DIHEDRAL SIEVING ON CLUSTER COMPLEXES

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Abstract. The *cyclic sieving phenomenon* of Reiner, Stanton, and White characterizes the stabilizers of cyclic group actions on finite sets using q-analogue polynomials. Eu and Fu demonstrated a cyclic sieving phenomenon on generalized cluster complexes of every type using the q-Catalan numbers. In this paper, we exhibit the *dihedral sieving phenomenon*, introduced for odd n by Rao and Suk, on clusters of every type, and on generalized clusters of type A. In the type A case, we show that the Raney numbers count both reflection-symmetric k-angulations of an n-gon and a particular evaluation of the q,t-Fuss-Catalan numbers. We also introduce a sieving phenomenon for the symmetric group, and discuss possibilities for dihedral sieving for even n.

1. Introduction

1.1. Examples of Sieving.

Definition 1.1 (Cyclic Sieving, [RSW04])**.** Suppose X is a finite set acted on by a cyclic group $C_n = \langle r \rangle$, and $X(q)$ is a polynomial in q. The pair $(X \circlearrowleft C_n, X(q))$ has the *cyclic sieving phenomenon* (CSP) if for all $\ell \in [n]$,

$$|\{x \in X : r^\ell x = x\}| = X(e^{2\ell\pi i/n}).$$

Implicit in this definition is the choice of the generator r of C_n which is sent to the primitive n-th root of unity $e^{2\pi i/n}$.

Cyclic sieving was generalized to actions by a product of two cyclic groups, or *bicyclic sieving*, by H. Barcelo, V. Reiner, and D. Stanton in [BRS08], and later described for any finite abelian group in [BER11]. In all cases, the value(s) plugged into the polynomials are specified by a particular representation of the group.

S. Rao and J. Suk [RS17] defined a notion of sieving for the dihedral group $I_2(n)$ for n odd, with the presentation $I_2(n) := \langle r, s | r^n = s^2 = 1,\ rs = sr^{-1} \rangle$. Our definition below appears to be quite different from theirs, but it is equivalent (see Lemma 2.5). Let the *defining representation* ρ_{def} be the two-dimensional representation of $I_2(n)$ sending

$$r \mapsto \begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}$$
$$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition 1.2 (Odd Dihedral Sieving, cf. [RS17, Proposition 4.3])**.** Suppose X is a finite set acted on by the dihedral group $I_2(n)$ with n odd, and $X(q,t)$ is a symmetric polynomial in q and t. The pair $(X \circlearrowleft I_2(n), X(q,t))$ has the *dihedral sieiving phenomenon* (DSP) if for all $g \in I_2(n)$ with eigenvalues $\{\lambda_1, \lambda_2\}$ for $\rho_{\text{def}}(g)$:

$$|\{x \in X : gx = x\}| = X(\lambda_1, \lambda_2).$$

Note that the requirement that X be symmetric is due to the fact that the eigenvalues of ρ_{def} are not ordered. Rao and Suk showed instances of dihedral sieving in several combinatorial settings.

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Theorem 1.3 ([RS17]). *Let n be odd and let $0 \leqslant k \leqslant n$. Then the following exhibit DSP:*

- $\left(X \circlearrowleft I_2(n), \left\{ {n \atop k} \right\}_{q,t}\right)$ *for X the set of order-k subsets of the positive integers $[n]$.*
- $\left(X \circlearrowleft I_2(n), \left\{ {n+k-1 \atop k} \right\}_{q,t}\right)$ *for X the set of order-k multisubsets of the positive integers $[n]$.*
- $\left(X \circlearrowleft I_2(n), \frac{1}{\{n+1\}_{q,t}} \left\{ {2n \atop n} \right\}_{q,t}\right)$ *for X the set of non-crossing partitions of the n-gon.*
- $\left(X \circlearrowleft I_2(n), \frac{1}{\{n\}_{q,t}} \left\{ {n \atop k} \right\}_{q,t} \left\{ {n \atop k+1} \right\}_{q,t}\right)$ *for X the set of non-crossing partitions of the n-gon using $n-k$ blocks.*
- $\left(X \circlearrowleft I_2(n), (qt)^{\binom{n-2}{2}} \operatorname{Cat}_{n-2}(q,t)\right)$ *for X the set of triangulations of the n-gon.*

In the above Theorem, we use

$$\{n\}_{q,t} := q^{n-1} + q^{n-2}t + \cdots + t^{n-1} = \sum_{i=0}^{n-1} q^i t^{n-1-i}$$

$$\{n\}!_{q,t} := \{n\}_{q,t} \cdot \{n-1\}_{q,t} \cdot \cdots \cdot \{1\}_{q,t} = \prod_{i=1}^{n} \{i\}_{q,t}$$

$$\left\{ {n \atop k} \right\}_{q,t} := \frac{\{n\}!_{q,t}}{\{k\}!_{q,t}\{n-k\}!_{q,t}}$$

as the *q,t-analogues* (Note that these correspond to [RS17]'s Fibonacci polynomials through substitutions of $q+t$ and $-qt$.) We also use the q,t-Catalan numbers $\operatorname{Cat}_n(q,t)$ as defined by Garsia and Haiman in [GH96].

1.2. **Statement of Results.** As noted in Theorem 1.3, dihedral sieving for triangulations on an n-gon has already been demonstrated in [RS17, Theorem 6.1], using the q,t-Catalan numbers. Our results generalize this in two different directions. First, we consider pentagonalizations, heptagonalizations, and so on. For this, we will use the q,t-Fuss-Catalan numbers (see Definition 4.13).

Theorem 1.4 (Theorem 4.24). *Let X be the set of k-angulations of an n-gon, for n odd and $n \equiv 2 \pmod{k-2}$. Then the pair $(X \circlearrowleft I_2(n), \operatorname{Cat}_{n-1,\frac{n-2}{k-2}}(q,t)$ exhibits dihedral sieving*

The above result can also be phrased in the language of the generalized cluster complex of type A. Next, we will consider triangulations of other types. For this we will introduce *cluster complexes* (see §3) as and algebraic interpretation of triangulations. The below result is phrased in this language. Let $X = \Delta(\Phi)$ be a cluster complex, with dihedral action generated by the reflections τ_+, τ_-. We will use a generalization of the q,t-Catalan numbers to arbitrary type (see Definition 5.2).

Theorem 1.5. *The pair $(\Delta(\Phi) \circlearrowleft I_2(n), \operatorname{Cat}(\Phi, q, t))$ exhibits dihedral sieving for all odd n and Φ of type $A, B/C, D, E, F$, or I.*

First, we will discuss various interpretations of sieving phenomena in §2. In §3 we will define the cluster complex for any root system, and discuss the realization of the cluster complex for types B and D. Then we will proceed to proving Theorem 1.4 in §4, and Theorem 1.5 in §5. Finally, in §6, we discuss some miscellaneous conjectures and describe possible notions of dihedral sieving for even n.

2. Sieving Phenomena

There is an equivalent definition of cyclic sieving using representation theory, which we now describe. In order to do so, first we will define the ring of representations.

Definition 2.1. Let G be a finite group, and let $\operatorname{Irr} G$ be the set of isomorphism classes of finite-dimensional $\mathbb{C}$-representations of G. The *representation ring* $\operatorname{Rep} G$ is the quotient ring $\mathbb{Z}[\operatorname{Irr} G]/I$, where I is the ideal generated by $\{[V \otimes W] - [V] \cdot [W]\}$ for $V, W \in \operatorname{Irr} G$. Here $[V \otimes W]$ denotes the decomposition of $V \otimes W$ as an element of $\mathbb{Z}[\operatorname{Irr} G]$ and the brackets refer to the isomorphism class.

The elements of $\operatorname{Rep} G$ are the finite-dimensional representations of G, addition is direct sum, and multiplication is the tensor product.

Let ρ be a one-dimensional representation of C_n, which can be specified by choosing which generator of C_n to send to $e^{2\pi i/n}$, as is done the definition of cyclic sieving.

Proposition 2.2 ([RSW04, Proposition 2.1])**.** A pair $(X \circlearrowleft C_n, X(q))$ has the CSP if and only if $X(\rho)$ and $\mathbb{C}[X]$ are isomorphic as C_n-representations.

This gives an equivalent description of the cyclic sieving phenomenon in the language of representations of C_n. Rao and Suk, motivated by the above proposition, defined sieving phenomena for any group with a finitely-generated ring of representations.

Definition 2.3 ([RS17, Definition 2.7])**.** Let G be a group acting on a finite set X, and $\rho_1, \dots, \rho_k$ be a generating set for $\operatorname{Rep} G$. Together with a k-variable polynomial $X(q_1, \dots, q_k)$, these form a triple $(X \circlearrowleft G, \rho_1, \dots, \rho_k, X(q_1, \dots, q_k))$ which exhibits *G-sieving* if and only if $X(\rho_1, \dots, \rho_k)$ and $\mathbb{C}[X]$ are isomorphic as G-representations.

This definition encompasses all known forms of sieving phenomena. We offer an alternative definition, which is roughly equivalent but slightly more restrictive. Let G be a group, and ρ be a chosen complex representation for G, of dimension d.

