Quivers of Period 2

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Abstract

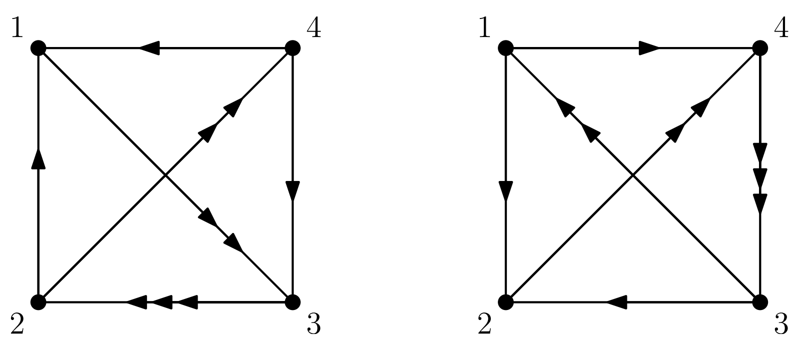
A quiver with vertices labeled from $1, \dots, n$ is said to have period 2 if the quiver obtained by mutating at 1 and then 2 is isomorphic to the original quiver under the permutation $(1, \dots, n) \to (n-1, n, 1, 2, \dots, n-2)$ of the vertices. In this paper, we classify period 2 quivers with 6 nodes, and we also construct infinite families of period 2 quivers in an attempt to move towards a complete classification. We also examine symmetries occuring in such quivers.

1 Introduction

There is a well known relation between cluster algebras and integer sequences which are Laurent polynomials in their initial terms [2]. For example, the terms in the Somos-4 sequence, given by the recurrence

$$x_n x_{n+4} = x_{n+1} x_{n+3} + x_{n+2}^2,$$

are given by the cluster exchange relation associated to mutating vertex 1 in Figure 1(a).



(a) The Somos-4 quiver S_4 (b) The mutation $\mu_1 S_4$

Figure 1: The Somos-4 quiver and its mutation at 1

Surprisingly, after mutating at 1, we obtain a quiver which is isomorphic to our original quiver under the permutation $(1, 2, 3, 4) \mapsto (4, 1, 2, 3)$ of the vertices, and so we can think of a sequence of mutations as iterations of the recurrence. In [3], Fordy and Marsh classify quivers satisfying this property, and they also consider a more general type of periodicity corresponding to Somos type sequences in higher dimensional spaces.

In this paper, we introduce several new results concerning quivers of period 2 in an attempt to move towards a complete classification. In Section 2, we recall the notion of a quiver and a quiver mutation, and we also define the notion of periodicity considered in [3]. We then recall in Section 3 the notion of a primitive quiver, and we briefly describe how quivers of period 1 and sink-type quivers of higher period can be described in terms of primitives. Afterwards, we will examine in Section 4 certain symmetries arising in period 2 quivers. We establish that period 2 quivers on 5 nodes are graph symmetric and find a sufficient condition for period 2 quivers to be graph symmetric. In Section 5, we classify the period 2 quivers on 6 nodes. Finally, in Section 6 we will introduce two new infinite families of period 2 quivers in Theorems 6.3 and 6.7.

2 Mutations and Periodicity

In this paper, we will only consider quivers containing no 1-cycles or 2-cycles, and we will assume that the vertices of a quiver Q lie on an N-sided polygon with vertices labeled $1, \ldots, N$ in clockwise order. We will identify a quiver Q with N nodes with the skew-symmetric $N \times N$ matrix whose (i, j)-entry is given by the number of arrows from i to j minus the number of arrows from j to i. We define quiver mutation as follows:

Definition 2.1. *[1] Given a quiver Q and a vertex k, we construct the mutation of Q at k, denoted by $\mu_k Q$, as follows:*

1. *Reverse all arrows which either begin or end at the node k.*
2. *Suppose there are p arrows from a node i to k and q arrows from k to a node j. Then we add pq arrows from i to j to any existing arrows between the two nodes.*
3. *We remove both arrows of any 2-cycles created in the second step.*

Remark 2.2. *We can also describe quiver mutation in terms of the adjacency matrix. If A and B are the skew-symmetric matrices corresponding to the quivers Q and $\overline{Q} = \mu_k P$ respectively and if a_{ij} and b_{ij} are the corresponding matrix entries, then the entries of B are given by the following formula.*

$$b_{i,j} = \begin{cases} -a_{ij} & \text{if i=k or j=k} \\ a_{ij} + \frac{1}{2}(|a_{ik}|a_{kj} + a_{ik}|a_{kj}|) & \text{otherwise} \end{cases}$$

We now consider the permutation ρ: (1, ..., N) $\mapsto$ (N, 1, ..., N-2, N-1) of the vertices of a quiver Q. This permutation acts on Q so that the number of arrows from i to j in ρ(Q) is the same as the number of arrows from ρ^{-1}(i) to ρ^{-1}(j) in Q. Thus the action can be viewed as rotating the arrows in a clockwise direction while leaving the vertices fixed. Note that the action of ρ on our quiver corresponds to the conjugation of our adjacency matrix by the permutation matrix:

$$\rho = \begin{pmatrix} 0 & \cdots & \cdots & 1 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}.$$

We now consider a sequence of mutations, starting at node 1, then at node 2, and so on.

Definition 2.3. *[3] We say that a quiver Q has period m if it satisfies $Q(m+1) = \rho^m Q(1)$ under the mutation sequence depicted by*

$$Q = Q(1) \xrightarrow{\mu_1} Q(2) \xrightarrow{\mu_2} \ldots \xrightarrow{\mu_{m-1}} Q(m) \xrightarrow{\mu_m} Q(m+1) = \rho^m Q(1).$$

Note that in the previous definition, if $Q(1)$ has period m, then each of the quivers $Q(2), \ldots, Q(m+1)$ must have period m. Also note that a quiver Q has period 2 if and only if its opposite, the quiver obtained from Q by reversing all arrows, is also a period 2 quiver.

3 Primitive Quivers

In [3], the authors are able to classify period 1 quivers in terms of a finite set of period 1 quivers which they call primitives. They also define primitives for higher periodicities and use them to classify a subset of periodic quivers which we will describe later in the section. First, we recall how the period 1 primitives are constructed. Consider the matrix

$$\tau = \begin{pmatrix} 0 & \dots & \dots & -1 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix},$$

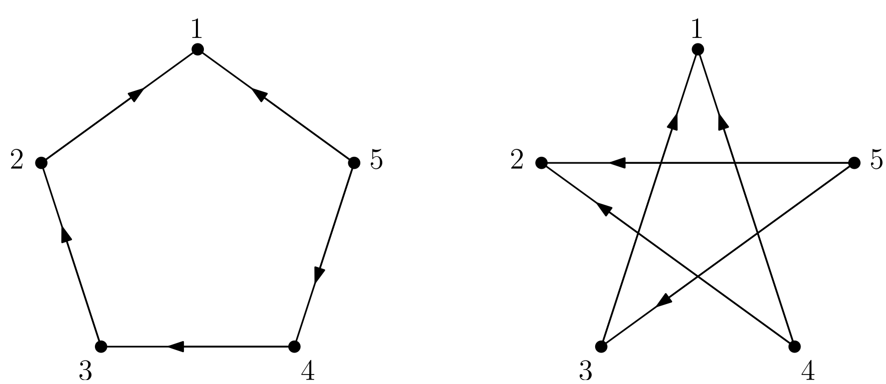
which we call a *skew-rotation*. We construct the primitives from this rotation as follows. First consider a quiver with a single arrow from $N-k+1$ to 1 and denote its adjacency matrix by $R_N^{(k)}$ so that $(R_N^{(k)})_{N-k+1,1} = 1$, $(R_N^{(k)})_{1,N-k+1} = -1$, and $(R_N^{(k)})_{ij} = 0$ otherwise. We then define a matrix $B_N^{(k)}$ by

$$B_N^{(k)} = \begin{cases} \sum_{i=0}^{N-1} \tau^i R_N^{(k)} \tau^{-i} & \text{if } N = 2r+1 \text{ and } 1 \leq k \leq r, \text{ or if } N = 2r \text{ and } 1 \leq k \leq r-1 \\ \sum_{i=0}^{r-1} \tau^i R_N^{(r)} \tau^{-i} & \text{if } k = r \text{ and } N = 2r. \end{cases}$$

We call the quiver $P_N^{(k)}$ associated to the matrix $B_N^{(k)}$ a period 1 primitive. It is shown in [3] that the period 1 primitives are in fact period 1 quivers, and it is also shown that all positive linear combinations of the period 1 primitives on N nodes are period 1. In fact, these positive linear combinations classify the sink-type period 1 primitives. Recall that a node j is called a sink if all arrows incident to j end at j.

Definition 3.1. *A quiver is said to be period m sink-type if it is period m and if the node k of $Q(k)$ is a sink for $1 \leq k \leq m$.*

Proposition 3.2. *([3, Proposition 3.6]) Let $N = 2r$ or $2r+1$, where r is an integer. Every period 1 sink-type quiver with N nodes has a corresponding matrix of the form $B = \sum_{k=1}^{r} m_k B_N^{(k)}$, where the m_k are arbitrary nonnegative integers.*



(a) $P_5^{(1)}$ (b) $P_5^{(2)}$

Figure 2: Period 1 primitives on 5 vertices

Fordy and Marsh defined in [3] period 2 primitives by "splitting" period 1 primitives and used them to classify the whole family of strictly period 2 sink-type quivers. (It was shown in [3, Section 4] that there are neither period 2 sink-type quivers nor strictly period 2 primitives on an odd number of vertices).

