GRAPH COVERINGS AND CRITICAL GROUPS OF SIGNED GRAPHS AND VOLTAGE GRAPHS

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Abstract. Given a graph covering, there is an induced surjection of critical groups. We will try to determine the structure of the kernel and give results at different levels of generality. A main tool will be dualizing the short exact sequence and working instead with the cokernel.

In the case of signed graphs and two-sheeted coverings, we will show that the kernel of the induced surjection can be described using the critical group of another signed graph. This, in particular, gives an interpretation of H. Bai's computation of the p-group component of the critical group of the n-cube, for p odd.

Contents

1. Introduction

Given a directed graph $G = (V, E)$, recall its directed incidence matrix $\partial \in \mathbb{Z}^{V \times E}$ is indexed by the rows and columns such that the column corresponding to edge e has a 1 in the entry corresponding to vertex u and a -1 in the entry corresponding to vertex v, if e is directed from v to u. Self-loops are allowed (and contribute a column of 0's) and so are multiple edges.

Recall that the critical group $K(G)$ of the graph G is defined to be $\mathbb{Z}^E/(\mathrm{im}(\partial) \oplus \ker(\partial))$ or equivalently $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$. As a starting point, Treumann [8] showed that if there is a "topological surjection" p from a graph $G' = (V', E')$ to a graph $G = (V, E)$, where he called p a Berman bundle, then there is an induced surjection of critical groups $K(G') \to K(G)$. A special case of these topological surjections are covering spaces.

Given the induced surjection $K(G') \to K(G)$, it is natural to ask what the kernel of the map is, so that we might determine more about the structure of the larger, more complicated graph G' in terms of $K(G)$ and the kernel.

In Section 2, we introduce the tool of Pontryagin duality, which we will use to dualize the surjection $K(G') \to K(G)$ to obtain an injection $K(G) \to K(G')$. As special cases, we will prove a natural isomorphism of $K(G) = \mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$ with its dual and an alternative presentation of the critical group of a signed graph, which will be defined later.

Then, we will use Pontryagin duality to turn our problem into finding the cokernel of the injection $K(G) \to K(G')$. Corollary 4.2 will tell us that the sequence $0 \to K(G) \to K(G') \to \mathrm{coker} \to 0$ is in fact split exact at all primes that do not divide n, where in the special case of G' being a cover of G, n is the number of sheets in the cover.

Proposition 5.1 gives a presentation for the cokernel of the injection $K(G) \to K(G')$ in general. In a more specific case (that still contains covers a special case), Proposition 5.2 gives only a slightly simpler presentation and Proposition 5.3 reduces it to finding the cokernel of a matrix. Even though these propositions are a little cumbersome, a corollary is Proposition 5.4, which gives the cokernel exactly in a special case. Given any connected graph G and its critical group, this special case helps determine the structure of the critical group of the graph obtained by replicating each vertex of G n times and forming a complete bipartite graph between two sets of n vertices for each edge in G. An example of an application of Proposition 5.4 is in Proposition 8.8 in Section 8.5.

Then, using the group algebra over the integers, Section 6 gives a description of the critical group $K(G')$ and the cokernel of the injection of $K(G)$ into $K(G')$ in the special case where G' is a regular cover of G. As an example, the results will apply to describing the critical groups of signed graphs without half self loops as cokernels of injections $K(G)$ into $K(G')$.

Finally, Section 7 covers critical groups of signed graphs. In particular, given an induced surjection $K(G') \to K(G)$ of signed graphs where G' is a 2-sheeted cover of G, the kernel can be described in terms of another signed graph. As an application, we provide an interpretation of Bai's proof [2] of the Sylow $p-$group of the critical group of n-cube for all odd primes p.

2. Pontryagin Duality

Let Λ_1 be a rational lattice in $\mathbb{R}^{n_1}$ and Λ_2 be a rational lattice in $\mathbb{R}^{n_2}$. Let $\Lambda_1^{\mathbb{R}}$ and $\Lambda_2^{\mathbb{R}}$ be the real vector spaces generated by Λ_1 and Λ_2 over $\mathbb{R}$. Suppose Λ_1 and Λ_2 have the same rank ($\dim(\Lambda_1^{\mathbb{R}}) = \dim(\Lambda_2^{\mathbb{R}})$).

Let M be a linear map from $\Lambda_1^{\mathbb{R}}$ to $\Lambda_2^{\mathbb{R}}$ such that $M\Lambda_2 \subset \Lambda_1$ and $M\Lambda_2$ is of full rank in Λ_1. In particular, M is an invertible map from $\Lambda_2^{\mathbb{R}}$ to $\Lambda_1^{\mathbb{R}}$. We have the following picture.

$$\begin{tikzcd}
\Lambda_2^{\mathbb{R}} \arrow[r, "\sim", "M"'] & \Lambda_1^{\mathbb{R}} \\
\Lambda_2 \arrow[u, hook] \arrow[r, hook] & \Lambda_1 \arrow[u, hook]
\end{tikzcd}$$

Taking duals, we claim we have the following diagram:

$$\begin{tikzcd}
(\Lambda_2^{\mathbb{R}})^* & (\Lambda_1^{\mathbb{R}})^* \arrow[l, "\sim"', "M^t"] \\
\Lambda_2^{\#} \arrow[u, hook] & \Lambda_1^{\#} \arrow[l, hook] \arrow[u, hook]
\end{tikzcd}$$

We also have to verify that $M^t\Lambda_1^{\#} \subset \Lambda_2^{\#}$. To do this, recall that M^t will take an element $v_1^* \in (\Lambda_1^{\mathbb{R}})^*$ and map it to $v_1^* \circ M$. Suppose $v_1^* \in \Lambda_1^{\#}$. Then, given any $v_2 \in \Lambda_2$, we see that $(v_1^* \circ M)(v_2) \in \mathbb{Z}$ as $Mv_2 \in \Lambda_1$. Therefore, $M^t\Lambda_1^{\#}$ is contained in $\Lambda_2^{\#}$

The group $\Lambda_1/M\Lambda_2$ is finite since M is invertible. Let $\widehat{\Lambda_1/M\Lambda_2}$ denote the Pontryagin Dual of $\Lambda_1/M\Lambda_2$. We also note that $\Lambda_2^{\#}/M^t\Lambda_1^{\#}$ is finite as M^t is an invertible map from $(\Lambda_1^{\mathbb{R}})^*$ to $(\Lambda_2^{\mathbb{R}})^*$.

Lemma 2.1. *There is a natural isomorphism between* $\Lambda_2^{\#}/M^t\Lambda_1^{\#}$ *and* $\widehat{\Lambda_1/M\Lambda_2}$.

Proof. Let κ be the quotient map from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$. Consider the map from $\Lambda_2^{\#}$ to $\mathrm{Hom}(\Lambda_1, \mathbb{R}/\mathbb{Z})$ (so the Pontryagin dual of Λ_1 with the discrete topology) that sends v_2^* to $\kappa \circ v_2^* \circ M^{-1}$. Note that $v_2^* \circ M^{-1}$ is $(\Lambda_1^{\mathbb{R}})^*$ but not necessarily in $\Lambda_1^{\#}$. First, we find the kernel.

Suppose that for all $v_1 \in \Lambda_1$, $(\kappa \circ v_2^* \circ M^{-1})(v_1) = 0 \Leftrightarrow (v_2^* \circ M^{-1})(v_1) \in \mathbb{Z}$. This is true if and only if $v_2^* \circ M^{-1} = (M^t)^{-1}v_2^* \in \Lambda_1^{\#}$ by the definition of the dual lattice. Therefore, the kernel is $M^t\Lambda_1^{\#}$. This means we have an inclusion of $\Lambda_2^{\#}/M^t\Lambda_1^{\#}$ into $\mathrm{Hom}(\Lambda_1, \mathbb{R}/\mathbb{Z})$.

Also, given $v_2^* \in \Lambda_2^{\#}$, we can also see the kernel of $\kappa \circ v_2^* \circ M^{-1}$ contains $M\Lambda_2$. To see this, suppose $v_1 \in M\Lambda_2$. Then, there exist $v_2 \in \Lambda_2$ such that $v_1 = Mv_2$. Then, $(\kappa \circ v_2^* \circ M^{-1}) \circ v_1 = (\kappa \circ v_2^* \circ M^{-1}) \circ Mv_2 = \kappa(v_2^*(v_2))$, which is 0 since $v_2^* \in \Lambda_2^{\#}$ and $v_2 \in \Lambda_2$.

Therefore, we have an inclusion of $\Lambda_2^{\#}/M^t\Lambda_1^{\#}$ into $\widehat{\Lambda_1/M\Lambda_2}$. We need to show that this is surjective. To see this, we will use a cardinality argument.

Since a finite abelian group is always isomorphic to its Pontryagin dual, we have $|\Lambda_2^{\#}/M^t\Lambda_1^{\#}| \leq |\widehat{\Lambda_1/M\Lambda_2}| = |\Lambda_1/M\Lambda_2|$.

However, we can repeat the exact same argument we used above, but instead replace Λ_1 with $\Lambda_2^{\#}$ and Λ_2 with $\Lambda_1^{\#}$. This would yield an injection from $(\Lambda_1^{\#})^{\#}/(M^t)^t(\Lambda_1^{\#})^{\#} = \Lambda_1/M\Lambda_2$ to $\widehat{\Lambda_2^{\#}/M^t\Lambda_1^{\#}}$. This means $|\Lambda_1/M\Lambda_2| \leq |\Lambda_2^{\#}/M^t\Lambda_1^{\#}|$.

Combining this with the inequalities above, we get $|\Lambda_2^{\#}/M^t\Lambda_1^{\#}| = |\widehat{\Lambda_1/M\Lambda_2}|$. This means our injection must also be a surjection and we have an isomorphism from $\Lambda_2^{\#}/M^t\Lambda_1^{\#}$ to $\widehat{\Lambda_1/M\Lambda_2}$, as desired. □

There are a couple of examples that will be important to us.

Example 1. Let $G = (V, E)$ be an undirected graph (possibly with multiple edges). Let $\partial\partial^t$ be its Laplacian. We define the critical group $K(G)$ to be $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$.

It can be shown that is S is a subset of V containing one vertex in each connected component of G, $L(G)_0$ is $\partial\partial^t$ with the columns and rows indexed by S deleted, then $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$ is isomorphic to $\mathbb{Z}^{V\setminus S}/\mathrm{im}(L(G)_0)$ by projecting away the coordinates corresponding to S. (It should not surprise the reader that we have to make this arbitrary choice of S, as the Pontryagin duality in the presentation of the critical group as $\mathbb{Z}^E/(\mathrm{im}(\partial) \oplus \ker(\partial))$ comes from projecting onto the bond space and taking the inner product. Choosing a basis for the bond space in terms of fundamental bonds involves choosing one fundamental bond in each connected component to omit.)

Then, if we let $\Lambda_1 = \Lambda_2 = \mathbb{Z}^{V\setminus S}$, then Proposition 2.1 gives us $\widehat{\mathbb{Z}^{V\setminus S}/\mathrm{im}(L(G)_0)}$ is isomorphic to $(\mathbb{Z}^{V\setminus S})^{\#}/L(G)_0(\mathbb{Z}^{V\setminus S})^{\#}$. If we identify $\mathbb{Z}^{V\setminus S}$ with $(\mathbb{Z}^{V\setminus S})^*$ with the standard inner product, then

$$(\mathbb{Z}^{V\setminus S})^{\#}/L(G)_0(\mathbb{Z}^{V\setminus S})^{\#} = \mathbb{Z}^{V\setminus S}/\mathrm{im}(L(G)_0).$$

However, we would like to eliminate the need to select the subset S, which we can do.

Proposition 2.2. *There is a isomorphism between* $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$ *to* $\widehat{\mathrm{im}(\partial)/\mathrm{im}}(\partial\partial^t$ *that sends a coset representative* $w \in \mathrm{im}(\partial)$ *of an element of* $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$ *to the character* $\langle u \cdot \rangle \pmod 1$, *where u is any preimage of w under the map* $\partial\partial^t : \mathbb{Z}^V \to \mathrm{im}(\partial)$.

Proof. We note that the map from $\mathbb{Z}^{V\setminus S}/\mathrm{im}(L(G)_0)$ to $\mathbb{Z}^{V\setminus S}\widehat{/\mathrm{im}(L}(G)_0)$ maps a coset representative $v \in \mathbb{Z}^{V\setminus S}$ to $\langle v, L(G)_0^{-1}\cdot\rangle \pmod 1 = \langle L(G)_0^{-1}v, \cdot\rangle \pmod 1$. If we pad $L(G)_0^{-1}v$ with 0's in the coordinates corresponding elements of S, then we get a character on $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$. Morever, this map from $\mathbb{Z}^{V\setminus S}\widehat{/\mathrm{im}(L}(G)_0)$ to $\mathrm{im}(\widehat{\partial)/\mathrm{im}(}\partial\partial^t)$ is injective (hence an isomorphism).

Furthermore, after padding $L(G)_0^{-1}v$ with zeros corresponding to elements of S, we can add any element of $\ker(\partial^t)(=\ker(\partial\partial^t)$, since $\ker(\partial^t) \perp \mathrm{im}(\partial)$. Equivalently, if we take the matrix after deleting only the rows corresponding to S of $\partial\partial^t$, then we can take any preimage u of v under this matrix instead of choosing $L(G)_0^{-1}v$ (padded with 0's). Finally, this is equivalent to taking the unique vector $w \in \mathrm{im}(\partial)$ whose coordinates corresponding to vertices in $V\setminus S$ agree with v and letting u be any preimage of w under the map $\partial\partial^t : \mathbb{Z}^V \to \mathrm{im}(\partial)$. □

Example 2. Let $G = (V, E)$ be a connected, signed graph and $\partial\partial^T$ be its Laplacian, where we assume ∂ has full rank. We will define what exactly ∂ and ∂^T are in Section 7, but it suffices to know that, if $\mathbb{Z}^V_{0 \pmod 2}$ is defined as the sublattice of $\mathbb{Z}^V$, where all the entries add up to an even number, then $\mathrm{im}(\partial) = \mathbb{Z}_{0 \pmod 2}$ and $\partial\partial^T$ is symmetric. We will define the critical group of the signed graph G to be $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^T)$.