Definition 2.4. Suppose G acts on finite set X, and $X(q_1, \dots, q_d)$ is a symmetric polynomial in d variables. The pair $(X \circlearrowleft G, \rho, X(q_1, \dots, q_d))$ exhibits *G-sieving* if and only if for all $g \in G$, if $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $\rho(g)$, then

$$|\{x \in X : gx = x\}| = X(\lambda_1, \dots, \lambda_d).$$

Definition 1.2 is a special case of the above definition, namely, with $\rho = \rho_{\text{def}}$ the defining representation and G a dihedral group $I_2(n)$ with n odd. It is also a special case of Definition 2.3, with $G = I_2(n)$ and $\operatorname{Rep} I_2(n)$ with choice of generators ρ_{def} and $-\det$. Here det is the one-dimensional representation of $I_2(n)$ which sends r to 1 and s to -1.

Lemma 2.5 (cf. [RS17, Proposition 4.3])**.** Suppose that n is odd, and $I_2(n)$ acts on a finite set X. Then for any polynomial $X(\cdot, \cdot)$, the $I_2(n)$-representations $\mathbb{C}[X]$ and $X(\rho_{\text{def}}, -\det)$ are isomorphic if and only if the pair $(X \circlearrowleft I_2(n), X(q+t, -qt))$ exhibits the DSP.

Proof. Fix n odd, and a finite set X with an $I_2(n)$ action. Then for all $g \in I_2(n)$, we have $|\{x \in X : gx = x\}| = \operatorname{tr}(\mathbb{C}[X](g))$, since the trace counts the number of fixed points of the permutation represention. Next, for a polynomial X in two variables, we have $\operatorname{tr} X(\rho_{\text{def}}, -\det)(g) = X(\operatorname{tr} \rho_{\text{def}}(g), -\operatorname{tr} \det(g)) = X(\lambda_1 + \lambda_2, -\lambda_1\lambda_2)$, where λ_1, λ_2 are the eigenvalues of $\rho_{\text{def}}(g)$ in some order. Finally, since $\mathbb{C}[X] \cong X(\rho_{\text{def}}, -\det)$ if and only if they have the same trace, the Lemma follows. ∎

With Lemma 2.5 in hand, we can show instances of dihedral sieving for odd n using our Definition 1.2, without needing to consider the representation-theoretic interpretation.

Part of the reason we are interested in dihedral sieving is because often when a set has a natural cyclic action, it also has a dihedral action. In a few cases, such as $X = \binom{[n]}{k}$ and $X = \left(\!\binom{[n]}{k}\!\right)$, the symmetric group S_n has the most natural action. We can define symmetric group sieving as another special case of Definition 2.4, using the *permutation representation* $\rho := \rho_{\text{perm}}$. This n-dimensional representation sends each $\pi \in S_n$ to its permutation matrix over $\mathbb{C}$.

Definition 2.6 (Symmetric Sieving)**.** Suppose that S_n acts on a finite set X, and $X(q_1, \ldots, q_n)$ is a symmetric polynomial in n variables. Then the pair $(X \circlearrowleft S_n, X(q_1, \ldots, q_n))$ has *symmetric sieving* if for all permutations $g \in S_n$ with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ for $\rho_{\text{perm}}(g)$:

$$|\{x \in X : gx = x\}| = X(\lambda_1, \ldots, \lambda_n).$$

The symmetric group S_n contains every group G acting on $[n]$ as a subgroup. For such groups G, instances of G-sieving on sets X with a symmetric action can be obtained from instances of symmetric sieving. We present one instance of symmetric sieving, which will make a reappearance in §6.

Proposition 2.7. Let $h_k(q_1, \ldots, q_n)$ be the degree-k complete homogeneous polynomial in n variables. Then the pair $\left(\left(\!\binom{[n]}{k}\!\right) \circlearrowleft S_n, h_k(q_1, \ldots, q_n)\right)$ exhibits symmetric sieving.

Proof. Let $X = \left(\!\binom{[n]}{k}\!\right)$ and ρ be the $\mathrm{GL}_n(\mathbb{C})$-representation on $\mathbb{C}[X]$ with basis given by the degree k-monomials in $x_1, \ldots, x_n$. Since S_n is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ via the representation ρ_{perm}, this is also the S_n-representation $\mathbb{C}[X]$. Fix $\pi \in S_n$, with eigenvalues $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ for $\rho_{\text{perm}}(\pi)$. We need to show that $\operatorname{tr} \rho(\pi) = h_k(\lambda_1, \ldots, \lambda_n)$. Well, $\rho_{\text{def}}(\pi)$ may be diagonalized to the matrix $\operatorname{diag}(\lambda_1, \ldots, \lambda_n) =: M$, and so $\operatorname{tr} \rho(M) = \operatorname{tr} \rho(\rho_{\text{perm}}(\pi))$, since $\rho_{\text{perm}}(\pi)$ and M are in the same conjugacy class in $\mathrm{GL}_n(\mathbb{C})$. Finally, note that the eigenvalues of $\rho(M)$ are exactly the terms in $h_k(x_1, \ldots, x_n)$, so the result follows. □

3. Combinatorial Interpretation of Cluster Complexes

3.1. Cluster Complexes.

In this section, we will be reviewing some key definitions and facts needed to define sieving for cluster complexes.

Definition 3.1. A subset Φ in a Euclidean space E is called a *reduced root system* if the following axioms are satisfied:

1. Φ is finite, spans E and $0 \notin \Phi$.
2. If $\alpha \in \Phi$, then the only multiples of α in Φ are $\pm\alpha$.
3. If $\alpha \in \Phi$, the reflection across α defined as $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ leaves Φ invariant.

Definition 3.2. A subset Π of a root system Φ is a set of *simple roots* if:

1. Π is a basis of E.
2. each root $\beta \in \Phi$ can be written as $\beta = \sum k_\alpha \alpha$ where $\alpha \in \Pi$ and k_α are integral coefficients all nonpositive or all nonnegative.

From now on, let Φ be an irreducible root system of rank n. Let $\Phi_{>0}$ denote the set of positive roots in Φ with respect to the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and $I = \{1, \ldots, n\}$ its index set. Correspondingly, let $-\Pi = \{-\alpha_1, \ldots, -\alpha_n\}$ denote the set of negative simple roots. Let $S = \{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$ denote the set of simple reflections corresponding to

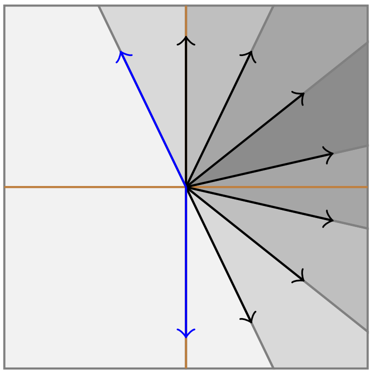

FIGURE 1. The action of τ_+ on $\Phi_{\geqslant -1}$, where each cluster is a pair of adjacent vectors and positive linear spans of the same shading are sent to each other under τ_+. (Blue roots are in $-\Pi$.)

reflecting across the simple roots α_i. Note that S generates a finite reflection group W that acts on Φ naturally. Let $\Phi_{\geqslant -1} = -\Pi \cup \Phi_{>0}$ denote the set of almost positive roots. Now we proceed to define an action on $\Phi_{\geqslant -1}$.

Definition 3.3 (Coxeter diagram)**.** Let Φ be a root system with simple roots $\{\alpha_1, \dots, \alpha_\ell\}$ and let $\langle \alpha, \beta \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. The Coxeter diagram of Φ is a graph with ℓ vertices, the i-th joined to the j-th $(i \neq j)$ by $\langle \alpha_i, \alpha_j \rangle \cdot \langle \alpha_j, \alpha_i \rangle$ edges.

Let $I = I_+ \cup I_-$ be a partition of I such that the sets I_+ and I_- are totally disconnected in the Coxeter diagram, or equivalently the roots in I_+ are mutually orthogonal (same with I_-).

$$\tau_\epsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i \text{ for } i \in I_{-\epsilon} \\ \left(\prod_{i \in I_\epsilon} s_i\right)(\alpha) & \text{otherwise} \end{cases}$$

for $\epsilon \in \{+, -\}$. See Figure 1 for an example of how τ_+ (or τ_-) might act on a two-dimensional $\Phi_{\geqslant -1}$. The product $R = \tau_- \tau_+$ generates a cyclic group $\langle R \rangle$ that acts on $\Phi_{\geqslant -1}$.

Then we can use the map R to define a relation of *compatibility* on $\Phi_{\geqslant -1}$ by

(1) $\alpha, \beta \in \Phi_{\geqslant -1}$ are compatible if and only if $R(\alpha), R(\beta)$ are compatible;
(2) $-\alpha_i \in -\Pi$ and $\beta \in \Phi_{>0}$ are compatible if and only if the simple root expansion of β does not involve α_i.

Note that compatibility is well-defined since for any pair simple roots $\alpha, \beta \in \Phi_{\geqslant -1}$, we can repeatedly let R act on them until one is a root in $\Phi_{>0}$ and the pair of roots then has to satisfy condition (2) above.

Now we are ready to define the cluster complex.

Definition 3.4. The cluster complex $\Delta(\Phi)$ of the root system Φ is the simplicial complex whose faces are the subsets of roots in $\Phi_{\geqslant -1}$ which are pairwise compatible.

