell(w) = 1$ clear. Suppose we have proved the statement for all $w' \in S_n$ with $\ell(w') < \ell$, and let $\ell(w) = \ell$. Let

$$C = (\mathrm{id} = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_\ell = w)$$

be a greedy chain in $[\mathrm{id}, w]$, and suppose $w = w_{\ell-1}t_{ab}$ for $a < b$. Since

$$C' = (\mathrm{id} = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{\ell-1})$$

is a greedy chain in $[\mathrm{id}, w_{\ell-1}]$, by the inductive hypothesis it suffices to show

$$\mathrm{GW}(w) = \mathrm{GW}(w_{\ell-1}) \cdot (x_a + x_{a+1} + \cdots + x_{b-1}).$$

We compare $\mathrm{Inv}(w)$ and $\mathrm{Inv}(w_{\ell-1})$. After switching $w(a)$ and $w(b)$ in $\mathrm{Inv}(w)$, the pair (a,b) is no longer in $\mathrm{Inv}(w_{\ell-1})$. It suffices to show that $\mathrm{Inv}(w_{\ell-1}) = \mathrm{Inv}(w) \setminus \{(a,b)\}$. Note that every pair $(c,d) \in \mathrm{Inv}(w)$ satisfying $c \neq a, b$ and $d \neq a, b$ is in $\mathrm{Inv}(w_{\ell-1})$. The remaining pairs in $\mathrm{Inv}(w)$ are of one of the following forms:

(i) (a,j) for $a < j < b$
(ii) (a,j) for $a < b < j$
(iii) (j,b) for $a < j < b$
(iv) (j,b) for $j < a < b$.

Since $\ell(wt_{ab}) = \ell(w) - 1$, we have $w(j) \notin [w(b), w(a)]$ for every $j \in [a+1, b-1]$. Thus, all ordered pairs under cases (i) and (iii) are in $\mathrm{Inv}(w_{\ell-1})$.

For case (ii), suppose $(a,j) \in \mathrm{Inv}(w) \setminus \mathrm{Inv}(w_{\ell-1})$, so $w(j) \in [w(b), w(a)]$. Let $r > b$ be the smallest integer such that $w(r) \in [w(b), w(a)]$; note that such an r exists because $j > b$ has this property. Then $\ell(wt_{ar}) = \ell(w) - 1$, which contradicts the fact that C is a greedy chain.

For case (iv), suppose $(j,b) \in \mathrm{Inv}(w) \setminus \mathrm{Inv}(w_{\ell-1})$, so $w(j) \in [w(b), w(a)]$. Let $r < a$ bt the largest integer such that $w(r) \in [w(b), w(a)]$; note that such an r exists because $j < a$ has this property. Then $\ell(wt_{rb}) = \ell(w) - 1$, which contradicts the fact that C is a greedy chain.

In conclusion, we have $\mathrm{Inv}(w_{\ell-1}) = \mathrm{Inv}(w) \setminus \{(a,b)\}$, which implies

$$\mathrm{GW}(w) = \mathrm{GW}(w_{\ell-1}) \cdot (x_a + x_{a+1} + \cdots + x_{b-1}),$$

as desired. □

Definition 4.21. We define a partial order $\preceq$ on $\{(a,b) \in \mathbb{N}^2 \mid a < b\}$ such that $(a,b) \preceq (c,d)$ if and only if $[a,b] \subseteq [c,d]$.

Given an integer ℓ, we define a partial order $\preceq_\ell$ on multisets with ℓ elements in $\{(a,b) \in \mathbb{N}^2 \mid a < b\}$ as follows: for two multisets G and H, we say $G \preceq_\ell H$ if and only if there exists a pairing of elements in G and H such that for each pair $((a_i,b_i),(c_i,d_i)) \in G \times H$ in this pairing, we have $(a_i,b_i) \preceq (c_i,d_i)$. We call such pairing a *dominant pairing*.

Example 4.22. Let $G = \{(2,6),(3,4)\}$ and $H = \{(2,4),(1,7)\}$. We have $(2,6) \preceq (1,7)$ and $(3,4) \preceq (2,4)$, so $G \preceq_\ell H$.

Definition 4.23. For a saturated chain $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$ in the Bruhat interval $[u_0, u_\ell]$ in S_n, we define its *generating set* G_C to be the multiset containing the pairs of the positions (a_i, b_i) swapped along edges in C, that is

$$G_C = \{(a_i, b_i) \in [n] \mid u_i = u_{i-1}t_{a_ib_i}, a_i < b_i, i \in [\ell]\}.$$

Example 4.24. For the saturated chain $123 \lessdot 132 \lessdot 231 \lessdot 321$, the generating set is $\{(2,3),(1,3),(1,2)\}$.