Let $\Lambda_1 = \mathbb{Z}^V_{0 \pmod 2}$ and $\Lambda_2 = \mathbb{Z}^V$ in Lemma 2.1. Then, Lemma 2.1 gives us a natural isomorphism between $\mathbb{Z}^V_{0 \widehat{\pmod 2)/}}\partial\partial^T\mathbb{Z}^V$ and $\Lambda_2^\#/(\partial\partial^T)^t\Lambda_1^\# = \mathbb{Z}^V/\partial\partial^T(\mathbb{Z}^V_{0 \pmod 2})^\#$. Here, it can be shown that $(\mathbb{Z}^V_{0 \pmod 2})^\# = \mathbb{Z}^V + \mathbb{Z}(\frac{1}{2}, \ldots, \frac{1}{2})$.

3. Covering spaces and Berman bundles

3.1. **Covering spaces.** In [4], the authors defined a *graph map* from a graph G' to a graph G to be a continuous function that maps the interior of each edge of G' homeomorphically to the interior of an edge of G. The authors also characterized when G' is actually a covering space of G.

In order to present their characterization here, we will reproduce their notion of a *permutation voltage graph* and a *derived graph* here, where the derived graph is the actual covering space and the permutation voltage graph is a recipe for how to build the derived graph. The notation is copied almost word for word, but there is a small issue. Gross and Tucker defined the graph coverings for directed graphs, and, since we are interested in the undirected case, we need to

(1) arbitrarily orient the edges of G,
(2) assign permutations to each edge of G with a function s,
(3) construct the derived graph G^s from the permutations,
(4) forget about the orientation on G^s

to construct a covering of an undirected graph.

Definition 1. Let S_n denote the symmetric group on $\{1, \ldots, n\}$. A permutation voltage assignment in S_n for a directed graph $G = (V, E)$ is a function s that assigns to each edge of G a permutation in S_n. The pair (G, s) is called a permutation voltage graph.

Definition 2. Given a permutation voltage graph (G, s), a derived graph G^s is constructed as follows:

(1) The vertex set is the cartesian product $V \times \{1, \ldots, n\}$. For convenience, the vertices will be denoted as v_i instead of (v, i).
(2) The edge set is the the cartesian product $E \times \{1, \ldots, n\}$. For convenience, the edges will be written as e_i instead of e_i.
(3) If edge e runs from u to v in G, then e_i runs from u_i to $v_{\pi(i)}$, where π is the permutation $s(e)$ associated to e by s.

An example of a covering space derived is in Figure 1.

Remark 1. To see why this is a characterization of the graph coverings, suppose we are given a graph G and a covering G^s that is an $n-$sheeted cover.

Since G^s is an $n-$sheeted cover, the fibre of each vertex v is of order n, and we can label these vertices as $v_1, \ldots, v_n$. It remains to understand the edges in G^s. Suppose there is an edge e between v and u and the vertices in the fibres of v and u are $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$.

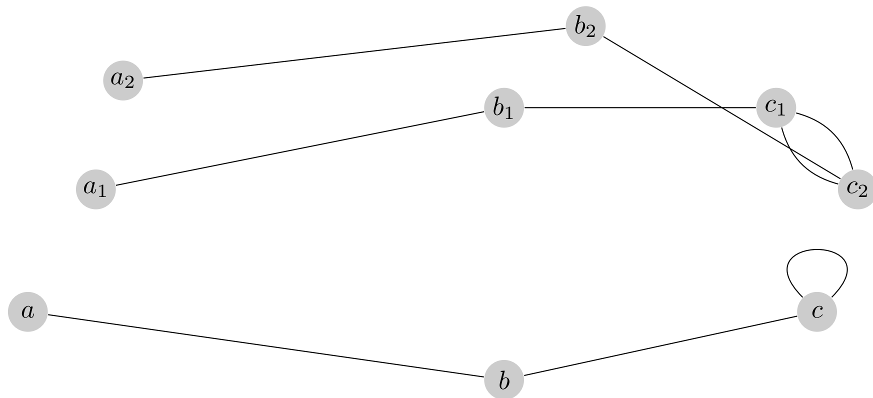

Figure 1. Vertices $\{a_1, a_2\}$ project to a, $\{b_1, b_2\}$ project to b, and $\{c_1, c_2\}$ project to c

From the lifting property of covering spaces, if we are given a vertex v_i in the fibre of v, the path from v to u along the edge e lifts to a path from v_i to a vertex in the fibre of u (say u_j). We can repeat this process for each vertex v_i in the fibre of v. This results in a map π from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ where the numbers correspond to indices of the vertices in the fibres of v and u.

This map also must be an surjection. To see this, given a vertex u_i in the fibre of u, we can lift a path from u to v by going along e in the opposite direction to a path from u_i to a vertex in the fibre of v. This means i is in the image of π. Since π is surjective, π is in fact a permutation. Therefore, we can encode any covering graph of a graph G by assigning permutations to the edges (and the orientation of the edges correspond to whether we are lifting a path from v to u or a path from u to v above).

3.2. **Berman Bundles.** Treumann introduced the notion of a "Berman Bundle" in [8]. We will use a slightly different definition that is equivalent.

Let G' and G be graphs, and G' be a covering space of G. Then, if $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and $G' = (V', E')$, the vertices of V' can be partitioned into the fibres $U_1, \ldots, U_n$ such that U_i maps to v_i for each $1 \leq i \leq n$.

Now, take G' and

(1) add new vertices to the fibres
(2) and then add edges within the fibres.

If the graph G' can be obtained in this way (by modifying a cover of G as above), then we call the graph G' *divisible* by G.

If G' is divisible by G, then there is a projection map from G' to G that sends all the vertices in U_i to v_i for each $1 \leq i \leq n$.

Definition 3. If G' is divisible by G and $p : G' \to G$ is the associated projection map, then we call p a *Berman bundle*.

Definition 4. If $p : G' \to G$ is a Berman bundle, let G'_p be the original covering space of G we used to construct G'.

An example of a Berman bundle is in Figure 2. The original covering space of the Berman bundle in Figure 2 is the covering space in Figure 1.

The reason for the interest in Berman Bundles is Proposition 19 in [8] which says that if $p : G' \to G$ is a Berman Bundle, then there is an induced surjection of critical groups from $K(G')$ to $K(G)$.

Since we will be working with the critical group in terms of the graph Laplacian instead of cycle and bond spaces as in [8], we will need to describe the surjection in our case. We first prove the surjection independently of Treumann. Then, in Proposition 3.3, we will show that Proposition 3.1 is equivalent to the one given by Treumann.

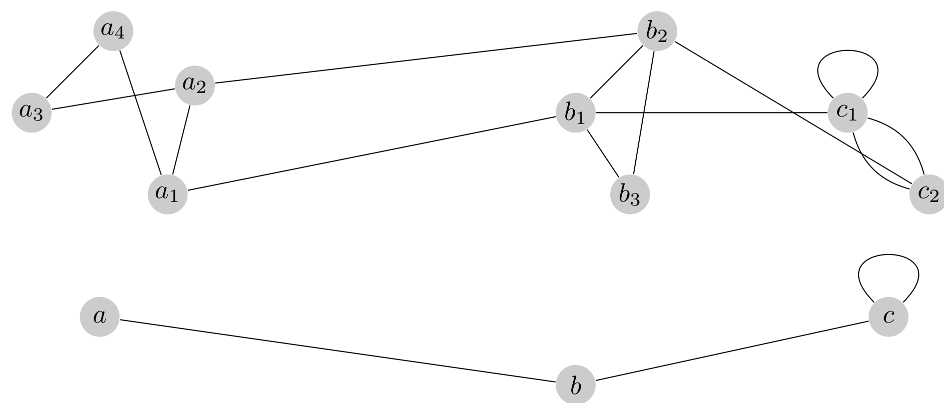

FIGURE 2. Vertices $\{a_1, a_2, a_3, a_4\}$ project to a, $\{b_1, b_2, b_3\}$ project to b, and $\{c_1, c_2\}$ project to c

Definition 5. Given a Berman bundle $p : G' = (V', E') \to G = (V, E)$, consider the map ϕ of lattices from $\mathbb{Z}^{V'}$ to $\mathbb{Z}^V$ that sends a standard basis element $e_{v'} \in \mathbb{Z}^{V'}$ to $e_{p(v)}$ in $\mathbb{Z}^V$. We call ϕ the *induced surjection of lattices.*

We will show in the proof of Proposition 3.1 that ϕ is indeed surjective and ϕ restricts to a surjection from $\operatorname{im}(\partial')$ to $\operatorname{im}(\partial)$, where ∂ and ∂' are the directed incidence matrices of G and G'.

Proposition 3.1. *Suppose $G = (V, E)$, $G' = (V', E')$ and $p : G' \to G$ is a Berman Bundle. Then, there is an induced surjection from $K(G')$ to $K(G)$.*

Proof. Let ϕ be the induced surjection of lattices as in Definition 5 from $\mathbb{Z}^{V'}$ to $\mathbb{Z}^V$ that sends a standard basis element $e_{v'} \in \mathbb{Z}^{V'}$ to $e_{p(v)}$ in $\mathbb{Z}^V$. Let $\operatorname{im}(\partial')$ be the sublattice of $\mathbb{Z}^{V'}$ consisting of elements in $\mathbb{Z}^{V'}$ orthogonal to the characteristic vector of each connected component of G' (e.g. sum of the coordinates corresponding to each connected component is zero). Let $\operatorname{im}(\partial) \subset \mathbb{Z}^V$ be defined similarly.

We want to show that our map ϕ of lattices of $\mathbb{Z}^{V'}$ to $\mathbb{Z}^V$ restricts to a map from $\operatorname{im}(\partial')$ that maps surjectively into $\operatorname{im}(\partial)$.

To show this, it suffices to show that the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z}^{E'} \arrow[r, two heads, "\psi"] \arrow[d, two heads, "\partial'"] & \mathbb{Z}^E \arrow[d, two heads, "\partial"] \\ \operatorname{im}(\partial') \arrow[r, "\phi"] & \operatorname{im}(\partial) \end{tikzcd}$$

where ψ maps $e_{k'}$ for $k' \in E'$ to $e_{p(k')}$ if k' was from the original covering space G'_p and maps $e_{k'}$ to 0 if k' was not an edge from the original covering space. This is well-defined since there is a natural n to 1 identification of the edges of G'_p with the edges of G under p, if G'_p is an $n-$sheeted covering of G.

To verify this, it suffices to chase $e_{k'}$ for $k' \in E'$ clockwise and counterclockwise in the picture from $\mathbb{Z}^{E'}$ to $\operatorname{im}(\partial)$. If k' was not in the original covering space, then going clockwise would result in zero since $\psi(e_{k'})$ is already zero. Going counterclockwise would also result in zero, since both edges of k' would be in the same fibre. Then, $\phi(\partial' e_{k'})$ would be zero.

If k' is in the original covering, then going clockwise would result in $\partial e_{p(k')}$. Now, we go counterclockwise. Suppose k' runs from v' to u' in G'. Then, $\partial' e_{k'}$ is the vector $e_{u'} - e_{v'} \in \operatorname{im}(\partial') \subset \mathbb{Z}^{V'}$. Then, $\phi(e_{u'} - e_{v'}) = e_{p(u')} - e_{p(v')} \in \operatorname{im}(\partial)$. This is exactly $\partial e_{p(k')}$ as desired.

Therefore, we have the following picture:

$$\begin{tikzcd}
\mathbb{Z}^{V'} \arrow[rr, two heads, "\phi"] & & \mathbb{Z}^{V} \\
\mathrm{im}(\partial') \arrow[u, hook] \arrow[rr, two heads, "\phi"'] \arrow[d, two heads, "\kappa_1"] & & \mathrm{im}(\partial) \arrow[u, hook] \arrow[d, two heads, "\kappa_2"] \\
K(G') := \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t) & & K(G) := \mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)
\end{tikzcd}$$

To finish, we need to show that $\kappa_2 \circ \phi$ factors through the quotient map κ_1. It suffices to show that every column of $\partial'\partial'^t$ is sent to zero by $\kappa_2 \circ \phi$. Recall that for $v' \in V'$, the column x of $\partial'\partial'^t$ corresponding to v' has the property that

(1) $-x_{v''}$ is the number of edges between v' and v'' if $v' \neq v''$
(2) $x_{v'}$ is the degree of v'.

First, we note that the component of $\phi(x)$ corresponding to a vertex v is $\phi(x)_v = \sum_{v'' \in p^{-1}(v)} x_{v''}$ by the definition of ϕ.

There are two cases to consider.

(1) If v' is not in the original covering space G'_p, then the image of the column x is identically zero. To see this, if $v \in V$ and $v \neq p(v')$, then $\phi(x)_v = \sum_{v'' \in p^{-1}(v)} x_{v''} = \sum_{v'' \in p^{-1}(v)} 0 = 0$.
   If $v \in V$ and $v = p(v')$, then $\phi(x)_v = \sum_{v'' \in p^{-1}(v)} x_{v''}$, but since the components of x corresponding to vertices in $p^{-1}(v)$ are the only nonzero entries of the vector x, this is zero since $x \in \mathrm{im}(\partial')$.
(2) Suppose that v' is in the original covering space G'_p. For convenience, we can assume that G has no self-loops. To see this, if G' is divisible by G, then G' is still divisible by G after we remove the self loops (which does not affect the critical group of G). In addition, none of the maps in this proof are affected by the self-loops.
   Then, by definition of ϕ, if $v \neq p(v')$, then $-\phi(x)_v = -\sum_{v'' \in p^{-1}(v)} x_{v''}$ is the number of edges between v' and any vertex in the fibre $p^{-1}(v)$. By the definition of a Berman Bundle, this must be the number of edges between $p(v')$ and v.
   If $v = p(v')$, then $\phi(x)_v = \sum_{v'' \in p^{-1}(v)} x_{v''}$ is the degree of v' minus the number of vertices in $p^{-1}(p(v'))$ adjacent to v'. Equivalently, this is the the number of edges between v' and all other vertices in a different fibre $(G' \backslash p^{-1}(p(v')))$. This is exactly the degree of $p(v') = v$ by the definition of a Berman Bundle.
   Therefore, this means $\phi(x)$ is exactly the column of the Laplacian $\partial\partial^t$ corresponding to $p(v')$.

