DIFFERENTIAL POWERS IN SEMIGROUP AND POLYNOMIAL RINGS

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Contents

Abstract. Prior research has explored several closely related notions of the power of an ideal, including the ordinary power, the symbolic power, and the Frobenius power. The differential power encodes similar data but offers relative ease of computation, and, for monomial ideals over affine semigroup rings, can be combinatorially interpreted using the lattice of the semigroup. Using the language of standard pairs introduced in [STV95], we prove a combinatorial formula for differential powers of radical ideals in polynomial rings. We outline a similar characterization of differential powers in semigroup rings. Generalizing the work of [Ken+21], we show that interior ideals' differential powers are eventually principal in polynomial rings. For semigroup rings corresponding to rational normal curves, we show that the generators of differential powers of interior ideals eventually have periodic behaviour. Finally, we provide Macaulay2 implementations of algorithms that allow for the conversion of standard pairs to their ideal for the computation of differential powers of ideals in polynomial rings.

1. Introduction

There are various notions of powers of an ideal I in a commutative ring R. For instance, the N^{th} *ordinary power* I^N, which is the ideal generated by the products of n elements of I, the *Frobenius power*, $I^{[p^e]}$, which is the ideal generated by $(p^e)^{th}$ powers of elements in I [HH89], and the symbolic power, the intersection of powers of associated primes, are three such notions.

The *differential power*, denoted $I^{\langle N\rangle}$ has not been as well studied, but has been the subject of a resurgence of recent interest. They offer as well connections to symbolic powers: for

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prime ideals in polynomial rings with characteristic zero, the differential power equals the symbolic power.

In this paper, we study the differential powers $I^{\langle N \rangle}$ of an ideal I from both asymptotic and combinatorial directions.

Outline. In 2, we describe the prerequisite notions and motivating questions necessary for our study of differential powers. In Section 3, we establisht the differential operators of semigroup rings. Section 4 contains a reinterpretation of the theory of standard pairs as used in [MY22] in terms of overlap classes. Section 5 gives a formula for the set of standard pairs of differential powers of radical ideals in polynomial rings. In Section 6, we explore the asymptotic behaviour of differential powers in polynomial rings, culminating in a proof of eventual principality for some principal ideals. We explore asymptotic behaviour of rational normal curves for semigroup rings corresponding to rational notmal curves in Section 7. Finally, we conclude by providing Macaulay2 code for the computation of differential powers in polynomial rings (Section 8) and for the computation of ideals given only their standard pairs (Section 9).

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2. Differential Operators and Differential Powers

For more information, see [Bjö81].

Let X be a $\mathbb{C}$-algebra. Denote by $\mathrm{End}_{\mathbb{C}}(X)$ the $\mathbb{C}$-algebra of $\mathbb{C}$-linear functions $f : X \to X$. For two functions $f, g \in \mathrm{End}_{\mathbb{C}}(X)$, define $[f, g] : X \to X$ to be their *commutator*:

$$[f, g](x) := f(g(x)) - g(f(x)).$$

We note that $[f, g] \in \mathrm{End}_{\mathbb{C}}(X)$.

We define the *algebra of differential operators* of X inductively as follows. Let

$$D^0(X) = \{f \in \mathrm{End}_{\mathbb{C}}(X) \mid \text{ there exists } r \in X \text{ such that } f(x) = rx \text{ for all } x \in X\},$$

and for $N \geq 0$, let

$$D^{N+1}(X) = \{\delta \in \mathrm{End}_{\mathbb{C}}(X) \mid [\delta, r] \in D^N(X) \text{ for all } r \in X\},$$

where $r \in X$ is identified with the linear function $x \mapsto rx$. A quick inductive argument verifies that $D^N(X) \subseteq D^{N+1}(X)$ for all $N \geq 0$. Then, let

Definition 2.1. The *algebra of differential operators* of X (over $\mathbb{C}$)

$$D(X) = \bigcup_{N \in \mathbb{N}} D^N(X).$$

Proposition 2.2. For all $\delta \in D^N(X)$ and $\mu \in D^M(X)$ for $N, M \in \mathbb{N}$, we have $\delta\mu \in D^{N+M}(X)$.

Proof. We prove this by induction on the *lexicographic order* on $\mathbb{N}\times\mathbb{N}$, in which $(a,b)<(c,d)$ if either $a<c$, or $a=c$ and $b<d$. First, if $N=M=0$, then δ and μ are given by multiplication by some constants $r,s\in X$, so $[\delta,\mu]=[r,s]=rs-sr\in D^0(X)$.

Now, suppose the statement is true for all $(N',M')<(N,M)$. To show that $\delta\mu\in D^{N+M}(X)$, we need to show that $[\delta\mu,r]\in D^{N+M-1}(X)$ for all $r\in X$. We can rewrite $[\delta\mu,r]$ in terms of $[\delta,r]$ and $[\mu,r]$:

$$\begin{aligned}[\delta\mu,r] &= \delta\mu r - r\delta\mu = \delta(\mu r - r\mu + r\mu) - r\delta\mu\\ &= \delta([\mu,r]+r\mu) - r\delta\mu = \delta[\mu,r] + \delta r\mu - r\delta\mu\\ &= \delta[\mu,r] + (\delta r - r\delta)\mu\\ &= \delta[\mu,r] + [\delta,r]\mu.\end{aligned}$$

By the definition of $D^{N-1}(X)$, we have $[\delta,r]\in D^{N-1}(X)$ and $[\mu,r]\in D^{M-1}(X)$ for all $r\in X$. As $(N-1,M)<(N,M-1)<(N,M)$, our inductive hypothesis implies that $\delta[\mu,r],[\delta,r]\mu\in D^{N+M-1}(X)$, which means $[\delta\mu,r]\in D^{N+M-1}(X)$ for all $r\in X$. But this means that $\delta\mu\in D^{N+M}(X)$. □

Definition 2.3. The *order* of a differential operator $\delta\in D(X)$, written $\operatorname{ord}_X(\delta)$, is defined to be the smallest N for which $\delta\in D^N(X)$.

As L is the field of fractions of R, we can define *differential operators* on R in terms of those on L. Let $\frac{\partial}{\partial t_i}=\partial_i$ and $\theta_i=t_i\partial_i$. Let the hyperplanes $h_i=0$ for $1\le i\le k$ be the boundaries of $\mathbb{N}A$ such that $h_i(\mathbb{N}A)\ge 0$. Denote the collection of these hyperplanes by $\mathcal{H}$. We assume without loss of generality that the equations h_i have integer coefficients.

In [ST01], the authors give a full description of the differential operators on R:

Theorem 2.4 ([ST01, Theorem 3.2.2])**.** The ring of differential operators $D(R)$ is a graded object:

$$D(R)=\bigoplus_{\mathbf{a}\in\mathbb{Z}A} D(R)_{\mathbf{a}} = \bigoplus_{\mathbf{a}\in\mathbb{Z}A}\mathcal{D}_{\mathbf{a}}\mathbb{C}[\theta_1,\dots,\theta_d],$$

where

$$\mathcal{D}_{\mathbf{a}} = t^{\mathbf{a}}\prod_{i=1}^{k}\prod_{j=1}^{-h_i(\mathbf{a})}(h_i(\theta_1,\dots,\theta_d)-j+1).$$

These operators are chosen so that an operator $\mathcal{D}_{\mathbf{a}}$ acts on a monomial in R geometrically via translation of its degree vector by $\mathbf{a}$ in $\mathbb{Z}A$. The operator annihilates a monomial if and only if this translation lands outside of $\mathbb{N}A$.

We have an important corollary to this theorem, which outlines how the operators $\mathcal{D}_{\mathbf{a}}$ interact with each other.

Corollary 2.5 ([ST01], Corollary 3.2.4)**.** For $\mathbf{a},\mathbf{b}\in\mathbb{Z}A$, the following are equivalent.

1. $\mathcal{D}_{\mathbf{a}}\mathcal{D}_{\mathbf{b}}=\mathcal{D}_{\mathbf{b}}\mathcal{D}_{\mathbf{a}}$,
2. $\mathcal{D}_{\mathbf{a}}\mathcal{D}_{\mathbf{b}}=\mathcal{D}_{\mathbf{a}+\mathbf{b}}$,
3. $h_i(\mathbf{a})h_i(\mathbf{b})\ge 0$ for all $i=1,\dots,k$; that is, $\mathbf{a}$ and $\mathbf{b}$ lie in the same chamber of the hyperplane arrangement $\mathcal{H}$.

In particular, if we take $\mathbf{a}_1,\dots,\mathbf{a}_\ell\in\mathbb{Z}^d$ such that each hyperplane chamber is generated by some of the $\mathbf{a}_i$, then $D(R)$ is generated by $\mathcal{D}_{\mathbf{a}_1},\dots,\mathcal{D}_{\mathbf{a}_\ell}$.

For the forwards inclusion, take any $f\in I^{\langle m\rangle}$, Because h_i are linear functions, the *order* of $\mathcal{D}_{\mathbf{a}}$ (as a differential of Laurent polynomials) is exactly $\sum_{i=1}^{k} \max\{-h_i(\mathbf{a}), 0\}$. For now, we write this as $\mathrm{ord}_L(\mathcal{D}_{\mathbf{a}}) = \sum_{i=1}^{k} \max\{-h_i(\mathbf{a}), 0\}$

In a special case, the differential operators of the polynomial ring $\mathbb{C}[x_1, \ldots, x_d]$ are

$$\langle x^{\alpha}\partial^{\beta} \mid \alpha, \beta \in \mathbb{N}^d \rangle.$$

Since all the rings we consider in this paper are Noetherian, we know their ideals are finitely generated. Furthermore, we assume that every ideal is *homogeneous*, that is, it can be generated by homogeneous elements. For an ideal $I \subseteq R$, we study a special type of its power:

Definition 2.6. The *N-th differential power* $I^{\langle N \rangle}$ of ideal $I \subseteq R$ is defined as

$$I^{\langle N \rangle} = \{f \in R : \delta(f) \in I \text{ for all } \delta \in D(R) \text{ of order at most } N-1\}.$$

Since the zero order differential operators are simply elements in R, we have $I^{\langle 1 \rangle} = I$.

Definition 2.7. For a polynomial ring $S = \mathbb{C}[x_1, x_2, ..., x_n]$, the ideal I is the set given by $I \subseteq S = \{f_1, f_2, ..., f_l\} := \{g_1 f_1 + ... + g_i f_i | g_i \in S\}$. In this context, we will assume our f_i are homogenous and thus that our ideal is homogenous.

The ideal of a variety V, notated $\mathbf{I}(V)$, is the set of polynomials in $\mathbb{C}[x_1, ..., x_n]$ in $\mathbb{P}^{n-1}$ such that for every $p \in I$, $f(p) = 0$, notated $V = \mathbf{V}(I)$.

Definition 2.8. A subset V of $\mathbb{P}^{n-1}$ over $\mathbb{C}$ is a variety if there exists some set of polynomials S in $\mathbb{C}[x_1, ..., x_n]$ such that $V = \{x \in \mathbb{C}^n \mid f(x) \forall f \in S\}$.

The variety V of an ideal I is the set of points in $\mathbb{P}^{n-1}$ such that for every $f \in I$, $f(p) = 0$, notated $V = \mathbf{V}(I)$.

3. Order of Differential Operators over Semigroup Rings

Though we know that the algebra of differential operators of the semigroup ring R injects nicely into that of the Laurent polynomial ring L, we do *not* necessarily know that the order of a differential operator $\delta \in D(R)$ over R is the same as its order considered as a differential operator over L. We prove that, in fact, for any two $\mathbb{C}$-algebras whose differential operators are related as those of R and L are, the two notions of order are equivalent.