Definition 3.3. *([3]) For the matrices $R_N^{(k)}$, where $N = 2r$, $1 \le k \le r-1$, we define the primitive $P_{N,2}^{(k,1)}$ to be the quiver with matrix given by*

$$B_{N,2}^{(k,1)} = \sum_{i=0}^{r-1} \tau^{2i} R_N^{(k)} \tau^{-2i}. \tag{3.1}$$

In addition, if N is divisible by 4, we obtain the additional primitive $P_{N,2}^{(r,1)}$ with matrix given by

$$B_{N,2}^{(r,1)} = \sum_{i=0}^{r/2-1} \tau^{2i} R_N^{(r)} \tau^{-2i}. \tag{3.2}$$

Finally, we obtain primitives $P_{N,2}^{(k,2)}$ with matrix given by

$$B_{N,2}^{(k,2)} = \tau B_{N,2}^{(k,1)} \tau^{-1}. \tag{3.3}$$

Notice that geometrically, the primitive $P_{N,2}^{(k,1)}$ is obtained from the period 1 primitive $P_N^{(k)}$ by removing half of the arrows, the ones corresponding to odd powers of τ, and the removed arrows form $P_{N,2}^{(k,2)}$.

Proposition 3.4. *([3, Proposition 4.5]) If $N = 2r$ where r is an integer, then every period 2 sink-type quiver with N nodes has a matrix of the form*

$$B = \begin{cases} \sum_{k=1}^{r} \sum_{j=1}^{2} m_{k,j} B_{N,2}^{(k,j)} & \text{if } 4 \mid N \\ \sum_{k=1}^{r-1} \sum_{j=1}^{2} m_{k,j} B_{N,2}^{(k,j)} + m_{r,1} B_N^{(r)} & \text{if } 4 \nmid N \end{cases} \tag{3.4}$$

where the $m_{k,j}$ are non-negative integers so that if 4 divides N, there is at least one k, $1 \le k \le r$, so that $m_{k,1} \neq m_{k,2}$, and so that if 4 does not divide N, there is at least one k, $1 \le k \le r-1$ so that $m_{k,1} \neq m_{k,2}$.

Fordy and Marsh [3] do not classify general period 2 quiver in terms of their primitives. However, assuming certain symmetries, they classify the period 2 quivers on 5 nodes or less, and they are able to construct an infinite family of period 2 quivers on an arbitrary number of nodes.

4 Graph Symmetry

Definition 4.1. *Let Q be a quiver on N nodes and let B_Q be its corresponding matrix with (i,j) entry $b_{i,j}$ for $1 \le i,j \le N$. Q is graph symmetric if for all $1 \le i,j \le N$, $(B_Q)_{i,j} = (B_Q)_{N+1-j,N+1-i}$.*

Fordy and Marsh [3] introduce the idea of graph symmetry in Remark 7.2; stating that period 2 quivers on 4 nodes were all graph symmetric, and classifying all period 2 quivers on 5 nodes which satisfy graph symmetry. In fact, not all period 2 quivers on 4 nodes are graph symmetric. Fordy and Marsh's classification of period 2 quivers on 4 nodes, in Section 7.2 of their paper, imposed the condition that node 1 is not a sink; all these are graph symmetric. By dropping the non-sink condition, we provide a complete classification of period 2 quivers on 4 nodes, some of which are graph symmetric, some of which are not graph symmetric, in the following section. Furthermore, we are able to show that all period 2 quivers on 5 nodes are graph symmetric, hence the classification of period 2 graph symmetric quivers on 5 nodes in Section 7.3 of Fordy and Marsh [3] is a complete classification of period 2 quivers on 5 nodes. However, for $N \neq 2, 3, 5$ we found that period 2 quivers on N nodes were not necessarily graph symmetric. In Sections 4.2 and 4.4, we provide a construction of period 2 quivers on N nodes for $N \geq 4$ even and $N \geq 7$ odd, which are not graph symmetric. In addition, in Sections 4.3 and 4.5 we show some relationships between the period 2 property and graph symmetry.

In this section and those to follow we will find it useful to use an alternative formulation of the period 2 property given in Fordy and Marsh [3], namely that a quiver Q is period 2 if and only if $\mu_1\rho Q = \rho^{-1}\mu_1 Q$.
It will also prove useful to note that for a quiver Q on N nodes whose corresponding matrix B_Q has entries $b_{i,j}$, for $i > j$, (i,j) entries of $\mu_1\rho B_Q$ and $\rho^{-1}\mu_1 B_Q$ are:

$$(\mu_1\rho B_Q)_{i,j} = \begin{cases} b_{i-1,j-1} + \varepsilon(b_{N,j-1}, b_{N,i-1}), & \text{if } j \neq 1 \\ b_{N,i-1}, & \text{if } j = 1 \end{cases} \tag{4.1}$$

$$(\rho^{-1}\mu_1 B_Q)_{i,j} = \begin{cases} b_{i+1,j+1} + \varepsilon(b_{i+1,1}, b_{j+1,1}) = b_{i+1,j+1} + \varepsilon_{i,j}, & \text{if } i \neq N \\ b_{j+1,1}, & \text{if } i = N \end{cases} \tag{4.2}$$

where $\varepsilon(x,y) = \frac{1}{2}(x|y| - y|x|)$ and $\varepsilon_{i,j} = \varepsilon(m_i, m_j)$, where $m_j = b_{j+1,1}$ for $1 \leq j \leq N-1$ by the convention in [3].
Note that when checking that $\mu_1\rho Q = \rho^{-1}\mu_1 Q$ it is sufficient to just check the lower triangular portion of the matrices as they are skew-symmetric.

4.1 Classification of 4 Nodes

Given B_Q, a general $N \times N$ skew-symmetric matrix that represents the quiver Q, we impose the relation

$$\mu_1\rho(B_Q) = \rho^{-1}\mu_1(B_Q) \tag{4.3}$$

so that Q is period 2. Given that a quiver Q is period 2, it is determined by the arrows incident with vertex 1 and vertex N , that is, B_Q is determined by the entries in the first column and the last row, and the number of arrows going from vertex 2 to vertex 1 is equal to the number of arrows going from vertex N to vertex $N-1$, that is, $b_{21} = b_{N,N-1}$ in B_Q. Thus for a 4 node quiver we let

$$b_{i1} = m_{i-1} \text{ for } i = 2, \ldots, 4 \quad \text{and} \quad b_{42} = p_2.$$

By imposing the relation (4.3), we get the following four conditions

$$\varepsilon_{31} = 0 \quad , \quad m_3 = b_{32} + \varepsilon(p_2, m_1) \quad , \quad b_{32} + \varepsilon_{21} = m_3 \quad , \quad \varepsilon_{32} = \varepsilon(m_3, p_2).$$

We choose $m_1 \geq 0$. Since $\varepsilon_{31} = 0$, $m_3 \geq 0$ if $m_1 > 0$.

Remark 4.2. *In Section 7.2 [3], the authors claimed that in order to find non-sink-type period 2 quivers, they must require $m_2 < 0$ for node 1 not to be sink and they had the result that all non-sink-type period 2 quivers with 4 nodes satisfied graph symmetry. However, node 1 not being sink is not equivalent to $B(1)$ not being a sink type and there are non-sink-type period 2 quivers with 4 nodes that do not satisfy graph symmetry. We give a classification of all period 2 quivers with 4 nodes (including period 1 quivers and period 2 sink-type quivers[1]).*

[1] We do not restrict our attention to strictly period 2 quivers or non-sink-type quivers by excluding period 1 quivers and sink-type quivers in our classification as it might be easier to observe the general pattern this way.

Case 4.2.1 ($m_2 = p_2$). *In this case,* $b_{32} = m_3 + \varepsilon_{12}$ *and*

$$B(1) = \begin{pmatrix} 0 & -m_1 & -m_2 & -m_3 \\ m_1 & 0 & -m_3 - \varepsilon_{12} & -m_2 \\ m_2 & m_3 + \varepsilon_{12} & 0 & -m_1 \\ m_3 & m_2 & m_1 & 0 \end{pmatrix},$$

where $m_2 = p_2$ *and either* $m_1 > 0$ *and* $m_3 \geq 0$*, or* $m_1 = 0$*.*

Case 4.2.2 ($m_2 \neq p_2$ $m_1 > 0$). *Since* $\varepsilon_{21} = \varepsilon(p_2, m_1)$*, it must be that* $m_2, p_2 \geq 0$*. Hence,* $b_{32} = m_3 + \varepsilon_{12} = m_3$ *and*

$$B(1) = \begin{pmatrix} 0 & -m_1 & -m_2 & -m_3 \\ m_1 & 0 & -m_3 & -p_2 \\ m_2 & m_3 & 0 & -m_1 \\ m_3 & p_2 & m_1 & 0 \end{pmatrix},$$

where $m_2, p_2 \geq 0$*,* $m_1 > 0$ *and* $m_3 \geq 0$*. Notice that all the quivers classified in this case are not graph-symmetric.*

Case 4.2.3 ($m_2 \neq p_2$, $m_1 = 0$ and $m_3 > 0$). *Since* $\varepsilon_{32} = \varepsilon(m_3, p_2)$*, it must be that* $m_2, p_2 \geq 0$*. Since* $m_1 = 0$*,* $b_{32} = m_3 + \varepsilon_{12} = m_3$ *and*

$$B(1) = \begin{pmatrix} 0 & 0 & -m_2 & -m_3 \\ 0 & 0 & -m_3 & -p_2 \\ m_2 & m_3 & 0 & 0 \\ m_3 & p_2 & 0 & 0 \end{pmatrix},$$

where $m_2, p_2 \geq 0$ *and* $m_3 > 0$*.*

Case 4.2.4 ($m_2 \neq p_2$, $m_1 = 0$ and $m_3 < 0$). *Since* $\varepsilon_{32} = \varepsilon(m_3, p_2)$*, it must be that* $m_2, p_2 \leq 0$*.* $b_{32} = m_3 + \varepsilon_{12} = m_3$ *and*

$$B(1) = \begin{pmatrix} 0 & 0 & -m_2 & -m_3 \\ 0 & 0 & -m_3 & -p_2 \\ m_2 & m_3 & 0 & 0 \\ m_3 & p_2 & 0 & 0 \end{pmatrix},$$

where $m_2, p_2 \leq 0$ *and* $m_3 < 0$*.*

Case 4.2.5 ($m_2 \neq p_2$, $m_1 = 0$ and $m_3 = 0$). *There are no restrictions on the sign of* m_2 *and* p_2*, and* $b_{32} = m_3 = 0$*, giving*

$$B(1) = \begin{pmatrix} 0 & 0 & -m_2 & 0 \\ 0 & 0 & 0 & -p_2 \\ m_2 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \end{pmatrix},$$

where $m_2 \neq p_2$*.*

Note that from the classification, sink-type quivers are not all graph-symmetric and there are non-sink-type quivers that do not satisfy graph symmetry.