Since $\phi(\mathrm{im}(\partial'\partial'^t)) \subset \mathrm{im}(\partial\partial^t)$, the map $\kappa \circ \phi$ factors through $K(G') := \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t)$, as desired. Since $\kappa \circ \phi$ is surjectve, the map from $K(G')$ to $K(G)$ is surjective. $\square$

Example 3. To illustrate how $\phi(\mathrm{im}(\partial'\partial'^t) \subset \mathrm{im}(\partial\partial^t)$, consider the Berman Bundle in Figure 2. The Laplacian of the larger graph G' is

$$\begin{array}{c c}
 & \begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & c_1 & c_2 \end{array} \\
\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_2 \end{array} &
\begin{pmatrix}
3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 4 & -1 & -1 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & -2 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 3
\end{pmatrix}
\end{array}$$

which after applying ϕ is

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & c_1 & c_2 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

The Laplacian of G is

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \end{array},$$

so the image of each column of $\partial'\partial'^t$ is either 0 or a column of $\partial\partial^t$

Remark 2. Not only is $\phi(\mathrm{im}(\partial'\partial'^t)) \subset \mathrm{im}(\partial\partial^t)$, but $\phi(\mathrm{im}(\partial'\partial'^t)) = \mathrm{im}(\partial\partial^t)$. To see this, consider any vertex $v \in G$. If v is not connected to any edges, then the column associated with v in $\partial\partial^t$ is zero.

Otherwise, v is connected to some other vertex u. By the extra condition we imposed on Berman bundles, there exists a vertex $v' \in p^{-1}(v)$ such that v' is adjacent to a vertex in $p^{-1}(u)$. Then, v' must fall under the second case in the proof of Proposition 3.1 above. This means the column of $\partial'\partial'^t$ associated with v' must map to the column of $\partial\partial^t$ associated with v. Since we can do this for any vertex v that is not isolated, the image of $\mathrm{im}(\partial'\partial'^t)$ under ϕ is in fact all of $\mathrm{im}(\partial\partial^t)$.

Proposition 3.2. *Given a Berman bundle $p : G' \to G$, let $\kappa : \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t) \to \mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$ be the induced surjection of critical groups from Proposition 3.1 above.*

Let $\Lambda^{V'}$ be the sublattice of $\mathrm{im}(\partial')$ where, for each vertex v of G, the sum of the coordinates corresponding to vertices in the fibre $p^{-1}(v)$ is zero. The kernel of κ as a subgroup of $\mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t)$ is $\Lambda^{V'}/\Lambda^{V'} \cap \mathrm{im}(\partial'\partial'^t) \cong \Lambda^{V'} + \mathrm{im}(\partial'\partial'^t)/\mathrm{im}(\partial'\partial'^t)$.

Proof. Reproducing the diagram in Proposition 3.1, we have the following picture

$$\begin{tikzcd} \Lambda^{V'} \arrow[r, hook, "\iota"] & \mathrm{im}(\partial') \arrow[r, two heads, "\phi"] \arrow[d, two heads, "\kappa_1"] & \mathrm{im}(\partial) \arrow[d, two heads, "\kappa_2"] \\ & K(G') := \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t) \arrow[r, two heads, "\kappa"] & K(G) := \mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t) \end{tikzcd}$$

To determine the kernel of κ, we first note that the kernel of κ_2 is by definition $\mathrm{im}(\partial\partial^t)$. Also, $\phi^{-1}(\mathrm{im}(\partial\partial^t)) = \ker(\phi) + \mathrm{im}(\partial'\partial'^t) = \Lambda^{V'} + \mathrm{im}(\partial'\partial'^t)$ since $\phi(\mathrm{im}(\partial'\partial'^t)) = \mathrm{im}(\partial\partial^t)$.

Finally, the image of $\Lambda^{V'} + \mathrm{im}(\partial'\partial'^t)$ under κ is $(\Lambda^{V'} + \mathrm{im}(\partial'\partial'^t))/\mathrm{im}(\partial'\partial'^t) \cong \Lambda^{V'}/\Lambda^{V'} \cap \mathrm{im}(\partial'\partial'^t)$, as desired. ☐

To show that the surjection in Proposition 3.1 is equivalent to the one from in Treumann [8], given graph $G' = (V', E')$, $G = (V, E)$, and a Berman bundle $p : G' \to G'$, there is a subgraph $G'_p = (V'_p, E'_p)$ of G' where p restricted to G' is a covering space. In particular, this means $p : E'_p \to E$ is an n to 1 map, if G'_p is an $n-$sheeted covering of G.

Now, that we have identified the covering space G'_p inside of G', we can define the induced surjection of critical groups $K(G') \to K(G)$ introduced by Treumann. Consider the map $\phi : \mathbb{Z}^{E'} \to \mathbb{Z}^E$, where if $a \in E'$, then $e_a \in \mathbb{Z}^{E'}$ (the standard basis element associated with a) is mapped by ϕ to zero if $a \notin E'_p$ and mapped to $e_{\phi(a)} \in \mathbb{Z}^E$ otherwise.

Then, as asserted by [8], if B' and Z' are the bond and cycle lattices of G' and B and Z are the bond and cycle lattices of G, then $\phi(B' \oplus Z') \subset B \oplus Z$. Note that defining the cycle and bond spaces require us to arbitrarily orient the edges of G' and G, so we need to require that p preserves orientation of the edges (if p does not preserve the orientation of an edge $a \in E'$, we can just reverse the orientation). This means we have the following picture

$$\begin{tikzcd}
 & \mathbb{Z}^{E'} \arrow[r, two heads, "\phi"] \arrow[d, "\kappa_1"] & \mathbb{Z}^{E} \arrow[d, "\kappa_2"] & \\
\operatorname{im}(\partial')/\partial'\partial'^t \arrow[r, "\sim"] & \mathbb{Z}^{E'}/(B' \oplus Z') \arrow[r, two heads] & \mathbb{Z}^{E}/(B \oplus Z) \arrow[r, "\sim"] & \operatorname{im}(\partial)/\partial\partial^t
\end{tikzcd}$$

where the surjection from $\mathbb{Z}^{E'}/(B' \oplus Z')$ to $\mathbb{Z}^{E}/(B \oplus Z)$ is induced.

Recall that $\mathbb{Z}^{E'}/(B' \oplus Z')$ is isomorphic to $\mathbb{Z}^{V'}/\partial'\partial'^t$ as they are both presentation of the critical group $K(G')$. Therefore, we need to chase the arrows from $\mathbb{Z}^{V'}/\partial'\partial'^t$ to $\mathbb{Z}^{V}/\partial\partial^t$ in the diagram. We first note that is suffices to perform the diagram chase for the case where G' and G are both connected. If G' is not connected, then we could restrict ourselves to each connected component S' of G'. Here, the image of S' under p must be a connected component of G. If we can understand the surjection from $K(S')$ to $K(p(S'))$ for each connected component S' of G', then that determines the map from $K(G')$ to $K(G)$ since $K(G') = \oplus_i^k K(S_i')$, where S_i' are the connected components of G'.

Therefore, we assume that G' and G are connected. First, we recall the isomorphism between $\mathbb{Z}^V/\partial\partial^t$ and $\mathbb{Z}^E/(B' \oplus Z)$.

Recall that if ∂ is a directed incidence matrix of G, then, if we delete the column corresponding to some vertex $v_0 \in G$ to obtain $\overline{\partial}$, $\overline{\partial}^t$ is a basis for the bond lattice B of G. Recall also that the projection $\overline{\partial}^t(\overline{\partial\partial}^t)^{-1}\overline{\partial}$ from $\mathbb{R}^E$ to the bond *space* ($\mathbb{R}$-linear span of the bond lattice) maps $\mathbb{Z}^E/(B \oplus Z)$ isomorphically unto $B^\#/B$, where $B^\#$ is identified as a sublattice of the bond space through the standard inner product.

Then, since $\overline{\partial}^t(\overline{\partial\partial}^t)^{-1}$ is a basis for $B^\#$ after making this identification of $B^\#$ inside of the bond space, we see that $B^\#/B = \overline{\partial}^t(\overline{\partial\partial}^t)^{-1}\mathbb{Z}^{V\setminus\{v_0\}}/\overline{\partial}^t\mathbb{Z}^{V\setminus\{v_0\}}$. This is isomorphic to $\mathbb{Z}^{V\setminus\{v_0\}}/\overline{\partial\partial}^t\mathbb{Z}^{V\setminus\{v_0\}} \cong \operatorname{im}(\partial)/\operatorname{im}(\partial\partial^t)$ by sending the coset representative $e_v \in \mathbb{Z}^{V\setminus\{v_0\}}$ to $\overline{\partial}^t(\overline{\partial\partial}^t)^{-1}e_v$.

Therefore, if $y \in \mathbb{Z}^{V\setminus\{v_0\}}$ is a coset representative of a coset in $\mathbb{Z}^{V\setminus\{v_0\}}/\overline{\partial\partial}^t\mathbb{Z}^{V\setminus\{v_0\}}$ and x is a preimage of y under the map $\overline{\partial} : \mathbb{Z}^E \to \mathbb{Z}^{V\setminus\{v_0\}}$, then the coset of x in $\mathbb{Z}^E/(B \oplus Z)$ maps to the coset of y in $\mathbb{Z}^{V\setminus\{v_0\}}/\overline{\partial\partial}^t\mathbb{Z}^{V\setminus\{v_0\}}$ since the image of x under the projection onto the Bond space is $\overline{\partial}^t(\overline{\partial\partial}^t)^{-1}\overline{\partial}x = \overline{\partial}^t(\overline{\partial\partial}^t)^{-1}y$. The coset corresponding to $\overline{\partial}^t(\overline{\partial\partial}^t)^{-1}y$ in $B^\#/B = \overline{\partial}^t(\overline{\partial\partial}^t)^{-1}\mathbb{Z}^{V\setminus\{v_0\}}/\overline{\partial}^t\mathbb{Z}^{V\setminus\{v_0\}}$ maps exactly to the coset represented by y in $\mathbb{Z}^{V\setminus\{v_0\}}/\overline{\partial\partial}^t\mathbb{Z}^{V\setminus\{v_0\}}$.

Finally, suppose y' is the element in $\operatorname{im}(\partial)$ obtained by filling in the last coordinate of y so that the coordinates of y' sum to zero. Then, since every element in the image of ∂ is in $\operatorname{im}(\partial)$, x is mapped by ∂ to y'. To summarize, in order to map an element of $\operatorname{im}(\partial)/\operatorname{im}(\partial\partial^t)$ to $\mathbb{Z}^E/(B \oplus Z)$, it suffices to

(1) take a coset representative y' of the element
(2) find a preimage $x \in \mathbb{Z}^E$ of y' under the map ∂
(3) take the coset of the preimage x

Proposition 3.3. *The surjection given in Proposition 3.1 is equivalent to the surjection given in* [8].

Proof. We will chase the arrows from $\operatorname{im}(\partial')/\partial'\partial'^t$ to $\operatorname{im}(\partial)/\partial\partial^t$. We will work with the coset representatives, and everything will be well-defined since all the arrows are isomorphisms.

We first go from $\operatorname{im}(\partial')/\partial'\partial'^t$ to $\mathbb{Z}^{E'}/(B' \oplus Z')$. First, fix an element $x \in \operatorname{im}(\partial')$ that we will view as a coset representative in $\operatorname{im}(\partial')/\partial'\partial'^t$. Then, if $y \in \mathbb{Z}^{E'}$ is a preimage of x under the map $\partial' : \mathbb{Z}^{E'} \to \operatorname{im}(\partial')$, then the coset represented by y in $\mathbb{Z}^{E'}/(B' \oplus Z')$ maps to the coset represented by x in $\operatorname{im}(\partial')/\partial'\partial'^t$.

The map ϕ above from $\mathbb{Z}^{E'}$ to $\mathbb{Z}^E$ maps y to $\phi(y) \in \mathbb{Z}^E$. Then, we apply ∂ to $\phi(y)$ to get a representative of a coset in $\operatorname{im}(\partial)/\partial\partial^t$. Consider a vertex $v \in V$. Let $\operatorname{in}^+(v)$ denote the edges directed into v and $\operatorname{in}^-(v)$ denote the edges directed out of v. Then, the component of $\partial\phi(y)$ corresponding to v is

$$\sum_{e\in\operatorname{in}^+(v)} \phi(y)_e - \sum_{e\in\operatorname{in}^-(v)} \phi(y)_e.$$

Let $\operatorname{in}^+(p^{-1}(v))$ denote the edges of G' directed into the fibre of v and $\operatorname{in}^-(p^{-1}(v))$ be defined similarly. Then, by how ϕ maps the edges,

$$\sum_{e\in\operatorname{in}^+(v)} \phi(y)_e - \sum_{e\in\operatorname{in}^-(v)} \phi(y)_e = \sum_{e\in\operatorname{in}^+(p^{-1}(v))} y_e - \sum_{e\in\operatorname{in}^-(p^{-1}(v))} y_e.$$

Since the edges within the fibre $p^{-1}(v)$, this is equal to

$$\sum_{v'\in p^{-1}(v)} (\partial y)_{v'} = \sum_{v'\in p^{-1}(v)} x_{v'}.$$

Therefore, in order to get the component corresponding to v in the image of x in $\mathrm{im}(\partial)/\partial\partial^t$, we sum all the components corresponding to vertices in $p^{-1}(v)$, which is exactly the same as Proposition 3.1 above. ☐

Because $\phi(\mathrm{im}(\partial'\partial'^t)) = \partial\partial^t$, we can also determine the kernel of the induced surjection $\kappa : \mathbb{Z}^{V'}/\mathrm{im}(\partial'\partial'^t) \to \mathbb{Z}^V/\mathrm{im}(\partial\partial^t)$ as a subgroup of $mbZ^{V'}/\mathrm{im}(\partial'\partial'^t)$.