Lemma 4.25. Given a permutation $w \in S_n$ with $\ell(w) = \ell$, for every saturated chain

$$C = (\mathrm{id} = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_\ell = w)$$

in $[\mathrm{id}, w]$, we have $G_C \preceq_\ell \mathrm{Inv}(w)$.

Example 4.26. Continuing Example 4.24, we get a generating set of $\{(2,3),(1,3),(1,2)\}$, which equals $\mathrm{Inv}(321)$.

Proof of Lemma 4.25. We induct on ℓ, with the base case of $\ell(w) = 1$ clear. Suppose we have proved the statement for all $w' \in S_n$ with $\ell(w') < \ell$, and let $\ell(w) = \ell$. Let $a < b$ satisfy $w = w_{\ell-1}t_{ab}$, and let

$$C' = (\mathrm{id} = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{\ell-1}).$$

Since $G_C = G_{C'} \cup \{(a,b)\}$, and we know that $G_{C'} \preceq_{\ell-1} \mathrm{Inv}(w_{\ell-1})$ from the inductive hypothesis, so we have

$$G_C = G_{C'} \cup \{(a,b)\} \preceq_\ell \mathrm{Inv}(w_{\ell-1}) \cup \{(a,b)\}.$$

It now suffices to show that $\mathrm{Inv}(w_{\ell-1}) \cup \{(a,b)\} \preceq_\ell \mathrm{Inv}(w)$, which we show by constructing a dominant pairing $\mathcal{P}$.

Since $(a,b) \notin \mathrm{Inv}(w_{\ell-1})$, the multisets $\mathrm{Inv}(w_{\ell-1}) \cup \{(a,b)\}$ and $\mathrm{Inv}(w)$ are in fact both sets. Note that every pair $(c,d) \in \mathrm{Inv}(w)$ satisfying $c \neq a, b$ and $d \neq a, b$ is in $\mathrm{Inv}(w_{\ell-1})$, and (a,b) is in $\mathrm{Inv}(w_{\ell-1}) \cup \{(a,b)\}$. The remaining pairs in $\mathrm{Inv}(w)$ are of one of the following forms:

(i) (a,j) for $a < j < b$
(ii) (a,j) for $a < b < j$
(iii) (j,b) for $a < j < b$
(iv) (j,b) for $j < a < b$.

Since $\ell(wt_{ab}) = \ell(w) - 1$, we have $w(j) \notin [w(b), w(a)]$ for every $j \in [a+1, b-1]$. Then the pairs from cases (i) and (iii) are also in $\mathrm{Inv}(w_{\ell-1})$. We put these pairs which appear in both $\mathrm{Inv}(w_{l-1}) \cup \{(a,b)\}$ and $\mathrm{Inv}(w)$ together in $\mathcal{P}$.

Next, suppose we have $(a,j) \in \mathrm{Inv}(w)$ falling under case (ii). If (b,j) is also in $\mathrm{Inv}(w)$, then $w(j) < w(b) < w(a)$. Hence (a,j) is in $\mathrm{Inv}(w_{\ell-1})$, and we let $((a,j),(a,j)) \in \mathcal{P}$. Otherwise if $(b,j) \notin \mathrm{Inv}(w)$, then $w(b) < w(j) < w(a)$. So $(b,j) \in \mathrm{Inv}(w_{\ell-1}) \setminus \mathrm{Inv}(w)$, and we can let $((b,j),(a,j)) \in \mathcal{P}$ since $[b,j] \subseteq [a,j]$.

Finally, we consider $(j,b) \in \mathrm{Inv}(w)$ falling under case (iv). If (j,b) is also in $\mathrm{Inv}(w)$, then $w(b) < w(a) < w(j)$. Hence (j,a) is in $\mathrm{Inv}(w_{\ell-1})$, and we let $((j,a),(j,a)) \in \mathcal{P}$. Otherwise if $(j,b) \notin \mathrm{Inv}(w)$, then $w(b) < w(j) < w(a)$. So $(j,a) \in \mathrm{Inv}(w_{l-1}) \setminus \mathrm{Inv}(w)$, and we can let $((j,a),(j,b)) \in \mathcal{P}$ since $[j,a] \subseteq [j,b]$.

We have considered all cases of pairs in $\mathrm{Inv}(w)$ and constructed a dominant pairing $\mathcal{P}$, which shows that $\mathrm{Inv}(w_{\ell-1}) \cup \{(a,b)\} \preceq_\ell \mathrm{Inv}(w)$, as desired. $\square$