Proposition 3.1. Suppose X is a $\mathbb{C}$-algebra and $Y \subseteq X$ is a $\mathbb{C}$-subalgebra. Suppose further that $D^N(Y) \subseteq D^N(X)$ for all $N \geq 0$. Then for each $N \geq 0$, $D(Y) \cap D^N(X) = D^N(Y)$.

Proof. Clearly, $D^N(Y) \subseteq D(Y) \cap D^N(X)$ for all $N \geq 0$. We prove the reverse containment by induction on N. Suppose $\delta \in D(Y) \cap D^0(X)$, so that $[\delta, f] = 0$ for all $f \in X$. In particular, $[\delta, r] = 0$ for all $r \in Y$, which means that $\delta \in D^0(Y)$, so the base case is satisfied. Now, suppose the result is true up to some $N \geq 0$, and suppose $\delta \in D(Y) \cap D^{N+1}(X)$. Then $[\delta, f] \in D^N(X)$ for all $f \in X$; in particular, $[\delta, r] \in D^N(X)$ for all $r \in Y$. As $D(Y)$ is closed under taking brackets, we have $[\delta, r] \in D(Y)$ as well, so by our inductive hypothesis, $[\delta, r] \in D(Y) \cap D^N(X) = D^N(Y)$. But this means precisely that $\delta \in D^{N+1}(Y)$, and we are done. $\square$

Corollary 3.2. Under the hypotheses of the previous proposition, we have $\mathrm{ord}_Y(\delta) = \mathrm{ord}_X(\delta)$ for all $\delta \in D(Y)$. In particular, the result holds if Y is a normal semigroup algebra and X is the ambient algebra of Laurent polynomials.

Proof. Let $\operatorname{ord}_Y(\delta) = N$ and $\operatorname{ord}_X(\delta) = M$. The first equation implies that $\delta \in D^N(Y)$, and as $D^N(Y) \subseteq D^N(X)$, we have $M \leq N$. The second equation implies that $\delta \in D^M(X)$. As $\delta \in D(Y)$, we have $\delta \in D(Y) \cap D^M(X) = D^M(Y)$ by the previous proposition, which implies that $N \leq M$. Thus $N = M$. $\square$

Lemma 3.3. For a monomial ideal $I \subseteq R$, the N^{th} differential power $I^{\langle N \rangle}$ is equal to the set of elements $f \in R$ such that $\mathcal{D}_a(f) \in I$ for all $a \in \mathbb{Z}A$ for which $\mathcal{D}_a$ is a Saito-Traves operator of order less than N. In symbols,

$$I^{\langle N \rangle} = \{f \in R \mid \mathbf{c} \in \mathbb{Z}A, \operatorname{ord}(\mathcal{D}_{\mathbf{c}}) < N \Longrightarrow \mathcal{D}_{\mathbf{c}}(f) \in I\}$$

Proof. Set

$$J = \{f \in R \;\big|\; a \in \mathbb{Z}A, \operatorname{ord}(\mathcal{D}_a) < N \Longrightarrow \mathcal{D}_a(f) \in I\}.$$

By definition,

$$I^{\langle N \rangle} = \{p \in R \;\big|\delta \in D(R), \operatorname{ord}(\delta) < N \Longrightarrow \delta(p) \in I\}.$$

To show $J \subseteq I$, suppose $p \in J$. It suffices for us to consider the case in which p is a monomial. Let $\delta \in D(R)$ be an order $< N$ differential operator. From Theorem 2.4, we have a decomposition

$$\delta = \sum_{\mathbf{a} \in V} \mathcal{D}_{\mathbf{a}} q_{\mathbf{a}},$$

where $V \subseteq \mathbb{Z}A$ is finite and $q_{\mathbf{a}} \in \mathbb{C}[\theta_1, \ldots, \theta_d]$. As each operator $\mathcal{D}_{\mathbf{a}} q_{\mathbf{a}}$ is homogeneous of degree $\mathbf{a}$, there can be no cancellation of terms between these summands. By Corollary 3.2, the order of δ as a differential operator on R is equal to that as a differential operator on L, so

$$\operatorname{ord}(\delta) = \max_{\mathbf{a} \in V}\{\operatorname{ord}(\mathcal{D}_{\mathbf{a}} q_{\mathbf{a}})\} < N,$$

as the order of differential operators on L corresponds to the maximum number of partial differentials appearing among its terms. This implies that $\operatorname{ord}(\mathcal{D}_{\mathbf{a}} q_{\mathbf{a}}) < N$, which further implies that $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) < N$ for all $\mathbf{a} \in V$.

Recall that the operators $q_{\mathbf{a}}$ either annihilate a monomial or preserve its degree, which means that for all monomials $m \in R$, we have $q_{\mathbf{a}}(m) = cm$ for some constant $c \in \mathbb{C}$. As differential operators are linear, we can write the following:

$$\delta(p) = \sum_{\mathbf{a} \in V} \mathcal{D}_{\mathbf{a}} q_{\mathbf{a}}(p) = \sum_{\mathbf{a} \in V} c_{\mathbf{a}} \mathcal{D}_{\mathbf{a}}(p).$$

Since $p \in J$ we have $D_{\mathbf{a}}(p) \in I$ for each $\mathbf{a} \in V$, showing $\delta(p) \in I$. Thus $p \in I^{\langle n \rangle}$, so $J \subseteq I$.

For the reverse inclusion, take any $p \in I^{\langle n \rangle}$. Then $\delta(p) \in I$ for all differential operators and thus Saito-Traves operators of order less than N, showing $p \in J$. Thus we can conclude that $I^{\langle N \rangle} = J$. $\square$

4. Standard Pairs and Differential Powers

In [MY22], the authors introduce the notion of a standard pair of a monomial ideal $I \subseteq \mathbb{C}[\mathbb{N}A]$. Standard pairs encode the algebraic data of I in combinatorial terms by describing the set of monomials outside of I, known as the *standard monomials* of I. We recount the basic facts of standard pairs as they appear in [MY22].

Definition 4.1 (Pairs of A)**.** Let $A \in \mathbb{Z}^{d \times n}$ be a matrix whose columns generate a normal strongly convex cone in $\mathbb{R}^d$.

- Let $F \subseteq A$ be the set of generators of $\mathbb{N}A$ lying a face of the cone $\mathbb{R}_{\geq 0}A$, and let $a \in \mathbb{N}A$. The pair (a, F) is called a *pair* of A.
- If (a, F) and (b, G) are face pairs of A, then we say that (a, F) is *contained* in (b, G), writing $(a, F) \prec (b, G)$, if $a + \mathbb{N}F \subseteq b + \mathbb{N}G$.
- Face pairs (a, F) and (b, F) are said to *overlap* if $a - b \in \mathbb{Z}F$, or equivalently, if there exists $f \in \mathbb{N}F$ for which $a + f + \mathbb{N}F \subseteq b + \mathbb{N}F$. Overlapping is an equivalence relation among face pairs, and we write $[a, F]$ for the equivalence classes.
- We say a face pair (a, F) *divides* (b, G) if there exists $c \in \mathbb{N}A$ for which $a + c + \mathbb{N}F \subseteq b + \mathbb{N}G$.

Definition 4.2. Let $I \subseteq \mathbb{C}[\mathbb{N}A] = \mathbb{C}[t^{a_1}, \ldots, t^{a_n}]$ be a monomial ideal. The pair (a, F) is called a *proper pair* of I if for all $c \in a + \mathbb{N}F$, we have $t^c \notin I$. A *standard pair* of I is a proper pair that is maximal with respect to containment among all proper pairs of I.

The set of standard pairs of a monomial ideal encodes information equivalent to its set of standard monomials.

In [MY22], the authors show that every monomial ideal $I \subseteq R$ has finitely many standard pairs. We include their precises results for use later.

Theorem 4.3 ([MY22], Corollary 3.15 and Theorem 3.16)**.** The number of overlap classes of standard pairs of a monomial ideal $I \subseteq R$ is finite. Moreover, there are finitely many standard pairs of I belonging to each such overlap class.

We wish to apply the results of [ST01] to study differential powers on semigroup rings. To do so, we analyze the behavior of standard pairs under differential powers.

Proposition 4.4. Let $(\mathbf{a}, F)$ be a proper pair of a monomial ideal $I \subseteq R$, so that $\mathbf{a} + \mathbb{N}F \subseteq \operatorname{stdMon}(I)$. If $(\mathbf{b}, F)$ is a face pair of I that overlaps $(\mathbf{a}, F)$, then $(\mathbf{b}, F)$ is also a proper pair of I.

Proof. If $(\mathbf{b}, F)$ overlaps $(\mathbf{a}, F)$, then there exists $\mathbf{c} \in \mathbb{N}F$ such that $\mathbf{c} + \mathbf{b} + \mathbb{N}F \subseteq \mathbf{a} + \mathbb{N}F$. In particular, we have $\mathbf{c} + \mathbf{b} + \mathbb{N}F \subseteq \operatorname{stdMon}(I)$, which means that for all $\mathbf{f} \in \mathbf{b} + \mathbb{N}F$, we have

$$t^{\mathbf{c}+\mathbf{f}} = t^{\mathbf{c}} t^{\mathbf{f}} \notin I.$$

But this means that $t^{\mathbf{f}} \notin I$; otherwise, we would have $t^{\mathbf{c}} t^{\mathbf{f}} \in I$, as I is an ideal. Thus $\mathbf{f} \in \operatorname{stdMon}(I)$ for all $\mathbf{f} \in \mathbf{b} + \mathbb{N}F$, so $(\mathbf{b}, F)$ is a proper pair of I. □

Thus, it makes sense to talk about whether an overlap class $[\mathbf{a}, F]$ is contained in $\operatorname{stdMon}(I)$. In analogy with standard pairs, we define a *standard overlap class* of I to be an overlap class of I that is maximal with respect to divisibility. Equivalently, standard overlap classes are overlap classes of standard pairs. Section 4 guarantees that each monomial ideal has finitely many standard overlap classes. We denote the set of standard overlap classes by $\operatorname{Overlap}(I)$.

We would like to describe how taking differential powers transforms the set of standard overlap classes.

By Lemma 3.3, the monomial elements of $I^{\langle N \rangle}$ consist of those monomials in R that are mapped into I by every Saito-Traves operator $\mathcal{D}_{\mathbf{a}}$ of order $< N$. Dually, this means that the standard monomials of $I^{\langle N \rangle}$ are the monomials in R that are mapped *out* of I by *any* Saito-Traves operator of order $< N$. Thus the standard monomials of $I^{\langle N \rangle}$ are precisely the preimages of standard monomials of I by Saito-Traves operators of order $< N$. As Saito-Traves operators act geometrically by translation, we have the following result:

Lemma 4.5. For a monomial ideal $I \subseteq R$ and a positive integer N,

$$\operatorname{stdMon}(I^{\langle N\rangle}) = \{\mathbf{a} - \mathbf{c} \mid \mathbf{a} \in \operatorname{stdMon}(I), \mathbf{c} \in \mathbb{Z}^d, \operatorname{ord}(\mathcal{D}_{\mathbf{c}}) < N, \text{ and } \mathbf{a} - \mathbf{c} \in \mathbb{N}A\}.$$

We'd like to understand the standard monomials of differential powers in terms of standard pairs. At first, it would seem that we should be able to obtain the standard pairs of $I^{\langle N\rangle}$ simply by translating the standard pairs of I as in the lemma above. But as these translations could cause standard pairs to land outside of $\mathbb{N}A$, this does not work. Thus, we need to extend our notion of pairs of A *outside* of $\mathbb{N}A$.

For a face F of A and a point $\mathbf{c} \in \mathbb{Z}^d$, we say that $(\mathbf{c}, F)$ *overlaps* $\mathbb{N}A$ if $(\mathbf{c}+\mathbb{N}F)\cap\mathbb{N}A \neq \emptyset$. Equivalently, this means that there exists some $\mathbf{f} \in \mathbb{N}F$ for which $\mathbf{f} + \mathbf{c} + \mathbb{N}F \subseteq \mathbb{N}A$.