4. Exact Sequences from Induced Surjections

From Proposition 3.1 we know that if $p : G' \to G$ is a Berman bundle, then there is an induced surjection $\kappa : K(G') \to K(G)$ of critical groups. We would like to understand the critical group $K(G')$ of the larger graph through knowledge of the kernel of κ and the group $K(G)$. We determined a presentation of $\ker(\kappa)$ in Proposition 3.2. This gives us the short exact sequence

$$0 \leftarrow K(G) \overset{\kappa}{\leftarrow} K(G') \leftarrow \ker(\kappa) \leftarrow 0$$

However, in order to understand $\ker(\kappa)$ better, it will often be useful to consider the cokernel to the short exact sequence we get after applying Pontryagin duals.

$$0 \to \widehat{K(G)} \overset{\widehat{\kappa}}{\to} \widehat{K(G')} \to \mathrm{coker}(\widehat{\kappa}) \to 0$$

4.1. **The dual map $\widehat{\kappa}$.** We want to understand the map $\widehat{\kappa}$ more concretely using Lemma 2.1. We will find an injection from $K(G)$ to $K(G')$ by the sequence of maps

$$K(G) \overset{\sim}{\to} \widehat{K(G)} \overset{\widehat{\kappa}}{\to} \widehat{K(G')} \overset{\sim}{\to} K(G'),$$

where the isomorphisms are from Pontryagin duality. By an abuse of notation, we will refer to the injection from $K(G)$ to $K(G')$ also as $\hat{\kappa}$.

Recall that from Example 1, that there is an isomorphism from $K(G) := \mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$ to $\widehat{K(G)}$ that sends an element of $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$ with coset representative $v \in \mathrm{im}(\partial)$ to the character $\langle u, \cdot\rangle \pmod 1 \in \widehat{K(G)}$, where u is any preimage of v under the map $\partial\partial^t : \mathbb{Z}^V \to \mathrm{im}(\partial)$. From proof of Lemma 2.1 and Proposition 2.2, this character is independent of the coset representative v chosen, independent of the preimage u, and independent of the coset representative at which to evaluate the function.

By definition, the map $\widehat{\kappa} : \widehat{K(G)} \to \widehat{K(G')}$ takes $\langle u, \cdot\rangle \pmod 1$ and maps it to $\langle u, \kappa\cdot\rangle \pmod 1$. We need to find a representative v' in $K(G') := \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t)$ such that $\langle u, \kappa\cdot\rangle \pmod 1 = \langle u', \cdot\rangle \pmod 1$, where u' is a preimage of v' under the map $\partial'\partial'^t : \mathbb{Z}^{V'} \to \mathrm{im}(\partial')$

The coset represented by v' will be the image of the coset represented by v.

To do so, we note that, given $y \in \mathrm{im}(\partial')$, $\langle u, \kappa y\rangle$ is equal to (where V' is the vertex set of G')

$$\sum_{w\in V'} u_{p(w)} y_w.$$

Let x be the vector in $\mathbb{R}^{V'}$ such that for all $w \in V'$, $x_w = u_{p(w)}$, so that $\langle u, \kappa\cdot\rangle \pmod 1 = \langle x, \cdot\rangle$. This means $L(G')x$ is the vector v' for which we are looking.

It only remains to compute $v' = \partial'\partial'^t x$. We determine the component of v' corresponding to w for each $w \in V'$. Let $\mathrm{in}(w)$ denote the vertices that are adjacent to w. Directly from the definition of matrix multiplication, we see that the component of v' corresponding to w is

$$\deg(w)x_w - \sum_{w'\in \mathrm{in}(w)} x_{w'} =$$

$$\deg(w)u_{p(w)} - \sum_{w'\in \mathrm{in}(w)} u_{p(w')} =$$

$$\deg(w)u_{p(w)} - \sum_{w'\in \mathrm{in}(w)\cap p^{-1}(p(w))} u_{p(w)} - \sum_{w'\in \mathrm{in}(w)\setminus p^{-1}(p(w))} u_{p(w')}.$$

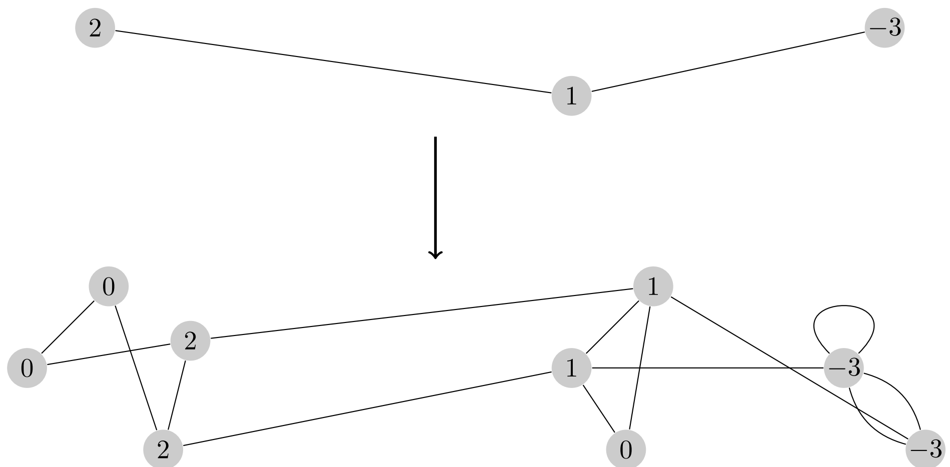

FIGURE 3. Coset representative of an element of critical group of the graph on top maps to coset representive in critical group of Berman bundle

There are two cases. If w is not in the original cover G'_p, then the sum becomes

$$\deg(w)u_{p(w)} - \sum_{w' \in \mathrm{in}(w) \cap p^{-1}(p(w))} u_{p(w)} = \\ \deg(w)u_{p(w)} - \deg(w)u_{p(w)} = 0.$$

If w is in the original cover G'_p, then the sum becomes

$$\left(\deg(w)u_{p(w)} - \sum_{w' \in \mathrm{in}(w) \cap p^{-1}(p(w))} u_{p(w)}\right) - \sum_{w' \in \mathrm{in}(w) \setminus p^{-1}(p(w))} u_{p(w')} = \\ |\mathrm{in}(v') \setminus p^{-1}(p(v'))| u_{p(w)} - \sum_{w' \in \mathrm{in}(w) \setminus p^{-1}(p(w))} u_{p(w')} = \\ |\mathrm{in}(p(w))| u_{p(w)} - \sum_{w' \in \mathrm{in}(p(w))} u_{w'} = \\ (\partial \partial^t u)_{p(w)} = v_{p(w)}.$$

Example 4. An example of this injection is shown in Figure 3.

4.2. **The cokernel of $\hat{\kappa}$.** As before, let $p : G' \to G$ be a Berman bundle and κ be the induced surjection $\kappa : K(G') \to K(G)$ of critical groups. Let $G' = (V', E')$ and $G = (V, E)$.

Let $\hat{\phi}$ be the injection from $\mathbb{Z}^V$ into $\mathbb{Z}^{V'}$, where, if e_v is the basis vector of $\mathbb{Z}^V$ corresponding to a vertex $v \in V$, then $\hat{\phi}(e_v) = \sum_{v' \in p^{-1}(v) \cap G'_p} e_{v'}$. Informally, we "replicate" e_v for each vertex v' in the fibre $p^{-1}(v)$ of v that was also in the original cover G'_p. Then, if we restrict $\hat{\phi}$ to $\mathrm{im}(\partial)$, we have the following picture

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{im}(\partial) \arrow[r, "\hat{\phi}"] \arrow[d, "\kappa_1"] & \mathrm{im}(\partial') \arrow[d, "\kappa_2"] \\ 0 \arrow[r] & K(G) := \mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t) \arrow[r, "\hat{\kappa}"] & K(G') := \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t) \end{tikzcd}$$

The commutative of the diagram is directly from the description of $\hat{\kappa}$ in the previous section. In injectivity of $\hat{\kappa}$ is from the surjectivity of κ. The image of $\hat{\kappa}$ in $\mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t)$ is the same as the image of $\kappa_2 \circ \hat{\phi}$. This is $\hat{\phi}(\mathrm{im}(\partial))/\hat{\phi}(\mathrm{im}(\partial)) \cap \mathrm{im}(\partial'\partial'^t)) = (\hat{\phi}(\mathrm{im}(\partial)) + \mathrm{im}(\partial'\partial'^t))/\mathrm{im}(\partial'\partial'^t)$.

Therefore, the cokernel of $\hat{\kappa}$ is $\mathrm{im}(\partial')/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathrm{im}(\partial)))$.

Since we will need the maps $\hat{\kappa}$ and $\hat{\phi}$ for future reference, we will give them a name.

Definition 6. Given a Berman bundle $p : G' \to G$ where $G' = (V', E')$ and $G = (V, E)$, let the map $\hat{\phi}$ that sends $e_v \in \mathbb{Z}^V$ corresponding to a vertex $v \in V$ to $\hat{\phi}(e_v) = \sum_{v' \in p^{-1}(v) \cap G'_p} e_{v'} \in \mathbb{Z}^{V'}$ be called the *induced injection of lattices*. This injection will also restrict to an injection from $\mathrm{im}(\partial)$ into $\mathrm{im}(\partial')$.

Let the map $\hat{\kappa}$ from $K(G)$ to $K(G')$ induced by κ_1 and κ_2 above be called the induced injection of critical groups. Since $\hat{\kappa}$ comes from dualizing the surjection from $K(G')$ to $K(G)$ in Proposition 3.1, $\hat{\kappa}$ is injective.

Remark 3. (added while writing the regular cover section) It will be useful to also understand the induced injection $\hat{\kappa}$ from $K(G)$ to $K(G')$ with the presentation of the critical group in terms of the edges.

To do so, we first recall that if $p : G' \to G$ is Berman bundle, then there is a graph $G'_p \subset G'$ such that p restricted to G'_p is a covering. Given an edge k of G, define $p^{-1}(k)$ to be all the edges in G'_p that map to k under the covering map from G'_p to G. Also, while we supressed this fact earlier, recall that to construct the covering graph G'_p, we oriented the edges of G and constructed the covering space G'_p so that the covering map also preserves direction of edges. This is possible due to the work of Gross and Tucker [4].

Define the map $\hat{\psi}$ from $\mathbb{Z}^E$ to $\mathbb{Z}^{E'}$ that sends e_k for k an edge of G to $\sum_{k' \in p^{-1}(k)} e_k$. Then, we claim that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z}^E \arrow[r, hook, "\hat{\psi}"] \arrow[d, "\partial"] & \mathbb{Z}^{E'} \arrow[d, "\partial'"] \\
\mathbb{Z}^V \arrow[r, hook, "\hat{\phi}"] & \mathbb{Z}^{V'}
\end{tikzcd}$$

It suffices to show that the image of $e_k \in \mathbb{Z}^E$, the standard basis element corresponding to the edge $k \in E$, maps to the same thing going clockwise and counterclockwise.

First, if we go counterclockwise, then if k is directed from v to u, then e_k maps to $e_u - e_v$. Then, the image under $\hat{\phi}$ is $\sum_{u' \in p^{-1}(u) \cap G'_p} e_{u'} - \sum_{v' \in p^{-1}(v) \cap G'_p} e_{v'}$.

Now, if we go clockwise, then k maps to $\sum_{k' \in p^{-1}(k)} e_{k'}$. This maps precisely to $\sum_{u' \in p^{-1}(u) \cap G'_p} e_{u'} - \sum_{v' \in p^{-1}(v) \cap G'_p} e_{v'}$ as long as our cover G'_p is also direction preserving (and it is).

Therefore, we have the following diagram

$$\begin{tikzcd}
& \mathbb{Z}^E \arrow[rr, hook, "\hat{\psi}"] \arrow[dl, "\partial"'] \arrow[dd, dotted, two heads, "\kappa_3"] & & \mathbb{Z}^{E'} \arrow[dl, "\partial'"'] \arrow[dd, two heads, "\kappa_4"] \\
\mathrm{im}(\partial) \arrow[rr, hook, "\hat{\phi}"] \arrow[dd, two heads, "\kappa_1"] & & \mathrm{im}(\partial') \arrow[dd, two heads, "\kappa_2"] & \\
& \mathbb{Z}^E/(\mathrm{im}(\partial) \oplus \ker(\partial)) \arrow[rr, dotted, hook] \arrow[dl, dotted, "\sim"', "\overline{\partial}"] & & \mathbb{Z}^{E'}/(\mathrm{im}(\partial') \oplus \ker(\partial')) \arrow[dl, "\sim"', "\overline{\partial'}"] \\
\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t) \arrow[rr, hook, "\hat{\kappa}"'] & & \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t) &
\end{tikzcd}$$

and the injection from $\mathbb{Z}^E/(\mathrm{im}(\partial) \oplus \ker(\partial))$ to $\mathbb{Z}^{E'}/(\mathrm{im}(\partial') \oplus \ker(\partial'))$ is induced by $\hat{\psi}$. To see this, if we want to go from $\mathbb{Z}^E/(\mathrm{im}(\partial) \oplus \ker(\partial))$ to $\mathbb{Z}^{E'}/(\mathrm{im}(\partial') \oplus \ker(\partial'))$, we following the arrows from $\mathbb{Z}^E/(\mathrm{im}(\partial) \oplus \ker(\partial))$ to $\mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t)$ to $im(\partial')/\mathrm{im}(\partial'\partial'^t)$ to $\mathbb{Z}^{E'}/(\mathrm{im}(\partial') \oplus \ker(\partial'))$.