4.2 Non-Graph-Symmetric Sink-type Quivers with Even Number of Nodes

Given N even, the period 2 primitives $P_{N,2}^{(k,1)}$ and $P_{N,2}^{(k,2)}$ do not have graph symmetry for certain values of k.

Proposition 4.3. *Given $N = 2r$, the period 2 primitives $P_{N,2}^{(k,1)}$ and $P_{N,2}^{(k,2)}$ are not graph-symmetric if k is even, i.e., if $k = 2, 4, \ldots, 2\lfloor \frac{r}{2} \rfloor$. More generally, for a sink-type quiver Q of the form specified in Equation (3.4), if there exist a k even (for $1 \leq k \leq r$ if $4 \mid N$ and $1 \leq k \leq r-1$ if $4 \nmid N$) such that $m_{k,1} \neq m_{k,2}$, then Q does not have graph symmetry.*

Proof. Given $N = 2r$, when $k \leq r-1$ and even, the $(N-k+1, 1)$ entry of $B_{N,2}^{(k,1)}$ is 1 which corresponds to the term $R_N^{(k)}$ in the sum by Equation (3.1). The (N, k) entry of $B_{N,2}^{(k,2)}$ is 1 which corresponds to the term $\tau^{k-1} R_N^{(k)} \tau^{1-k}$ in the sum by Equation (3.3). Since $B_{N,2}^{(k,1)}$ and $B_{N,2}^{(k,2)}$ add up to $B_N^{(k)}$ whose entries are at most 1, the (N, k) entry of $B_{N,2}^{(k,1)}$ is 0 which is not equal to its $(N-k+1, 1)$ entry and thus breaks the symmetry. Analogous argument shows that $B_{N,2}^{(k,2)}$ is not graph-symmetric. In addition, if N is divisible by 4, when $k = r$ the $(r+1, 1)$ entry of $B_{N,2}^{(r,1)}$ is 1 which corresponds to the term $R_N^{(r)}$ in the sum by Equation (3.2) and the (N, r) entry of $B_{N,2}^{(r,2)}$ is 1 which corresponds to the term $\tau^{r-1} R_N^{(r)} \tau^{1-r}$ in the sum by Equation (3.3). This implies that $B_{N,2}^{(r,1)}$ and $B_{N,2}^{(r,2)}$ do not satisfy graph symmetry.

If B_Q has the form described in Equation (3.4), whenever $m_{k1} \neq m_{k2}$ for $1 \leq k \leq r$ if $4 \mid N$ and $1 \leq k \leq r-1$ if $4 \nmid N$, $b_{N-k+1,1} = m_{k1}$ and $b_{N,k} = m_{k2} \neq b_{N-k+1,1}$ which breaks the graph symmetry. $\square$

4.3 5 Nodes

In [3], Fordy and Marsh classify period 2 quivers on 5 nodes by imposing graph symmetry. We confirm that their classification is complete by showing that all period 2 quivers on 5 nodes are indeed graph symmetric.

Theorem 4.4. *All period 2 quiver on 5 nodes are graph symmetric.*

Proof. Consider a period 2 quiver Q on 5 nodes and let the (i, j) entry of the matrix associated with this quiver be $b_{i,j}$ for $1 \leq i, j \leq 5$. The lower triangular portion of the matrix is:

$$\begin{array}{ccccc} 0 & & & & \\ b_{2,1} & 0 & & & \\ b_{3,1} & b_{3,2} & 0 & & \\ b_{4,1} & b_{4,2} & b_{4,3} & 0 & \\ b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & 0. \end{array}$$

Since Q is period 2, $(\mu_1 \rho Q)_{i,j} = \left(\rho^{-1} \mu_1 Q\right)_{i,j}$ by (4.3). Following (4.1) and (4.2), we have
$i = 5, j = 1$ gives $b_{2,1} = b_{5,4}$.
$i = 5, j = 2$ and $i = 4, j = 1$ give $b_{4,1} = b_{3,1} + \varepsilon(b_{5,4}, b_{5,1})$ and $b_{5,3} + \varepsilon(b_{2,1}, b_{5,1}) = b_{5,2}$.
$i = 2, j = 1$ and $i = 5, j = 4$ give $b_{5,1} + \varepsilon(b_{2,1}, b_{3,1}) = b_{3,2}$ and $b_{4,3} = b_{5,1} + \varepsilon(b_{5,4}, b_{5,3})$.
We show $b_{3,1} = b_{5,3}$ as this would imply $b_{4,1} = b_{5,2}$ and $b_{3,2} = b_{4,3}$.
$i = 4, j = 2$ gives $b_{3,1} + \varepsilon(b_{5,1}, b_{5,3}) = b_{5,3} + \varepsilon(b_{5,1}, b_{3,1})$.
Without loss of generality let $b_{5,1} \geq 0$. Note that then $\varepsilon(b_{5,1}, x) \geq 0$ for all real numbers x and equality holds if $x \geq 0$.

Case 4.4.1 ($b_{3,1} \geq 0$)**.** *In this case, since* $\varepsilon(b_{5,1}, b_{5,3}) \geq 0$ *and* $\varepsilon(b_{5,1}, b_{3,1}) = 0$*,* $b_{3,1} \leq b_{3,1} + \varepsilon(b_{5,1}, b_{5,3}) = b_{5,3}$*. Hence* $b_{5,3} \geq 0$ *and* $b_{3,1} + \varepsilon(b_{5,1}, b_{5,3}) = b_{5,3}$ *reduces to* $b_{3,1} = b_{5,3}$*.*

Similarly the case $b_{5,3} \geq 0$ leads to $b_{3,1} = b_{5,3}$.

Case 4.4.2 ($b_{3,1}, b_{5,3} < 0$)**.** *In this case* $b_{3,1} - b_{5,1}b_{5,3} = b_{5,3} - b_{5,1}b_{3,1}$ *hence* $(b_{3,1} - b_{5,3})(b_{5,1}+1) = 0$*. Since* $b_{5,1} \geq 0$ *we again have* $b_{5,3} = b_{3,1}$ *and the quiver is symmetric.*

□

4.4 Breaking Graph Symmetry for N odd, $N > 5$

For $N \geq 7$, N odd, there are infinitely many period 2 quivers on N nodes which are not graph symmetric.

Definition 4.5. *Let* $N = 2k+1, k \geq 3$*. We define* F_N *to be the quiver on* N *nodes associated with matrix* B_{F_N} *whose* (i,j) *entry for* $i > j$ *is as follows:*

$$(B_{F_N})_{i,j} = \begin{cases} -1, & \text{if } i = j+3, j = 2w-1, 1 \leq w \leq k-1 \\ 1, & \text{if } i = j+1, j = 2w+1, 1 \leq w \leq k-1 \\ -1, & \text{if } i = N, j = 2 \text{ or } i = N, j = N-2 \text{ or } i = N-2, j = 1 \\ 1, & \text{if } i = N-2, j = 2 \\ 0, & \text{else.} \end{cases}$$

Theorem 4.6. *Let* $N \geq 7$ *be odd and* n *be non-negative integer. Then* $B_N^{(1)} + nB_{F_N}$ *is the matrix corresponding to a period 2 quiver.*

This is a subset of $\mathcal{F}_{S_2}$ defined in Section 6.2 so the quivers of this form are indeed period 2.

4.5 Sufficient Condition for Graph Symmetry of a period 2 quiver

Theorem 4.7. *Let* Q *be a 2-periodic quiver on* N *nodes and let the* (i,j) *entries of the corresponding matrix be* $b_{i,j}$ *for* $1 \leq i,j \leq N$*. If*

$$b_{i,1} = b_{N,N+1-i} \text{ for all } 2 \leq i \leq N, \tag{4.4}$$

then the quiver is graph symmetric.