For Φ in type A_n, B_n, or D_n, the cluster complex $\Delta(\Phi)$ can be realized as dissections of a regular polygon such that $\langle R \rangle$ corresponds to a group action on the dissections by rotating the given polygon. We will present the bijections between cluster complexes of these types and dissections of regular polygons in §5.

3.2. **Generalized Cluster Complexes.** Let m be a positive integer. For every positive root $\alpha \in \Phi_{>0}$, let $\alpha^1, \dots, \alpha^m$ denote the m "colored" copies of α. Then we define

$$\Phi^m_{\geqslant -1} = \{\alpha^k : 1 \leq k \leq m, \alpha \in \Phi_{>0}\} \cup \{(-\alpha_i)^1 : \alpha_i \in \Pi\}$$

i.e. $\Phi^s_{\geqslant -1}$ contains m copies of $\Phi_{>0}$ and one copy of Π. Similarly we can define the m-analogue R_m of R acting on $\Phi^m_{\geq -1}$. For $\alpha^k \in \Phi^m_{\geqslant -1}$, we define

$$R_m(\alpha^k) = \begin{cases} \alpha^{k+1} & \text{if } \alpha \in \Phi_{>0} \text{ and } k < m; \\ (R(\alpha))^1 & \text{otherwise.} \end{cases}$$

The map R_m also induces the relation of compatibility on $\Phi^m_{\geqslant -1}$ as the followng:

(1) α^k is compatible with β^l if and only if $R_m(\alpha^k)$ is compatible with $R_m(\beta^l)$;
(2) $(-\alpha_i)^i$ is compatible with β^l if and only if the simple root expansion of β does not involve α.

Furthermore, the above two conditions uniquely determine the compatibility relation. The generalized cluster complex $\Delta^m(\Phi)$ associated to the root system Φ is defined to be the simplicial complex whose vertices are the elements in $\Phi^m_{\geqslant -1}$ and faces are the subsets of $\Phi^m_{\geqslant -1}$ that are pairwise compatible.

4. k-angulations of an n-gon

4.1. **Generalized Clusters of Type A.** In this subsection we describe the bijection between diagonals of a regular polygon and faces of the generalized cluster complex for type A_n root systems given in [FR05]. Let Φ be a root system of type A_n. Then note that the roots in $\Phi_{\geqslant -1}$ are of the form $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$ for $1 \leq i < j \leq n$ and negative roots $-\alpha_i$. Let P be a regular polygon with $(n+1)m+2$ vertices labeled $\{1, \ldots, n+3\}$ in counterclockwise. A diagonal of P is called *m-allowable* if it cuts P into two polygons which can be dissected into $(m+2)$-gons. The roots in $\Phi_{\geqslant -1}$ identifies with diagonals of P as follows.

(1) For $1 \leq i \leq \frac{n+1}{2}$, the root $-\alpha_{2i-1} \in \Pi$ is identified with the diagonal connecting vertices $(i-1)m+1$ and $(n+1-i)m+2$.
(2) For $1 \leq i \leq \frac{n}{2}$, the root $-\alpha_{2i} \in \Pi$ is identified with the diagonal connecting vertices $im+1$ and $(n+1-i)m+2$.
(3) For each $\alpha_{ij} \in \Phi_{>0}$, there are exactly m diagonals which are m-allowable and intersects the diagonal $-\alpha_1, \ldots, -\alpha_j$ and no other diagonals. This collection of diagonals is of the form $R^0_m D, R^1_m D, \ldots, R^{m-1}_m D$ for some diagonal D. For $1 \leq k \leq m$, we identify α^k_{ij}.

In addition we call the diagonals corresponding to the negative simple roots a m-snake on P. Notice that under this bijection, the action of $\langle R \rangle$ corresponds to a clockwise $\frac{2\pi}{(n+1)m+2}$ rotation of polygon about the center. Each pair of compatible roots are sent to a pair of noncrossing diagonals and the $(m+2)$-angulations of P corresponds exactly to maximal faces of $\Delta^m(\Phi)$.

4.2. **Combinatorial preliminaries.**

Definition 4.1. The Raney numbers are defined as follows:

$$R_{p,r}(k) = \frac{r}{kp+r}\binom{kp+r}{k}.$$

Remark 4.2. Note that when $r = 1$, the Raney number is exactly the Fuss–Catalan number, denoted as

$$C_p(k) = R_{p,1}(k) = \frac{1}{(p-1)k+1}\binom{kp}{k}.$$

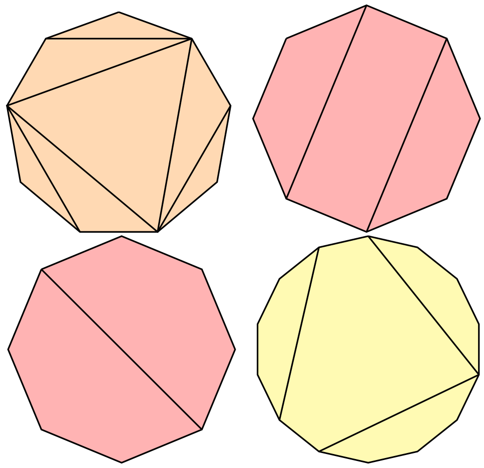

FIGURE 2. Four examples of k-angulations.

If $r = 1, p = 2$, the Raney number is the usual Catalan number, denoted as

$$C_k = R_{2,1}(k) = \frac{1}{k+1}\binom{2k}{k}.$$

The following two propositions are stated and proved in [ZY17].

Proposition 4.3. Let p be a positive integer and let r, k be nonnegative integers. Then we have

$$R_{p,r}(k) = \sum_{i_1+\cdots+i_r=k} C_p(i_1)C_p(i_2)\cdots C_p(i_r).$$

Proposition 4.4. The Raney numbers satisfy the following recurrences:

$$R_{p,1}(k) = \sum_{i=0}^{k-1} R_{p,1}(i)R_{p,p-1}(k-1-i)$$

$$R_{p,r}(k) = \sum_{i=0}^{k} R_{p,r}(i)R_{p,r-1}(k-i) \qquad \text{for } r > 1.$$

4.3. **Counting Symmetric k-angulations.** In this section, we prove dihedral sieving for k-angulations of n-gons for odd n, using the standard dihedral action on the n-gon extended to the chords constituting the k-angulation. This proof will suffice to complete the case of dihedral sieving on A_n cluster complexes.

Definition 4.5 (k-angulation)**.** A *k-angulation* of an n-gon is a partition of the interior of a regular n-gon in the plane, using non-intersecting segments connecting vertices of the n-gon, such that each part of the interior is a k-gon.

Lemma 4.6. n-gons that admit a k-angulation satisfy $n \equiv 2 \pmod{k-2}$.

Proof. Induction on the number of parts of the k-angulation. □

We will often write $n = m(k-2)+2$, where we may interpret m as the number of k-gons resulting from any k-angulation of the n-gon. We will also frequently use $s := k-2$ to keep expressions cleaner.

Corollary 4.7. For odd n, no n-gon admits a k-angulation for even k.

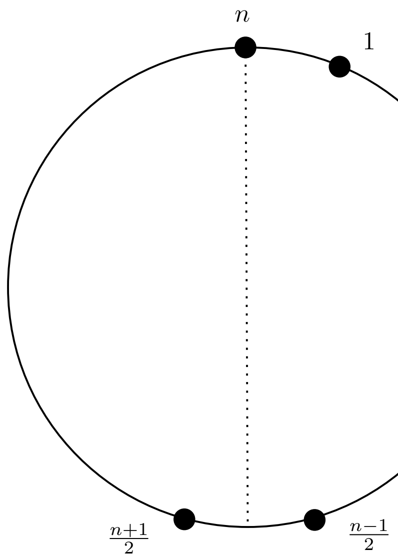

FIGURE 3. Four vertices on a centrally-symmetric n-gon that we k-angulate.

The dihedral group element $g \in I_2(n)$ acts on a k-angulation by sending vertex i to $g(i)$ and correspondingly sends the chord connecting i and j to the chord connecting $g(i)$ and $g(j)$. It is evident that this action sends k-angulations to k-angulations. Recall that in the case of n odd, there is a unique "type" of reflection in $I_2(n)$, which has precisely one fixed point and may be viewed geometrically as reflection about the perpendicular bisector of some side of the n-gon. (This stands in contrast to the case of n even, where the two "types" of reflection has zero or two fixed points.)

Theorem 4.8. *Let $S_k(n)$ be the number of k-angulations of an n-gon fixed by the reflection across a specific axis for n and k odd. Then*

$$S_k(n) = R_{k-1,\frac{k-1}{2}}\left(\frac{m-1}{2}\right)$$

where $m = \frac{n-2}{k-2}$.

Proof. We show the statement is true by showing that $S_k(n)$ equals the right-hand side of Proposition 4.3 with $p = k-1, r = \frac{k-1}{2}, m = \frac{m-1}{2}$. Notice that the total number of k-angulations of an n-gon is exactly the Fuss–Catalan number $C_{k-1}(m)$.

Refer to Figure 3. We want to find the number of k-angulations of the n-gon fixed under the reflection across the y-axis. From now on, we restrict ourselves to the right half of the polygon to the y-axis since we can reflect the polygon and yield a valid k-angulation fixed by the reflection. We want to find a path from vertex n to vertex $(n-1)/2$ such that the length of the path (number of edges) is exactly $\frac{k-1}{2}$ and the polygons with one edge in the path can be k-angulated. Then the polygon defined by the path reflected on both sides and the edge between the vertex $(n-1)/2$ and vertex $(n+1)/2$ is exactly a k-gon.