If we work instead with coset representatives, this is the same as going from $\mathbb{Z}^E$ to $\mathrm{im}(\partial)$ to $\mathrm{im}(\partial')$ and then taking a preimage under ∂' to get to $\mathbb{Z}^{E'}$. Since the top face of the cube commutes, this is the same as going from $\mathbb{Z}^E$ to $\mathbb{Z}^{E'}$ through $\hat{\phi}$. (end of remark)

There is a slightly different form of the cokernel that will be useful for us.

Proposition 4.1. *Suppose $p : G' \to G$ is a Berman bundle, $G = (V, E)$, $G' = (V', E')$, G' is connected (so G is connected), and G'_p is an $n-$sheeted cover of G.*

Let $\mathbb{Z}^V_{0 \pmod n}$ be the sublattice of $\mathbb{Z}^V$ where all the coordinates add up to a multiple of n. Define $\mathbb{Z}^{V'}_{0 \pmod n}$ similarly.

The cokernel of $\hat{\kappa} = \mathrm{im}(\partial')/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathrm{im}(\partial)))$ can be expressed as $\mathbb{Z}^{V'}_{0 \pmod n}/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))$.

Remark 4. Maybe the notation should be changed, but right now I'm using

(1) $\mathrm{im}(\partial)$ is the sublattice of $\mathbb{Z}^V$ where the coordinates add to zero for *each* connected component of G. Equivalently, this is $\partial\mathbb{Z}^E$.
(2) $\mathbb{Z}^V_{0 \pmod n}$ is the sublattice of $\mathbb{Z}^V$ where the coordinates (all of them) add up to a multiple of n. The definition is the same regardless of the connected components.

Proof. Given the inclusion ι of $\mathrm{im}(\partial')$ into $\mathbb{Z}^{V'}_{0 \pmod n}$, we have the following diagram

$$\begin{tikzcd} \mathrm{im}(\partial') \arrow[r, hook, "\iota"] \arrow[d, "\kappa_1"] & \mathbb{Z}^{V'}_{0 \pmod n} \arrow[d, "\kappa_2"] \\ \mathrm{im}(\partial')/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathrm{im}(\partial))) \arrow[r, "\overline{\iota}"] & \mathbb{Z}^{V'}_{0 \pmod n}/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) \end{tikzcd}$$

The map $\overline{\iota}$ is induced as $\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathrm{im}(\partial)) \subset \mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)$. We want to show that $\overline{\iota}$ is actually an isomorphism.

To show that $\overline{\iota}$ is an injection, we need to show that $\mathrm{im}(\partial') \cap (\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) \subset \mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathrm{im}(\partial))$ (which means we have equality since $\mathrm{im}(\partial') \cap (\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V) \supset \mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathrm{im}(\partial))$ was necessary for ι to factor through $\mathrm{im}(\partial')/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathrm{im}(\partial)))$ as $\overline{\iota}$).

Since $\mathrm{im}(\partial'\partial'^t) \subset \mathrm{im}(\partial')$, $\mathrm{im}(\partial') \cap (\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) = \mathrm{im}(\partial'\partial'^t) + \mathrm{im}(\partial') \cap \hat{\phi}(\mathbb{Z}^V)$. This follows from a general fact of abelian groups. If A_1, A_2 and A_3 are abelian groups with $A_1 \subset A_3$, then if $a_1 \in A_1$ and $a_2 \in A_2$ with $a_1 + a_2 \in A_3$, then a_2 must be in A_3 since $a_1 \in A_3$.

To finish the proof that $\overline{\iota}$ is an injection, we need to show that $\mathrm{im}(\partial') \cap \hat{\phi}(\mathbb{Z}^V) \subset \hat{\phi}(\mathrm{im}(\partial))$.

If G and G' are connected and G'_p is an $n-$sheeted covering of G, then the sum of the coordinates of $\hat{\phi}(x)$, where $x \in \mathbb{Z}^V$ is n times the sum of the coordinates of x. Therefore, the only way for $\hat{\phi}(x)$ to be in $\mathrm{im}(\partial')$ is if $x \in \mathrm{im}(\partial)$. Therefore, $\mathrm{im}(\partial') \cap \hat{\phi}(\mathbb{Z}^V) \subset \hat{\phi}(\mathrm{im}(\partial))$ and $\overline{\iota}$ is injective, as desired.

Now we show that $\overline{\iota}$ is surjective. The image of $\overline{\iota}$ is $\mathrm{im}(\partial')/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) \cap \mathrm{im}(\partial') = (\mathrm{im}(\partial') + \mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))$. To show that this is all of $\mathbb{Z}^{V'}_{0 \pmod n}/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))$, it suffices to show that $(\mathrm{im}(\partial') + \mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) \supset \mathbb{Z}^{V'}_{0 \pmod n}$.

We show that $(\mathrm{im}(\partial') + \hat{\phi}(\mathbb{Z}^V)) \supset \mathbb{Z}^{V'}_{0 \pmod n}$. Fix an element $x \in \mathbb{Z}^{V'}_{0 \pmod n}$. Let c be the sum of the coordinates of x. Let y be any element $\hat{\phi}(\mathbb{Z}^V)$ where the sum of the coordinates is $\frac{c}{n}$. Then, $x - \hat{\phi}(y) \in \mathrm{im}(\partial')$, so $x \in \mathrm{im}(\partial') + \hat{\phi}(\mathbb{Z}^V)$, as desired.

Therefore, $\overline{\iota}$ is both injective and surjective, so $\hat{\kappa} = \mathrm{im}(\partial')/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathrm{im}(\partial))) = \mathbb{Z}^{V'}_{0 \pmod n}/(\mathrm{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))$. □

4.3. **Splitting backmaps of $\hat{\kappa}$.** Returning to the motivation of determine the critical group of the cover in terms of the kernel and cokernel in the exact sequence

$$0 \to K(G) \xrightarrow{\widehat{\kappa}} K(G') \to \mathrm{coker}(\widehat{\kappa}) \to 0$$

it is desirable to know if the sequence is split exact, where $\hat{\kappa}$ is the induced injection of critical groups.

We will show that the sequence is split exact at all the Sylow $p-$groups where p does not divide n.

Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & K(G) := \mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t) \arrow[r, "\hat{\kappa}"] & K(G') := \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t) \arrow[r] \arrow[d, "\wr"] & \mathrm{coker}(\widehat{\kappa}) \arrow[r] & 0 \\
0 & K(G) := \mathrm{im}(\partial)/\mathrm{im}(\partial\partial^t) \arrow[l] & K(G') := \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^t) \arrow[l, "\kappa"'] & \ker(\kappa) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the isomorphism in the middle is the identity map. We want to consider the image of $K(G)$ under the composition $\kappa \circ \hat{\kappa}$. If G'_p is an n-sheeted cover of G, then by the definition of the maps κ and $\hat{\kappa}$ this composition will take an element of $K(G)$ with coset representative x and map it to the element of $K(G)$ with coset representative nx.

Therefore, the image of $\kappa \circ \hat{\kappa}$ is $n(\mathrm{im}(\partial))/(\mathrm{im}(\partial\partial^t) \cap n(\mathrm{im}(\partial))) = (n(\mathrm{im}(\partial)) + \mathrm{im}(\partial\partial^t))/\mathrm{im}(\partial\partial^t)$. Then, the index $[K(G) : \kappa \circ \hat{\kappa}(K(G))] = [\mathrm{im}(\partial) : n(\mathrm{im}(\partial)) + \mathrm{im}(\partial\partial^t]$.

Since $n(\mathrm{im}(\partial)) \subset n(\mathrm{im}(\partial))+\mathrm{im}(\partial\partial^t, \mathrm{im}(\partial)$ we can quotient out to find the index is also $[\mathrm{im}(\partial)/(n(\mathrm{im}(\partial))) : \mathrm{im}(\partial\partial^t)/(n(\mathrm{im}(\partial)))]$. Let S be a subset of V containing one vertex in each connected component of G. Using the isomorphism π from $\mathrm{im}(\partial)$ to $\mathbb{Z}^{V\setminus S}$ that deletes the coordinates corresponding to vertices in S, we deduce that the index is $[\pi(\mathrm{im}(\partial))/\pi(n(\mathrm{im}(\partial))) : \pi(\mathrm{im}(\partial\partial^t))/\pi(n(\mathrm{im}(\partial)))] = [(\mathbb{Z}/n\mathbb{Z})^{V\setminus S} : \pi(\mathrm{im}(\partial\partial^t))/\pi(n(\mathrm{im}(\partial)))]$. This means the index $[K(G) : \kappa \circ \hat{\kappa}(K(G))]$ divides $n^{|V\setminus S|}$.

Equivalently, the kernel of $\kappa \circ \hat{\kappa}$ is a subgroup H of $K(G)$ whose order divides $n^{|V\setminus S|}$. This means we can quotient out by H to obtain

$$\begin{tikzcd}
0 \arrow[r] & K(G)/H \arrow[r, "\overline{\hat{\kappa}}"] & K(G')/\hat{\kappa}(H) \arrow[d, "\wr"] \\
0 & K(G) \arrow[l] & K(G')/\hat{\kappa}(H) \arrow[l, "\overline{\kappa}"']
\end{tikzcd}$$

Then, we can restrict $\overline{\kappa}$ to $\overline{\kappa^{-1}}(\mathrm{im}(\kappa \circ \hat{\kappa}))$ to get

$$\begin{tikzcd}
0 \arrow[r] & K(G)/H \arrow[r, "\overline{\hat{\kappa}}"] & \kappa^{-1}(\mathrm{im}(\kappa \circ \hat{\kappa}))/\hat{\kappa}(H) \arrow[d, "\wr"] \\
0 & \mathrm{im}(\kappa \circ \hat{\kappa}) \arrow[l] & \kappa^{-1}(\mathrm{im}(\kappa \circ \hat{\kappa}))/\hat{\kappa}(H) \arrow[l, "\overline{\kappa}"']
\end{tikzcd}$$

Since H is by definition the kernel of $\kappa \circ \hat{\kappa}$, $K(G)/H$ is isomorphic to $\mathrm{im}(\kappa \circ \hat{\kappa})$ under the map $\psi = \overline{\kappa} \circ \overline{\hat{\kappa}}$. Then, $\overline{\hat{\kappa}}$ is an injection of $K(G)/H$ into $\kappa^{-1}(\mathrm{im}(\kappa \circ \hat{\kappa}))/\hat{\kappa}(H)$ and $\psi^{-1} \circ \overline{\kappa}$ is a splitting backmap, which means the exact sequence

$$0 \to K(G)/H \xrightarrow{\overline{\hat{\kappa}}} \kappa^{-1}(\mathrm{im}(\kappa \circ \hat{\kappa}))/\hat{\kappa}(H) \to \mathrm{coker}(\overline{\hat{\kappa}}) \to 0$$

splits. In particular, this shows the original exact sequence splits at all primes p that do not divide n.

Corollary 4.2. *At each prime p not dividing n, $\mathrm{Syl}_p(G') = \mathrm{Syl}_p(G) \oplus \mathrm{Syl}_p(\mathrm{coker}(\hat{\kappa}))$.*

Proof. In going from

$$0 \to K(G) \xrightarrow{\widehat{\kappa}} K(G') \to \mathrm{coker}(\widehat{\kappa}) \to 0$$

to the split exact sequence

$$0 \to K(G)/H \xrightarrow{\overline{\hat{\kappa}}} \kappa^{-1}(\mathrm{im}(\kappa \circ \hat{\kappa}))/\hat{\kappa}(H) \to \mathrm{coker}(\overline{\hat{\kappa}}) \to 0,$$

we replaced $K(G)$ with the quotient group $K(G)/H$. Since p does not divide the order of H, $\mathrm{Syl}_p(K(G)) = \mathrm{Syl}_p(K(G)/H)$. Similarly, we first replaced $K(G')$ with the quotient $K(G')/\kappa(\hat{H})$, which does not affect the Sylow p subgroup $\mathrm{Syl}_p(K(G'))$. Then, we replaced $K(G')/\kappa(\hat{H})$ with $\kappa^{-1}(\mathrm{im}(\kappa \circ \hat{\kappa}))/\hat{\kappa}(H)$ which again does not affect the Sylow p-group since we are taking a subgroup of index $|H|$.

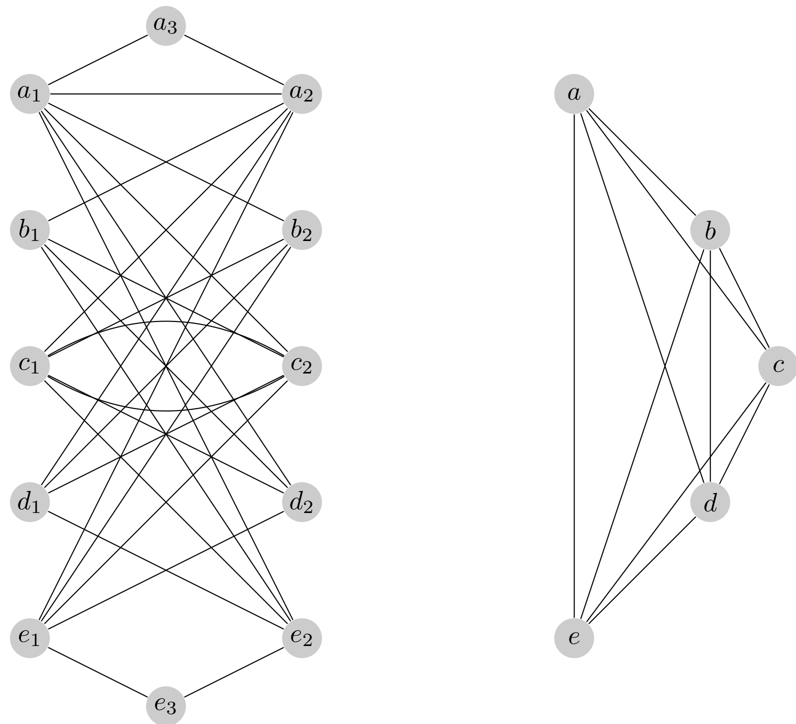

FIGURE 4. Berman Bundle of K_5

Finally, we replaced $\mathrm{coker}(\hat{\kappa})$ with the image when we restrict to $\kappa^{-1}(\mathrm{im}(\kappa \circ \hat{\kappa}))$, which is a subgroup of index $|H|$. Therefore, we have not altered the Sylow $p-$ groups in the exact sequence and find our split exact sequence implies

$$0 \to \mathrm{Syl}_p(K(G)) \xrightarrow{\widehat{\kappa}} \mathrm{Syl}_p(K(G')) \to \mathrm{Syl}_p(\mathrm{coker}(\widehat{\kappa})) \to 0$$

splits, as desired. $\square$

Remark 5. If H is trivial above, then the sequence

$$0 \to K(G) \xrightarrow{\widehat{\kappa}} K(G') \to \mathrm{coker}(\widehat{\kappa}) \to 0$$

is already split exact. From the computations above, this is equivalent to $nK(G)$ to be all of $K(G)$ or, if we reduce the laplacian $\partial\partial^t \pmod n$ that the image is all of $\mathrm{im}(\partial)/(n(\mathrm{im}(\partial)))$. If this is true, then the image of $\partial\partial^t$, when the entries are reduced $\pmod p$ for any p dividing n, is all the vectors in $\mathbb{F}_p^V$ that are orthogonal to the characteristic vectors of the connected components of G.