Proof. It suffices to prove $b_{i,j} = b_{N+1-j,N+1-i}$ for $i > j$ since the matrix of the quiver is skew-symmetric.
If $i = N, j = k$ and $i = N-k+1, j = 1$, for $2 \leq k \leq N-1$, $(\mu_1\rho Q)_{i,j} = \left(\rho^{-1}\mu_1 Q\right)_{i,j}$ and (4.4) give us:

$$b_{N-1,k-1} = b_{k+1,1} + \varepsilon(b_{N,N-1}, b_{N,k-1}) = b_{N,N-k} + \varepsilon(b_{2,1}, b_{N-k+2,1}) = b_{N-k+2,2}. \tag{4.5}$$

In addition ,

$$b_{i+1,j+1} - b_{i-1,j-1} = \varepsilon(b_{j+1,1}, b_{i+1,1}) + \varepsilon(b_{N,j-1}, b_{N,i-1}).$$

By (4.3),

$$= \varepsilon(b_{N,N-j}, b_{N,N-i}) + \varepsilon(b_{N-j+2,1}, b_{N-i+2,1}) = b_{N-j,N-i} - b_{N-j+2,N-i+2}$$

for $2 \leq j < i \leq N-1$.
Hence if $b_{i-1,j-1} = b_{N-j+2,N-i+2}$ then $b_{i+1,j+1} = b_{N-j,N-i}$.
By induction on k with base cases $b_{k+1,1} = b_{N,N-k}$ and $b_{k+2,2} = b_{N-1,N-k-1}$, which follows from (4.5), $1 \leq k \leq N-1$ and $1 \leq k \leq N-2$ respectively, we have that Q is graph symmetric. □

5 6 Node Quivers of Period 2 Classification

We did a case by case analysis for 6 node period 2 quivers (as was done in [3, Section 7.2]). As we shall see that for 6 node period 2 quivers, all the non-graph-symmetric quivers are integer linear combinations of period 2 primitives, i.e., there are no additional ε terms in the matrix that represents these quivers.

We again impose the relation

$$\mu_1\rho(B_Q) = \rho^{-1}\mu_1(B_Q)$$

and let

$$b_{i1} = m_{i-1} \text{ for } i = 2,\dots,6 \quad \text{and} \quad b_{6j} = p_{6-j} \text{ for } j = 2,\dots,4.$$

We get the following 14 conditions that must hold:

$$\begin{aligned}
m_5 &= b_{32} + \varepsilon_{21}, &(5.1)\\
p_4 &= b_{42} + \varepsilon_{31}, &(5.2)\\
p_3 &= b_{52} + \varepsilon_{41}, &(5.3)\\
p_2 &= p_4 + \varepsilon_{51}, &(5.4)\\
m_1 + \varepsilon(m_5, p_4) &= b_{43} + \varepsilon_{32}, &(5.5)\\
m_2 + \varepsilon(m_5, p_3) &= b_{53} + \varepsilon_{42}, &(5.6)\\
m_3 + \varepsilon(m_5, p_2) &= p_3 + \varepsilon_{52}, &(5.7)\\
m_4 + \varepsilon_{51} &= m_2, &(5.8)\\
b_{32} + \varepsilon(p_4, p_3) &= b_{54} + \varepsilon_{43}, &(5.9)\\
b_{42} + \varepsilon(p_4, p_2) &= p_2 + \varepsilon_{53}, &(5.10)\\
b_{52} + \varepsilon(p_4, m_1) &= m_3, &(5.11)\\
b_{43} + \varepsilon(p_3, p_2) &= m_1 + \varepsilon_{54}, &(5.12)\\
b_{53} + \varepsilon(p_3, m_1) &= m_4, &(5.13)\\
b_{54} + \varepsilon(p_2, m_1) &= m_5. &(5.14)
\end{aligned}$$

We set $m_1 \geq 0$ and consider all the cases based on the signs of $m_2,\dots,m_5$. This is a natural way to classify all 6 node period 2 quivers since $\varepsilon(x,y)$ vanishes whenever x and y are of the same sign, simplifying the conditions above written in the most general form. For example, whenever $m_1m_5 \geq 0$, $m_2 = m_4$ and $p_4 = p_2$. The following seven cases contain all the 6 node period 2 quivers (including period 1 quivers and period 2 sink-type quivers).

5.1 The case $m_1 \geq 0$, $m_2 \geq 0$, $m_3 \geq 0$, $m_5 \geq 0$, $p_4 \geq 0$

$$B(1) = \begin{pmatrix} 0 & -m_1 & -m_2 & -m_3 & -m_2 & -m_5 \\ m_1 & 0 & -m_5 & -p_4 & -m_3 & -p_4 \\ m_2 & m_5 & 0 & -m_1 & -m_2 & -m_3 \\ m_3 & p_4 & m_1 & 0 & -m_5 & -p_4 \\ m_2 & m_3 & m_2 & m_5 & 0 & -m_1 \\ m_5 & p_4 & m_3 & p_4 & m_1 & 0 \end{pmatrix}.$$

Note that $B(1) = m_1B_{6,2}^{(1,1)} + m_5B_{6,2}^{(1,2)} + m_2B_{6,2}^{(2,1)} + p_4B_{6,2}^{(2,2)} + m_3B_6^{(3)}$ with all the coefficients nonnegative. By Proposition 3.4, this case is exactly the 6 node period 2 sink-type quivers.

5.2 The case $m_2 < 0$, $m_5 \le 0$

$$B(1) = \begin{pmatrix} 0 & 0 & -m_2 & 0 & -m_2 & -m_5 \\ 0 & 0 & -m_5 & -p_4 & 0 & -p_4 \\ m_2 & m_5 & 0 & 0 & -m_2 & 0 \\ 0 & p_4 & 0 & 0 & -m_5 & -p_4 \\ m_2 & 0 & m_2 & m_5 & 0 & 0 \\ m_5 & p_4 & 0 & p_4 & 0 & 0 \end{pmatrix}.$$

5.3 The case $m_2 \le 0, m_3 < 0$, $m_5 \le 0$, $p_4 \le 0$

$$B(1) = \begin{pmatrix} 0 & 0 & -m-2 & -m_3 & -m_2 & -m_5 \\ 0 & 0 & -m_5 & -p_4 & -m_3 & -p_4 \\ m_2 & m_5 & 0 & 0 & -m_2 & -m_3 \\ m_3 & p_4 & 0 & 0 & -m_5 & -p_4 \\ m_2 & m_3 & m_2 & m_5 & 0 & 0 \\ m_5 & p_4 & m_3 & p_4 & 0 & 0 \end{pmatrix}.$$

5.4 The case $m_2 \ge 0$, $m_3 < 0$, $m_5 < 0$

$$B(1) = \begin{pmatrix} 0 & 0 & -m_2 & -m_3 & -m_2 & -m_5 \\ 0 & 0 & -m_5 & -m_2 & -m_3 & -m_2 \\ m_2 & m_5 & 0 & -m_2(m_5-m_3) & -m_2 & -m_3 \\ m_3 & m_2 & m_2(m_5-m_3) & 0 & -m_5 & -m_2 \\ m_2 & m_3 & m_2 & m_5 & 0 & 0 \\ m_5 & m_2 & m_3 & m_2 & 0 & 0 \end{pmatrix}.$$

5.5 The case $m_1 \ge 0$, $m_2 \ge 0$, $m_3 < 0$

$$B(1) = \begin{pmatrix} 0 & -m_1 & -m_2 & -m_3 & -m_2 & -m_1 \\ m_1 & 0 & -m_1 & -m_2+m_1m_3 & -m_3 & -m_2 \\ m_2 & m_1 & 0 & -m_1+m_2m_3 & -m_2+m_1m_3 & -m_3 \\ m_3 & m_2-m_1m_3 & m_1-m_2m_3 & 0 & -m_1 & -m_2 \\ m_2 & m_3 & m_2-m_1m_3 & m_1 & 0 & -m_1 \\ m_1 & m_2 & m_3 & m_2 & m_1 & 0 \end{pmatrix}.$$

Notice that all the quivers covered in this case are period 1 and not period 2.

5.6 The case $m_1 \ge 0$, $m_2 < 0$, $m_5 \ge 0$

$$B(1) = \begin{pmatrix} 0 & -m_1 & -m_2 & 0 & -m_2 & 0 \\ m_1 & 0 & -m_5+m_1m_2 & -m_2 & -m_5+m_1m_2 & -m_2 \\ m_2 & m_5-m_1m_2 & 0 & -m_1+m_2m_5 & -m_2 & 0 \\ 0 & m_2 & m_1-m_2m_5 & 0 & -m_5+m_1m_2 & -m_2 \\ m_2 & -m_1m_2 & m_2 & m_5-m_1m_2 & 0 & -m_1 \\ m_5 & m_2 & 0 & m_2 & m_1 & 0 \end{pmatrix}.$$

5.7 The case $m_1 \ge 0$, $m_2 < 0$, $m_3 \ge 0$, $m_5 \ge 0$

$$B(1) = \begin{pmatrix} 0 & -m_1 & -m_2 & -m_3 & -m_2 & -m_5 \\ m_1 & 0 & -m_5+m_1m_2 & -m_2 & -m_3+m_1m_2 & -m_2 \\ m_2 & m_5-m_1m_2 & 0 & -m_1-m_2(m_3-m_5) & -m_2 & -m_3 \\ m_3 & m_2 & m_1+m_2(m_3-m_5) & 0 & -m_5+m_1m_2 & -m_2 \\ m_2 & m_3-m_1m_2 & m_2 & m_5-m_1m_2 & 0 & -m_1 \\ m_5 & m_2 & m_3 & m_2 & m_1 & 0 \end{pmatrix}.$$

6 Infinite Families

6.1 Family $\mathcal{F}_{S_1}$

We define four types of quivers on odd number of nodes which we will utilize.