Now we show how the $r = \frac{k-1}{2}$ partition of the number $\frac{m-1}{2} = i_1 + \cdots + i_r$ corresponds to the desired path from the n-th vertex to the $(n-1)/2$-th vertex. Start from vertex n, at the j-th step for $1 \leq j \leq r$, suppose we are at the vertex v, then connect the vertex v to vertex $(v + j(k-2) + 1) \bmod n$ (where vertices are computed modulo n). Note that the length of the path is exactly r as desired and we end up exactly at vertex $r + \frac{m-1}{2} \cdot (k-2) = (n-1)/2$.

Now consider the polygons with the edge corresponding to i_j in the path. Notice that these polygons have $2 \pmod{k-2}$ vertices and thus they can be k-angulated. The number of k-angulations of these polygons are the Fuss–Catalan numbers $C_{k-1}(i_j)$. Hence the number of k-angulations of the polygon with respect to one path $i_1, \ldots, i_r$ is $\prod_{j=1}^{r} C_{k-1}(i_i)$.

Therefore we have, for $r = \frac{k-1}{2}$

$$S_k(n) = \sum_{i_1+\cdots+i_r=\frac{m-1}{2}} C_{k-1}(i_1)C_{k-1}(i_2)\cdots C_{k-1}(i_r).$$

We can now conclude that

$$S_k(n) = R_{k-1,\frac{k-1}{2}}\left(\frac{m-1}{2}\right)$$

as desired. □

Now we proceed to count the number of k-angulations of an n-gon fixed by a rotation of order d.

Theorem 4.9. *Let $T(d,s,m)$ denote the number of $(s+2)$-angulations of an $(sm+2)$-gon which are fixed under rotation by a primitive d-th root of unity, where $d|(s+2)$. Then*

$$T(d,s,m) = \frac{sm+2}{s+2}R_{s+1,\frac{s+2}{d}}\left(\frac{m-1}{d}\right) = \binom{\frac{m(s+1)+1}{d}-1}{\frac{m-1}{d}}.$$

Proof. We show this by a similar approach to Theorem 4.8 and we label the polygons counterclockwise using $\{1,\ldots,sm+2\}$. We want to force the inner $(s+2)$-gon to be fixed under the rotation of order d. First we fix the inner polygon to contain the vertex $sm+2$. Now this is equivalent to finding a path from vertex $sm+2$ to vertex $\frac{sm+1}{d}$ of length $\frac{s+2}{d}$ such that the polygons with one edge in the path can also be $(s+2)$-angulated, since then we can rotate the path to force the inner polygon to be fixed under the rotation of order d. Therefore we find the $r := \frac{s+2}{d}$-partitions $i_1+\cdots+i_r = \frac{m-1}{d}$ where each such partition corresponds to a path from vertex $sm+2$ to vertex d as follows Start from vertex $sm+2$, at the j-th step for $1 \leq j \leq r$, suppose we are at the vertex v, then connect the vertex v to vertex $(v+js+1) \bmod (sm+2)$.

Now for the polygons with one edge in the path forming the inner polygon, the number of $(s+2)$-angulations are exactly the Fuss–Catalan numbers $C_{s+1}(i_j)$ and we can rotate these $(s+2)$-angulations to yield the $(s+2)$-angulations of the $(sm+2)$-gon fixed under the rotation of order d. In addition, note that there are $\frac{sm+2}{s+2}$ valid starting points for the path in total, since we were only counting the case where the vertex $sm+2$ is a vertex of the inner polygon. Therefore we can conclude that for $r = \frac{s+2}{d}$,

$$T(d,s,m) = \frac{sm+2}{s+2}\sum_{i_1+\cdots+i_r=\frac{m-1}{d}} C_{s+1}(i_1)\cdot C_{s+1}(i_2)\cdots C_{s+1}(i_r) = \frac{sm+2}{s+2}R_{s+1,\frac{s+2}{d}}\left(\frac{m-1}{d}\right).$$

Then by manipulating the binomial expression for $R_{s+1,\frac{s+2}{d}}\left(\frac{m-1}{d}\right)$ and after canceling, we obtain the desired result. □

4.4. q,t-Fuss-Catalan Numbers.

Definition 4.10 (Dyck path)**.** In a grid a boxes high and b boxes wide, an (a,b)-*Dyck path* is a path, entirely going either north or east,along the edges in the graph that do not cross below the long diagonal from the bottom-left corner to the upper-right corner.

A Dyck path gives rise to a Young diagram by considering the squares in the grid completely above the path; see, for instance, Figure 4.

Definition 4.11 (Area)**.** The *area* of an (a, b)-Dyck path is the number of full grid squares between the path and the diagonal. Equivalently, it is the number of squares in the Young diagram below the path.

Definition 4.12 (Sweep, cf. [ALW16])**.** The *sweep* function `sweep` is a function on (a, b)-Dyck paths, defined recursively as follows. Assign the level $l_0 = a$ to the first edge in the path λ. Each subsequent edge has level $l_i = l_{i-1} + a$ if the i^{th} step is North and $l_i = l_{i-1} - b$ if the i^{th} step is East. The edges are then sorted by increasing level to obtain a new path in the grid.

By [ALW15; TW18], `sweep` is a bijection on (a, b)-Dyck paths.

Definition 4.13. The *rational* q, t*-Catalan numbers* $\mathrm{Cat}_{a,b}(q, t)$ are defined by the following sum of as (a, b)-Dyck paths λ:,

$$\mathrm{Cat}_{a,b}(q, t) := \sum_{\lambda} q^{\texttt{area}(\lambda)} t^{\texttt{area}(\texttt{sweep}(\lambda))}.$$

Note that when $t = 1$, $\texttt{sweep}(\lambda)$ is irrelevant in the evaluation of $\mathrm{Cat}_{a,b}(q, 1)$.
We assume henceforth that s and m are odd positive integers.

Definition 4.14. Let $Y_s(\ell, m)$ be the Young diagram for the partition $(\ell+\lambda_1, \ldots, \ell+\lambda_m)$ for $\lambda_k := (m - k)s$.

See Figure 5 for $Y_3(2, 5)$.

Definition 4.15. $Y_s(\ell, m)$ has the indexing (i, j, k) of its boxes as follows. $1 \leqslant i \leqslant m$ tells the component of the partition, equivalently the row in the Young diagram. $0 \leqslant j \leqslant m-i$ tells the "block" in the row, from the left: either the block of width ℓ, or one of the blocks of width s. $1 \leqslant k \leqslant \begin{cases} \ell & j = 0 \\ s & j > 0 \end{cases}$ tells the position in the block, from the left.

For instance, in Figure 4, highlighted squares include: $(1, 4, 2)$, $(2, 2, 3)$, $(2, 3, 2)$, $(3, 2, 3)$, $(4, 1, 1)$, and in Figure 5 the southwesternmost square is indexed as $(5, 0, 1)$.

Lemma 4.16. The zero-area Dyck path in the $(ms + 1) \times m$ grid corresponds to the Young diagram $Y_s(0, m)$: zero squares on the bottom-most row, s squares on the next, etc., and $(m - 1)s$ squares on the top row.

Proof. The number of squares on row i (indexing from 0 to $m-1$) is equal to $\lfloor n_i \rfloor$ where n_i satisfies $\frac{i}{n_i} = \frac{m}{ms+1}$, the slope of the long diagonal, thus $n_i = \left(\frac{ms+1}{m}\right) i = si + \frac{i}{m}$. Since $0 \leqslant i < m$, $\lfloor n_i \rfloor = si$, as desired. □

See Figure 4 for an example of the diagram for $m = 5$ and $s = 7$.

Theorem 4.17. *$D_s(m)$, defined as the difference between the number of even-area and odd-area $(ms + 1, m)$-Dyck paths, is equal to*

$$R_{s+1,\frac{s+1}{2}}\binom{m-1}{2} = \frac{1}{m}\binom{\frac{(s+1)m}{2}}{\frac{m-1}{2}}.$$

We shall first prove a sequence of results about $D_s(\ell, m)$, defined as the analogous quantity for paths in $Y_s(\ell, m)$. We begin with a pair of recurrences, and massage these in Proposition 4.19 and Proposition 4.20 to obtain a Raney recurrence.