Lemma 4.27. Given a permutation $w \in S_n$, for every saturated chain C in $[\mathrm{id}, w]$, we have

$$\mathrm{Newton}(m_C) \subseteq \mathrm{Newton}(\mathrm{GW}(w)).$$

Proof. Let $\ell = \ell(w)$. By Lemma 4.25, we have $G_C \preceq_\ell \mathrm{Inv}(w)$, so there exists a dominant pairing $\mathcal{P}$ of G_C and $\mathrm{Inv}(w)$ such that $[a, b] \subseteq [c, d]$ for each pair $((a, b), (c, d)) \in \mathcal{P}$. Observe that each pair $(a, b) \in G_C$ corresponds to a linear factor

$$x_a + x_{a+1} + \cdots + x_{b-1}$$

of m_C, and each pair $(c, d) \in \mathrm{Inv}(w)$ corresponds to a linear factor

$$x_c + x_{c+1} + \cdots + x_{d-1}$$

of $\mathrm{GW}(w)$. As a result, each monomial of m_C is also in $\mathrm{GW}(w)$, so

$$\mathrm{Newton}(m_C) \subseteq \mathrm{Newton}(\mathrm{GW}(w)),$$

as desired. □

Theorem 4.28. Every greedy chain of $[\mathrm{id}, w]$ is a dominant chain of D^w, and

$$\mathrm{Newton}(D^w) = \mathrm{Newton}(\mathrm{GW}(w)).$$

As a result, for all $w \in S_n$, the dual Schubert polynomial D^w has SCNP.

Proof. By Lemma 4.27, every saturated chain C in the Bruhat interval $[\mathrm{id}, w]$ satisfies $\mathrm{Newton}(m_C) \subseteq \mathrm{Newton}(\mathrm{GW}(w))$. Then for the sum

$$D^w(x) = \frac{1}{(\ell(w))!} \sum_C m_C(x),$$

we have by Proposition 2.24 that

$$\mathrm{Newton}(D^w) \subseteq \mathrm{Newton}(\mathrm{GW}(w)).$$

Conversely, by Lemma 4.20, any greedy chain C' in the interval $[\mathrm{id}, w]$ satisfies

$$\mathrm{Newton}(m_{C'}) = \mathrm{Newton}(\mathrm{GW}(w)).$$

Since

$$\mathrm{Newton}(m_{C'}) \subseteq \mathrm{Newton}(D^w)$$

for all chains C' in the interval $[\mathrm{id}, w]$, we obtain the reverse inclusion

$$\mathrm{Newton}(\mathrm{GW}(w)) \subseteq \mathrm{Newton}(D^w).$$

Combining the two inclusions yields

$$\mathrm{Newton}(D^w) = \mathrm{Newton}(\mathrm{GW}(w)).$$

For all greedy chains C' of the interval $[\mathrm{id}, w]$, since

$$\mathrm{Newton}(m_{C'}) = \mathrm{Newton}(\mathrm{GW}(w)) = \mathrm{Newton}(D^w),$$

C' is a dominant chain of D^w. Finally by Lemma 4.17, there exists a greedy chain C' in the interval $[\mathrm{id}, w]$, which shows that D^w has SCNP. □

Corollary 4.29. For all $w \in S_n$, the dual Schubert polynomial D^w has SNP.

Proof. This follows from Proposition 4.3 and Theorem 4.28. □

Corollary 4.30. A saturated chain C in $[\mathrm{id}, w]$ is a greedy chain if and only if C is a dominant chain of D^w.

Sketch of Proof. The forward direction is given by Theorem 4.28. For the backwards direction, we use proof by contradiction, based on the definition of greedy chains. □

4.3. **Postnikov-Stanley Polynomials.** In this section, we wish to generalize Corollary 4.29 to all Postnikov-Stanley polynomials. Our main conjecture is the following.

Conjecture 4.31. For all intervals $[u, w]$ in S_n, the Postnikov-Stanley polynomial D_u^w has SNP.

We have verified Conjecture 4.31 with SageMath for all Bruhat intervals in S_n for $3 \leq n \leq 6$. We have also proven the following partial results.

4.3.1. *Proofs for Small Ranks.* In this subsubsection, we prove Conjecture 4.31 for Postnikov-Stanley polynomials of some small ranks.

Proposition 4.32. All Postnikov-Stanley polynomials of rank 2 have SNP.

Proof. This follows from Lemma 4.7 and Proposition 4.3. □

Lemma 4.33. All Postnikov-Stanley polynomials of 2-crowns have SNP.

Proof. This follows from Lemma 4.8 and Proposition 4.3. □

Lemma 4.34. All Postnikov-Stanley polynomials of cubes have SNP.

Proof. This follows from Lemma 4.9 and Proposition 4.3. □

Conjecture 4.35. All Postnikov-Stanley polynomials of rank 3 have SNP.

Remark. By Proposition 4.3, it suffices to prove Conjecture 4.10, which can be reduced to proving the conjecture for 4-crowns.

4.3.2. *Proofs for "Near" Dual Schubert Polynomials.* In this section, we prove Conjecture 4.31 for some Postnikov-Stanley polynomials that are closely related to dual Schubert polynomials.

Corollary 4.36. For all $w \in S_n$, the Postnikov-Stanley polynomial $D_w^{w_0}$ has SCNP and SNP, where $w_0 = n(n-1)\cdots 1$ is the longest permutation.