This is equivalent to the Laplacian $\partial\partial^t$ not decreasing in rank when the entries interpreted as elements in $\mathbb{F}_p$, which is equivalent to having no elements on the diagonal of the Smith Normal Form that are divisible by p. This means we need $K(G)$ to have no p-group component, where p does not divide n.

In this case, Corollary 4.2 already shows us the sequence is split exact.

Example 5. Consider the bipartite covering of the complete graph K_5. We can add vertices and edges so that it is no longer a covering, but it is still a Berman Bundle to get Figure 4. Since the critical group of K_5 is $(\mathbb{Z}/5\mathbb{Z})^3$ and 5 does not divide 2, we would expect the sequence

$$0 \to K(G) \xrightarrow{\widehat{\kappa}} K(G') \to \mathrm{coker}(\widehat{\kappa}) \to 0$$

to be split exact. In particular, if we write the critical group of the graph shown in Figure 4 as a direct product of $p-$groups, we would expect to see three copies of $\mathbb{Z}/5\mathbb{Z}$ in the 5-group component. Indeed, SAGE tells us the critical group is $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/11220\mathbb{Z} = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/5\mathbb{Z})^3 \oplus \mathbb{Z}/11\mathbb{Z} \oplus \mathbb{Z}/17\mathbb{Z}$.

5. Reformulation of Cokernel of $\hat{\kappa}$ for Berman Bundles

Given a Berman bundle $p : G' \to G$ and the induced injection $\hat{\kappa} : K(G) \to K(G')$, we want to determine information on the critical group $K(G')$ based on $K(G)$ and $\text{coker}(\hat{\kappa})$. However, in order to do so, we need to be able to find the cokernel. We will focus on the case where G' is connected (so G is connected).

Consider the fibres of the covering space $G'_p \subset G'$ when p is restricted to G'_p. Since p restricted to G'_p is a covering space and G is connected, p must be an n-sheeted cover for some integer n. Let $S \subset V'$ be of set of $|V|$ vertices where one vertex is selected from each fibre of G'_p.

Consider the injection ι of $\mathbb{Z}_{0 \pmod n}^{V' \setminus S}$ into $\mathbb{Z}_{0 \pmod n}$ that "fills in the missing coordinates with zeros". More precisely, let $e_{v'} \in \mathbb{Z}_{0 \pmod n}^{V' \setminus S}$ where $v' \in V' \setminus S$ be sent to $e_{v'} \in \mathbb{Z}_{0 \pmod n}^{V'}$.

From Proposition 4.1, if $\hat{\kappa}$ is the induced injection of critical groups and $\hat{\phi}$ is the induced injection of lattices, then the cokernel of $\hat{\kappa}$ is $\mathbb{Z}_{0 \pmod n}^{V'}/(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))$.

From the injection $\iota : \mathbb{Z}_{0 \pmod n}^{V' \setminus S} \to \mathbb{Z}_{0 \pmod n}$, we have the subgroup $\iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S})/(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) \cap \iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S}) = (\iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S}) + \text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))/(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))$ inside $\text{coker}(\hat{\kappa}) = \mathbb{Z}_{0 \pmod n}^{V'}/(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))$.

We will show that this subgroup is actually the entire group, giving another presentation of $\text{coker}(\hat{\kappa})$. To do so, it suffices to show that $\iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S}) + \text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V) \supset \mathbb{Z}_{0 \pmod n}^{V'}$.

We will show that $\iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S}) + \hat{\phi}(\mathbb{Z}^V) \supset \mathbb{Z}_{0 \pmod n}^{V'}$. To do so, consider an element $x \in \mathbb{Z}_{0 \pmod n}^{V'}$. Suppose $V = \{v_1, \ldots, v_k\}$, $S = \{v'_1, \ldots, v'_k\}$, $p(v'_i) = v_i$ for each $1 \le i \le k$, and the coordinates of x that correspond to $v'_1, \ldots, v'_k$ are $c_1, \ldots, c_k$ respectively. Then, consider the element $y \in \mathbb{Z}^V$ where the coordinate corresponding to v_i is c_i.

If we subtract $x - \hat{\phi}(y)$, we see that result is an element of $\iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S})$ since the components of $x - \hat{\phi}(y)$ corresponding to v'_i for any $1 \le i \le k$ is zero. Also, the coordinates of $x - \hat{\phi}(y)$ sum to a multiple of n since the coordinates of both x and $\hat{\phi}(y)$ sum to a multiple of n. This means $\iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S}) + \hat{\phi}(\mathbb{Z}^V) \supset \mathbb{Z}_{0 \pmod n}^{V'}$, as desired. Furthermore, this element y is the unique element in $\mathbb{Z}^V$ such that $x - \hat{\phi}(y) \in \iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S})$.

Finally, since $\iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S})/(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) \cap \iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S}) = (\iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S}) + \text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))/(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) = \text{coker}(\hat{\kappa}) = \mathbb{Z}_{0 \pmod n}^{V'}/(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V))$, we only need to understand the intersection $(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) \cap \iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S})$.

Given a column x of $\partial'\partial'^t$ (or any element of $\iota(\mathbb{Z}_{0 \pmod n}^{V'})$), there exists a unique element $y \in \mathbb{Z}_{0 \pmod n}^{V'}$ such that $x - \phi(\hat{y}) \in \iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S})$ from the argument above. We can do this for each column of the matrix $\partial'\partial'^t$ to find a new matrix R whose vectors span the lattice $(\text{im}(\partial'\partial'^t) + \hat{\phi}(\mathbb{Z}^V)) \cap \iota(\mathbb{Z}_{0 \pmod n}^{V' \setminus S})$.

Then, if $\pi : \mathbb{Z}^{V'} \to \mathbb{Z}^{V' \setminus S}$ projects away the coordinates corresponding to S, we can apply π to $\text{coker}(\hat{\kappa}) = \iota(\mathbb{Z}_{0 \pmod n}^{V'})/\text{im}(R)$ to find the cokernel is $\mathbb{Z}_{0 \pmod n}^{V'}/(\text{im}(\pi(R)))$.

In summary, our reformulation of the cokernel is in the following Proposition. An example of this is in Example 6

Proposition 5.1. *If $p : G' \to G$ is a Berman bundle, $G' = (V', E')$ is connected, $G = (V, E)$ and $S \subset V'$ contains one element in fibre of G'_p (so $|S| = |V|$), $\hat{\phi}$ is the induced injection of lattices, then we can construct the matrix $R(G')_S \in \mathbb{Z}^{V' \setminus S \times V'}$ by*

(1) Add elements of $\hat{\phi}(\mathbb{Z}^V)$ to each column of $\partial'\partial'^t$ such that the coordinates corresponding to any vertex in S is zero. For each column, there is an unique element of $\hat{\phi}(\mathbb{Z}^V)$ that will do this.

(2) Delete the rows of the resulting matrix corresponding to vertices in S. Each of these rows should be zero.

The cokernel $\hat{\kappa}$ is $\mathbb{Z}_{0 \pmod n}^{V' \setminus S}/\text{im}(R(G')_S)$.

Definition 7. For convenience, we give call the matrix $R(G')_S$ the $S-$row reduced Laplacian.

If G is just a vertex, then $S = \{s\}$ is just one vertex. Then, the $S-$row reduced Laplacian $R(G')_S$ is the Laplacian $\partial'\partial'^t$ after we delete the row corresponding to s. Then, since the columns still add to the zero column, we can delete the column s to get the usual reduced Laplacian when we regard s as the sink to find the S-row reduced Laplacian in this case has the same image as the usual reduced Laplacian.

Example 6. For example, in the Berman bundle in Figure 2, let $S = \{a_1, b_1, c_1\}$. Given the Laplacian

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_2 \end{array}\begin{array}{c} \begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & c_1 & c_2 \end{array} \\ \begin{pmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 3 \end{pmatrix} \end{array},$$

the critical group is $\mathbb{Z}/249\mathbb{Z}$. Since the critical group of the graph G is a line graph, we would expect the cokernel $\hat{\kappa}$ to be $\mathbb{Z}/249\mathbb{Z}$. The cokernel is $\mathbb{Z}_{0 \pmod 2}^{\{a_2,a_3,a_4,b_2,b_3,c_2\}}$ quotiented out by the image of R. To get R, we take the laplacian $\partial'\partial'^t$ and add elements of $\hat{\phi}(\mathbb{Z}^V)$ so that the columns are in $\mathbb{Z}_{0 \pmod 2}^{\{a_2,a_3,a_4,b_2,b_3,c_2\}}$. This yields,

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_2 \end{array}\begin{array}{c} \begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & c_1 & c_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -5 & 5 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -5 & 5 \end{pmatrix} \end{array}.$$

Now, we delete the rows corresponding to a_1, b_2, or c_1 to get

$$\begin{array}{c} \\ a_2 \\ a_3 \\ a_4 \\ b_2 \\ b_3 \\ c_2 \end{array}\begin{array}{c} \begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & c_1 & c_2 \end{array} \\ \begin{pmatrix} -4 & 4 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -5 & 5 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -5 & 5 \end{pmatrix} \end{array}.$$

The cokernel of this matrix when regarded as map into $\mathbb{Z}^{V' \setminus S}$ is $\mathbb{Z}/498\mathbb{Z}$. Since the image of the matrix is contained in $\mathbb{Z}_{0 \pmod 2}^{V' \setminus S}$ and $\mathbb{Z}_{0 \pmod 2}^{V' \setminus S}$ is a sublattice of index 2 of $\mathbb{Z}^{V' \setminus S}$, the cokernel of $\hat{\kappa}$ must be an index 2 subgroup of $\mathbb{Z}/498\mathbb{Z}$. The only possibility for this is $\mathbb{Z}/249\mathbb{Z}$, which is what we expected.

5.1. **Reformulation in a special case.** For this entire section, we preserve the assumptions of Proposition 5.1. So $p : G' \to G$ is a Berman bundle, $G' = (V', E')$ is connected, $G = (V, E)$ and $S \subset V'$ contains one element in fibre of G'_p (so $|S| = |V|$), $\hat{\phi}$ is the induced injection of lattices.

Proposition 5.1 gives us the cokernel of the induced injection $\hat{\kappa} : K(G) \to K(G')$ in the form $\mathbb{Z}_{0 \pmod n}^{V' \setminus S}/\mathrm{im}(R(G')_S)$. However, the condition that the lattice being quotiented is $\mathbb{Z}_{0 \pmod n}^{V' \setminus S}$ instead of $\mathbb{Z}^{V' \setminus S}$ makes computing the cokernel harder.

Recall that in constructing the Berman bundle $p : G' \to G$, we constructed G' by taking an $n-$sheeted covering of G and

(1) added new vertices to the fibres of the coverings
(2) added new edges within the fibres.

Suppose we did not add any vertices (skipped step 1). Then, the fibre of each vertex in G would would still contain n vertices, but there might be more connections within the fibres.

Definition 8. Define the Berman bundles $p : G' \to G$, where, in the process of constructing G', step 1 above is skipped *augmented covering spaces*.

In particular, a covering space is an augmented covering space.

Definition 9. Given the S-row reduced Laplacian $R(G')_S$ of G', call the S-reduced Laplacian $L(G')_S$ the result when the columns of $R(G')_S$ corresponding to S are also deleted.

In particular, combined with the remark given in Definition 7, the definition of a S-reduced Laplacian coincides with the usual definition of a reduced Laplacian.

Proposition 5.2. *Preserving the assumptions of Proposition 5.1, if $R(G')_S$ is the $S-$row reduced Laplacian, then the characteristic vector of the fibre of any vertex in G is in the kernel of $R(G')_S$.*

In particular, this means the image of $R(G')_S$ is the same as the image of $L(G')_S$ and the cokernel of the induced injection $\hat{\kappa}$ is $\mathbb{Z}_{0 \pmod n}^{V'\setminus S}/\mathrm{im}(L(G')_S)$.

Proof. Let $\chi_{p^{-1}(v)}$ be the characteristic vector of a fibre of a vertex v of G. It suffices to show that $\partial'\partial'^t\chi_{p^{-1}(v)}$ is an element of $\hat{\phi}(\mathbb{Z}^V)$. To show this, we will show that $(\partial'\partial'^t\chi_{p^{-1}(v)})_{v'}$ (the component corresponding to v') is $(\partial\partial^t e_v)_{p(v')}$.