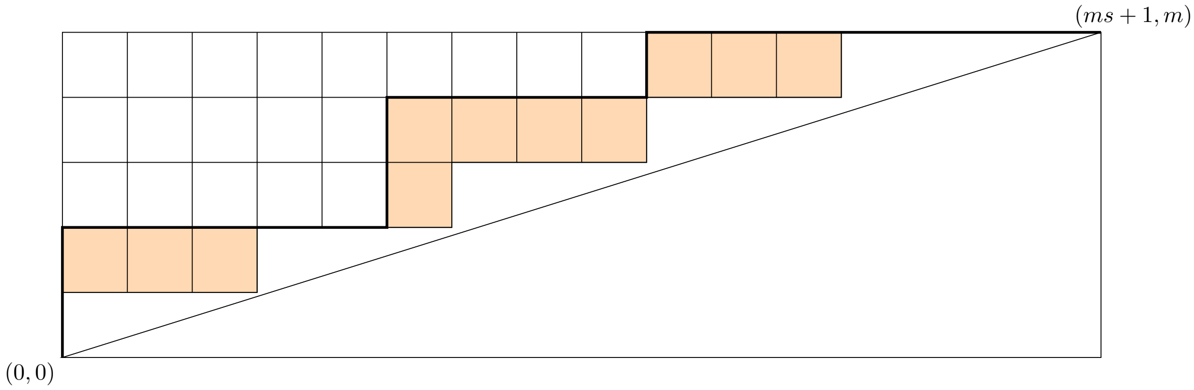

FIGURE 4. Young diagram $Y_3(0,5)$ with a Dyck path and corresponding area shown. Note that since $\gcd(ms+1, m) = 1$, the diagonal lies strictly below the diagram.

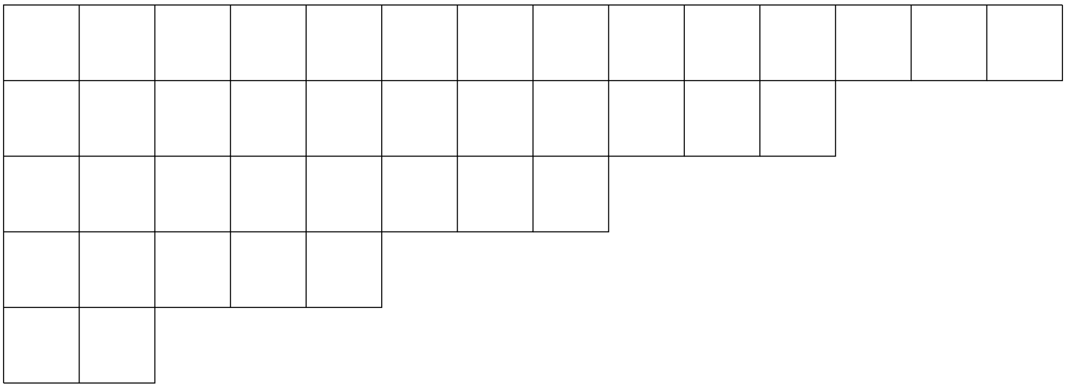

FIGURE 5. Young diagram $Y_3(2,5)$.

Proposition 4.18. For odd ℓ and odd $s > 3$, $D_s(\ell, m)$ satisfies the recurrences

$$D_s(1,m) = \sum_{y=0}^{m-2} (-1)^{y+1} D_s(1,y) D_s(2s-1, m-(y+2)) \tag{1}$$

$$D_s(\ell,m) = \sum_{y=0}^{m} (-1)^{(m+1)y} D_s(\ell-2,y) D_s(1,m-y) \tag{2}$$

along with the base case $D_s(\ell, 0) = 1$.

Proof of (1). Our approach is inspired by the proof of [RTY18, Proposition 1.5], where markers are placed on the Young diagram which form the basis for a recursion.

In $Y_s(\ell, m)$, place a marker y_i for $2 \leqslant i \leqslant m-1$ in the box $(m-i, i-1, 1)$ (so, with $2s-1$ empty boxes to its right) and place a marker y_1 in the box $(m-1, 0, 1)$. See Figure 6.

We sum over the southwesternmost marker contained above a given Dyck path in the diagram:

$$\sum_{y=2}^{m-1} (-1)^{e_s(1,m,y)} D_s(1, y-2) D_s(2s-1, m-y).$$

Here, the term $D_s(1, y-2)D_s(2s-1, m-y)$ obtains its first multiplicand from the possibilities for the path below the rectangle to the marker and its second from the possibilities for the path from the marker to the northeast corner. The sign is determined

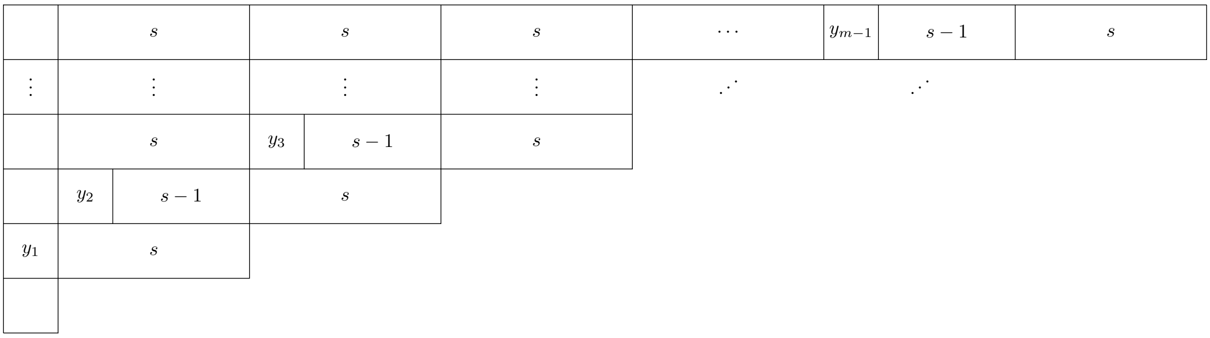

FIGURE 6. Young diagram $Y_s(1, m)$.

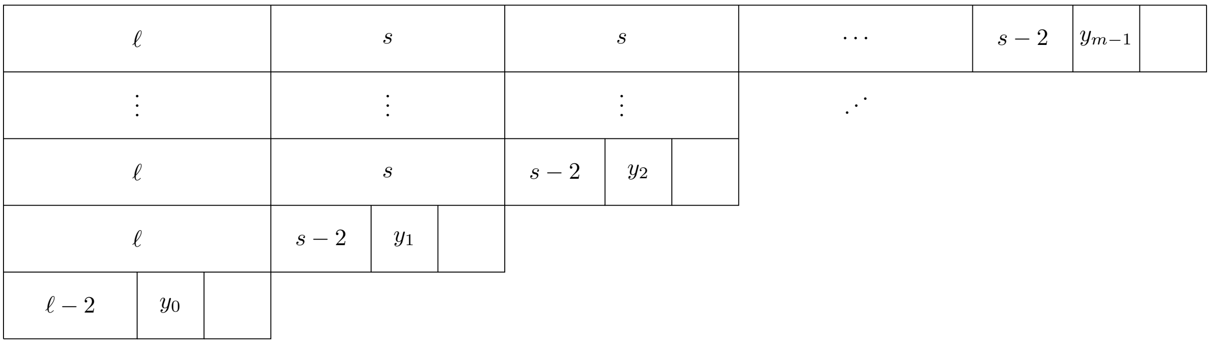

FIGURE 7. Young diagram $Y_s(\ell, m)$.

by the size of the $(m-y) \times (2+(y-1)s)$ rectangle included above the path. The $y=1$ term vanishes because any path containing the marker y_1 may be obtained by any path in $Y_s(1, m-2)$ prefixed by either EN or NE, yielding paths whose areas cancel.

A path in $Y_s(1, m)$ need not contain a marker; these are counted by noting that the first two directions are NN followed by any path (of the same parity area) in $Y_s(1, m-2)$. Noting that $D_s(2s-1, 0) = 1$ and that $e_s(1, m, m) \equiv 0 \pmod 2$, we find

$$\begin{aligned}
D_s(1, m) &= \sum_{y=2}^{m-1} (-1)^{e_s(1,m,y)} D_s(1, y-2) D_s(2s-1, m-y) + D_s(1, m-2) \\
&= \sum_{y=2}^{m-1} (-1)^{(m-y)\cdot(1+(y-1)s)} D_s(1, y-2) D_s(2s-1, m-y) + D_s(1, m-2) D_s(2s-1, 0) \\
&= \sum_{y=2}^{m} (-1)^{y+1} D_s(1, y-2) D_s(2s-1, m-y) \\
&= \sum_{y=0}^{m-2} (-1)^{(m+y)(y+1)} D_s(1, y) D_s(2s-1, m-(y+2)).
\end{aligned}$$

□

Proof of (2). Place a marker y_i for $1 \leqslant i \leqslant m-1$ in the box $(m-i, i, s-1)$ and place a marker y_0 in the box $(m, 0, \ell-1)$ (so, in all cases with 1 empty box to its right). See Figure 7.

We sum over the southwesternmost marker contained above a given Dyck path in the diagram: $\sum_{y=0}^{m-1} (-1)^{e_s(\ell,m,y)} D_s(\ell-2, y) D_s(1, m-y)$, where each summand arises from the necessary inclusion of the $(m+1-y)\times(\ell+sy-1)$ rectangle (yielding the exponent e_s of -1) along with a path below the rectangle to the marker (yielding the first multiplicand) and a path from the marker (yielding the second multiplicand).

A path in $Y_s(\ell,m)$ need not contain a marker; these are counted by noting that the first direction is N followed by any path (of opposite parity area) in $Y_s(s,m-1)$ (since each marker y_i is avoided). Noting that $D_s(1,0)=1$ and that $e_s(1,m,m)\equiv 0 \pmod 2$, we find

$$\begin{aligned} D_s(\ell,m) &= \sum_{y=0}^{m-1}(-1)^{e_s(\ell,m,y)}D_s(\ell-2,y)D_s(1,m-y)+D_s(\ell-2,m) \\ &= \sum_{y=0}^{m-1}(-1)^{(m+1-y)\cdot(\ell+sy-1)}D_s(\ell-2,y)D_s(1,m-y)+D_s(\ell-2,m)D_s(1,0) \\ &= \sum_{y=0}^{m}(-1)^{my}D_s(\ell-2,y)D_s(1,m-y). \end{aligned}$$

□

Proposition 4.19. For all odd ℓ, s and m, with $s>3$, $D_s(\ell,m)$ vanishes.