Proof. This follows from Theorem 4.28 and Proposition 3.5. □

Proposition 4.37. For all Bruhat intervals $[s_i, w]$ in S_n, where s_i is a simple transposition, the Postnikov-Stanley polynomial $D_{s_i}^w$ has SNP.

Proof (sketch). We can use Monk's formula for Schubert polynomials to obtain the generalized Littlewood-Richardson coefficients associated with simple transpositions. The proposition then follows from Theorem 2.20 and Theorem 5.3. □

Remark. The Postnikov-Stanley polynomial $D_{s_i}^w$ does not necessarily have SCNP: see Example 4.5.

Corollary 4.38. For all Bruhat intervals $[u, s_i w_0]$ in S_n, where s_i is a simple transposition and w_0 is the longest permutation, the Postnikov-Stanley polynomial $D_u^{s_i w_0}$ has SCNP and SNP.

Proof. This follows from Proposition 4.37 and Proposition 3.5. □

Conjecture 4.39. Given a Bruhat interval $[u, w]$ in S_n, if either w avoids 4231, or u avoids 1324, then D_u^w has SCNP and SNP.

Remark. By Proposition 4.3, Conjecture 4.39 is implied by Conjecture 4.14.

4.3.3. *A Conjecture Using the EL Shelling Order.* By Proposition 4.2, given a Bruhat interval $[u, w]$, for each saturated chain, m_C has SNP. To prove SNP for D_u^w, it suffices to find an appropriate order $(C_i)_{1\leq i\leq k}$ of all saturated chains in the interval $[u, w]$ such that $\sum_{i=1}^{j} m_{C_i}$ has SNP for all $1 \leq j \leq k$. One candidate is the EL-shelling order.

Definition 4.40. For $v_1 \lessdot v_2$ in the Bruhat order, and v_2 differs from v_1 by the transposition of numbers (i, j) (not t_{ij}), then $\lambda(v_1, v_2) \coloneqq (i, j)$. For a chain

$$C = (u = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_n = w),$$

define

$$\pi(C) = (\lambda(w_0, w_1), \lambda(w_1, w_2), \dots, \lambda(w_{n-1}, w_n)).$$

The *EL-shelling order* of all saturated chains in $[u, w]$ is given by the lexicographical order of $\pi(C)$.

Example 4.41. In Figure 8 which displays the Bruhat order of S_4, the EL-shelling order is given by the following. For every two saturated chains, beginning at the bottom vertex, if one chain C_1 goes to the left prior to another chain C_2, then $C_1 > C_2$ in the EL-shelling order.

Remark. If we view each saturated chain as a simplex whose vertices are given by the element it passes through, then the EL-shelling order gives a shelling of the complex given by all the chains in the Bruhat interval.

Intuitively, if two saturated chains share many common vertices, then their Newton polytopes will not differ significantly, and the sum of their chain weights will likely possess SNP. The EL-shelling order ensures that the Newton polytope with the newly added chain "behaves well" when considering the complex composed of the simplices attached to the simplex corresponding to the new chain. However, there is no guarantee regarding the relationship between the new chain and the chain whose corresponding simplex is not attached to it, and their Newton polytopes could differ substantially, which may still violate SNP.

Conjecture 4.42. If we relabel the saturated chains $(C_i)_{1\leq i\leq k}$ in a Bruhat interval using the EL-shelling order, then $\sum_{i=1}^{j} m_{C_i}$ has SNP for all $1 \leq j \leq k$.

We have verified Conjecture 4.42 using SageMath for all Bruhat intervals in S_3, S_4, and S_5. If Conjecture 4.42 is true, then it implies Conjecture 4.31, that all Postnikov-Stanley polynomials have SNP.

5. Newton Polytopes of Dual Schubert Polynomials

5.1. **Newton Polytopes as Generalized Permutahedra.** In this section, we give a characterization of the Newton polytope of a dual Schubert polynomial as a generalized permutahedron.

Definition 5.1. For $1 \leq a < b \leq n$, let $M_{ab} = ([n], \mathcal{B})$ be the matroid on $[n]$ with bases $\mathcal{B} = \{\{a\}, \{a + 1\}, \dots, \{b - 1\}\}$.