Definition 6.1. [2] *Let $N = 2k+1, k \geq 3$ be the number of nodes of the quiver. The (i,j), $i > j$ entries of matrices for the quivers A_N,C_N,D_N,E_N are as follows:*

$$(B_{A_N})_{i,j} = \begin{cases} 1, & \text{if } i = j+1, 1 \leq j \leq N-1 \\ -1, & \text{if } i = j+2, 1 \leq j \leq N-2 \\ -1, & \text{if } i = N-1, j = 1 \text{ or } i = N, j = 2 \\ 1, & \text{if } i = N-1, j = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$(B_{C_N})_{i,j} = \begin{cases} 1, & \text{if } i = j+1, 2 \leq j \leq N-1 \\ 1, & \text{if } i = N, j = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$(B_{D_N})_{i,j} = \begin{cases} -1, & \text{if } i = 2w_1+1, j = 2w_2-1, 1 \leq w_2 < w_1 \leq k, (i,j) \neq (N,1) \\ 0, & \text{otherwise} \end{cases}$$

$$(B_{E_N})_{i,j} = \begin{cases} 1, & \text{if } i = 2w_1+1, j = 2w_2-1, 2 \leq w_2 \leq w_1 \leq k-1 \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 6.2. *Let n and m be non-negative integers. For N odd and greater or equal to 7, the quiver corresponding to the matrix $B_{A_N} + nB_{C_N} + mB_{D_N} + nmB_{E_N}$ is period 2.*

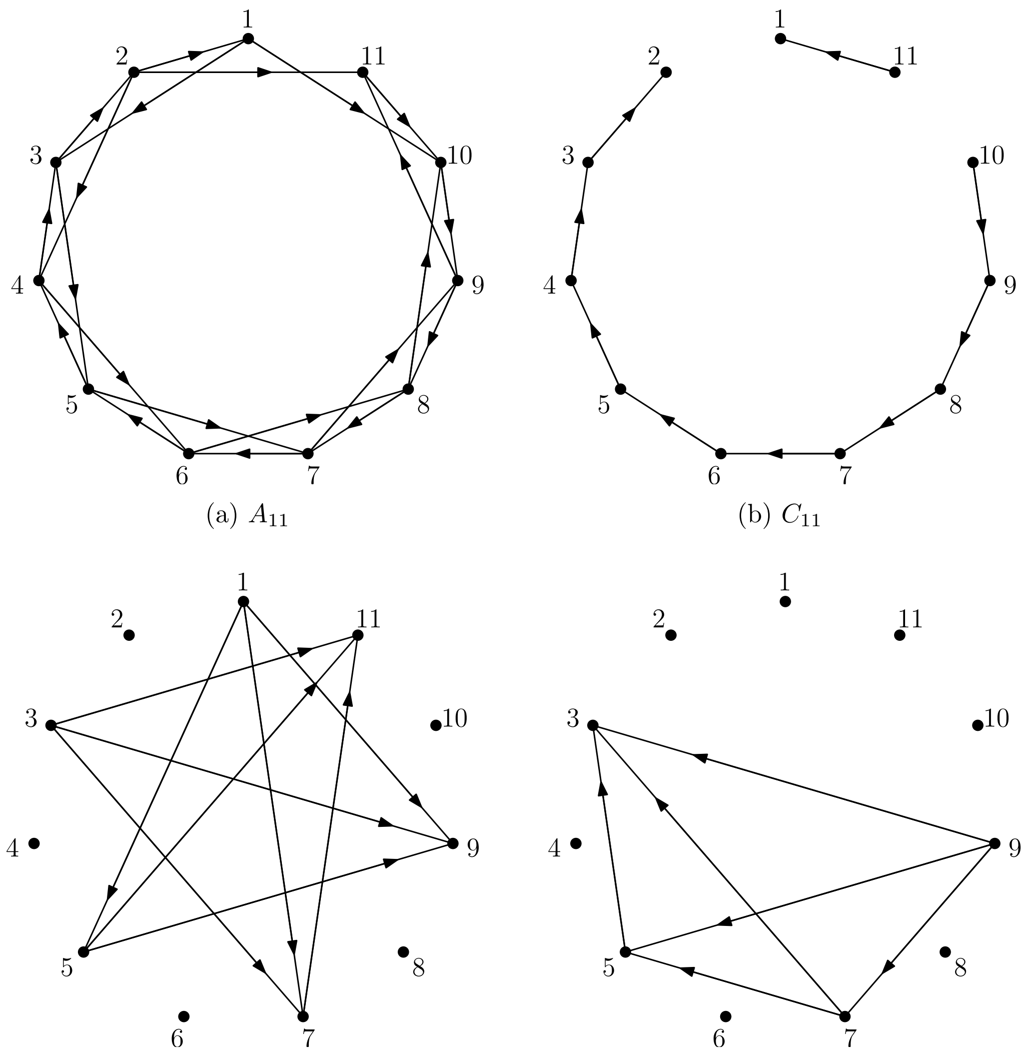
Remark 6.3. *Note that in the 5 node case that A_5 and C_5 make sense and the resulting quiver when a non-negative number of copies of C_5 are added to A_5 is period 2.*

Remark 6.4. *If $m = 0$ and $n \neq 1$, then the resulting matrix falls within the family $\mathcal{F}$ in [3, Section 7.2] which will be discussed later on.*

Proof. The lower triangle portion of the matrix corresponding to the resulting quiver Q is:

$$\begin{array}{cccccccccccccc}
0 \\
1 & 0 \\
-1 & 1+n & 0 \\
0 & -1 & 1+n & 0 \\
-m & 0 & nm-1 & 1+n & 0 \\
0 & 0 & 0 & -1 & 1+n & 0 \\
-m & 0 & nm-m & 0 & nm-1 & 1+n & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1+n & 0 \\
-m & 0 & nm-m & 0 & nm-m & 0 & nm-1 & 1+n & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 1+n & 0 \\
-m & 0 & nm-m & 0 & nm-m & 0 & nm-m & 0 & \dots & 0 & nm-1 & 1+n & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 1+n & 0 \\
n & -1 & -m & 0 & -m & 0 & -m & 0 & \dots & 0 & -m & 0 & -1 & 1 & 0.
\end{array}$$

[2] We do not define a quiver B_N as we reserve subscripts of B to denote matrices.



(a) A_{11} (b) C_{11} (c) D_{11} (d) E_{11}

Figure 3: Quivers on 11 nodes

We now show the (i,j) entries of $\mu_1\rho B_Q$ and $\rho^{-1}\mu_1 B_Q$ are equal for $1 \le j < i \le N$ by using relations (1) and (2). Let the $b_{i,j}$ be the (i,j) entry of B_Q.

Case 6.4.1 $(3 \le j < i \le N-2)$**.** *In this case* $\varepsilon(b_{N,j-1}, b_{N,i-1}) = \varepsilon(b_{i+1,1}, b_{j+1,1}) = 0$*, as* $b_{N,j-1}, b_{N,i-1}, b_{i+1,1}$*, and* $b_{j+1,1}$ *are all nonpositive, so relations (1) and (2) reduce to* $b_{i+1,j+1} = b_{i-1,j-1}$ *which is true for all* $3 \le j < i \le N-2$.

Note that Q is graph symmetric so $(\rho^{-1}\mu_1 B_Q)_{N-j+1,N-i+1} = (\mu_1\rho B_Q)_{i,j}$ for all $1 \le j < i \le N$. Hence $(\rho^{-1}\mu_1 B_Q)_{i,j} = (\mu_1\rho B_Q)_{i,j}$ if and only if $(\rho^{-1}\mu_1 Q)_{N-j+1,N-i+1} = (\mu_1\rho B_Q)_{N-j+1,N-i+1}$. It now suffices to check $(\rho^{-1}\mu_1 B_Q)_{i,j} = (\mu_1\rho B_Q)_{i,j}$ for $i = 2, N-1$, and N; the cases $j = N-1, 2, 1$ follow directly.

Case 6.4.2 $(i = N-1)$**.** *If* $j = 1$ *then* $b_{N,2} + \varepsilon(b_{N,1}, b_{2,1}) = -1 + \varepsilon(n, 1) = -1 = b_{N,N-2}$ *as desired. If* $j = N-3$ *then* $b_{N,N-2} + \varepsilon(b_{N,1}, b_{N-2,1}) = -1 + \varepsilon(n, -m) = -1 + nm = nm - 1 + \varepsilon(-m, -1) = b_{N-2,N-4} + \varepsilon(b_{N,N-4}, b_{N,N-2})$.
If $j = N-2$ *then* $b_{N,N-1} + \varepsilon(b_{N,1}, b_{N-1,1}) = 1 + \varepsilon(n, -1) = 1 + n = n + 1 + \varepsilon(0, -1) = b_{N-2,N-3} +$

$\varepsilon(b_{N,N-3}, b_{N,N-2})$.
If $1 < j < N-3$ and j odd then $b_{N,j+1} + \varepsilon(b_{N,1}, b_{j+1,1}) = 0 + \varepsilon(n, 0) = 0 = 0 + \varepsilon(b_{N,j-1}, -1) = b_{N-2,j-1} + \varepsilon(b_{N,j-1}, b_{N,N-2})$.
If $j = 2$ then $b_{N,3} + \varepsilon(b_{N,1}, b_{3,1}) = -m + \varepsilon(n, -1) = b_{N-2,1} + \varepsilon(b_{N,1}, b_{N,N-2})$.
If $2 < j < N-3$ and j even then $b_{N,j+1} + \varepsilon(b_{N,1}, b_{j+1,1}) = -m + \varepsilon(n, -m) = -m + nm = nm - m + \varepsilon(-m, -1) = b_{N-2,j-1} + \varepsilon(b_{N,j-1}, b_{N,N-2})$.