We are ready to describe differential powers in terms of standard overlap classes.

Proposition 4.6. Let I be a monomial ideal in a normal affine semigroup ring $\mathbb{C}[\mathbb{N}A]$, and let $\operatorname{Overlap}(I)$ be the set of overlap classes of standard pairs of I. Then, for all $N > 1$,

$$\begin{aligned}\operatorname{Overlap}(I^{\langle N\rangle}) = \{[\mathbf{a} - \mathbf{c}, F]_A \mid [\mathbf{a}, F] \in \operatorname{Overlap}(I), \mathbf{c} \in \mathbb{Z}^d, \operatorname{ord}(\mathcal{D}_{\mathbf{c}}) < N,\\ \text{and } (\mathbf{a} - \mathbf{c} + \mathbb{N}F) \cap \mathbb{N}A \neq \emptyset\}.\end{aligned}$$

Proof. Forthcoming. ☐

Note that the polynomial ring S is the semigroup ring associated to the $n \times n$ identity matrix $\mathbb{I}_n$, which generates the cone defined by hyperplane equations

$$h_i := x_i = 0, \quad 1 \leq i \leq n.$$

Given this understanding of S, we should expect that the Saito-Traves operators of S relate nicely to its standard differential operators, which we verify in the next proposition.

Proposition 4.7. The Saito-Traves operators of the polynomial ring S have the following form: for $\mathbf{a} \in \mathbb{Z}^d$, we have $\mathcal{D}_{\mathbf{a}} = d_{\mathbf{a},1} \cdots d_{\mathbf{a},n}$, where

$$d_{\mathbf{a},i} = \begin{cases} x_i^{a_i}, & a_i \geq 0; \\ \partial_i^{-a_i}, & a_i < 0. \end{cases}$$

Proof. We build up the proof by cases.

Case 1. First, note that $\mathbf{a} = \mathbf{0}$ is the unique point lying on every facet of the cone $\mathbb{R}_{\geq 0}\mathbb{I}_n$, so $\mathcal{D}_{\mathbf{0}} = x^{\mathbf{0}} = 1 = d_{\mathbf{0},1} \cdots d_{\mathbf{0},n}$.

Case 2. Now suppose $\mathbf{a} = \pm\mathbf{e}_i$ for some $1 \leq i \leq n$, where $\mathbf{e}_i$ is the i^{th} standard basis vector of $\mathbb{R}^n$. Then $h_j(\pm\mathbf{e}_i) = 0$ if $j \neq i$ and $h_i(\pm\mathbf{e}_i) = \pm 1$. Therefore, we have

$$\mathcal{D}_{\mathbf{e}_i} = x_i \quad \text{ and } \quad \mathcal{D}_{-\mathbf{e}_i} = x_i^{-1}(x_i\partial_i) = \partial_i,$$

which we rewrite as $\mathcal{D}_{\pm\mathbf{e}_i} = d_{\pm\mathbf{e}_i,i} = d_{\pm\mathbf{e}_i,1} \cdots d_{\pm\mathbf{e}_i,n}$.

Case 3. Now, consider $\mathbf{a} = a\mathbf{e}_i$ for some $a \in \mathbb{Z}$, $a \neq 0$ and $1 \leq i \leq n$. If we let $s = \pm 1$ be the sign of a, then

$$\mathbf{a} = \underbrace{s\mathbf{e}_i + \cdots + s\mathbf{e}_i}_{|a| \text{ times}},$$

which by case 2 and repeated application of Corollary 2.5 yields

$$\mathcal{D}_{\mathbf{a}} = (\mathcal{D}_{s\mathbf{e}_i})^{|a|} = (d_{s\mathbf{e}_i,i})^{|a|} = d_{|a|s\mathbf{e}_i,i} = d_{\mathbf{a},i} = d_{\mathbf{a},1} \cdots d_{\mathbf{a},n}.$$

Case 4. Now, let $\mathbf{a} \in \mathbb{Z}^n$ be arbitrary. Write $\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. Then the $a_i\mathbf{e}_i$ all lie in the same chamber of $\mathbb{R}^d$ with respect to the hyperplanes $h_j = x_j = 0$: for all $1 \leq i, j, k \leq n$, we have

$$h_j(a_i\mathbf{e}_i)h_j(a_k\mathbf{e}_k) = \begin{cases} a_i^2, & i = j = k; \\ 0, & \text{otherwise.} \end{cases}$$
$$\geq 0.$$

Appealing once again to Corollary 2.5 and applying case 3, we have

$$\mathcal{D}_{\mathbf{a}} = \mathcal{D}_{a_1\mathbf{e}_1} \cdots \mathcal{D}_{a_n\mathbf{e}_n} = d_{a_1\mathbf{e}_1,1} \cdots d_{a_n\mathbf{e}_n,n} = d_{\mathbf{a},1} \cdots d_{\mathbf{a},n}.$$

□

5. Differential Powers of Radical Ideals in Polynomial Rings

For a polynomial ring $S = k[x_1, ..., x_n]$, all prime monomial ideals are generated by subsets of generators. Without loss of generality, we can assume an arbitrary prime ideal I is generated by the first s generators for some $s < n \in \mathbb{N}$.

Thus,

$$I = \langle x_1, ..., x_s \rangle.$$

Theorem 5.1. Given two ideals I and J, $\text{stdMonomials}(I_1 \cap I_2) = \text{stdMonomials}(I_1) \cup \text{stdMonomials}(I_2)$.

Proof. By definition,

$$\text{stdMon}(\bigcap_{i=1}^{m} \langle A_i \rangle) = \{b \in k[X] \mid b \in A_i \ \forall i\}$$
$$= \bigcup_{i=1}^{m} \text{stdMon}(A_i)$$

□

Lemma 5.2. For two ideals I and J, $\text{stdPairs}(I) = \text{stdPairs}(J)$ if and only if $I = J$.

Lemma 5.3. For a radical monomial ideal $I = \langle A_1 \rangle \cap \cdots \cap \langle A_m \rangle$ where each $\langle A_i \rangle$ is prime,

$$\text{stdPairs}(I) = \{(1, \{X \setminus A_1\}), (1, \{X \setminus A_2\}), \ldots, (1, \{X \setminus A_m\})\}.$$

Proof. For the forward inclusion, consider any standard pair (b, Z) of I. Since $b \notin I$, we have that $b \in \text{stdMonomial}(I)$. Note, that for any $z \in Z$, we have that $x^z \notin I$, and therefore $x^{b+z} \in \text{stdMonomial}(I)$. Thus (b, Z) describes a set of standard monomials of I.

By definition, every standard monomial of I is contained in a standard pair. Thus, $\text{stdMonomial}(I) = \text{stdPairs}(I)$.

We will show that $\text{stdMonomials}(I) = \bigcup_{i=1}^{m} \{(1, \{X \setminus A_i\})$.

Consider an arbitrary f in the set of standard monomials of I. Then $f \notin I$, so $f \notin \langle A_i \rangle$ for all A_i, which implies $f \in \langle X \setminus A_i \rangle$ for some fixed i. Then $f = x_1^{a_1} ... x_n^{a_n}$, where $a_i = 0$ for $x_i \in A_i$. This is the form required for $f \in (1, X \setminus A_i)$. Thus $f \in \bigcup_{i=1}^{m} \{(1, \{X \setminus A_i\})$.

Take any $f \in \bigcup_{i=1}^{m}\{(1, \{X \setminus A_i\})\}$. Then $f \in (1, \{X \setminus A_i\})$ for some i. Thus we have $f = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ where $a_i = 0$ if $x_i \in A_i$. Thus $f \in k[X \setminus A_i]$, so $f \in \text{stdMonomials}(\langle A_i \rangle)$, and thus $f \in \bigcap_{i=1}^{m} \text{stdMon}(\langle A_i \rangle) = \text{stdMon}\, I$. □

Corollary 5.4. For radical I, if $I = P_1 \cap P_2 \cap ... \cap P_n$ for prime P_i,

$$I^{\langle \ell \rangle} = P_1^{\langle \ell \rangle} \cap ... \cap P_n^{\langle \ell \rangle}.$$

Proof. The statement is true for symbolic powers, and differential and symbolic powers are equal when I is radical. □

Lemma 5.5. Let P_F be a prime monomial ideal corresponding to a face F given by some subset of the columns $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ of I^n. Then $P_f = \langle x_i \mid x_i \notin F \rangle$, and

$$\text{stdPairs}(P_F^{\langle \ell \rangle}) = \{(\mathbf{a}, F) \mid |\mathbf{a}| < \ell, a_i = 0 \text{ if } \mathbf{e}_i \in F\}.$$

.

Proof. For any P_F, $P_F^{\langle \ell \rangle} = P_F^{\ell}$, since P_F is a complete intersection ideal.

We will show that the set of standard pairs $\{(\mathbf{a}, F) \mid |\mathbf{a}| < \ell, a_i = 0 \text{ if } \mathbf{e}_i \in F\}$ corresponds to the ideal P_F.

We know that this set of standard pairs describes the intersection given by

$$\bigcap_{(\mathbf{a}, Z)} \langle x_i^{a_i+1} \mid \mathbf{e}_i \notin F \rangle.$$

Without loss of generality, assume $P_F^{\ell} = \langle x_1, ..., x_d \rangle^{\ell}$, where $R = \mathbb{C}[x_1, \ldots, x_d, x_{d+1}, \ldots, x_n]$. Then $F = \{\mathbf{e}_{d+1}, \ldots, \mathbf{e}_n\}$. Note that $P_F^{\ell} = \langle \mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}^d, |\mathbf{a}| = \ell \rangle$.

Then our intersection is given by

$$\bigcap_{(\mathbf{a} \in \mathbb{N}, F)} \langle x_i^{a_i+1} \mid 1 \leq i \leq d, \text{where } |\mathbf{a}| < \ell\}.\rangle$$

Consider some $\mathbf{x}^{\mathbf{b}}$ in the set of generators in ${P_F}^{\ell}$. Then $|\mathbf{b}| < \ell$. Consider an arbitrary standard pair $(\mathbf{a}, F)$, such that $|\mathbf{a}| < \ell$. Then we want to show $x_1^{b_1} \ldots x_d^{b_d} \in \langle x_1^{a_1+1} \rangle$ We claim there exists an i such $b_i > a_i$ - if for all $b_i \leq a_i$, then $|\mathbf{b}| < |\mathbf{a}|$, but we need $\ell = |\mathbf{b}| < |\mathbf{a}| < \ell$. So $b_i > a_i$, and thus $b_i \geq a_i + 1$, so $x_i^{a_i+1} \mid \mathbf{x}^{\mathbf{b}}$.

We will prove that $P_F^{\ell} \supseteq \bigcap_{(\mathbf{a} \in \mathbb{N}, F)} \langle x_i^{a_i+1} \mid 1 \leq i \leq d, \text{where } |\mathbf{a}| < \ell \rangle$ using the contrapositive: if $\mathbf{x} \notin P_F^{\ell}$, then $\mathbf{x} \notin \bigcap_{(\mathbf{a} \in \mathbb{N}, F)} \langle x_i^{a_i+1} \mid 1 \leq i \leq d, \text{where } |\mathbf{a}| < \ell \rangle$.

Well, if $\mathbf{x}^{\mathbf{a}}$ is not in ${P_F}^{\ell}$, we know $\mathbf{x}^{\mathbf{a}}$ is a standard monomial of ${P_F}^{\ell}$. The standard monomials of P_F^{ℓ} are given by $\{\mathbf{x}^{\mathbf{a}} \cdot \mathbf{xc} \mid \forall\, \mathbf{a} \in \mathbb{N}^d \text{ such that } |\mathbf{a}| < \ell \text{ and } \forall\, \mathbf{c} \in F\}$.