The motivation for defining such a matroid is the following observation about its matroid polytope: letting $e_i \in \mathbb{R}^n$ denote the unit vector with a 1 in the ith coordinate, we have

$$P(M_{ab}) = \mathrm{conv}\{e_a, e_{a+1}, \dots, e_{b-1}\} = \mathrm{Newton}(x_a + x_{a+1} + \cdots + x_{b-1}).$$

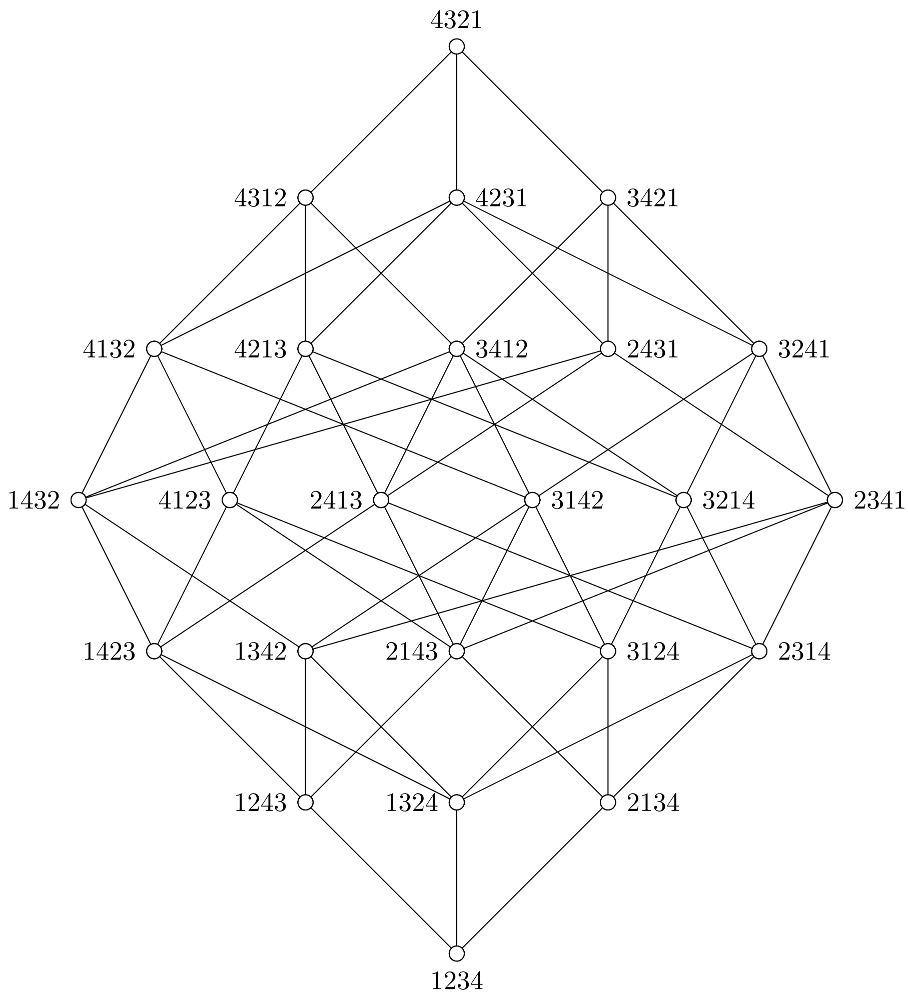


FIGURE 8. The Bruhat order of S_4

Now we may apply theorems about matroid polytopes from Section 2.5 to obtain a characterization of Newton(D^w) as a generalized permutahedron. In the following, let $I \supseteq [a, b)$ denote $I \supseteq \{a, a+1, \dots, b-1\}$, for $I \subseteq [n]$.

Theorem 5.2. For $w \in S_n$, Newton(D^w) is a generalized permutahedron $P_n^z(\{z_I\})_{I \subseteq [n]}$ with

$$z_I = \sum_{(a,b) \in \mathrm{Inv}(w)} \mathbb{1}_{I \supseteq [a,b)}$$

for all $I \subseteq [n]$.

Proof. By Definition 2.32, we have for $I \subseteq [n]$ that

$$r_{M_{ab}}([n] \setminus I) = \begin{cases} 0 & \text{if } I \supseteq [a,b) \\ 1 & \text{if } I \not\supseteq [a,b). \end{cases}.$$

Then from Proposition 2.33, we have

$$\begin{aligned} P(M_{ab}) &= P_n^z(\{r_{M_{ab}}([n]) - r_{M_{ab}}([n]\setminus I)\})_{I\subseteq[n]} \\ &= P_n^z(\{1 - r_{M_{ab}}([n]\setminus I)\})_{I\subseteq[n]} \\ &= P_n^z(\{\mathbb{1}_{I\supseteq[a,b)}\})_{I\subseteq[n]}. \end{aligned}$$

Recall from Theorem 4.28 and Definition 4.18 that $\mathrm{Newton}(D^w) = \mathrm{Newton}(\mathrm{GW}(w))$, where

$$\mathrm{GW}(w) = \prod_{(a,b)\in\mathrm{Inv}(w)} (x_a + x_{a+1} + \cdots + x_{b-1}).$$

Then by Proposition 2.24, we have

$$\begin{aligned} \mathrm{Newton}(D^w) &= \sum_{(a,b)\in\mathrm{Inv}(w)} \mathrm{Newton}(x_a + x_{a+1} + \cdots + x_{b-1}) \\ &= \sum_{(a,b)\in\mathrm{Inv}(w)} P(M_{ab}) \\ &= \sum_{(a,b)\in\mathrm{Inv}(w)} P_n^z(\{\mathbb{1}_{I\supseteq[a,b)}\})_{I\subseteq[n]} \\ &= P_n^z\Big(\Big\{\sum_{(a,b)\in\mathrm{Inv}(w)} \mathbb{1}_{I\supseteq[a,b)}\Big\}\Big)_{I\subseteq[n]}, \end{aligned}$$

where the last equality is by Proposition 2.31. $\square$

5.2. **Vertices of Newton Polytopes.** In this section, we give a simple characterization of the vertices of the Newton polytope of a dual Schubert polynomial.