Proof. We fix s and induct on pairs of positive odd integers (ℓ,m) in the poset where $(\ell,m)\leqslant(\ell',m')$ iff $\ell\leqslant\ell'$ and $m\leqslant m'$. We prove the result by direct computation for elements in the set $I:=\{(\ell,1)\mid \text{odd } \ell\in\mathbb{N}\}$, in which case $Y_s(\ell,1)$ is simply ℓ blocks in a (horizontal) row. There are $\ell+1$ paths, characterized by the position of the N, and as that position increases their areas alternate parity. I's upwards-closed ideal equals the entire poset, so we invoke strong induction and use (1) and (2): we observe that for odd m and any $y\in\mathbb{Z}$:

- exactly one of y and $m-y$ is odd; and
- exactly one of y and $m-y-2$ is odd.

The result follows. □

Proposition 4.20. By Proposition 4.4, we conclude that for even m,

$$D_s(\ell,m)=R_{s+1,\frac{\ell+1}{2}}\left(\frac{m}{2}\right).$$

Proof. Since in (1) and (2) the first argument of D_s is always odd, for odd y the summands vanish. Hence we restrict our attention to even y. Letting $D'_s(\ell,m):=D_s(\ell,2m)$, we observe that

$$\begin{aligned} D'_s\left(1, \frac{m}{2}\right) &= D_s(1, m) \\ &= \sum_{y=0}^{\frac{m}{2}-1} (-1)^{2y+1} D_s(1, 2y) D_s(2s-1, m-(2y+2)) \\ &= -\sum_{y=0}^{\frac{m}{2}-1} D'_s(1, y) D'_s\left(2s-1, \frac{m}{2} - y - 1\right) \\ D'_s\left(\ell, \frac{m}{2}\right) &= D_s(\ell, m) \\ &= \sum_{y=0}^{\frac{m}{2}} (-1)^{(m+1)2y} D_s(\ell-2, 2y) D_s(1, m-2y) \\ &= \sum_{y=0}^{\frac{m}{2}} D'_s(\ell-2, y) D'_s\left(1, \frac{m}{2} - y\right). \end{aligned}$$

These precisely match the recurrences of Proposition 4.4. □

Proof of Theorem 4.17. The case of $s = 1$ is proved in [RS17, Theorem 6.1], as the Raney numbers $R_{2,1}(n)$ are precisely the Catalan numbers.

We observe that $D_s(m) = D_s(0, m) = D_s(s, m-1)$. Since m is odd, from Proposition 4.20, we are done. □

From this result, we are able to claim a proof of the reflection case of dihedral sieving for the instance of k-angulations of an n-gon, since we have shown equality between the number of k-angulations fixed by a given reflection and the appropriate evaluation of the Catalan q,t-analogue. We now cite a sequence of prior results that provide to us the rotational case of dihedral sieving.

Theorem 4.21 ([Loe05])**.** *The q,t-Fuss Catalan $FC_n^m(q,t)$ numbers satisfy*

$$q^{mn(n-1)/2} FC_m^n(q, q^{-1}) = \frac{1}{[mn+1]_q} \begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q.$$

Conjecture 4.22 ([ALW16])**.** The rational q,t-Catalan numbers satisfy

$$q^{(a-1)(b-1)/2}\, \mathrm{Cat}_{a,b}(q, q^{-1}) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q.$$

Proposition 4.23 ([EF08])**.** For $d \geq 2$ a divisor of $sm+2$, let ω be a primitive d-th root of unity. Then

$$\mathrm{Cat}_{sm+1,m}(\omega, \omega^{-1}) = \begin{cases} \binom{\frac{m(s+1)+1}{d} - 1}{\frac{m-1}{d}} & \text{if } d \geq 2 \text{ and } d \mid k, \\ 0 & \text{otherwise} \end{cases}$$

Theorem 4.24. *Let $X_{s,m}$ with s and m odd be the set of $(s+2)$-angulations of an n-gon for $n := sm+2$, with the natural dihedral action of $I_2(n)$. Then $(X_{s,m} \circlearrowleft I_2(n), \mathrm{Cat}_{sm+1,m}(q,t))$ exhibits dihedral sieving.*

Proof. For reflections on the polygon, by Theorem 4.8 and Theorem 4.17, we have

$$S_{s+2}(n) = D_s(m) = \mathrm{Cat}_{sm+1,m}(1, -1)$$

where $S_{s+2}(n)$ is the number of $(s+2)$-dissections of an n-gon and $D_s(m)$ is the difference between the number of even-area and odd-area $(ms+1,m)$-Dyck paths.

For rotations on the polygon, by Theorem 4.9 and Proposition 4.23, we have for ω a primitive d-th root of unity

$$T(d,s,m) = \mathrm{Cat}_{sm+1,m}(\omega,\omega^{-1})$$

where $T(d,s,m)$ denotes the number of $(s+2)$-angulations of an $(sm+2)$-gon fixed under rotation by ω. Now we can conclude dihedral sieving for $(X_{s,m} \circlearrowleft I_2(n), \mathrm{Cat}_{sm+1,m}(q,t))$. □

5. Clusters of Other Types

In this section we exhibit dihedral sieving for the cluster complexes $\Delta(\Phi)$ of type $A, B/C, D, E, F$, or I. First, we should define the polynomial we will use. In [EF08], the polynomial:

$$\mathrm{Cat}(\Phi,q) := \prod_{i=1}^{n} \frac{[h+e_i+1]_q}{[e_i+1]_q}$$

is used to exhibit cyclic sieving. Here h is the Coxeter number and $e_1,\ldots,e_n$ are the exponents of Φ (but these details do not matter for us).

Theorem 5.1 ([EF08, Theorem 1.2])**.** *The pair $(\Delta(\Phi) \circlearrowleft C_n, \mathrm{Cat}(\Phi,q))$ exhibits the cyclic sieving phenomenon for all root systems Φ.*

We need to introduce a t parameter in order to exhibit dihedral sieving. We'll use the following "definition".

Definition 5.2 (cf. [Stu10, Conjectures 3 and 4])**.** Let Φ be a root system with root poset P. The q,t-Cat(Φ) polynomial is defined to satisfy the following two specializations:

$$\mathrm{Cat}(\Phi,q,q^{-1}) = \mathrm{Cat}(\Phi,q)$$

and

$$\mathrm{Cat}(\Phi,q,1) = \sum_{I\in J(P)} q^{|I|}.$$

In [Stu10] this polynomial is actually explicitly defined, and it is conjectured that it has these properties. We will only be using the above specializations, however. The first specialization takes care of the cyclic part for us, due to Theorem 5.1. It only remains for us to show the following claim, for each type Φ.

Claim 5.3. For a root system Φ with root poset P, and $\Delta(\Phi)$ with odd-n dihedral action, we have that $\sum_{I\in J(P)} q^{|I|}$ counts the number of facets in $\Delta(\Phi)$ fixed by τ_+, or by τ_-.

For Φ of type F or I, there is still a nice choice of poset P which we can call the root poset, so we will do these cases too. The following subsections verify the above claim for each type in turn. For types B and D, we first describe a realization of facets in $\Delta(\Phi)$ as fancy triangulations.

5.1. Type B_n/C_n Cluster Complex.

In [FR05], a combinatorial realization of $\Delta(\Phi)$ for type B_n is given as follows (and it is the same for C_n). Let P be a centrally symmetric regular polygon with $2n+2$ vertices. The vertices of the complex are of two types:

- The diameters of P, i.e., the diagonals connecting antipodal vertices;
- The pairs (D,D') of distinct diagonals of P such that D is related to D' by a half-turn about the center of P.

Two vertices are called noncrossing if no diagonal representing one vertex crosses a diagonal representing the second vertex. The faces of the complex are the sets of pairwise noncrossing vertices. Therefore the maximal faces correspond to centrally symmetric triangulations of P.

Moreover, the action of the reflections τ_-, τ_+ on a type B_n face is in fact reflection in the usual sense [FZ03, Proposition 3.15 (1)].

Lemma 5.4. If F is a facet for $\Delta(B_n)$ for n even and $\varepsilon = \pm 1$, then $\tau_\varepsilon(F) \neq F$.

Proof. We realize F as a centrally symmetric triangulation of a $(2n+2)$-gon. Then F must contain some diameter D, otherwise the central triangle would not be centrally symmetric by a half-turn.

Let A be the axis of reflection of τ_ε. If A does not go through vertices, then it must be perpendicular to the diameter contained in F, otherwise the reflection about A cannot fix F. Now, any triangulation of the $n+2$ vertices not on the right of D must contain some diagonal which crosses the axis A (because n is even). This diagonal will not be sent to another diagonal under τ_ε, since if it was, then the configuration would not be non-crossing.