Because the set of standard pairs describes exactly the set $\{\mathbf{x}^{\mathbf{a}} \cdot \mathbf{xc} \mid \forall\, \mathbf{a} \in \mathbb{N}^d$ such that $|\mathbf{a}| < \ell$ and $\forall\, \mathbf{c} \in F\}$, it suffices to show that if $\in \{\mathbf{x}^{\mathbf{a}} \cdot \mathbf{xc} \mid \forall \mathbf{a} \in \mathbb{N}^{\mathbf{d}}$ such that $|\mathbf{a}| < \ell$ and $\forall \mathbf{c} \in F\}$, then $\in \mathbf{P_F^l}$ and the converse: if $\in {\mathbf{P_F}}^{\ell}$, then $\in \{\mathbf{x}^{\mathbf{a}} \cdot \mathbf{xc} \mid \forall\, \mathbf{a} \in \mathbb{N}^{\mathbf{d}}$ such that $|\mathbf{a}| < \ell$ and $\forall\, \mathbf{c} \in F\}$.

Let $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}}$ be in $\{\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{c}} \mid \forall\, \mathbf{a} \in \mathbb{N}^d$ such that $|\mathbf{a}| < \ell$ and $\forall\, \mathbf{c} \in F\}$. Assume towards a contradiction that there exists a generator $\mathbf{y}^{\mathbf{b}}$ such that $\mathbf{y}^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}}$. Since $\mathbf{c} \in \mathbb{N}^{n-(d+1)}$ and $\mathbf{x}$ is some product of $\{x_{d+1}, ..., x_n\}$, then $\mathbf{y}^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{c}}$, a contradiction.

Now we show the reverse containment by contrapositive. Let $\mathbf{y}^{\mathbf{c}} \notin \{\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{c}} \mid \forall\, \mathbf{a} \in \mathbb{N}^d$ such that $|\mathbf{a}| < \ell$ and $\forall\, \mathbf{c} \in F\}$. This is true if and only if $\mathbf{y}^{\mathbf{c}} = \mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}}$, and $|a| < \ell$. This is precisely the condition necessary for $\mathbf{y}^{\mathbf{b}} \notin P_F^l : \mathbf{x}^{\mathbf{c}}$ can be chosen freely but $|a| > \ell$ for $a \in P_F^\ell$. □

Theorem 5.6. Let I be a radical ideal. Then I is given by the intersection of prime ideals, that is, $I = P_{F_1} \cap P_{F_2} \cap \cdots \cap P_{F_n}$. Then

$$\operatorname{stdPairs}(I^{\langle \ell \rangle}) = \bigcup_{i=1}^{n} \operatorname{stdPairs}(P_F^{\langle \ell \rangle}).$$

Proof. First, note that the ideal I is a minimal intersection of primes.

in the intersection, that is, there are no primes P_{F_i}, P_{F_j} such that $P_{F_i} \subsetneq P_{F_j}$. Then the complement of P_{F_i} F_i cannot contain F_j nor can F_j contain F_i. Since the standard pairs of these primes describe these faces, no standard pair of P_{F_i} contains a standard pair of P_{F_j} (and vice versa). Thus

$$\operatorname{stdPairs}(P_{F_m} \cap P_{F_n}) = \operatorname{stdPairs}(P_{F_m}) \cup \operatorname{stdPairs}(P_{F_n}),$$

and this result can be extended to give

$$\operatorname{stdPairs}(I) = \operatorname{stdPairs}(P_{F_1} \cap ... \cap \operatorname{stdPairs}(P_{F_n})).$$

From Lemma 5.5, $P_F^{\langle \ell \rangle} = \{\mathbf{x}^{\mathbf{a}} \mid |\mathbf{a}| < \ell, a_i = 0 \text{ if } \mathbf{e}_i \in F\}$. Since $I = P_{F_1} \cap P_{F_2} \cap \cdots \cap P_{F_n}$, Corollary 5.4 implies that $I^{\langle \ell \rangle} = P_{F_1}^{\langle \ell \rangle} \cap \cdots \cap P_{F_n}^{\langle \ell \rangle}$.

Then $\operatorname{stdPairs}(I) = \operatorname{stdPairs}(P_{F_1}^{\langle \ell \rangle}) \cap \cdots \cap \operatorname{stdPairs}(P_{F_n}^{\langle \ell \rangle})$.

By Lemma 5.5, for each P_{F_i}, $\operatorname{stdPairs}(P_{F_i}^{\langle \rangle} = \{(\mathbf{a}, F) \mid |\mathbf{a}| < \ell, a_i = 0 \text{ if } \mathbf{e}_i \in F\}$.

Then $\operatorname{stdPairs}(I^{\langle \ell \rangle}) = \bigcup_{i=1}^{m} \{(\mathbf{a}, \{F_i\}), |\mathbf{a}| < \ell, a_i = 0 \text{ if } \mathbf{e}_i \in F\}$.

□

6. Asymptotic Behavior of Differential Powers in Polynomial Rings

In this section, we always assume that $I \subseteq S = \mathbb{C}[t_1, \ldots, t_d]$ is a monomial ideal on m generators, that is, I is of the form $I = \langle t^{\beta^j} \mid 1 \leq j \leq m \rangle$, where $\beta^j = (\beta_1^j, \ldots, \beta_d^j) \in \mathbb{N}^d$. By abuse of notation, we sometimes drop j when there is no ambiguity in β. We also use the notation $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_d) \in \mathbb{N}^d$ and $\beta + N = (\beta_1 + N, \ldots, \beta_d + N)$ for $N \in \mathbb{N}$ repeatedly in statements. At the end of the subsection, we will prove that if $\sqrt{I} = \langle t_1 \cdots t_d \rangle$, then there exists $N \in \mathbb{N}$ such that $I^{\langle n+1 \rangle}$ is principal for all $n \geq N$.

Before proving this statement, we need some preliminary lemmas. Using the characterization of the differential operators in the polynomial ring, one can show the following lemma, which allows us to apply inductive arguments on $I^{\langle n+1 \rangle}$ if we can compute $I^{\langle 2 \rangle}$ explicitly. This lemma actually works for arbitrary ideal of S, not necessarily monomial.

Lemma 6.1. [Ken+21] $(I^{\langle k+1 \rangle})^{\langle \ell+1 \rangle} = I^{\langle k+\ell+1 \rangle}$ for all $k, \ell \in \mathbb{N}$.

Lemma 6.2. If $I = \langle t^\beta \rangle$ with $\beta_i > 0$ for all $1 \leq i \leq d$, then $I^{\langle 2 \rangle} = \langle t^{\beta+1} \rangle$.

Proof. ($\Rightarrow$) Since 1 is a zero order differential operator of S, $I^{\langle 2 \rangle} \subseteq I$. We may thus assume that every element of $I^{\langle 2 \rangle}$ is of the form $t^\beta f \in I^{\langle 2 \rangle}$ for some $f \in S$. For the differential operator $\partial_i \in D(S)$ of order 1, since $\beta_i > 0$ for all $1 \leq i \leq d$, we can compute

$$\partial_i(t^\beta f) = \beta_i t_1^{\beta_1} \cdots t_i^{\beta_i - 1} \cdots t_d^{\beta_d} f + t^\beta \partial_i(f) \in I.$$

Because $t^\beta\partial_i(f) \in I$, $\beta_i t_1^{\beta_1}\cdots t_i^{\beta_i-1}\cdots t_d^{\beta_d} f \in I$, which implies that t_i divides f. Since this argument holds for all $1 \le i \le d$, we know $t_1\cdots t_d$ divides f and thus $t^\beta f \in \langle t^{\beta+1}\rangle$.

($\Leftarrow$) For the other containment, we need to show that $\delta(t^{\beta+1}) \in I$ for any $\delta \in D(S)$ of order zero or one, which would imply $t^{\beta+1} \in I^{\langle 2\rangle}$. If δ has order zero, then $\delta \in S$ and $\delta t^{\beta+1} = t^\beta(\delta t_1\cdots t_d) \in I$. If δ has order one, then it is a linear combination of ∂_i and 1 with coefficients from S. Because

$$\partial_i(t^{\beta+1}) = (\beta_i+1)t_1^{\beta_1+1}\cdots t_i^{\beta_i}\cdots t_d^{\beta_d+1} \in I,$$

for any $\delta \in D(S)$ of order one, $\delta(t^{\beta+1}) \in I$. Therefore, $t^{\beta+1} \in I^{\langle 2\rangle}$ and $\langle t^{\beta+1}\rangle \subseteq I^{\langle 2\rangle}$. $\square$

By Lemma 6.1 and Lemma 6.2, an inductive argument implies the next result. It further implies that once some differential power of I is principal such that the generator has positive exponent in every t_i, the higher differential powers remain principal. To prove Theorem 6.7, it suffices to find a particular N such that $I^{\langle N\rangle}$ is principal.

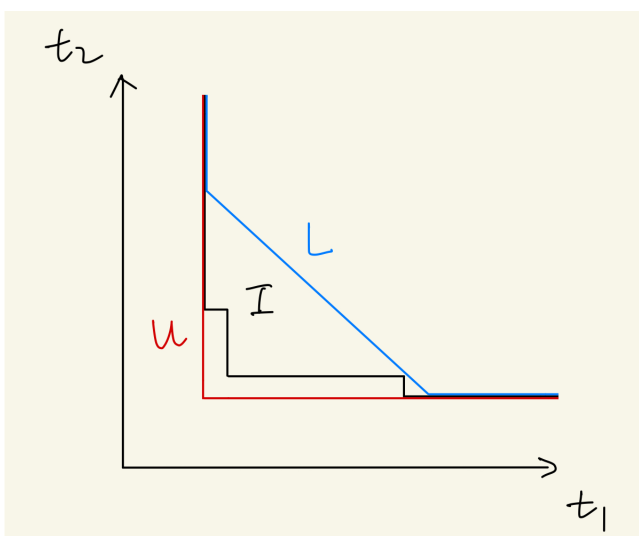
Lemma 6.3. If $I = \langle t^\beta\rangle$ with $\beta_i > 0$ for all $1 \le i \le d$, then $I^{\langle N+1\rangle} = \langle t^{\beta+N}\rangle$ for all $N \in \mathbb{N}$.

The following proposition holds in any semigroup ring R and thus holds for S.

Proposition 6.4. If $J \subseteq K$ are ideals of R, then $J^{\langle N+1\rangle} \subseteq K^{\langle N+1\rangle}$ for all $N \in \mathbb{N}$.

Proof. Let $f \in J^{\langle N+1\rangle}$. By definition, for any differential operator $\delta \in D(R)$ of order at most N, $\delta(f) \in J \subseteq K$, which implies that $f \in K^{\langle N+1\rangle}$. $\square$

This proposition gives the proof structure of Theorem 6.7. We will find ideals $L \subseteq I \subseteq U$ squeezing I such that the $L^{\langle N+1\rangle}$ and $U^{\langle N+1\rangle}$ are the same principal ideal. Then Proposition 6.4 forces $I^{\langle N+1\rangle}$ to be a principal ideal. In the two dimensional case, I is generated by monomials in a staircase shape as shown in the figure below. The upper bound U is a principal ideal, whose generator is the intersection obtained by extending the infinite rays bounding I. The lower bound L is described formally in the next lemma. Geometrically, it is generated by elements lying on the same segment of $t_1+t_2 =$ constant. We further require that the infinite rays bounding L, I, and U eventually overlap.



For higher dimensional cases, U is still a monomial ideal whose generator is the intersection by extending the $d-1$ dimensional boundaries of I, and L is generated by elements on the simplex $t_1 + \cdots + t_d =$ constant. We still require that the $d-1$ dimensional boundaries of L, I, and U eventually overlap.