Theorem 5.3. For $w \in S_n$, the point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\mathrm{Newton}(D^w)$ if and only if the monomial x^α has a coefficient of 1 in $\mathrm{GW}(w)$.

As $\mathrm{GW}(w)$ is a product of linear polynomials with all coefficients equal to 0 or 1, we prove the following more general statement to obtain Theorem 5.3.

Proposition 5.4. Let $q \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ be a polynomial of the form

$$q = \prod_{i=1}^{m} (c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n)$$

for positive integers m, n and coefficients $c_{ij} \in \{0, 1\}$ for all $1 \leq i \leq m, 1 \leq j \leq n$. The point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\mathrm{Newton}(q)$ if and only if the monomial x^α has a coefficient of 1 in q.

We introduce the following definitions and lemmas to prove Proposition 5.4.

Definition 5.5. For a polynomial $f \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$, let $\mathrm{Ver}(f)$ denote the set of vertices of $\mathrm{Newton}(f)$.

Lemma 5.6. For two polynomials $f, g \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$, we have

$$\mathrm{Ver}(fg) \subseteq \mathrm{Ver}(f) + \mathrm{Ver}(g).$$

Proof. For $\alpha \in \mathrm{Ver}(fg)$, write $\alpha = \beta + \gamma$ for some $\beta \in \mathrm{Newton}(f)$ and $\gamma \in \mathrm{Newton}(g)$. If $\gamma \notin \mathrm{Ver}(g)$, then we have

$$\begin{aligned}\beta + \gamma &\in \beta + \mathrm{Newton}(g)\\ \beta + \gamma &\notin \beta + \mathrm{Ver}(g),\end{aligned}$$

which implires $\alpha = \beta + \gamma$ is not a vertex of $\mathrm{Newton}(fg)$, a contradiction. Thus, we have $\gamma \in \mathrm{Ver}(g)$. Similarly, $\beta \in \mathrm{Ver}(f)$. ☐

Lemma 5.7. Let $f \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ be a linear polynomial with all coefficients 1. Let $g \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ be a polynomial such that the coefficient of x^γ in g is 1 for all $\gamma \in \mathrm{Ver}(g)$. Then for all $\alpha \in \mathrm{Ver}(fg)$, the coefficient of x^α in fg is 1.

Proof. We write

$$f = x_{r_1} + x_{r_2} + \cdots + x_{r_s}$$

and denote the sum of coefficient 1 terms in g by

$$h = x^{\alpha^1} + x^{\alpha^2} + \cdots + x^{\alpha^k},$$

where $\alpha^1, \alpha^2, \ldots, \alpha^k$ are multi-indices. The coefficient 1 terms of fg are precisely the coefficient 1 terms of the product

$$fh = (x_{r_1} + x_{r_2} + \cdots + x_{r_s})(x^{\alpha^1} + x^{\alpha^2} + \cdots + x^{\alpha^k}).$$

By Proposition 2.24, we have

$$\begin{aligned}\mathrm{Newton}(fg) &= \mathrm{Newton}(f) + \mathrm{Newton}(g)\\ &= \mathrm{Newton}(f) + \mathrm{Newton}(h)\\ &= \mathrm{Newton}(fh).\end{aligned}$$

Let $\alpha \in \mathrm{Ver}(fg)$, and write $\alpha = \beta + \gamma$ for some $\beta \in \mathrm{Newton}(f)$ and $\gamma \in \mathrm{Newton}(g)$. By Lemma 5.6, we have $\gamma \in \mathrm{Ver}(g)$. As elements of $\mathrm{Ver}(g)$ correspond to (not necessarily all) monomials in h, it suffices to show that the coefficient of x^α in fh is not greater than or equal to 2.