Case 6.4.3 $(i = N)$**.** *If $j = 1$ then $b_{2,1} = 1 = b_{N,N-1}$ as desired.*
If $j = 2$ then $b_{N-1,1} + \varepsilon(b_{N,1}, b_{N,N-1}) = -1 + \varepsilon(n, 1) = -1 = b_{3,1}$.
If $j = 3$ then $b_{N-1,2} + \varepsilon(b_{N,2}, b_{N,N-1}) = 1 + \varepsilon(-1, 1) = 0 = b_{4,1}$.
If $j = N-2$ then $b_{N-1,N-3} + \varepsilon(b_{N,N-3}, b_{N,N-1}) = -1 + \varepsilon(0, 1) = -1 = b_{N-1,1}$.
If $j = N-1$ then $b_{N-1,N-2} + \varepsilon(b_{N,N-2}, b_{N,N-1}) = 1 + n + \varepsilon(-1, 1) = n = b_{N,1}$.
If $3 < j < N-2$ then $b_{N-1,j-1} + \varepsilon(b_{N,j-1}, b_{N,N-1}) = \varepsilon(b_{N,j-1}, 1) = b_{N,j-1} = b_{j+1,1}$.

Case 6.4.4 $(i = 2)$**.** *Since $i > j$ we have to only check the case $i = 2, j = 1$.*
$b_{3,2} + \varepsilon(b_{3,1}, b_{2,1}) = 1 + n + \varepsilon(-1, 1) = n = b_{N,1}$ as desired.

□

6.2 Family $\mathcal{F}_{S_2}$

Definition 6.5. *. Let $N \geq 5$ be odd and integers a_i, t_i, c_i, g_i satisfy*

- $a_1, t_1 \leq 0$
- $a_{\lfloor \frac{N}{4} \rfloor} = t_{\lfloor \frac{N}{4} \rfloor}$ *if* $N \equiv 1 \bmod 4$
- $g_i + a_j, g_i + t_j, c_i + a_j, c_i + t_j \leq 0$ *for* $j = i, i+1$, *for all* $1 \leq i \leq \lfloor \frac{N}{4} \rfloor$

The first and last condition ensure that entries in the first column, except first two entries and last entry, and last row, except the first entry and last two entries, are non-positive.
We define the (i, j) entries, $i > j$, of a quiver on N nodes,where N is odd, as follows:

$$\underset{2 \leq m \leq \frac{N+1}{2}}{b_{m,1}} = \begin{cases} 1 & \text{if } m = 2 \\ a_1 & \text{if } m = 3 \\ g_n + t_n & \text{if } m = 2n+2, n \geq 1 \\ c_n + a_{n+1} & \text{if } m = 2n+3, n \geq 1 \end{cases}$$

$$\underset{\frac{N+3}{2} \leq m \leq N}{b_{m,1}} = \begin{cases} 1 & \text{if } m = N \\ a_1 & \text{if } m = N-1 \\ c_n + t_n & \text{if } m = N+2-(2n+2), n \geq 1 \\ g_n + a_{n+1} & \text{if } m = N+2-(2n+3), n \geq 1 \end{cases}$$

$$\underset{1 \leq m \leq \frac{N-1}{2}}{b_{N,m}} = \begin{cases} 1 & \text{if } m = 1 \\ t_1 & \text{if } m = 2 \\ c_n + a_n & \text{if } m = 2n+1, n \geq 1 \\ g_n + t_{n+1} & \text{if } m = 2n+2, n \geq 1 \end{cases}$$

$$\underset{\frac{N+1}{2}\leq m\leq N-1}{b_{N,m}} = \begin{cases} 1 & \textit{if } m = N-1 \\ t_1 & \textit{if } m = N-2 \\ g_n + a_n & \textit{if } m = N-(2n+1), n \geq 1 \\ c_n + t_{n+1} & \textit{if } m = N-(2n+2), n \geq 1. \end{cases}$$

Define

$$\begin{aligned} b_{m,2} &= b_{N,m-2} - b_{m,1} \textit{ for } 3 \leq m \leq N-1 \\ b_{m,3} &= b_{m-2,1} - b_{N,m-2} \textit{ for } 4 \leq m \leq N-1 \\ b_{i+1,j+1} &= b_{i-1,j-1} \textit{ for } 3 \leq j < i \leq N-2. \end{aligned} \tag{6.1}$$

The (i,j) entries for $i=j$ are 0, and for $j>i$, $b_{i,j} = -b_{N+1-i,N+1-j}$.

Remark 6.6. *Note that relations (6.1) can be replaced with*

$$\begin{aligned} b_{N-1,m} &= b_{m+2,1} - b_{N,m} \textit{ for } 2 \leq m \leq N-2 \\ b_{N-2,m} &= b_{N,m+2} - b_{m+2,1} \textit{ for } 2 \leq m \leq N-3 \\ b_{i+1,j+1} &= b_{i-1,j-1} \textit{ for } 3 \leq j < i \leq N-2. \end{aligned} \tag{6.2}$$

(6.2), along with defintions of $b_{i,1}$, for $2 \leq i \leq N$, and $b_{N,j}$ for $1 \leq j \leq N-1$ define the same quiver Q.

The lower triangle portions of the matrices on 11 and 13 nodes are shown below, with $c_0 = g_0 = 0, a_0 = t_0 = 1$.

$$\begin{array}{ccccccccccc}
0 \\
1=g_0+t_0 & 0 \\
c_0+a_1 & a_0-a_1 & 0 \\
g_1+t_1 & g_0-g_1 & t_0-t_1 & 0 \\
c_1+a_2 & a_1-a_2 & c_0-c_1 & a_0-a_1 & 0 \\
g_2+t_2 & g_1-g_2 & t_1-t_2 & g_0-g_1 & t_0-t_1 & 0 \\
c_2+t_2 & a_2-t_2 & c_1-c_2 & a_1-a_2 & c_0-c_1 & a_0-a_1 & 0 \\
g_1+a_2 & g_2-g_1 & t_2-a_2 & g_1-g_2 & t_1-t_2 & g_0-g_1 & t_0-t_1 & 0 \\
c_1+t_1 & t_2-t_1 & c_2-c_1 & a_2-t_2 & c_1-c_2 & a_1-a_2 & c_0-c_1 & a_0-a_1 & 0 \\
g_0+a_1 & g_1-g_0 & a_2-a_1 & g_2-g_1 & t_2-a_2 & g_1-g_2 & t_1-t_2 & g_0-g_1 & t_0-t_1 & 0 \\
1 & g_0+t_1 & c_1+a_1 & g_1+t_2 & c_2+a_2 & g_2+a_2 & c_1+t_2 & g_1+a_1 & c_0+t_1 & 1=g_0+a_0 & 0
\end{array}$$

$$\begin{array}{ccccccccccccc}
0 \\
1=g_0+t_0 & 0 \\
c_0+a_1 & a_0-a_1 & 0 \\
g_1+t_1 & g_0-g_1 & t_0-t_1 & 0 \\
c_1+a_2 & a_1-a_2 & c_0-c_1 & a_0-a_1 & 0 \\
g_2+t_2 & g_1-g_2 & t_1-t_2 & g_0-g_1 & t_0-t_1 & 0 \\
c_2+a_3 & a_2-a_3 & c_1-c_2 & a_1-a_2 & c_0-c_1 & a_0-a_1 & 0 \\
g_2+a_3 & 0 & t_2-t_3 & g_1-g_2 & t_1-t_2 & g_0-g_1 & t_0-t_1 & 0 \\
c_2+t_2 & t_3-t_2 & 0 & a_2-a_3 & c_1-c_2 & a_1-a_2 & c_0-c_1 & a_0-a_1 & 0 \\
g_1+a_2 & g_2-g_1 & a_3-a_2 & 0 & t_2-t_3 & g_1-g_2 & t_1-t_2 & g_0-g_1 & t_0-t_1 & 0 \\
c_1+t_1 & t_2-t_1 & c_2-c_1 & t_3-t_2 & 0 & a_2-a_3 & c_1-c_2 & a_1-a_2 & c_0-c_1 & a_0-a_1 & 0 \\
g_0+a_1 & g_1-g_0 & a_2-a_1 & g_2-g_1 & a_3-a_2 & 0 & t_2-t_3 & g_1-g_2 & t_1-t_2 & g_0-g_1 & t_0-t_1 & 0 \\
1 & g_0+t_1 & c_1+a_1 & g_1+t_2 & c_2+a_2 & g_2+t_3 & c_2+t_3 & g_2+a_2 & c_1+t_2 & g_1+a_1 & c_0+t_1 & 1=g_0+a_0 & 0
\end{array}$$

Theorem 6.7. *$\mathcal{F}_{S_2}$ is a family of period 2 quivers.*

Proof. Let Q be a quiver of this family. We claim that $\mu_1\rho$ swaps a_k with t_k and g_k with c_k for each k in B_Q, henceforth known as the swapping property.