Lemma 6.5. Let $N \in \mathbb{N}$. If $t^\beta \in I^{\langle N+1\rangle}$ with $\beta_i > N$ for all $1 \leq i \leq d$, then $t^{\beta - N + \mathbf{a}} \in I$ for all $0 \leq \mathbf{a}_i \leq N$ and $\sum_{i=1}^d \mathbf{a}_i = (d-1)N$. Conversely, if

$$J = \langle t^{\beta+\mathbf{a}} \mid 0 \leq \mathbf{a}_i \leq N \text{ and } \sum_{i=1}^d \mathbf{a}_i = (d-1)N\rangle$$

with $\beta_i > 0$ for all $1 \leq i \leq d$, then $J^{\langle N+1\rangle} = \langle t^{\beta+N}\rangle$.

Notice that $N - \mathbf{a}$ is a vector in $\mathbb{N}^d$ whose sum of entries is exactly N. The current notation will make the proof of Theorem 6.7 easier to read.

Proof. Let $\mathbf{a} \in \mathbb{N}^d$ such that $0 \leq \mathbf{a}_i \leq N$ and $\sum_{i=1}^d \mathbf{a}_i = (d-1)N$. Let $\mathbf{a}' = N - \mathbf{a}$. Notice that $\mathbf{a}' \in \mathbb{N}^d$, $\mathbf{a}'_i \leq N$, and $\sum_{i=1}^d \mathbf{a}'_i = N$. Then $\partial^{\mathbf{a}'} = \partial_1^{\mathbf{a}'_1} \cdots \partial_d^{\mathbf{a}'_d} \in D(S)$ is a differential operator of order N. Since $t^\beta \in I^{\langle N+1\rangle}$ and $\beta_i > N \geq \mathbf{a}'_i$,

$$\partial^{\mathbf{a}'}(t^\beta) = ct^{\beta - \mathbf{a}'} \in I$$

for some non-zero constant c. Therefore, $t^{\beta-\mathbf{a}'} = t^{\beta-N+\mathbf{a}} \in I$.

Now we prove the second half of this lemma: $J^{\langle N+1\rangle} = \langle t^{\beta+N}\rangle$. Notice that since t^β divides every generator of J, t^β divides every element of J. We will use this fact repeatedly in this proof.

($\Rightarrow$) Let $t^\gamma \in J^{\langle N+1\rangle}$ be a monomial with $\gamma \in \mathbb{N}^d$. Our goal is to show that $\gamma_i \geq \beta_i + N$ for all $1 \leq i \leq d$, which would imply $t^\gamma \in \langle t^{\beta+N}\rangle$. Because 1 is a zero order differential operator of $D(S)$, $t^\gamma \in J^{\langle N+1\rangle} \subseteq J$. Because t^β divides every generator of J, t^β divides every element of J including t^γ, which implies that $\gamma_i \geq \beta_i$ for all $1 \leq i \leq d$. If $\gamma_i < N$, then $\partial_i^{\gamma_i}(t^\gamma) \in J$. Notice that the power of t_i is zero in $\partial_i^{\gamma_i}(t^\gamma)$ and $\partial_i^{\gamma_i}(t^\gamma)$ itself is nonzero. This is a contradiction as $\partial_i^{\gamma_i}(t^\gamma) \in J$ implies that the power of t_i is at least $\beta_i > 0$. Hence, $\gamma_i \geq N$. Consider the differential operator $\partial_i^N \in D(S)$ of order N. Since $t^\gamma \in J^{\langle N+1\rangle}$,

$$\partial_i^N(t^\gamma) = ct_1^{\gamma_1} \cdots t_i^{\gamma_i - N} \cdots t_d^{\gamma_d} \in J$$

for some nonzero constant c. Since t^β divides every element of J, $\gamma_i - N \geq \beta_i$ and thus $t^\gamma \in \langle t^{\beta+N}\rangle$. For arbitrary $f \in J^{\langle N+1\rangle}$, the argument in this paragraph works for each of its monomial. Therefore, $J^{\langle N+1\rangle} \subseteq \langle t^{\beta+N}\rangle$.

($\Leftarrow$) To show the other containment, we need to show that $\delta(t^{\beta+N}) \in J$ for every $\delta \in D(S)$ of order at most N, which would imply $t^{\beta+N} \in J^{\langle N+1\rangle}$. Recall that δ is a linear combination of $\partial_i^{\mathbf{a}'}$, where $\mathbf{a}' \in \mathbb{N}^d$ and $\sum_{i=1}^d \mathbf{a}'_i \leq N$. Let $\mathbf{a} = N - \mathbf{a}'$. Then $0 \leq \mathbf{a}_i \leq N$ and $\sum_{i=1}^d \mathbf{a}_i \geq (d-1)N$. By direct computation, the power of t_i in $\partial_i^{\mathbf{a}'}(t^{\beta+N})$ is at least $\beta_i + N - \mathbf{a}' = \beta_i + \mathbf{a}$. By the definition of J, $\partial_i^{\mathbf{a}'}(t^{\beta+N}) \in J$. Because this argument works for all $\partial_i^{\mathbf{a}'}$, $t^{\beta+N} \in J^{\langle N+1\rangle}$ and thus $\langle t^{\beta+N}\rangle \subseteq J^{\langle N+1\rangle}$. □

Remark. The lemma above implies that J is the smallest ideal such that $J^{\langle N+1\rangle} = \langle x^{\beta+N}\rangle$.

We have obtained all the information we need for the upper bound and lower bound of I. The last step before proving Theorem 6.7 is to understand $\sqrt{I} = \langle t_1 \cdots t_d\rangle$. The geometric interpretation of this condition is that the monomials of I lie in the interior of the nonnegative orthant.

Lemma 6.6. Let $I = \langle t^{\beta^j} \mid 1 \leq j \leq m \rangle$ be a monomial ideal. Then $\sqrt{I} = \langle t_1 \cdots t_d \rangle$ if and only if every $\beta_i^j > 0$.

Proof. ($\Rightarrow$) Suppose $\sqrt{I} = \langle t_1 \cdots t_d \rangle$. For each $1 \leq j \leq m$, since $t^{\beta^j} \in I \subseteq \sqrt{I}$, there exists some positive power $k \in \mathbb{N}$ such that $t^{k\beta^j} \in \langle t_1 \cdots t_d \rangle$. Then there exists $f \in S$ such that $t^{k\beta^j} = t_1 \cdots t_d f$. Since k is non-zero, and since the exponents of t_i are all positive on the right hand side, we must have $\beta_i^j > 0$ for all $1 \leq i \leq d$.

($\Leftarrow$) Suppose $\beta_i^j > 0$ for all $1 \leq i \leq d$ and $1 \leq j \leq m$. Since $t_1 \cdots t_d$ divides every generator of I, $I \subseteq \langle t_1 \cdots t_d \rangle$. Because $\langle t_1 \cdots t_d \rangle$ is a prime ideal, it is the radical of itself. Hence, $\sqrt{I} \subseteq \langle t_1 \cdots t_d \rangle$. For the other containment, choose n large enough so that $n \geq \beta_i^1$ for all $1 \leq i \leq d$. Since t^{β^1} divides $(t_1 \cdots t_d)^n$, $(t_1 \cdots t_d)^n \in I$. Hence, $t_1 \cdots t_d \in \sqrt{I}$ and thus $\langle t_1 \cdots t_d \rangle \subseteq \sqrt{I}$. $\square$

This lemma allows us to apply Lemma 6.3 and Lemma 6.5 in the proof of Theorem 6.7.

Theorem 6.7. Let $I = \langle t^{\beta^j} \mid 1 \leq j \leq m \rangle$ be a monomial ideal. If $\sqrt{I} = \langle t_1 \cdots t_d \rangle$, then there exists $N \in \mathbb{N}$ such that $I^{\langle n+1 \rangle}$ is principal for all $n \geq N$.

Proof. By Lemma 6.3, it suffices to find one $N \in \mathbb{N}$ such that $I^{\langle N+1 \rangle}$ is principal. Motivating by Proposition 6.4, we will find an "upper bound" U and "lower bound" L sequeezing I such that their $(N+1)$-th differential power is the same principal ideal.

Let $\beta_i^{\min} = \min_j \{\beta_i^j \mid 1 \leq j \leq m\}$ and $\beta_i^{\max} = \max_j \{\beta_i^j \mid 1 \leq j \leq m\}$. Choose $N = \max_{i \neq j}\{\beta_i^{\max} + \beta_j^{\max} - \beta_i^{\min} - \beta_j^{\min}\}$. Let $\beta^{\min} = (\beta_1^{\min}, \ldots, \beta_d^{\min})$. By Lemma 6.6, $\beta_i^{\min} > 0$ for all $1 \leq i \leq d$.

Define $U = \langle t^{\beta^{\min}} \rangle$ and

$$L = \langle t^{\beta^{\min} + \mathbf{a}} \mid 0 \leq \mathbf{a}_i \leq N \text{ and } \sum_{i=1}^{d} \mathbf{a}_i = (d-1)N \rangle.$$

The geometric interpretation of I, U, and L is the the following. The ideal I is generated by finitely many lattice points in the interior of the nonnegative orthant. It is the union of the orthant-shaped cones placed on its generators. The upper bound U is the smallest orthant-shaped cone containing I. The lower bound L is generated by lattice points on some simplex contained by I. The algebraic formula for this simplex is $t_1 + \cdots + t_d = (\beta^{\min} + d - 1)N$ for $\beta^{\min} \leq t_i \leq \beta^{\min} + N$.

By Lemma 6.3 and Lemma 6.5, $U^{\langle N+1 \rangle} = L^{\langle N+1 \rangle} = \langle t^{\beta^{\min} + N} \rangle$. By Proposition 6.4, it suffices to show $L \subseteq I \subseteq U$. Because $t^{\beta^{\min}}$ divides every generator of I by construction, every element of I can be generated by $t^{\beta^{\min}}$. In other words, $I \subseteq U$.

Now we will prove $L \subseteq I$. For the sake of contradiction, suppose $L \not\subseteq I$. Then there is a particular $\mathbf{a}$ such that $t^{\beta^{\min} + \mathbf{a}} \notin I$ with $0 \leq \mathbf{a}_i \leq N$ and $\sum_{i=1}^{d} \mathbf{a}_i = (d-1)N$. Since $t^{\beta^{\min} + \mathbf{a}}$ cannot be divisible by any generator t^{β^j} of I, for each $1 \leq j \leq m$, there exists some $1 \leq i_j \leq d$ such that the power of t_{i_j} in $t^{\beta^{\min} + \mathbf{a}}$ is strictly less than its power in t^{β^j}. In other words, $\beta_{i_j}^{\min} + \mathbf{a}_{i_j} < \beta_{i_j}^j$. Because there are m generators in I, there are m such inequalities, indexed by j. The index i_j is any integer between 1 and d depending on j.

If all i_j are the same, then $\beta_{i_j}^{\min} < \beta_{i_j}^j$ for all j, which is a contradiction because at least one of them achieves equality by the definition of $\beta_{i_j}^{\min}$. Hence, there are at least two distinct

i_j, denoted by i_{j_1} and i_{j_2}. Summing their corresponding inequalities, we get

$$\beta_{i_{j_1}}^{\min} + \mathbf{a}_{i_{j_1}} + \beta_{i_{j_2}}^{\min} + \mathbf{a}_{i_{j_2}} < \beta_{i_{j_1}}^{j_1} + \beta_{i_{j_2}}^{j2},$$
$$\mathbf{a}_{i_{j_1}} + \mathbf{a}_{i_{j_2}} < \beta_{i_{j_1}}^{j_1} + \beta_{i_{j_2}}^{j2} - \beta_{i_{j_1}}^{\min} - \beta_{i_{j_2}}^{\min}.$$

Recall that $0 \leq \mathbf{a}_i \leq N$ and $\sum_{i=1}^{d} \mathbf{a}_i = (d-1)N$. These two conditions imply $\mathbf{a}_{i_{j_1}} + \mathbf{a}_{i_{j_2}} \geq N$, which gives the contradiction by the definition of N.