Next, suppose that A does go through vertices. In order to fix the diameter, the axis A and the diameter D must coincide. Then the left and right sides of D must be symmetric, in addition to having the half-turn central symmetry. This implies that the F must also be fixed by the reflection perpendicular to A, which was covered in the previous case. Therefore F cannot be fixed by τ_ε. □

Now it just remains to show that there are equally many order ideals $B_n^+ = T_{n,2n}$ of each parity.

Lemma 5.5. For $T_{n,2n}$ a trapezoid poset and $J(T_{n,2n})$ its set of order ideals,

$$\sum_{I \in J(T_{n,2n})} (-1)^{|I|} = 0.$$

Proof. We proceed by induction on n. The case $n = 1$ follows by computation. We note that the upper $|T_{n-1,2n-2}|$-many elements of $T_{n,2n}$ are precisely the subposet $T \cong T_{n-1,2n-2}$, say via the map f. For $I \in J(T_{n-1,2n-2})$, let $F(I) \subset T_{n,2n} \setminus T$ be defined such that for each $\alpha \in F(I)$, there is $\beta \in I$ with $\alpha \leqslant f(\beta)$. (Elements of $F(I)$ are "forced" to appear in the order ideal of $T_{2n,n}$ corresponding to $f(I)$.) For $I \in J(T_{n-1,2n-2})$, there are two cases: $F(I) \subsetneq T_{n,2n} \setminus T$, and $F(I) = T_{n,2n} \setminus T$.

In the former case, we can "toggle" any element $\alpha \in T_{n,2n} \setminus (T \cup F(I))$, i.e. we consider all order ideals containing $f(I) \cup F(I)$ as well as additional elements of $T_{n,2n} \setminus T$ but not α, as well as all order ideals containing $f(I) \cup F(I) \cup \{\alpha\}$ as well as additional elements of $T_{n,2n} \setminus T$. It is clear that these sets are of the same size and their elements pair up, with sizes of opposing parities.

In the latter case, we may toggle any element of the bottom row of T other that one with down-degree 1 to again obtain two sets of the same size whose elements pair up, having sizes of opposing parities. □

5.2. **Type D_n Cluster Complex.** In [FR05], a combinatorial realization of $\Delta(\Phi)$ for type D_n is given as follows. Let P be a regular polygon with $2n$ vertices. The vertices of the combinatorial realization of $\Delta(D_n)$ will fall into two groups. The vertices in the first group corresponds one-to-one to pairs of distinct non-diameter diagonals in P related by a half turn. In the second group, each vertex is indexed by a diameter of P, together with two "flavors", which we call "dashed" and "gray". Thus each diameter occurs twice, in each of the two flavors. We label the vertices of P counter clockwise by $\{1, \dots, n, -1, \dots, -n\}$

and we call $[1, -1]$ the primary diameter. By construction, the map R acts by rotating 2 clockwise to 1 and switching the flavors of certain diagonals. Specifically, R preserves flavor when applied to a diameter of form $[k, -k]$ for $1 \leq k \leq n$ unless $k = 1$. If $k = 1$, the flavor is switched.

The notion of compatibility for the cluster complex of type D_n is defined as follows. Two vertices at least one of which is not a diameter are compatible or not according to precisely the same rules as in the type B_n case. A more complicated condition determines whether two diameters are compatible. Two diameters with the same location and different flavors are compatible. Two diameters at different locations are compatible if and only if applying R repeatedly until either of them is in position $[1, -1]$ results in diameters of the same flavor. Explicitly, let D be a flavored diameter and let $\overline{D}$ denote D with its flavor reversed. Then for $1 \leqslant k \leqslant n-1$, the flavored diameter $R(D)$ is compatible with D (and incompatible with $\overline{D}$) if and only if k is even.

By definition, the maximal faces of the complex are the diagonals of P that satisfy the following. Within the flavored diagonals, if they are not of same flavor, they are allowed to overlap; if they are of the same flavor, they are allowed to cross. Aside from within the flavored diagonals, flavored diagonals cannot cross non-flavored diagonals and non-flavored diagonals should be pairwise noncrossing.

In this case, the action of the reflections τ_-, τ_+ on a type D_n face is not just to reflect the polygon, but also to reverse all of the flavors [FZ03, Proposition 3.16 (1)].

Lemma 5.6. If F is a facet for $\Delta(D_n)$ for n odd and $\varepsilon = \pm 1$, then $\tau_\varepsilon(F) \neq F$.

Proof. Suppose F contains a diameter which exists in one flavor but not the other. Then the reflection τ_ε causes there to exist a diameter in the other flavor with the same property. But only one flavor can have such a diameter in the original configuration, so τ_ε does not fix F.

There must be at least one diameter, for the same reason as for type B_n. This diameter exists in both flavors. Then no other diameter may exist, because differently-flavored diameters cannot cross. Therefore we are in exactly the same setting as in the proof of Lemma 5.4, so we can conclude. □

Again we are left to count the order ideals of the root poset by parity. The type-D poset is as described in the following Lemma.

Lemma 5.7. For $\widehat{T}_n$ a *tubular triangle poset*—i.e. two copies of a triangle poset, reflective copies of each other, affixed along the long end—and $J\left(\widehat{T}_n\right)$ its set of order ideals,

$$\sum_{I \in J\left(\widehat{T}_n\right)} (-1)^{|I|} = 0.$$

Proof. There is a copy of $\widehat{T}_{n-1}$, T, lying inside of $\widehat{T}_n$, which is $\widehat{T}_n$ without its minimal and rightmost elements. Let $F(I)$ be defined as in the proof of Lemma 5.5, and let $O(I) \subset \widehat{T}_n \setminus \widehat{T}_{n-1}$ be defined such that for each $\alpha \in O(I)$, there is $\alpha \in I$ with $f(\beta) \leqslant \alpha$. (Elements of $O(I)$ are "optional" to include in an order ideal of $\widehat{T}_n$ containing I.) For $I \in J\left(\widehat{T}_{n-1}\right)$, there are two cases: $F(I) \subsetneq \widehat{T}_n \setminus T$ and $F(I) = \widehat{T}_n \setminus T$.

In the former case, we can "toggle" any minimal element $\alpha \in \widehat{T}_n \setminus (T \cup F(I))$.

In the latter case, we may toggle any element of the bottom row of T other than the one with down-degree 1.

We have free rein to toggle any element of $O(I)$. □

W	E_6	E_7	E_8	F_4
$\mathrm{Cat}(W, 1, -1)$	-5	0	14	1
# fixed by τ_ε	5	$0, 24$	14	1

FIGURE 8. Data for symmetric case of dihedral sieving for exceptional types.

5.3. **Type I.** We view the action on Type I's cluster complex simply as the action of $\langle \tau_-, \tau_+ \rangle$ on $\Phi_{\geqslant -1}$, without any further combinatorial equivalences. The clusters are radially consecutive pairs of roots.

Arbitrarily label the simple roots α_- and α_+. We shall show, without loss of generality, that τ_- has exactly one fixed cluster. Note that $-\alpha_+$ is the only element of $\Phi_{\geqslant -1}$ on the same half-plane delineated by $\mathrm{span}\{\alpha_-\}$. $\tau_- = \sigma_{\alpha_-}$ since $I_- = \{\alpha_-\}$, so $-\alpha_+$ is fixed while α_- and $-\alpha_-$ exchange places, in turn exchanging their clusters. Similarly, the other cluster containing α_+ switches places with the other cluster containing $-\alpha_+$. Indeed, the only instance not nontrivially affected is the cluster whose positive span contains the normal vector to $\mathrm{span}\{\alpha_-\}$ not lying in the same half-plane as $-\alpha_+$. Thus precisely this one cluster is fixed.

It remains to count the order ideals by parity for type I. Here the poset comes from the construction in e.g. [CS15].

Lemma 5.8. Let $I^{(n)}$ be the line poset with two incomparable minimal elements, having n total elements, with $J(I^{(n)})$ its set of order ideals. For odd n,

$$\sum_{I \in J(I^{(n)})} (-1)^{|I|} = -1.$$

Proof. There is one order ideal of size 0 (the empty one), two of size 1 (one for each minimal element), one of size 3 (the minimal ones along with their parent), one of size 4, etc. Summing, we have

$$1 - 2 + \underbrace{1 - 1 + \cdots + 1 - 1}_{n-1 \text{ terms}} = -1$$

□

5.4. **Exceptional Types.** For the exceptional types E_6, E_7, E_8, F_4, we were able to compute both the polynomial and the number of facets fixed by reflection using Sage. This data is found in Figure 8.

Since we are only able to show odd dihedral sieving, the disparity for type E_7 is no surprise. We have now verified (up to sign) Claim 5.3 for all the desired types, which completes the proof of Theorem 1.5.

6. EVEN MORE SIEVING

6.1. **Extensions of Theorem 1.4.** There are three possible directions to further extend Theorem 1.4 on k-angulations of an n-gon (type A). The first is to consider *partial* k-angulations, also called $(k-2)$-*divisible dissections*. These correspond to non-maximal faces of the generalized cluster complex $\Delta^{(s)}(A_n)$. In [EF08], a cyclic sieving phenomenon is shown for this case. Rao and Suk conjectured in [RS17, Conjecture 6.2] that the little q, t-Shröder polynomials can give dihedral sieving for the case $s = 1$ (that is, partial triangulations). Unfortunately, there is a counterexample to that conjecture, for $n = 7$.