The lower triangular portion of $\mu_1\rho B_Q$ for Q on 9 nodes, with $c_0 = g_0 = 0$ and $a_0 = t_0 = 1$, is as follows:

$$\begin{array}{ccccccccccc}
0 \\
1 = c_0 + a_0 & 0 \\
g_0 + t_1 & t_0 - t_1 & 0 \\
c_1 + a_1 & c_0 - c_1 & a_0 - a_1 & 0 \\
g_1 + t_2 & t_1 - t_2 & g_0 - g_1 & t_0 - t_1 & 0 \\
c_2 + a_2 & c_1 - c_2 & a_1 - a_2 & c_0 - c_1 & a_0 - a_1 & 0 \\
g_2 + a_2 & t_2 - a_2 & c_1 - c_2 & t_1 - t_2 & g_0 - g_1 & t_0 - t_1 & 0 \\
c_1 + t_2 & c_2 - c_1 & a_2 - t_2 & g_1 - g_2 & a_1 - a_2 & c_0 - c_1 & a_0 - a_1 & 0 \\
g_1 + a_1 & a_2 - a_1 & g_2 - g_1 & a_2 - t_2 & c_1 - c_2 & t_1 - t_2 & g_0 - g_1 & t_0 - t_1 & 0 \\
c_0 + t_1 & c_1 - c_0 & t_2 - t_1 & c_2 - c_1 & t_2 - a_2 & g_1 - g_2 & a_1 - a_2 & c_0 - c_1 & a_0 - a_1 & 0 \\
1 & c_0 + a_1 & g_1 + t_1 & c_1 + a_2 & g_2 + t_2 & c_2 + t_2 & g_1 + a_2 & c_1 + t_1 & g_0 + a_1 & 1 = c_0 + t_0 & 0.
\end{array}$$

Recall that for $i > j$,

$$(\mu_1\rho B_Q)_{i,j} = \begin{cases} b_{i-1,j-1} + \varepsilon(b_{N,j-1}, b_{N,i-1}), & \text{if } j \neq 1 \\ b_{N,i-1}, & \text{if } j = 1. \end{cases}$$

Denote $(\mu_1\rho B_Q)_{i,j}$ by $b'_{i,j}$ for each i, j. It proves convinent to think of $b_{N,1}$ as $c_0 + a_0$ and as $c_0 + t_0$. We will show that the swapping property is true for the last column and first row, and that relations (6.1) are satisfied by $\mu_1\rho B_Q$, hence the swapping property holds for the entire matrix.

Case 6.7.1 ($j = 1$)**.** *In this instance $b'_{i,1} = b_{N,i-1}$ which corresponds to swapping a_k with t_k and g_k with c_k for each k in $b_{i,1}$ to obtain $b'_{i,1}$.The first column of $\mu_1\rho B_Q$ satisfies the swapping property.*

Case 6.7.2 ($i = N$, $j > 2$)**.** *In this instance, $\varepsilon(b_{N,j-1}, b_{N,i-1}) = b_{N,j-1}$. Hence $b'_{N,j} = b_{N-1,j-1} + b_{N,j-1} = b_{j+1,1}$ by (6.2). This which equates to swapping a_k with t_k and g_k with c_k for each k in $b_{N,j}$ to obtain $b'_{N,j}$.*

Case 6.7.3 ($i = N$,$j = 2$)**.** *$b'_{N,2} = b_{N-1,1} + \varepsilon(b_{N,1}, b_{N,N-1}) = b_{N-1,1} = g_0 + a_1$ which equates to swapping a_1 with t_1 and g_0 with c_0, since $g_0 = c_0$. Note too that $b'_{N,2} = c_0 + a + 1 = b_{3,1}$.*

We have established that the first column and last row satisfy the swapping property and that

$$\begin{gathered} b'_{i,1} = b_{N,i-1} \text{ for all } 2 \leq i \leq N \\ b'_{N,j} = b_{j+1,1} \text{ for all } 1 \leq j \leq N-1. \end{gathered} \tag{6.3}$$

It now sufficies to show that the recursive relations (6.1) hold for the $b'_{i,j}$. Using (6.3) we obtain:
For $3 \leq m \leq N-1$:

$$b'_{m,2} = b_{m-1,1} + \varepsilon(b_{N,1}, b_{N,m-1}) = b_{m-1,1} - b_{N,m-1} = b'_{N,m-2} - b'_{m,1}.$$

For $4 \leq m \leq N-1$:

$$b'_{m,3} = b_{m-1,2} + \varepsilon(b_{N,2}, b_{N,m-1}) = b_{m-1,2} = b_{N,m-3} - b_{m-1,1} = b'_{m-2,1} - b'_{N,m-2}.$$

For $3 < j < i \leq N-2$:

$$b'_{i+1,j+1} = b_{i,j} + \varepsilon(b_{N,j}, b_{N,i}) = b_{i,j} = b_{i-2,j-2} + \varepsilon(b_{N,j-2}, b_{N,i-2}) = b'_{i-1,j-1}.$$

For $j = 3, i \leq N-2$:

$$b'_{i+1,4} = b_{i,3} = b_{i-2,1} - b_{N,i-2} = b_{i-1,2}.$$

Hence the recurrences are satisfied and B_Q has the swapping property. $\mu_1\rho$ is an involution and Q has period 2.

□

6.3 Other Family

In this section, we describe a family of 2-periodic quivers (with at least five vertices) which enjoys a particularly simple description in terms of period 1 primitives. This family can be seen as a subset of a family described in [3, Section 7.4], which we'll call $\mathcal{F}$; in particular, a subset $\mathcal{F}_D$ of nonnegative linear combinations of a specific set of quivers. Before continuing, we will briefly review Fordy and Marsh's family $\mathcal{F}$. It consists of quivers on N vertices with matrices which are functions of $N-1$ parameters m_r (which we write as a vector $\mathbf{m}$),

$$B(\mathbf{m}) := \begin{pmatrix} 0 & -m_1 & \cdots & -m_{N-1} \\ m_1 & 0 & & * \\ \vdots & & 0 & \\ m_{N-1} & * & & 0 \end{pmatrix}, \tag{6.4}$$

where the entries in the regions marked $*$ are functions $b_{ij}(\mathbf{m})$. To be in $\mathcal{F}$, these quivers must additionally satisfy $m_i = m_{N-i}$ for $2 \leq i \leq N-2$, and the following identity (which ensures 2-periodicity):

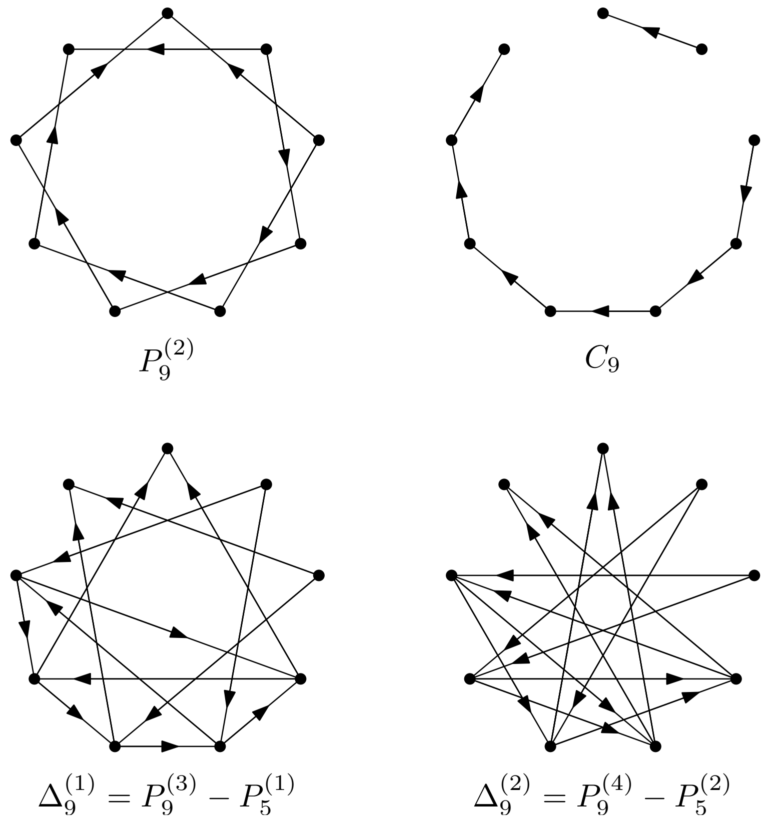
$$\rho^{-1}\mu_1\left(B(\mathbf{m})\right) = B\left(\sigma(\mathbf{m})\right), \tag{6.5}$$

where σ is the involution $(m_1, \ldots, m_{N-1}) \mapsto (m_{N-1}, \ldots, m_1)$. Because of the symmetry requirement for the m_i, σ can be considered as exchanging m_1 and m_{N-1}, so strictness of the 2-periodicity is equivalent to the condition $m_1 \neq m_{N-1}$. In their paper [3], Fordy and Marsh showed that membership in this family was equivalent to requiring

$$b_{ij} = \sigma^{j-1}(b_{i-j+1,1}) + \sum_{s=1}^{j-1} \sigma^{j-1-s}(\varepsilon_{s,i-j+s}), \tag{6.6}$$

where $\varepsilon_{x,y} = \varepsilon(m_x, m_y)$. It should be noted that σ acts formally on these terms as expressions in the m_i.