□

Remark. By Lemma 6.3, we can additionally ask for the smallest N such that $I^{\langle N+1\rangle}$ is principal. The minimality of J in Lemma 6.5 implies that such N is the smallest one guaranteeing $L \subseteq I$ in the proof of Theorem 6.7. We can find a combinatorial formula for the smallest N if S is two dimensional as in [Ken+21]. However, this is generally a hard question in higher dimensions.

The condition $\sqrt{I} = \langle t_1 \cdots t_d\rangle$ in Theorem 6.7 is in fact very important, without which the theorem fails immediately. For example, for $S = \mathbb{C}[x, y]$, the differential powers of $\langle x, y\rangle$ are exactly the ordinary powers, which are never principal. Furthermore, we may be able to adopt a similar squeezing argument to show that for a monomial ideal I with $\sqrt{I} \neq \langle t_1 \cdots t_d\rangle$, the differential powers of I are never principal unless I is principal. In the latter case, all differential powers of I is principle.

Conjecture 6.8. Let $I \subseteq S$ be a monomial ideal. Suppose I is not principal. Then $I^{\langle N\rangle}$ is principal for some N if and only if $\sqrt{I} = \langle t_1 \cdots t_d\rangle$.

7. Asymptotic Behavior of Differential Powers for Rational Normal Curves

For the next step, we will prove similar results in some semigroup rings with particularly nice structure. Unfortunately, ideals are not eventually principal even in the following simplest examples, but we can still prove some asymptotical behaviors of differential powers.

For simplicity, we first look at $R = \mathbb{C}[\mathbb{N}A]$ when $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & n \end{bmatrix}$. We may assume that $n \geq 2$ because when $n = 1$, R is the 2-dimensional polynomial ring.

In Theorem 6.7, we assume that $\sqrt{I} = \langle t_1 \cdots t_d\rangle$, the geometric meaning of which is that the monomials of I does not contain elements on the axes. Therefore, for the rest of this section, we will assume that all the ideals are generated by monomials in the interior of $\mathbb{R}_{\geq 0}A$.

To compute differential powers, we need to understand Saito-Traves differential operators. Since we are studying two dimensional semigroup rings, they have very clear patterns.

(ZACH: consolidating everything. heavy construction zone for the next few lemmas/props)

Label the chambers of $\mathbb{R}^2$ determined by the hyperplanes $h_1 = nx - y$ and $h_2 = y$ with Roman numerals I, II, III, and IV counter-clockwise as in the figure (forthcoming), so that the cone $\mathbb{R}_{\geq 0}A$ is chamber I. In terms of h_1 and h_2, chamber I consists of those points satisfying $h_1 \geq 0$ and $h_2 \geq 0$; chamber II, $h_1 \leq 0$ and $h_2 \geq 0$; chamber III, $h_1 \leq 0$ and $h_2 \leq 0$; and chamber IV, $h_1 \geq 0$ and $h_2 \leq 0$.

Lemma 7.1. The chambers of $\mathbb{R}^2$ are generated as cones by the following integer vectors:

- Chamber I by $(1, 0), \ldots, (1, n)$, the columns of A.

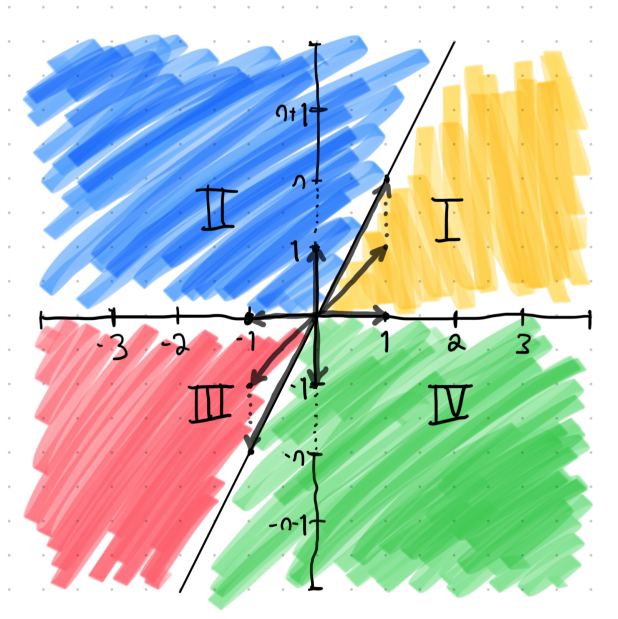


FIGURE 1. The chambers of $\mathbb{R}^2$ corresponding to the semigroup generated by A

- Chamber II by $(1, n), (0, 1)$, and $(-1, 0)$.
- Chamber III by $(-1, 0), \ldots, (-1, -n)$, the inverses of columns of A.
- Chamber IV by $(-1, -n), (0, -1)$, and $(1, 0)$.

Proof. That chambers I and III are generated by the specified vectors is immediate from the fact that $\mathbb{R}_{\geq 0}A$ is saturated.

To show that $(1, n), (0, 1)$, and $(-1, 0)$ generate chamber II, suppose $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}^2$ satisfies $h_1(\mathbf{a}) \leq 0$ and $h_2(\mathbf{a}) \geq 0$, *i.e.* $na_1 - a_2 \leq 0$ and $a_2 \geq 0$. If $a_1 \leq 0$, then $\mathbf{a} = -a_1(-1, 0) + a_2(0, 1)$. Otherwise, if $a_1 > 0$, then $\mathbf{a} = a_1(1, n) + (a_2 - na_1)(0, 1)$. Either way, $\mathbf{a}$ is a nonnegative integer combination of $(-1, 0), (0, 1)$, and $(1, n)$, so these vectors generate chamber II.

The argument for chamber IV is analogous to that for chamber II. □

Proposition 7.2. Let $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}^2$. The order of $\mathcal{D}_{\mathbf{a}}$ has the following characterization:

- $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = 0$ if and only if $\mathbf{a} \in \mathbb{N}A$.
- $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = kn$ for some $k > 0$ if and only if one of the following holds:
  1. $a_1 = -k$ and $0 \geq a_2 \geq -nk$; or
  2. there exists $\ell > 0$ for which $a_1 = \ell - k$ and either $a_2 = \ell n$ or $a_2 = -kn$.
- $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = kn + r$ for some $k \geq 0, 0 < r < n$ if and only if there exists $\ell \geq 0$ such that $a_1 = \ell - k$ and either $a_2 = \ell n + r$ or $a_2 = -kn - r$.

Diagrammatically, (figure X) shows the geometric arrangement of the (will write later. too high)

Proof. We prove the statement by proving the following statements phrased in terms of the chambers of $\mathbb{R}^2$:

1. If $\mathbf{a}$ lies in chamber I, then $\mathcal{D}_{\mathbf{a}}$ has order 0.

2. If $\mathbf{a}$ lies in chamber II, then there exist unique integers $k, r, \ell \in \mathbb{N}$ with $0 \leq r < n$ for which $\mathbf{a} = k(-1,0) + r(0,1) + \ell(1,n) = (\ell - k, \ell n + r)$. In this case, $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = kn + r$.
3. If $\mathbf{a}$ lies in chamber III, then $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = -a_1 n$.
4. If $\mathbf{a}$ lies in chamber IV, then there exist unique integers $k, r, \ell \in \mathbb{N}$ with $0 \leq r < n$ for which $\mathbf{a} = k(-1,-n) + r(0,-1) + \ell(1,0) = (\ell - k, -kn - r)$. In this case, $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = kn + r$.

Note these statements agree when $\mathbf{a}$ lies on the boundary of two chambers. As each $\mathbf{a}$ must lie in one of these chambers, this implies the statement of the proposition.

By Corollary 2.5, if $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}^2$ can be written as $\mathbf{a} = m_1\mathbf{b}_1 + \cdots + m_k\mathbf{b}_k$, where $m_i \geq 0$ are integers and the $\mathbf{b}_i$ generate the chamber containing $\mathbf{a}$, then

$$\mathcal{D}_{\mathbf{a}} = (\mathcal{D}_{\mathbf{b}_1})^{m_1} \cdots (\mathcal{D}_{\mathbf{b}_k})^{m_k}.$$

We first argue in cases on the chamber containing $\mathbf{a}$.

Case 1. If $\mathbf{a}$ lies in chamber I, then $h_1(\mathbf{a}), h_2(\mathbf{a}) \geq 0$, which means $\mathcal{D}_{\mathbf{a}} = t^{\mathbf{a}}$, which has order 0.

Case 2. If $\mathbf{a}$ lies in chamber II, then $\mathbf{a}$ can be expressed as $\mathbf{a} = k(-1,0) + r(0,1) + \ell(1,n)$ for some $k, r, \ell \in \mathbb{N}$. Then,

$$\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = \operatorname{ord}((\mathcal{D}_{(-1,0)})^k (\mathcal{D}_{(0,1)})^r (\mathcal{D}_{(1,n)})^\ell) = kn + r,$$

as $\operatorname{ord}(\mathcal{D}_{(1,n)}) = 0$. We show that we can assume $0 \leq r < n$. Otherwise, applying the division algorithm to r, there exist $q > 0, 0 \leq m < n$ such that $r = qn + m$. Then,

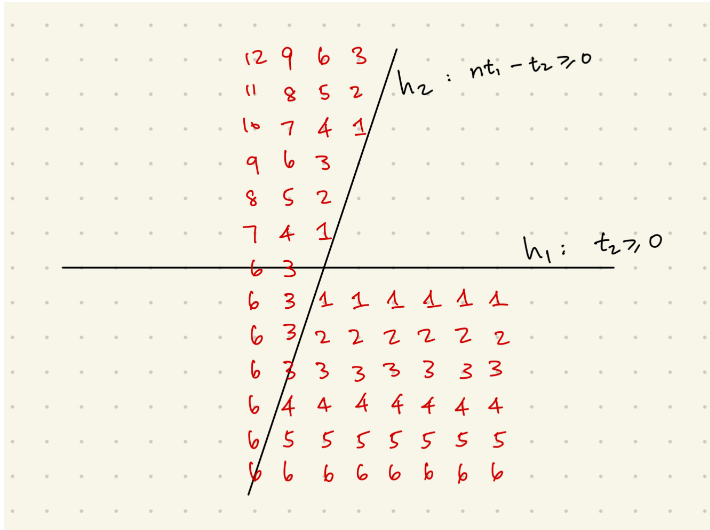
$$\begin{aligned}
\mathbf{a} &= k(-1,0) + r(0,1) + \ell(1,n) = (\ell - k, \ell n + r) \\
&= (\ell - k, n(\ell + q) + m) = ((\ell + q) - (k + q), n(\ell + q) + m) \\
&= (k + q)(-1,0) + m(0,1) + (\ell + q)(1,n).
\end{aligned}$$

Replacing k with $k + q$, r with m, and ℓ with $\ell + q$, we have the desired decomposition. The uniqueness property of the division algorithm guarantees that r is unique, from which it follows that k and ℓ must also be unique.

Case 3. If $\mathbf{a}$ lies in chamber III, then $\mathbf{a} = k_0(-1,0) + \cdots + k_n(-1,-n)$ for some $k_i \in \mathbb{N}$ by Lemma 7.1, so $a_1 = -k_0 - \cdots - k_n$. Then $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = (k_0 + \cdots + k_n)n = -a_1 n$.