To see this, we compute $(\partial'\partial'^t\chi_{p^{-1}(v)})_{v'}$ for each vertex $v' \in V'$. There are two cases:

(1) If $v' \in p^{-1}(v)$, then by the definition of matrix multiplication $(\partial'\partial'^t\chi_{p^{-1}(v)})_{v'}$ is
$$\deg(v') - |\mathrm{in}(v')|,$$
where $\mathrm{in}(v')$ is vertices adjacent to v' (not including v' if there is a self loop). Therefore, the number above is exactly $|\mathrm{in}(v')\backslash p^{-1}(v)|$. Since $p : G' \to G$ is an augmented covering space (covering space except for additional edges within the fibres), $|\mathrm{in}(v')\backslash p^{-1}(v)|$ is exactly $|\mathrm{in}(p(v'))|$ (the number of vertices other than $p(v')$ in G adjacent to $p(v')$). Therefore, the component of $\partial'\partial'^t\chi_{p^{-1}(v)}$ corresponding to v' is exactly $\deg(p(v'))$ which is the component of $\partial\partial^t e_v$ corresponding to $p(v')$, as desired.
(2) If $v' \notin p^{-1}(v)$, then by the definition of matrix multiplication, the component of $\partial'\partial'^t\chi_{p^{-1}(v)}$ corresponding to v' is
$$|\mathrm{im}(v') \cap p^{-1}(v)|.$$
Since G' is an augmented covering space, the number of edges from $p(v')$ to v is exactly the number of edges from any vertex in $p^{-1}(v)$ to v'. Therefore, the component corresponding to v' of $\partial'\partial'^t\chi_{p^{-1}(v)}$ is precisely the component corresponding to $p(v')$ of $\partial\partial^t e_v$.

Therefore, we have shown that $(\partial'\partial'^t\chi_{p^{-1}(v)})_{v'}$ (the component corresponding to v') is $(\partial\partial^t e_v)_{p(v')}$ for each $v' \in V'$. This means in particular that components of $\partial'\partial'^t\chi_{p^{-1}(v)}$ corresponding to vertices of the same fibre have the same value. This is precisely what is needed for $\partial'\partial'^t\chi_{p^{-1}(v)}$ to be in $\hat{\phi}(\mathbb{Z}^V)$. Therefore, the characteristic vector of the fibre of v is in the kernel of $R(G')_S$.

Therefore, we can delete the columns of $R(G')_S$ without changing the image, so we can replace $R(G')_S$ by $L(G')_S$ in the presentation of $\mathrm{coker}(\hat{\kappa})$, as desired. □

Remark 6. (For Reiner) In the case of a double cover, Proposition 5.2 is precisely the presentation of the signed graph. The example after the propositions won't be exactly a double cover (the number of edges going between vertices of the same fibre is not necessarily even), but it will try to get the point across.

Finally, we can rewrite $\mathrm{coker}(\hat{\kappa})$ as the cokernel of a map into $\mathbb{Z}^{V'\setminus S}$ instead of $\mathbb{Z}_{0 \pmod n}^{V'\setminus S}$ using Lemma 2.1.

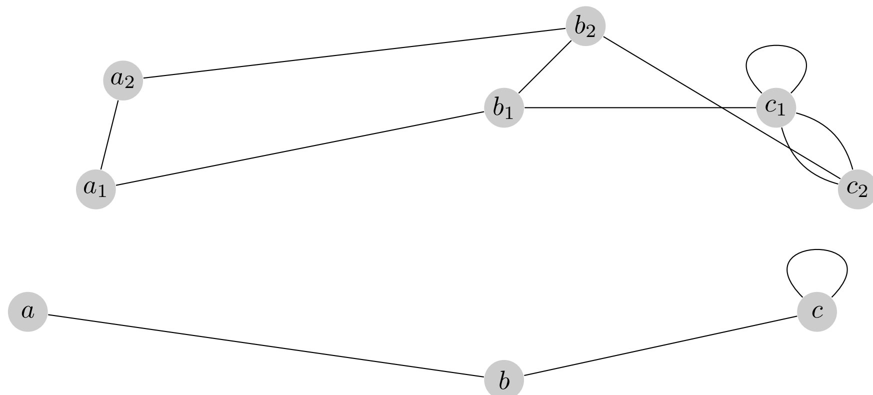

FIGURE 5. Vertices $\{a_1, a_2, a_3, a_4\}$ project to a, $\{b_1, b_2, b_3\}$ project to b, and $\{c_1, c_2\}$ project to c

Proposition 5.3. *Preserving the assumptions of Proposition 5.2 (that $p : G' \to G$ is an augmented covering, $G' = (V', E')$ is connected, and $\hat{\phi}$ is the induced injection of lattices), the cokernel of the induced injection $\hat{\kappa}$ of critical groups is*

$$\mathbb{Z}^{V'\setminus S}/L(G')_S^t(\mathbb{Z}_{0\ \ (\mathrm{mod}\ n)}^{V'\setminus S})^{\#} = \mathbb{Z}^{V'\setminus S}/L(G')_S^t(\mathbb{Z}^{V'\setminus S} + (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t).$$

Proof. From Proposition 5.2,

$$\mathrm{coker}(\hat{\kappa}) = \mathbb{Z}_{0\ \ (\mathrm{mod}\ n)}^{V'\setminus S}/\mathrm{im}(L(G')_S) = \mathbb{Z}_{0\ \ (\mathrm{mod}\ n)}^{V'\setminus S}/L(G')_S\mathbb{Z}^{V'\setminus S}.$$

Directly from Lemma 2.1, there is an isomorphism

$$\widehat{\mathbb{Z}_{0\ \ (\mathrm{mod}\ n)}^{V'\setminus S}/L(G')_S\mathbb{Z}^{V'\setminus S}} \cong (\mathbb{Z}^{V'\setminus S})^{\#}/L(G')_S^t(\mathbb{Z}_{0\ \ (\mathrm{mod}\ n)}^{V'\setminus S})^{\#} \cong \mathbb{Z}^{V'\setminus S}/L(G')_S^t(\mathbb{Z}^{V'\setminus S} + (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t),$$

as desired. □

Example 7. Consider the augmented covering space shown in Figure 5. The Laplacian is

$$\begin{array}{c} \\ a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \end{array} \\ \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 0 & 3 & -2 \\ 0 & 0 & 0 & -1 & -2 & 3 \end{pmatrix} \end{array},$$

From this, we can compute the critical group of the augmented cover G' is $\mathbb{Z}/26\mathbb{Z}$. Since the critical group of G is trivial, we would expect the cokernel to have critical group $\mathbb{Z}/26\mathbb{Z}$. To compute the cokernel using Proposition 5.3, we first compute the S-reduced Laplacian, where $S = \{a_1, b_1, c_1\}$. This yields $R(G')_S$ is

$$\begin{array}{c} \\ a_2 \\ b_2 \\ c_2 \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \end{array} \\ \begin{pmatrix} -3 & 3 & 1 & -1 & 0 & 0 \\ 1 & -1 & -4 & 4 & 1 & -1 \\ 0 & 0 & 1 & -1 & -5 & 5 \end{pmatrix} \end{array},$$

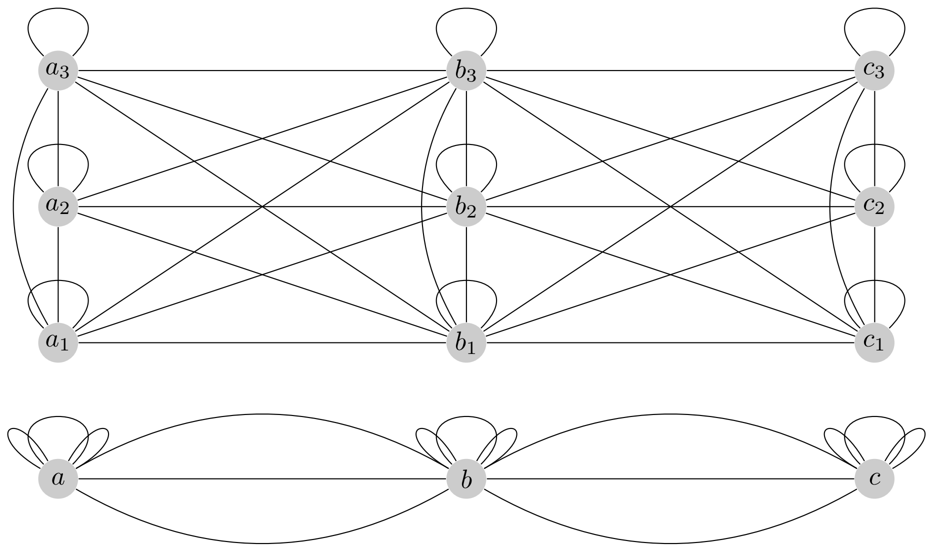

FIGURE 6. Construction of covering space

.

Note that the columns corresponding to each fibre of a vertex in G do sum to zero as asserted by Proposition 5.2. Then, the S-reduced Laplacian $L(G')_S$ is

$$\begin{array}{c c} & \begin{array}{ccc} a_2 & b_2 & c_2 \end{array} \\ \begin{array}{c} a_2 \\ b_2 \\ c_2 \end{array} & \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 5 \end{pmatrix}. \end{array}$$

It is true that when $n = 2$, $L(G')_S$ is symmetric (I should be writing other stuff up though). This is not true for larger n, so taking the transpose does make a difference. The image $L(G')_S^t(\mathbb{Z}^{V'\setminus S} + (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t)$ is generated by the columns of $L(G')_S^t$ and the column $L(G')_S^t(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t$. This yields the matrix

$$\begin{array}{c c} & \begin{array}{cccc} a_2 & b_2 & c_2 & \end{array} \\ \begin{array}{c} a_2 \\ b_2 \\ c_2 \end{array} & \begin{pmatrix} 3 & -1 & 0 & 1 \\ -1 & 4 & -1 & 2 \\ 0 & -1 & 5 & 2 \end{pmatrix}. \end{array}$$

The cokernel of this matrix is indeed $\mathbb{Z}/26\mathbb{Z}$.

5.2. **An exact formula for** $\operatorname{coker}(\hat{\kappa})$ **for a specific case.** Let $G = (V, E)$ be a connected graph where self loops are allowed. We will construct a specific covering space where we can determine $\operatorname{coker}(\hat{\kappa})$ exactly.

The steps are as follows:

(1) Multiply each edge in G by n to get the graph nG. For example, if G is a path with 3 vertices with self loops at each vertex and $n = 3$, then we would get the bottom graph in Figure 6.

(2) Take an $n-$covering of the graph nG, where if $v, u \in V$ are vertices of G and $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are the fibres of u and v, there is an edge between v and u in G, then there is an edge between v_i and u_j for all $1 \leq i \leq n$.

Continuing our example, our cover G' for the case where $n = 3$ and G is the path with 3 vertices is the top graph in Figure 6.

The cokernel of the injection from nG to the graph G' construct in this manner is as follows.

Proposition 5.4. *Given a vertex $v \in V$, let d_v be the outdegree of v. (So normal edges and self loops both count as 1 towards the count.)*

Consider the abelian group $\bigoplus_{v\in V}(\mathbb{Z}/d_v\mathbb{Z})^{n-1}$. Write the group in terms of the invariant factors so that $\bigoplus_{v\in V}(\mathbb{Z}/d_v\mathbb{Z})^{n-1} = \bigoplus_{i=1}^{(n-1)|V|}\mathbb{Z}/a_i\mathbb{Z}$, where $a_i|a_{i+1}$ for each $1 \leq i < n$ (many of the a_i's might be 1).

Then, the cokernel of the induced injection $\hat{\kappa}$ from nG to the cover G' is $\mathbb{Z}/a_1\mathbb{Z} \oplus \bigoplus_{i=2}^{(n-1)|V|}\mathbb{Z}/na_i\mathbb{Z}$.

Proof. The reason this proposition is true is the presentation of $\operatorname{coker}(\hat{\kappa})$ from Proposition 5.3 is particularly simple. Let the vertices of G be $V = \{v_1, \ldots, v_k\}$ and the fibre over a vertex $v_i \in V$ be $\{v_{i1}, \ldots, v_{in}\}$. Let $S = \{v_{1n}, \ldots, v_{kn}\}$.

Then, we claim the matrix $L_S(G')$ is diagonal. To see this, consider a column of $L_S(G')$ corresponding to a vertex $v_{qr} \in V'\backslash S$, where $1 \leq q \leq k$ and $1 \leq r \leq n-1$.

For each $1 \leq i, j, \leq k$, let ℓ_{ij} be the number of edges between v_i and v_j in G. If $i = j$, then ℓ_{ij} is the number of self loops at $i = j$. We claim that, if we take the column $x_{v_{pq}} \in \operatorname{im}(\partial')$ corresponding to v_{qr} of $\partial'\partial'^t$, where ∂' is the directed incidence matrix of G', then $x_{v_{pq}} + \hat{\phi}(\sum_{i=1}^k \ell_{pi}e_{v_i})$ has the property that the coordinate of v_{in} for any $1 \leq i \leq n$ corresponding to $x_{v_{pq}} + \hat{\phi}(\sum_{i=1}^k \ell_{pi}e_{v_i})$ is zero.

In fact more is true. From how we constructed G', the coordinate of $x_{v_{pq}}$ corresponding to v_{ij} is

$$(x_{v_{pq}})_{v_{ij}} = \begin{cases} -\ell_{ij} & \text{if } (i,j) \neq (p,q) \\ nd_v - \ell_{p,p} & \text{if } (i,j) = (p,q). \end{cases}$$

Note that the term $-\ell_{p,p}$ is there if $(i,j) = (p,q)$ because self loops don't count towards the element on the diagonal. Therefore,

$$x_{v_{pq}} + \hat{\phi}(\sum_{i=1}^k \ell_{pi}e_{v_i}) = \begin{cases} 0 & \text{if } (i,j) \neq (p,q) \\ nd_v & \text{if } (i,j) = (p,q). \end{cases}$$

Therefore $L_S(G')$ is an $(n-1)k$ by $(n-1)k$ matrix with $n-1$ copies of nd_v on the diagonal for each $v \in V$.

However, we are not quite done since we want to quotient out by $L(G')^t_S(\mathbb{Z}^{V'\backslash S} + (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t)$ not just $L(G')^t_S\mathbb{Z}^{V'\backslash S}$. To do so, we first note that $L(G')^t_S(\mathbb{Z}^{V'\backslash S} + (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t)$ is generated by the $(n-1)k$ columns of $L_S(G)$ and the column $x = L(G')^t_S(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t$ where $x_{v_{ij}} = d_{v_i}$ for each i.