We now describe $\mathcal{F}_D$, in a way that is independent of the characterization of $\mathcal{F}$; then we will show that indeed $\mathcal{F}_D \subset \mathcal{F}$. On N vertices, $\mathcal{F}_D$ consists of quivers obtained by adding a nonnegative linear combination of a finite number of quivers which are not themselves 2-periodic to a single copy of a certain period 1 primitive. To describe them, we introduce the following notation: let $P_N^{(\ell)}$ denote the period 1 primitives on N vertices as in [3]. Then take $\Delta_N^{(\ell)} = P_N^{(\ell+2)} - P_{N-4}^{(\ell)}$, where the difference is the difference of matrices obtained by adding two rows and columns of zeros around the matrix for $P_{N-4}^{(\ell)}$. Note that these are defined for $\ell + 2 < N/2$. Furthermore, take C_N to be the



$P_9^{(2)}$ C_9

$\Delta_9^{(1)} = P_9^{(3)} - P_5^{(1)}$ $\Delta_9^{(2)} = P_9^{(4)} - P_5^{(2)}$

Figure 4: Examples of C_N and $\Delta_N^{(\ell)}$ for $N = 9$.

quiver obtained by removing the arrows $2 \to 1$ and $N \to N-1$ from $P_N^{(1)}$. For examples of these quivers on 9 vertices, see figure 4.
With this, we have the following proposition:

Proposition 6.8. *With $P_N^{(\ell)}$, $\Delta_N^{(\ell)}$ and C_N defined as above, we have that the quivers (which constitute $\mathcal{F}_D$) given by*

$$Q_D(k, c_1, \dots, c_L) := P_N^{(2)} + kC_N + \sum_{\ell=1}^{L} c_\ell \Delta_N^{(\ell)}, \tag{6.7}$$

for any nonnegative $c_1, \dots, c_L$ (where $L = \lfloor N/2 \rfloor - 2$) and $k \geq 1$, are in the family $\mathcal{F}$, and therefore strictly 2-periodic.

Proof. Our approach is to show that the formula (6.6) agrees with the entries of the matrix obtained from (6.7) when we choose the appropriate values for the m_j. To begin, we consider the first column of the matrix B_D of $Q_D(k, c_1, \dots, c_L)$. For $N = 2r$, it is given by $b_{11} = b_{21} = 0$, $b_{31} = -1$, $b_{41} = c_1$, $b_{51} = c_2$, $\dots$, $b_{r+1,1} = c_L$, $b_{r+2,1} = c_{L-1}$, $\dots$, $b_{2r-2,1} = c_1$, $b_{2r-1,1} = -1$, and $b_{2r,1} = k$. For $N = 2r+1$, the sequence is the same, except there is an extra c_L in the middle, so $b_{r+2,1} = c_L$, after which it continues as it did for the even N case. To show membership in $\mathcal{F}$, we therefore set $m_i = b_{i+1,1}$ for $j = 1, \dots, N-1$, and check the conditions as described above. First, we see that indeed $m_i = m_{N-i}$ for $2 \leq i \leq N-2$, and $m_1 \neq m_{N-1}$, since $k > 0$. All that remains is to check the formula (6.6).

Note that, due to the nonnegativity of the m_i except for m_2 and m_{N-2} (which are -1), we have that $\varepsilon_{x,y} = 0$ unless exactly one of x and y is equal to 2 or $N-2$. Now, we compute with the formula (6.6), considering different cases for the value of $i-j$.

Case 6.8.1. $i-j=1$.

In this case, for $j=2$, (6.6) gives $b_{32} = \sigma(m_1) + \varepsilon_{1,2} = m_{N-1} = k$ (we have $\varepsilon_{1,2} = 0$ because $m_1 = 0$), which is what we expect from (6.7). For $2 < j < N-2$ and j odd, the first term on the right side of (6.6) vanishes since $m_1 = 0$, and the sum over s contributes two terms; the $s=1$ term, given by $\varepsilon_{N-1,2} = k$ and the $s=2$ term, given by $\varepsilon_{2,3} = -c_1$, giving $b_{ij} = k - c_1$. On the other hand, for j even in this range, the first term contributes $\sigma(m_1) = m_{N-1} = k$, while the $s=1$ term no longer contributes, meaning we again have $b_{ij} = k - c_1$. These results both agree with (6.7). Finally, for $j = N-2$, the $\sigma^{j-1}(b_{i-j+1,1})$ contributes a $\sigma(m_1) = k$, while the $s=1$ term vanishes, and the $s=2$ and $s=N-3$ terms cancel, so we find $b_{N-1,N-2} = k$; for $j = N-1$, the first term contributes 0, and the $s=1$ term cancels with the $s=N-2$ term, while the $s=2$ term cancels with the $s=N-3$ term, so we find $b_{N,N-1} = 0$; both of these agree with (6.7).

Case 6.8.2. $1 < i-j = d < N-1$.

In this case, we never have $i-j+s = 2$ in the sum, and we also never have $s = N-2$, since $j \leq N-d$ and s is at most $j-1$. So the only terms that can possibly contribute are $s=2$ and $s = N-2-d$. Neither term appears for $j=2$, so we always have $b_{i2} = \sigma(b_{i-1,1}) = m_d$. For $2 < j < N-1-d$, only the $s=2$ term appears, which contributes $-m_{d+2}$, so here we have $b_{ij} = m_d - m_{d+2}$ (note that the σ's have no effect because $1 < d < N-1$). For $j = N-1-d$ and $N-d$, both nonzero terms in the sum appear, but cancel each other out, due to the fact that $m_i = m_{N-i}$, so we once again obtain $b_{ij} = m_d$. Note that for $d > N-4$, there is no region $2 < j < N-1-d$, so we only obtain the outer two regions, where $b_{ij} = m_d$. Again, all of these results agree with (6.7).

Case 6.8.3. $i-j = N-1$.

In this case, we are only considering the entry $b_{N,1}$, which the formula trivially gives to be $m_{N-1} = k$, again in agreement with (6.7).

The proof for odd N is almost identical, so it is omitted. So, we indeed have that the quivers $Q_D(k, c_1, \ldots, c_L)$ described are all in the family $\mathcal{F}$, and therefore strictly 2-periodic.

□

In [3], Fordy and Marsh additionally showed that a specification of $m_1, \ldots, m_{N-1}$ within the stated constraints ($m_i = m_{N-i}$ for $2 \leq i \leq N-2$) only yielded a quiver in $\mathcal{F}$ if the m_i also satisfied $m_i \geq 0$ for $2 < i < N-2$ with i odd, as well as $m_2 = -1$ for N odd. However, for N odd, the nonnegativity of m_i for odd i with $2 < i < N-2$ implies that, in fact, we have $m_i \geq 0$ for all i with $2 < i < N-2$, since we require $m_i = m_{N-i}$ and either i or $N-i$ will be odd for N odd. So, we obtain the following:

Remark 6.9. *For N odd, the quivers $Q_D(k, c_1, \ldots, c_L)$ and their images under the involution $\rho^{-1}\mu_1$ account for all of the quivers in $\mathcal{F}$ on N vertices with either $m_1 = 0$ or $m_{N-1} = 0$.*

Because the quivers in $\mathcal{F}_D$ are in $\mathcal{F}$, we can easily find the image $\rho^{-1}\mu_1 Q_D(k, c_1, \ldots, c_L)$ in general. This is because $Q_D(k, c_1, \ldots, c_L) = B(0, -1, c_1, \ldots, c_L, \ldots, c_1, -1, k)$, where B here is the function in (6.4) (again, there is an extra c_L when N is odd), and by definition, this satisfies $[\rho^{-1}\mu_1]B(0, -1, c_1, \ldots, c_L, \ldots, c_1, -1, k) = B(k, -1, c_1, \ldots, c_L, \ldots, c_1, -1, 0)$. Because the c_ℓ are not affected by this operation, we can immediately conclude that the image $\tilde{Q}_D(k, c_1, \ldots, c_L)$ will be of the form

$$\tilde{Q}_D(k, c_1, \ldots, c_L) = P_N^{(2)} + k\tilde{C}_N + \sum_{\ell=1}^{L} c_\ell \Delta_N^{(\ell)}$$

for some $\tilde{C}_N$. We can then find $\tilde{C}_N$ simply by computing $[\rho^{-1}\mu_1]C_N - P_N^{(2)}$, which is evidently

$$\left(\tilde{C}_N\right)_{ij} = \begin{cases} -1 & i-j=1 \quad \text{or} \quad i=N-1 \text{ and } j=2, \\ 1 & j-i=1 \quad \text{or} \quad i=2 \text{ and } j=N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, the formula for the entries of $B(m_1, \ldots, m_{N-1})$ found by Fordy and Marsh shows that if $m_2 = m_{N-2} = -1$ with the rest of the m_j positive, then each entry is a linear combination of the parameters $m_1, \ldots, m_{N-2}$, because the $\varepsilon_{i,j}$ vanish except when one of i and j is either 2 or $N-2$, in which case $\varepsilon_{i,j}$ is equal to m_i or $-m_j$. Since the operation $\rho^{-1}\mu_1$ has the effect of exchanging m_1 and m_{N-1} for quivers in $\mathcal{F}$, this means that if we set $m_{N-1} = 0$, then the occurrences of m_1 in $\rho^{-1}\mu_1 B(m_1, \ldots, m_{N-2}, 0)$ exactly coincide with the occurrences of m_{N-1} in $B(0, m_2, \ldots, m_{N-1})$. In our case, this means we can recover the general form of the quivers in $\mathcal{F}$ from Q_D and $\tilde{Q}_D$:

Corollary 6.10. *For N odd, the quivers*

$$Q_A(k, \tilde{k}, c_1, \ldots, c_L) := P_N^{(2)} + kC_N + \tilde{k}\tilde{C}_N + \sum_{\ell=1}^{L} c_\ell \Delta_N^{(\ell)}$$

account for all of the quivers in $\mathcal{F}$ on N vertices.

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References

[1] S. Fomin and A. Zelevinsky. Cluster Algebras I: Foundations. *J. Amer. Math. Society*, 15 (2002) 497-529

[2] S. Fomin and A. Zelevinsky. The Laurent Phenomenon. *Advances in Applied Mathematics*, 28 (2002) 119-144

[3] A. Fordy and R. Marsh. Cluster Mutation-Periodic Quivers and Associated Laurent Sequences. *Journal of Algebraic Combinatorics*, 34 (2011) 19-66