Case 4. If $\mathbf{a}$ lies in chamber IV, an argument analogous to that in case 2 shows that $\mathcal{D}_{\mathbf{a}} = (\mathcal{D}_{(-1,-n)})^k (\mathcal{D}_{(0,-1)})^r (\mathcal{D}_{(1,0)})^\ell$ for some $k, \ell \geq 0$ and $0 \leq r < n$. In this case, $\operatorname{ord}(\mathcal{D}_{\mathbf{a}}) = kn + r$. □

The figure below is an example of R when $n = 3$, on which we label each vector $\mathbf{a}$ by the order of $\mathcal{D}_{\mathbf{a}}$. This visualization helps us better understand the algebraic description from Proposition 7.2: the order of $\mathbf{a} \in -\mathbb{R}_{\geq 0}A$ is the sum of vertical distances from $\mathbf{a}$ to h_1 and h_2.



By Proposition 7.2, for $m = kn$ with $k \in \mathbb{N}$, we know the m-th order Saito-Traves operators are

$$\{(-k, i) \mid -m \leq i \leq 0\} \cup \{(-k+i, ni) \mid i \in \mathbb{N}\} \cup \{(-k+i, -m) \mid i \in \mathbb{N}\};$$

for $m = kn + r$ with $0 \leq r < n$, the m-th order Saito-Traves operators are

$$\{(-k+i, ni+r) \mid i \in \mathbb{N}\} \cup \{(-k+i, -m-r) \mid i \in \mathbb{N}\}.$$

By Lemma 3.3, when we compute differential powers in R, it suffices to consider Saito-Traves operators corresponding to these vectors. Moreover, because $(1, 0), (1, n) \in A$, we know that for any $f \in I$, if $\mathcal{D}_{\mathbf{a}}(f) \in I$, then it's necessarily true that $\mathcal{D}_{\mathbf{a}+(1,0)}(f), \mathcal{D}_{\mathbf{a}+(1,n)}(f) \in I$. Therefore, we can further reduce the differential operators that we need to consider. To compute m-th differential power of I, for $m = kn$, it suffices to consider differential operators of the form

$$\{(-k, i) \mid -m \leq i \leq 0\};$$

and for $m = kn + r$ with $0 \leq r < n$, it suffices to consider differential operators of the form

$$\{(-k, i) \mid -m \leq i \leq 0\} \cup \{(-k, i), (-k, -m-i) \mid 0 \leq i \leq r\}.$$

This information helps us to prove the following lemmas and corollaries:

Corollary 7.3. For $m \geq 2$ with $m \not\equiv 0 \mod n$, $I^{\langle m+1 \rangle} = (I^{\langle m \rangle})^{\langle 2 \rangle}$.

Proof. Set $m = nk + r$ as in Proposition 7.2. Since $\mathcal{D}_{(-k,r-1)}$ and $\mathcal{D}_{(0,1)}$ belong to the same chamber, we have

$$\mathcal{D}_{(-k,r-1)}\mathcal{D}_{(0,1)} = \mathcal{D}_{(-k,r)}$$
$$\mathcal{D}_{(-k,-m+1)}\mathcal{D}_{(0,-1)} = \mathcal{D}_{(-k,-m)}$$

. Take any $f \in (I^{\langle m \rangle})^{\langle 2 \rangle}$. For any operators $\mathcal{D}_a$ with $\mathrm{ord}(\mathcal{D}_a) < m$, we have $\mathcal{D}_a(f) \in I$. By Proposition 7.2 and Corollary 2.5, it suffices to show $\mathcal{D}_{(-k,r)}(f), \mathcal{D}_{(-k,-m)}(f) \in I$, which in turn follows from the above equalities.

For the reverse inclusion, take any $f \in I^{\langle m+1 \rangle}$, then $\mathcal{D}_{(-k,r)}(f), \mathcal{D}_{(-k,-m)}(f) \in I$. Similarly $\mathcal{D}_{(0,1)}(f), \mathcal{D}_{(0,-1))}(f) \in I^{\langle m \rangle}$ and thus $f \in (I^{\langle m \rangle})^{\langle 2 \rangle}$.

□

Corollary 7.4. For $m \in \mathbb{N}$, $(I^{\langle m+1 \rangle})^{\langle n+1 \rangle} = I^{\langle m+n+1 \rangle}$.

These corollaries imply that if we want to compute the m-th differential power with $m = kn + r$, we can first find the next $(n+1)$-th power k times and the next second power r times. In order to prove results about differential powers asymptotically, we first apply these results to ideals of R generated by r consecutive monomials on the same line.

Definition 7.5. Let $\mathbf{a} = (a_1, a_2) \in \mathbb{N}A$, and suppose $r \geq 1$ satisfies $\mathbf{a} + (0, r-1) \in \mathbb{N}A$. Define the *vertical strip ideal* $I_{\mathbf{a},r} \subseteq R$ to be the following monomial ideal:

$$I_{\mathbf{a},r} = \langle s^{a_1}t^{a_2}, s^{a_1}t^{a_2+1}, \ldots, s^{a_1}t^{a_2+r-1} \rangle.$$

Geometrically, the vertical strip ideal $I_{\mathbf{a},r}$ is generated by the lattice points that lie on the vertical line segment connecting $\mathbf{a}$ to $\mathbf{a}+(0, r-1)$. These ideals provide a convenient language for arguing on the asymptotic behavior of monomial ideals under differential powers.

Lemma 7.6. Suppose $I_{\mathbf{a},r}$ is a vertical strip ideal lying in the interior of R, meaning that $(st^n)^k, s^k \notin I_{\mathbf{a},r}$ for all $k \in \mathbb{N}$. Then,

$$(I_{\mathbf{a},r})^{\langle 2 \rangle} = tI_{\mathbf{a},r} \cap t^{-1}I_{\mathbf{a},r}, \text{ and } (I_{\mathbf{a},r})^{\langle N+1 \rangle} = sI_{\mathbf{a},r} \cap stI_{\mathbf{a},r} \cap \cdots \cap st^n I_{\mathbf{a},r},$$

where $pI = \{p\alpha \mid \alpha \in I\}$.

Proof. We prove only the first identity; the second is analogous, though more involved.

As $I_{\mathbf{a},r}$ is in the interior of R, the sets $tI_{\mathbf{a},r}$ and $t^{-1}I_{\mathbf{a},r}$ are well-defined ideals of R. By Lemma 3.3, $f \in I_{\mathbf{a},r}^{\langle 2 \rangle}$ if and only if $\mathcal{D}_{\mathbf{b}} f \in I_{\mathbf{a},r}$ for all $\mathbf{b} \in \mathbb{Z}^2$ satisfying $\operatorname{ord}(\mathcal{D}_{\mathbf{b}}) < 2$. Proposition 7.2 allows us to determine exactly where the order 0 and 1 differential operators are in $\mathbb{Z}^d$. The order 0 operators correspond to vectors $\mathbf{b} \in \mathbb{N}A$, so that $\mathcal{D}_{\mathbf{b}} = s^{b_1}t^{b_2}$. As $I_{\mathbf{a},r}$ is an ideal, $\mathcal{D}_{\mathbf{b}} f = s^{b_1}t^{b_2} f \in I_{\mathbf{a},r}$ if and only if $f \in I_{\mathbf{a},r}$, so we need to check whether $f \in I_{\mathbf{a},r}$.

The order 1 operators correspond to the points $(1,0) + \ell(1,n)$ and $(0,-1) + \ell(1,0)$ for all $\ell \in \mathbb{N}$. By Corollary 2.5 and Lemma 7.1, these points correspond to the differential operators

$$(\mathcal{D}_{(1,n)})^\ell \mathcal{D}_{(0,1)} = (st^n)^\ell \mathcal{D}_{(0,1)} \text{ and } (\mathcal{D}_{(1,0)})^\ell \mathcal{D}_{(0,-1)} = s^\ell \mathcal{D}_{(0,-1)};$$

by an argument analogous to the order 0 case, it suffices to consider only $\mathcal{D}_{(0,1)}$ and $\mathcal{D}_{(0,-1)}$. Using Theorem 2.4, we calculate:

$$\mathcal{D}_{(0,1)} = nst\partial_s - t^2\partial_t, \text{ and } \mathcal{D}_{(0,-1)} = \partial_t.$$

Now, $f = s^{a_1}t^{a_2} \in (I_{\mathbf{a},r})^{\langle 2 \rangle}$ if and only if $f, \mathcal{D}_{(0,1)} f$, and $\mathcal{D}_{(0,-1)} f \in I_{\mathbf{a},r}$, that is,

$$(nst\partial_s - t^2\partial_t)s^{a_1}t^{a_2} = (na_1 - a_2)s^{a_1}t^{a_2+1} \text{ and } \partial_t s^{a_1}t^{a_2} = a_2 s^{a_1}t^{a_2-1}$$

are in $I_{\mathbf{a},r}$. We can rewrite these expressions as $\mathcal{D}_{(0,1)} f = ctf$ and $\mathcal{D}_{(0,-1)} f = dt^{-1}f$, where $c, d \in \mathbb{C}$ are constants. Note that $\mathcal{D}_{(0,1)}$ and $\mathcal{D}_{(0,-1)}$ only annihilate monomials of the form $(st^n)^k$ and s^k, respectively. Since $I_{\mathbf{a},r}$ is an interior ideal, no $f \in I_{\mathbf{a},r}$ is annihilated by these operators, so $c, d \neq 0$. Therefore, we have shown that $f \in (I_{\mathbf{a},r})^{\langle 2 \rangle}$ if and only if $f, tf, t^{-1}f \in I_{\mathbf{a},r}$, *i.e.* $(I_{\mathbf{a},r})^{\langle 2 \rangle} = tI_{\mathbf{a},r} \cap t^{-1}I_{\mathbf{a},r} \cap I_{\mathbf{a},r}$.

It remains to show that $tI_{\mathbf{a},r} \cap t^{-1}I_{\mathbf{a},r} \subseteq I_{\mathbf{a},r}$, which would imply the desired result. If $f \in tI_{\mathbf{a},r} \cap t^{-1}I_{\mathbf{a},r}$, then $f = t\alpha = t^{-1}\beta$ for some $\alpha, \beta \in I_{\mathbf{a},r}$. Then $\beta = t^2\alpha$ is in $I_{\mathbf{a},r}$. If we write α in terms of the generators of $I_{\mathbf{a},r}$ as $\alpha = ms^{a_1}t^{a_2+k}$ for some $m \in R$, $0 \leq k \leq r-1$, then the fact that $t^2\alpha = ms^{a_1}t^{a_2+k+2} \in I_{\mathbf{a},r}$ implies $f = t\alpha = ms^{a_1}t^{a_2+k+1} \in I_{\mathbf{a},r}$. $\square$

Proposition 7.7. For an interior vertical strip ideal $I_{\mathbf{a},r} \subseteq R$, the following hold:

$$(I_{\mathbf{a},r})^{\langle 2\rangle} = \begin{cases} I_{\mathbf{a}+(1,1),n-1}, & \text{if } r=1; \\ I_{\mathbf{a}+(1,1),n}, & \text{if } r=2; \\ I_{\mathbf{a}+(0,1),r-2}, & \text{if } r\geq 3. \end{cases}, \quad \text{and} \quad (I_{\mathbf{a},r})^{\langle n+1\rangle} = \begin{cases} I_{\mathbf{a}+(2,n),r}, & \text{if } 1\leq r\leq n; \\ I_{\mathbf{a}+(1,n),r-n}, & \text{if } r>n. \end{cases}$$

Proof. Case 1. Suppose $r=1$.

Case 2. Suppose $r=2$.