Suppose otherwise, then we have

$$x^\alpha = x_{r_a} x^{\alpha^p} = x_{r_b} x^{\alpha^q}$$

for some integers a, b, p, q such that $x_{r_a} \neq x_{r_b}$. Hence there exists some x^δ, where δ is a muiti-index such that

$$x_{r_b} x^\delta = x^{\alpha^p},\ x_{r_a} x^\delta = x^{\alpha^q}.$$

So there exists a monomial

$$x_{r_b} x^{\alpha^p} = x_{r_b}^2 x^\delta$$

and a monomial

$$x_{r_a} x^{\alpha^q} = x_{r_a}^2 x^\delta$$

in

$$fh = (x_{r_1} + x_{r_2} + \cdots + x_{r_s})(x^{\alpha^1} + x^{\alpha^2} + \cdots + x^{\alpha^k}).$$

But now x^α corresponds to the midpoint of the edge given by $x_{r_b}^2 x^\delta$ and $x_{r_a}^2 x^\delta$, since

$$x^\alpha = x_{r_a} x^{\alpha^p} = x_{r_a} x_{r_b} x^\delta.$$

Moreover, since $x_{r_b} \neq x_{r_a}$, the points corresponding to $x_{r_b}^2 x^\delta$ and $x_{r_a}^2 x^\delta$ are not the same. This contradicts with the fact that $\alpha \in \mathrm{Ver}(fg)$. Hence the coefficient of x^α in fh is not greater than or equal to 2, as desired. □

Definition 5.8. We call a polynomial $f \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ *substantial* if f has SNP, and furthermore $\alpha \in \mathrm{Ver}(f)$ if and only if the coefficient of x^α in f is 1.

Example 5.9. The polynomial $x_1^2 + 2x_1x_2 + x_2^2$ is substantial, while $x_1^2 + x_1x_2 + x_2^2$ is not.

Definition 5.10. For a polynomial $f \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$, its *1-skeleton* $X_1(f)$ is the set of all edges that lie on the boundary of $\mathrm{Newton}(f)$ with endpoints in $\mathrm{Ver}(f)$.

The 1-skeleton we refer to here differs from the definition of a 1-skeleton in algebraic topology: our 1-skeleton does not include 0-cells.

Definition 5.11. Let $e_i \in \mathbb{R}^n$ denote the unit vector with a 1 in the ith coordinate. A edge with endpoints $v_1, v_2 \in \mathbb{R}^n$ is *fundamental* if $v_1 - v_2$ is in the same direction as $e_i - e_j$ for some $1 \leq i, j \leq n$ with $i \neq j$.

Definition 5.12. A polynomial $f \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ is *regular* if all edges in $X_1(f)$ are fundamental.

Example 5.13. The polynomial $f = x_1^2 + x_2^2$ is regular because the only edge in $X_1(f)$ has endpoints $(2, 0)$ and $(0, 2)$, and $(2, 0) - (0, 2) = (2, -2)$ is in the direction $e_1 - e_2$.

Example 5.14. The substantial polynomial $f = x_1^2 + x_2x_3$ is not regular because the edge in $X_1(f)$ with endpoints $(2, 0, 0)$ and $(0, 1, 1)$ has vector $(2, -1, -1)$ not in the direction of $e_i - e_j$ for any $1 \leq i, j \leq 3$ with $i \neq j$.

The product fg of a linear polynomial f with all coefficients 1 and a substantial polynomial g is not necessarily substantial when g is not regular, as seen in the following example.

Example 5.15. Let $f = x_1 + x_2 + x_3$ and $g = x_1^2 + x_2x_3$. Then in

$$\begin{aligned} fg &= (x_1 + x_2 + x_3)(x_1^2 + x_2x_3) \\ &= x_1^3 + x_1x_2x_3 + x_1^2x_2 + x_2^2x_3 + x_1^2x_3 + x_2x_3^2, \end{aligned}$$

the monomial $x_1x_2x_3$ has coefficient 1. However, the point $(1, 1, 1)$ is not a vertex of

$$\mathrm{Newton}(fg) = \mathrm{conv}((3, 0, 0), (2, 1, 0), (2, 0, 1), (1, 1, 1), (0, 2, 1), (0, 1, 2))$$

because $(1, 1, 1) = \frac{1}{2}(2, 1, 0) + \frac{1}{2}(0, 1, 2)$. As a result, fg is not substantial.

Lemma 5.16. Let $f \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ be a linear polynomial with all coefficients 1. Let $g \in \mathbb{Z}_{\geq 0}[x_1, x_2, \ldots, x_n]$ be regular. Then fg is regular, and every integer point of $\mathrm{Newton}(f) + \mathrm{Ver}(g)$ lies on an edge of $X_1(fg)$.

Proof. Note that all edges of $\mathrm{Newton}(f)$ are fundamental by the definition of f. Since g is regular, all edges $\ell' \in X_1(g)$ are fundamental. Then for all points $\beta \in \mathrm{Newton}(f)$ and edges $\ell' \in X_1(g)$, the translated edge $\beta + \ell'$ is also fundamental. Recalling that

$$\mathrm{Newton}(fg) = \mathrm{Newton}(f) + \mathrm{Newton}(g)$$

by Proposition 2.24, all edges $\ell \in X_1(fg)$ must be of one of the following forms:

(*i*) $\ell = \beta + \ell'$ for $\beta \in \mathrm{Ver}(f)$ and $\ell' \in X_1(g)$

(*ii*) $\ell = \ell'' + \gamma$ for $\ell'' \in X_1(f)$ and $\gamma \in \mathrm{Ver}(g)$

(*iii*) $\ell = \{\beta_1, \beta_2\} + \ell'$ for $\beta_1, \beta_2 \in \mathrm{Ver}(f)$ and $\ell' \in X_1(g)$ such that ℓ' is fundamental for $\beta_1 - \beta_2$.