Another direction for possible generalization is to consider the generalized cluster complex $\Delta^{(s)}(\Phi)$ for root systems Φ of type other than A. The appropriate polynomial in this case would be the $\mathrm{FussCat}(\Phi, q, t)$ polynomials, defined in [Stu10]. This polynomial will be harder to work with, and it is also harder to count the number of generalized clusters fixed by reflection (it will not always be zero in this case).

The last generalization is to find some sort of object with a dihedral action which is counted by the rational q,t-Catalan numbers, $\mathrm{Cat}_{a,b}(q,t)$. One promising candidate might be the *rational associahedra* discussed in [ARW13]. While these objects do allow a cyclic action, it is not immediately clear what the action of reflection might be in this case.

We have, however, already done much of the work needed to evaluate the rational q,t-Catalan polynomials at $q=1, t=-1$. It may be possible to adapt the proof method of Theorem 4.17 to all (a,b)-Dyck paths, not just in the Fuss case. There is an alternative approach that may be easier, and is also of independent interest. The polynomial $\mathrm{Cat}_{a,b}(q,t)$ can be interpreted as a *flagged schur function*, which has a determinant formula given in [Wac85, Equation (3.1) and Lemma 3.3]. In computing this determinant in the Fuss case, we came across the following identity, which is more general than the case we were dealing with.

Proposition 6.1. Let $a_1, a_2, \ldots, a_{n+1}$ be a sequence of integers. Let M be the $(n+1)\times(n+1)$ matrix with (i,j)-entry $M_{ij} := \binom{a_i+j-1}{i-1}$. Then $\det M = 1$.

Proof. Let U be the $(n+1)\times(n+1)$ matrix with (i,j)-entry $U_{ij} := \binom{j-1}{i-1}$. Let L be the $(n+1)\times(n+1)$ matrix with (i,j)-entry $L_{ij} := \binom{a_i}{i-j}$. Then it can be checked using the Vandermonde identity that $L\cdot U = M$. Now, L is lower-triangular with diagonal entries all 1, and U and upper-triangular with diagonal entries all 1 as well. Therefore $\det M = \det L \cdot \det U = 1$. □

The above proposition can also be proven using row reduction. An alternate proof of Theorem 4.17 could have been given had we proven the following claim directly.

Conjecture 6.2. Suppose a and b are coprime positive integers. Let N be the $b\times b$ matrix with (i,j)-entry $N_{ij} := \binom{j+a+(2a-1)(b-i)}{i}$. Then $\det N = \frac{1}{2b+1}\binom{a(2b+1)}{b}$.

Note that this matrix is a minor of a matrix of the form in Proposition 6.1. There does not appear to be such a nice LU-decomposition, however, and row reduction has not had any success. We suspect that the above identity is a special case of something more general. If one could prove a more general fact, then combining with Proposition 6.1 could give a nice evaluation of the polynomial $\mathrm{Cat}_{a,b}(q,t)$.

6.2. **Even-n Dihedral Sieving.** As Rao and Suk remarked [RS17, Section 7], the dihedral group $I_2(n) = \langle r,s | r^n = s^2 = 1,\ rs = sr^{-1}\rangle$ is quite different for n even. Most notably, there are two distinct types of reflection s and rs, and they are not conjugate. Viewed as planar reflections on an n-gon, say that s is a reflection which goes through two vertices, and rs is the reflection which bisects two sides. The first instance of even-n dihedral sieving ought to be the simplest $I_2(n)$-action, namely, acting on the set $X=[n]$. In the case of the cyclic group and the dihedral group $I_2(n)$, this example gave us the q-analogue and the q,t-analogue for n, respectively. For even n, the we need to distinguish between the group elements s and rs somehow. The one-dimensional representation which does this is called χ_b in [RS17], and is defined as follows:

$$\chi_b(g) := \begin{cases} 1 & g\in\langle r^2,s\rangle \\ -1 & g\notin\langle r^2,s\rangle \end{cases}.$$

When n is odd, we have $\langle r^2,s\rangle = I_2(n)$, so this representation is nontrivial only in the even-n case. The need to utilize this representation as well suggests that we need to use three-variable polynomials for even-n dihedral sieving. The following definition does not quite fit into the framework of Definition 2.4, because it uses multiple representations, but it does still generalize Definition 1.2 because in the odd case, we always have $b = 1$.

Definition 6.3. Suppose X is a finite set acted on by the dihedral group $I_2(n)$, and $X(q, t, b)$ is a polynomial which is symmetric in q and t. The pair $(X \circlearrowleft I_2(n), X(q, t, b))$ has the *dihedral sieving phenomenon* if for all $g \in I_2(n)$ with eigenvalues $\{\lambda_1, \lambda_2\}$ for $\rho_{\mathrm{def}}(g)$ and λ_3 for $\chi_b(g)$:

$$|\{x \in X : gx = x\}| = X(\lambda_1, \lambda_2, \lambda_3).$$

With this definition in mind, we can define a sort of q, t, b-analogue of n as follows:

$$\langle n \rangle_{q,t,b} := \begin{cases} \{n\}_{q,t} & \nu_2(n) = 0 \\ \left\{\frac{n}{2}\right\}_{q,t} + b \cdot \left\{\frac{n}{2}\right\}_{q,t} & \nu_2(n) = 1 \\ \left\{\frac{n}{2}\right\}_{q,t} + b \cdot \left\{\frac{n}{4}\right\}_{q,t} + b^2 \cdot \left\{\frac{n}{4}\right\}_{q,t} & \nu_2(n) = 2 \\ \left\{\frac{n}{2}\right\}_{q,t} + b \cdot \left\{\frac{n}{4} + 1\right\}_{q,t} + b^2 \cdot \left\{\frac{n}{4} - 1\right\}_{q,t} & \nu_2(n) \geq 3 \end{cases}.$$

Since $I_2(n)$ is a subgroup of S_n, we can use instances of symmetric sieving to get instances of dihedral sieving. This q, t, b-analogue was defined with the aim of having the n terms in the summation $\langle n \rangle_{q,t,b}$ be exactly the eigenvalues of $\rho_{\mathrm{perm}}(g)$, when evaluated at $(q, t, b) = (\lambda_1, \lambda_2, \lambda_3)$ as in Definition 6.3. For a summation S, the *plethystic substitution* of S is $[S]$, the list consisting of the terms in S.

Proposition 6.4. The pair $\left(\left(\binom{[n]}{k}\right) \circlearrowleft I_2(n), h_k([\langle n \rangle_{q,t,b}])\right)$ exhibits dihedral sieving for all $n \geq k \geq 0$. Here h_k is the complete homogeneous polynomial in n variables and $[\langle n \rangle_{q,t,b}]$ denotes the plethystic substitution of $\langle n \rangle_{q,t,b}$.

Proof Sketch. A straightforward exhaustive check shows that for all $g \in I_2(n)$ the list $[\langle n \rangle_{\lambda_1,\lambda_2,\lambda_3}]$ is exactly the list of eigenvalues of g considered as an $n \times n$ permutation matrix. The result then follows from Proposition 2.7. □

While the above result is encouraging, it does not seem that the polynomial $h_k([\langle n \rangle_{q,t,b}])$ lends itself very well to product formulas such as the q, t binomial coefficient $\left\{ {n \atop k} \right\}_{q,t}$. Further work is needed to understand what, if any, is the "correct" notion of dihedral sieving when n is even. One major shortcoming of the current definition of $\langle n \rangle_{q,t,b}$ is that the expression seems particularly arbitrary when $\nu_2(n) \geq 3$. We would rather have a definition which is either independent of 2-valuation or is defined recursively on the 2-valuation.

6.3. **Symmetric Sieving for k-subsets.** In §2 we defined symmetric sieving and used the complete homogeneous polynomials for the case of k-multisubsets of $[n]$. A natural guess for symmetric sieving on k-subsets would be to use the elementary symmetric polynomials, but in fact these fail even in small cases. The polynomial we believe is correct is more complicated.

For an n-dimensional vector $\vec{v} = (v_1, \ldots, v_n)$ and non-negative integers a and b, we define $s_{a,b}(v_1, \ldots, v_n)$ to be the *Schur polynomial* $s_\lambda(v_1, \ldots, v_n)$, for the (hook) shape $\lambda = (a, 1, 1, \ldots, 1)$ with b total parts, unless a or b is zero, in which case we set $s_{a,0} = s_{0,b} = 0$. Then let $p_k(v_1, \ldots, v_n)$ be the symmetric polynomial defined by

$$p_k(v_1, \ldots, v_n) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \left(s_{k-2j+1,j}(v_1, \ldots, v_n) + s_{k-2j,j+1}(v_1, \ldots, v_n)\right).$$

We do not know of a more natural way to express this polynomial, though one may exist.

Conjecture 6.5. The pair $\left(\binom{[n]}{k} \circlearrowleft S_n, p_k(q_1, \ldots, q_n)\right)$ exhibits symmetric sieving for all $n \geq k \geq 0$.

We have directly checked the above conjecture whenever $k \leq 4$, and have some experimental evidence for it whenever $k \leq 7$. Using the plethystic substitution as we did for Proposition 6.4, this conjecture would imply an instance of dihedral sieving for even n on $\binom{[n]}{k}$.

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