Case 3. Suppose $r\geq 3$. Label the generators of $I_{\mathbf{a},r}$ as $g_1 = s^{a_1}t^{a_2}, \dots, g_r = s^{a_1}t^{a_2+r-1}$. Notice that for $2\leq i\leq r-1$, $g_i \in (I_{\mathbf{a},r})^{\langle 2\rangle}$, for $g_i = tg_{i-1} = t^{-1}g_{i+1}$. The generators $g_2,\dots,g_{r-1}$ correspond to the lattice points along the line segment connecting $\mathbf{a}+(0,1)$ to $\mathbf{a}+(0,r-2) = \mathbf{a}+(0,1)+(r-3)$, which means that $g_2,\dots,g_{r-1}$ generate $I_{\mathbf{a}+(0,1),r-2}$. Thus, we have shown $I_{\mathbf{a}+(0,1),r-2} \subseteq (I_{\mathbf{a},r})^{\langle 2\rangle}$.

For the other inclusion, suppose $f\in (I_{\mathbf{a},r})^{\langle 2\rangle}$ is a monomial. Then $f = t\alpha = t^{-1}\beta$ for some $\alpha,\beta \in I_{\mathbf{a},r}$ (ZACH: etc.) □

Theorem 7.8. Let $I\subseteq R$ be an interior monomial ideal for $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & n \end{bmatrix}$. If n is odd, then there exists an $N\in\mathbb{N}$ so that $I^{\langle N+1\rangle}$ is principal. If n is even, then there exists an $N\in\mathbb{N}$ so that either $I^{\langle N+1\rangle}$ is principal or $I^{\langle N+1\rangle}$ is of the form $I_{\mathbf{a},2}$.

Proof. Similar to Theorem 6.7, we want to find two ideals squeezing I so that their differential power are eventually the same. As shown in the Section 7, for an ideal (black), we find an lower bound (red) and an upper bound (blue) of the form $I_{\mathbf{a},r}$. If we can show that their number of generators are the same modulo n, then this theorem is implied by the previous five lemmas.

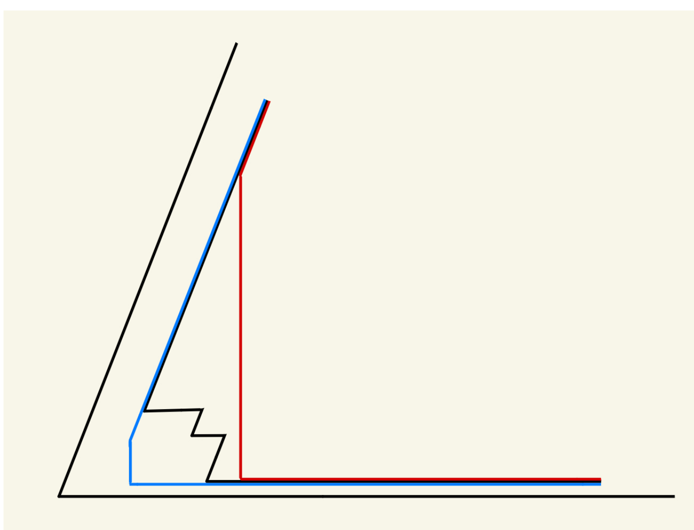


FIGURE 2. Caption

Let $I = \langle t^{\beta^j} \mid 1\leq j\leq m\rangle$, where $\beta^j = (\beta^j_1,\beta^j_2)\in\mathbb{N}^2$. Let $n\beta^k_1 - \beta^k_2 = \min_j\{n\beta^j_1 - \beta^j_2\}$ and $\beta^\ell_2 = \min_j\{\beta^j_2\}$. Pick $1\leq r\leq n$ such that $\beta^\ell_2 - \beta^k_2 + r - 1 \equiv 0 \mod n$. Let $\mathbf{a} = (\beta^k_1 + \frac{1}{n}(\beta^\ell_2 - \beta^k_2 + r - 1), \beta^\ell_2)$. By the choice of r, we know $\mathbf{a}\in\mathbb{N}^2$. We claim that $I\subseteq I_{\mathbf{a},r}$. Geometrically, the two infinite boundaries of I are parametrized by $t_2 = nt_1 - (n\beta^k_1 - \beta^k_2)$ and

$t_2 = \beta_2^\ell$, and they intersect at $(\beta_1^k + \frac{1}{n}(\beta_2^\ell - \beta_2^k), \beta_2^\ell)$. However, they do not always intersect at a lattice point. Thus, we choose the closest lattice point on the left of this intersection, which is obtained by shifting $\frac{r-1}{n}$. Because the slope of the first boundary is n, we know the finite boundary of the upper bound ideal contains exactly r lattice points, that is, $I_{\mathbf{a},r}$.

Now we will find the lower bound (red) ideal. Choose $\beta_1^{\max} = \max_j\{\beta_1^j\}$. Let $\mathbf{b} = (\beta_1^{\max}, \beta_2^\ell)$ and $s = n(\beta_1^{\max} - \beta_1^k) + \beta_2^k - \beta_2^\ell + 1$. Geometrically, the finite boundary of $I_{\mathbf{b},s}$ is the shortest vertical segment contained in I, so it contains exactly s lattice points. Thus, $I_{\mathbf{b},s} \subseteq I$.

Choose $N = n(\beta_1^{\max} - \beta_1^k) + \beta_2^k - \beta_2^\ell + 1 - r$, that is, n times the difference between the first entry of $\mathbf{a}$ and $\mathbf{b}$. By Proposition 7.2 and Proposition 7.7, we have

$$I^{\langle N+1\rangle} = I_{\mathbf{a},r}^{\langle N+1\rangle} = I_{\mathbf{b},s}^{\langle N+1\rangle} = I_{\mathbf{a}+(\frac{N}{n},N),r}.$$

Now we can apply Proposition 7.7 to reduce r to either 1 or 2. If n is even, we are done; if n is odd and r reduces to 2, we apply Proposition 7.7 repeatedly to obtain a principal ideal. $\square$

Conjecture 7.9. Let $A \in \mathbb{Z}^{d\times n}$. Suppose $\mathbb{N}A$ is saturated and $\mathbb{R}_{\geq 0}A$ is a strongly convex cone. Let $h_1, \dots, h_k$ be the hyperplane equations for the boundaries of $\mathbb{R}_{\geq 0}A$ with $h_i(\mathbb{N}A) > 0$. Let

$$D_n = \{\mathcal{D}_{\mathbf{a}} \mid h_i(\mathbf{a}) < 0 \text{ for all } 1 \leq i \leq k \text{ and } \sum_{i=1}^k h_i(\mathbf{a}) = -n\}.$$

Then there exists some $N \in \mathbb{N}$ such that

$$I^{\langle N\rangle} = \{f \in R \mid \delta(f) \in I \text{ for all } \delta \in D_N\}.$$

This conjecture is enough to show the following claim.

Conjecture 7.10. If $I \subseteq R$ is a monomial ideal such that the generators of I are in the interior of $\mathbb{R}_{\geq 0}A$, then there exists $n \in \mathbb{N}$ such that $I^{\langle n\rangle}$ is generated by a set of elements satisfying $\sum_i h_i = N$ for some $N \in \mathbb{Z}$.

8. Code for Differential Powers in Polynomial Rings

The following Macaulay2 code computes differential powers of arbitrary ideals. In the case where the ideal is radical, this corresponds with the ideal's symbolic power. This code is adapted from code provided in Appendix C of [Bah99].

```
nextDifferentialPower = method();
nextDifferentialPower (Ideal) := I -> (
    numRingGens := numgens ring I;
    numIGens := numgens I;
    jacobMat := map(jacobian I, Degree => -1);
    ident := id_((ring I)^numRingGens);
    idealGens := gens I;
    tenProd := ident ** idealGens;
    syzMat :=  syz(jacobMat|tenProd);
    subMat := submatrix(syzMat,{0..(numIGens-1)},);
    ideal (mingens ideal (idealGens*subMat))
)
```

```
differentialPower = method();
differentialPower (Ideal, ZZ) := (I,r)-> (
    powerIdeal:=ideal();
    if r==1 then (powerIdeal = I)
        else (powerIdeal = nextDifferentialPower(differentialPower(I,r-1))
    );
    powerIdeal
)
```

First, we compute the number of generators of the ring and the number of generators of the ideal. We calculate the jacobian matrix of the ideal.

```
    numRingGens := numgens ring I;
    numIGens := numgens I;
    jacobMat := map(jacobian I, Degree => -1);
```

We take the tensor product of an identity matrix of appropriate dimension, and then calculate the syzygy matrix of the Jacobian matrix concatenated with this tensor product.

```
    ident := id_((ring I)^numRingGens);
    idealGens := gens I;
    tenProd := ident ** idealGens;
    syzMat :=  syz(jacobMat|tenProd);
```

We take the submatrix consisting of columns of the syzygy matrix corrresponding to the generators of the ideal. Then we generate the new ideal generated by the generators of the ideal transformed by the submatrix. We return this new ideal.

```
    subMat := submatrix(syzMat,{0..(numIGens-1)},);
    ideal (mingens ideal (idealGens*subMat))
```

The function

```
    differentialPower
```

iterates this function as many times as is needed to calculate the nth differential power.

9. Code for Standard Pairs

The following code uses Lemma 3.3 in [STV95], reproduced here as Lemma 9.1, to generate an ideal from a liat of its standard pairs.

Lemma 9.1. For a monomial ideal I in a ring S generated by a set of variables X:

$$I = \bigcap_{(m,Z)\in \mathrm{stdPairs}(I)} \left(x_i^{\deg_{x_i}(m)+1} : x_i \in X \setminus Z \right).$$

```
    stdPairsToIdeal := (standardPairsList, ambientRing) -> (
    idealsToIntersect := {};
    for i in (0..(#standardPairsList)-1) do(
        newIdealGens:= {};
        currentPair := standardPairsList_i;
        point := currentPair_0;
        pointVars := support point;
        direction := currentPair_1;
        varSpace := gens(ambientRing) - set direction;
```

```
        for j in (0..(#varSpace)-1) do(
            currentDegree := degree(varSpace_j,point)+1;
            newGen := varSpace_j^(currentDegree);
            newIdealGens = append(newIdealGens,newGen);
            );
        newIdeal := ideal(newIdealGens);
        newIdeal = promote(newIdeal,ambientRing);
        idealsToIntersect = append(idealsToIntersect, newIdeal);
        );
    intersectionIdeal := intersect(idealsToIntersect);
    intersectionIdeal
)
```

Here is an explanation of how the code translates this data.

```
    stdPairsToIdeal := (standardPairsList, ambientRing) -> (
    idealsToIntersect := {};
```

We begin by initiating an empty list of ideals to intersect.

We iterate through the list of standard pairs.

```
    for i in (0..(#standardPairsList)-1) do(
```

Each standard pair will contribute a specific ideal to the list of ideals to intersect, so for each pair we initiate an empty list of generators for its ideal.

```
        newIdealGens:= {};
```

For each pair in the list, we calculate $X \setminus Z$.

```
        currentPair := standardPairsList_i;
        point := currentPair_0;
        direction := currentPair_1;
        varSpace := gens(ambientRing) - set direction;
```

For every variable x_i in $X \setminus Z$, we calculate the new generator it contributes to the ideal by $x_i^{\deg_{x_i}(m)+1}$ and add it to our list of ideal generators.

```
        for j in (0..(#varSpace)-1) do(
            currentDegree := degree(varSpace_j,point)+1;
            newGen := varSpace_j^(currentDegree);
            newIdealGens = append(newIdealGens,newGen);
            );
```

From the list of generators, we generate a new ideal. We must promote the new ideal to the ring provided (Macaulay2 assumes an integer ring otherwise). We add this new ideal to our list of ideals to intersect.

```
        newIdeal := ideal(newIdealGens);
        newIdeal = promote(newIdeal,ambientRing);
        idealsToIntersect = append(idealsToIntersect, newIdeal);
        );
```

We calculate the intersection of these ideals and return the result.

```
    intersectionIdeal := intersect(idealsToIntersect);
    intersectionIdeal
```

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