By checking the definition of regularity, we have fg is regular, as desired.

Note that every edge of $\mathrm{Newton}(f) + X_1(g)$ lies on an edge of $X_1(fg)$ (the case (i) or (iii) described above), so every point in $\mathrm{Newton}(f) + \mathrm{Ver}(g)$ lies on an edge of $X_1(fg)$. □

Theorem 5.17. Let $f \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ be a linear polynomial with all coefficients 1, and let $g \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ be substantial and regular such that fg has SNP. Then fg is substantial and regular.

Proof. By Lemma 5.16, it suffices to show that fg is substantial. By Lemma 5.7, for any $\alpha \in \mathrm{Ver}(fg)$, the coefficient of x^α in fg is 1. Since fg has SNP, it suffices to show that the coefficient 1 terms x^α of fg must correspond to a vertex α of $\mathrm{Newton}(fg)$.

By Proposition 2.24, we have

$$\mathrm{Newton}(fg) = \mathrm{Newton}(f) + \mathrm{Newton}(g).$$

Let $\alpha \in \mathrm{Ver}(fg)$, (not sure about this, but just skip straight to $\mathrm{Newton}(f) + \mathrm{Ver}(g)$) and $\alpha = \beta + \gamma$ for some $\beta \in \mathrm{Newton}(f)$, $\gamma \in \mathrm{Newton}(g)$. By Lemma 5.6, we have $\gamma \in \mathrm{Ver}(g)$. Moreover, by Lemma 5.16, every point of $\mathrm{Newton}(f) + \mathrm{Ver}(g)$ lies in an edge of $X_1(fg)$, so it suffices to show: if $\alpha = \beta + \gamma$ for $\beta \in \mathrm{Newton}(f)$, $\gamma \in \mathrm{Ver}(g)$ lies in an edge of $X_1(fg)$ (excluding the endpoint), then x^α has coefficient more than 1 in fg.

Suppose α lies in an edge ℓ of $X_1(fg)$ (excluding the endpoint), and ℓ is connected by a sequence the fundamental moves $e_r - e_s$.

Then there exists an edge $\ell' \in X_1(g)$ such that ℓ' is connected by a sequence the fundamental moves $e_r - e_s$, and both x^{e_r} and x^{e_s} are terms in f. Since $\beta + \gamma$ lies in the edge ℓ (excluding the endpoint), there are two possible cases:

(i) $\beta = e_r$, γ is an endpoint of ℓ' such that $\gamma + (e_r - e_s) \in \ell'$; or
(ii) $\beta = e_s$, γ is an endpoint of ℓ' such that $\gamma + (e_s - e_r) \in \ell'$.

For case (i), since the monomial x^{e_s} is in f, and the monomial $x^{\gamma+(e_r-e_s)}$ is in g, we have

$$x^\alpha = x^\beta x^\gamma \text{ and } x^\alpha = x^{e_s} x^{\gamma+(e_r-e_s)},$$

so x^α has coefficient more than 1 in fg.

For case (ii), since the monomial x^{e_r} is in f, and the monomial $x^{\gamma+(e_s-e_r)}$ is in g, we have

$$x^\alpha = x^\beta x^\gamma \text{ and } x^\alpha = x^{e_r} x^{\gamma+(e_s-e_r)},$$

so x^α has coefficient more than 1 in fg. □

Remark. We believe that the requirement that fg has SNP in Theorem 5.17 may be removed but have not proven this yet.

Corollary 5.18. For a matrix $M = \{c_i^j\}_{i\in[n]}^{j\in[m]} \in \{0,1\}^{m\times n}$, the polynomial $q(M)$ is a substantial and regular polynomial.

Proof. Proposition 4.2 and Theorem 5.17 allows us to apply induction on m, with the case $m = 1$ clear. □

Remark. Proposition 5.4 is the "substantial" part of Corollary 5.18.

Proof of Theorem 5.3. This follows from Theorem 4.28 and Proposition 5.4. □

Conjecture 5.19. The Newton polytope uniquely determines the product of coefficient 1 linear factors.

Remark. If Conjecture 5.19 is true, then the proof is anticipated to follow from Proposition 5.4.

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Massachusetts Institute of Technology
Email address: anser@mit.edu

Harvard University
Email address: katherinetung@college.harvard.edu

University of Michigan
Email address: zongxun@umich.edu