Therefore, we want to find the invariant factors $b_1, \ldots, b_{(n-1)k}$ of $[L_S(G')|x]$. To do so, we will use the fact that $b_1b_2\cdots b_i$ is the greatest common denominator of the determinants of all i by i minors of $[L_S(G')|x]$. We claim that the greatest common denominator of the determinants of all i by i minors of $[L_S(G')|x]$ is $n^{i-1}a_1\cdots a_i$.

First, the greatest common denominator is at most $n^ia_1\cdots a_i$ by considering the i by i minors of $L_S(G')$. Now, suppose we have a minor A that contains the rows and columns indexed by $p_1 < p_2 < \cdots < p_i$ and $q_1 < \cdots < q_i$ respectively. If $q_i \neq (n-1)k+1$, then A is a minor of $L_S(G')$, so we only need to consider the case where $q_i = (n-1)k+1$.

For any $1 \leq j < i$, in order for the column indexed by q_j to be nonzero, we must have a row index $p_{j'} = q_j$. Therefore, all but one row index, say p_{j_0} is matched to the column indices. Then, for each $1 \leq j < i$, the column of A indexed by q_j has a unique nonzero element $L_S(G')_{q_j,q_j}$.

Then, in the permutation expansion of the determinant of A, there is only one nonzero term. This term will select the unique nonzero element of the column indexed by q_j for each $1 \leq j < i$ and the element of the column indexed by q_i that is in row p_{j_0}. The determinant corresponds exactly to picking i elements on the diagonal of $L_S(G')$, taking their product, and multiplying the product by n^{i-1}. The exponent of n is $i-1$ instead of i because the elements in the last column of A are the elements on the diagonal of $L_S(G')$ divided by n.

Therefore, the greatest common denominator of all the determinants is precisely the greatest denominator of all the determinants of the i by i minors of the matrix $L_S(G')$ divided by n. This means the greatest common denominator is $n^{i-1}a_1\cdots a_i$, which implies that $b_1 = a_1$ and $b_j = ma_j$ for each $2 \leq j \leq (n-1)k$, as desired. □

In summary, given a connected graph $G = (V,E)$ with critical group $\bigoplus_{i=1}^{|V|}\mathbb{Z}/c_i\mathbb{Z}$ and the invariant factors $a_1, \ldots, a_{(n-1)k}$ of $\bigoplus_{v\in V}(\mathbb{Z}/d_v\mathbb{Z})^{n-1}$, where d_v is the outdegree of v for each vertex $v \in V$, the exact sequence

$$0 \to K(nG) \xrightarrow{\hat{\kappa}} K(G') \to \operatorname{coker}(\hat{\kappa}) \to 0,$$

where nG is the graph G with each edge multiplied n times and G' is the cover of nG defined in this section, is

$$0 \to \bigoplus_{i=1}^{|V|} \mathbb{Z}/nc_i\mathbb{Z} \xrightarrow{\hat{\kappa}} K(G') \to \mathbb{Z}/a_1\mathbb{Z} \oplus \bigoplus_{i=2}^{(n-1)|V|} \mathbb{Z}/na_i\mathbb{Z} \to 0.$$

Example 8. In the example in Figure 6, the critical group of nG is $(\mathbb{Z}/3\mathbb{Z})^2$. The outdegrees are 2, 3, and 2. Writing $(\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/3\mathbb{Z})^2$ in terms of the invariant factors yields $(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/6\mathbb{Z})^2$. This means the cokernel is $\mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^2 \oplus (\mathbb{Z}/18\mathbb{Z})^2$. Therefore, the exact sequence for our specific graph G is

$$0 \to (\mathbb{Z}/3\mathbb{Z})^2 \to K(G') \to \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^2 \oplus (\mathbb{Z}/18\mathbb{Z})^2 \to 0$$

or

$$0 \to (\mathbb{Z}/3\mathbb{Z})^2 \to K(G') \to (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/3\mathbb{Z})^3 \oplus (\mathbb{Z}/9\mathbb{Z})^2 \to 0.$$

Since we have a specific case, we can compute $K(G')$ directly, which yields $K(G') = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus (\mathbb{Z}/18\mathbb{Z})^2 \oplus \mathbb{Z}/54\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/9\mathbb{Z})^2 \oplus \mathbb{Z}/27\mathbb{Z}$. It can be checked using Littlewoord Richardson coefficients that the decomposition of $K(G')$ does not contradict our exact sequence.

6. Regular Covers

6.1. **Definitions.** Gross and Tucker [4] introdcued the notion of a *regular covering*, which, informally, is a covering G' of a graph G with a group H that acts freely on the graph G'. We will reproduce their characterization of regular covers here.

Let $G = (V, E)$ be a directed graph and H be a group. (We are working with undirected graphs, but we can arbitrary assign orientations to the edges.)

Definition 10. A *voltage assignment* in H is a function s that assigns to each edge e of G a group element $s(e) \in H$ called the *voltage* on e. The pair (G, s) is called an *ordinary voltage graph*.

To an ordinary voltage graph (G, s), we can defined a *derived graph* G^s as follows:

(1) Its vertex set is $V \times H$. It will be convenient to refer to the vertex (v, h) for $v \in V, h \in H$ as v_h instead.
(2) Its edge set is $E \times H$. It will be convenient to refer to the edge (e, h) for $e \in E$ and $h \in H$ as e_h instead. If edge e of G runs from vertex u to vertex v, then the edge e_h runs from vertex v_h to $u_{s(e)h}$.

Remark 7. Gross and Tucker [4] defined the edge e_h to run from vertex v_h to $u_{hs(e)}$ instead of $u_{s(e)h}$. This is seen to be equivalent to our definition by replacing H with H^{-1}. We changed the definition slightly so that the "Laplacian" will act on column vectors instead of row vectors.

Definition 11. Let $\mathbb{Z}[H]$ be the group algebra of H over $\mathbb{Z}$ and $\{e_h : h \in H\}$ be the basis elements.

Define the *derived directed incidence matrix* $\partial^s \in \mathbb{Z}[H]^{V \times E}$ of an ordinary voltage graph (G, s) to be such that

$$\partial^s_{v,k} = \begin{cases} 1 (= e_1) & \text{if } k \text{ is not a loop and is directed away from } v \\ -e_{s(k)} & \text{if } k \text{ is not a loop and is directed into } v \\ 1 - e_{s(k)} & \text{if } k \text{ is a loop at } v \\ 0 & \text{else.} \end{cases}$$

Definition 12. Given a derived directed indicidence matrix $\partial^s \in \mathbb{Z}[H]^{V \times E}$, let $\partial^{s*} \in \mathbb{Z}[H]^{E \times V}$ be defined by first taking the transpose of ∂^s and then applying the map that sends e_h to $e_{h^{-1}}$ to each element of ∂^s.

Definition 13. Define the *derived Laplacian* $L(G, s)$ of an ordinary voltage graph (G, s) to be $\partial^s \partial^{s*}$.

Remark 8. Given a vertex $v \in V$, let $\text{Deg}^+(v)$ denote the edges going out of v and $\text{Deg}^-(v)$ denote the edges going into v. Let $\text{Deg}(v)$ be the union $\text{Deg}^+(v) \cup \text{Deg}^-(v)$.

Self loops at v count as 2 towards $\deg(v)$ and 1 towards both $\deg^+(v)$ and $\deg^-(v)$.

It can be verified directly by matrix multiplication that

$$L(G, s)_{v,u} = \begin{cases} -\sum_{k \in \text{Deg}^+(v) \cap \text{Deg}^-(u)} e_{s(k)} - \sum_{k \in \text{Deg}^-(v) \cap \text{Deg}^+(u)} e_{s(k)^{-1}} & v \neq u \\ |\text{Deg}(v)| - \sum_{k \in \text{Deg}^+(v) \cap \text{Deg}^-(v)} e_{s(k)} + e_{s(k)^{-1}} & v = u. \end{cases}$$

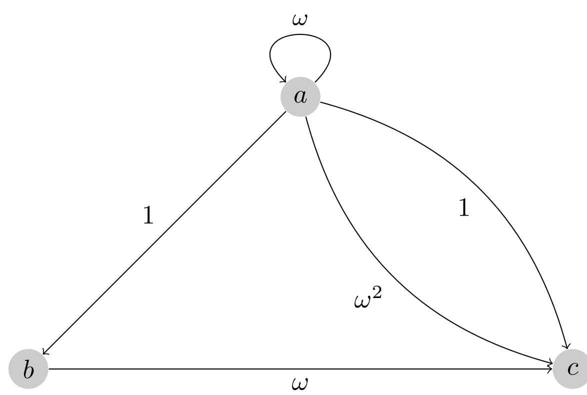

FIGURE 7. Ordinary voltage graph

Note that $\sum_{k\in \mathrm{Deg}^+(v)\cap \mathrm{Deg}^-(v)} e_{s(k)} + e_{s(k)^{-1}}$ is just a sum over self-loops.

Note that in the case that H is trivial, $-\sum_{k\in \mathrm{Deg}^+(v)\cap \mathrm{Deg}^-(u)} e_{s(k)} - \sum_{k\in \mathrm{Deg}^-(v)\cap \mathrm{Deg}^+(u)} e_{s(k)^{-1}}$ is negative the number of edges going between v and u and $|\mathrm{Deg}(v)| - \sum_{k\in \mathrm{Deg}^+(v)\cap \mathrm{Deg}^-(v)} e_{s(k)} + e_{s(k)^{-1}}$ is the degree of v neglecting self-loops (as self loops contribute 2 to $|\mathrm{Deg}(v)|$ and -2 to the sum $\sum_{k\in \mathrm{Deg}^+(v)\cap \mathrm{Deg}^-(v)} e_{s(k)} + e_{s(k)^{-1}}$.

Also, while $L(G, s)$ is not symmetric, it is invariant under the operation $*$ we defined on ∂^s.

Example 9. For example, consider the graph G and ordinary voltage assignments in Figure 7, where $H = \mathbb{Z}/3\mathbb{Z} = \langle\omega\rangle$. Then,

$$\partial^s = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1-e_\omega & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -e_{\omega^2} & -1 & -e_\omega \end{pmatrix} \end{array}, \partial^{s*} = \begin{pmatrix} 1-e_{\omega^2} & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -e_\omega \\ 1 & 0 & -1 \\ 0 & 1 & -e_{\omega^2} \end{pmatrix}$$

and

$$L(G, s) = \partial^s\partial^{s*} = \begin{pmatrix} 5-e_\omega-e_{\omega^2} & -1 & -1-e_\omega \\ -1 & 2 & -e_{\omega^2} \\ -1-e_{\omega^2} & -e_\omega & 3. \end{pmatrix}$$

Proposition 6.1. *If (G, s) is an ordinary voltage graph and G^s is the derived graph, then the critical group of G^s is equal to*

$$\mathbb{Z}[H]^E/(\mathrm{im}(\partial^{s*}) + \ker(\partial^s)) = \mathrm{im}(\partial^s)/\mathrm{im}(\partial^s\partial^{s*}).$$

Proof. Given $\mathbb{Z}[H]^V$ and $v \in V$, let $\iota_v : \mathbb{Z}[H] \to \mathbb{Z}[H]^V$ be the injection that takes an element $x \in \mathbb{Z}[H]$ and sends it to the element in $\mathbb{Z}[H]^V$ with x in the component corresonding to v and 0's everywhere else. In particular, this yields and isomorphism ψ from $\mathbb{Z}[H]^V$ with $\mathbb{Z}^{H\times V}$ that sends $\iota_v(e_h)$ to $e_{v,h}$. (Here, e_h is a basis element of the group algebra while $e_{v,h}$ is standard basis element of $\mathbb{Z}^{H\times V}$).

Similarly, there is an isomorphism between $\mathbb{Z}[H]^V$ and $\mathbb{Z}^{H\times V}$. Let G^s be the derived graph of the ordinary voltage graph (G, s) and ∂' be the directed incidence matrix of G^s (we would like to use ∂^s, but that is already taken). We will show that ∂^s and ∂' are really the same thing.

More precisely, the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z}[H]^E \arrow[r, "\sim"] \arrow[d, "\partial^s"] & \mathbb{Z}^{E\times H} \arrow[d, "\partial'"] \\ \mathbb{Z}[H]^V \arrow[r, "\sim"] & \mathbb{Z}^{V\times H} \end{tikzcd}$$

It suffices to show that $\iota_k(e_h)$, where $k \in E$ and $h \in H$, maps to the same element in $\mathbb{Z}^{V\times H}$ in both ways. If we go clockwise first, we see that $\iota_k(e_h)$ maps to $e_{k,h}$. This maps to the column of ∂' corresponding to the edge k_h in G^s.

Suppose k runs from v to u in G. Then, k_h runs from v_h to $u_{s(k)h}$. Therefore, the image of $\iota_k(e_h)$ is the column vector in $\mathbb{Z}^{V\times H}$ with a 1 in the component corresponding to (v,h), a -1 in the component corresponding to $(u, s(k)h)$, and 0's everywhere else.

Now, we chase the diagram counterclockwise. Applying ∂^s to $\iota_k(e_h)$ yields the column vector $\iota_v(e_h) - \iota_u(e_{s(k)h})$ in $\mathbb{Z}[H]^V$. This maps to $e_{v,h} - e_{u,s(k)h} \in \mathbb{Z}^{V\times H}$, which is the same as the column vector in $\mathbb{Z}^{V\times H}$ with a 1 in the component corresponding to (v,h), a -1 in the component corresponding to $(u, s(k)h)$, and 0's everywhere else, as desired.

In addition, we claim that $\operatorname{im}(\partial^{s*})$ and $\ker(\partial^s)$ inside of $\mathbb{Z}[H]^E$ map to $\operatorname{im}(\partial')$ and $\ker(\partial')$ inside of $\mathbb{Z}^{E\times H}$.

The fact that $\ker(\partial^s)$ maps isomorphically to $\ker(\partial')$ should be clear from the commutativity of the diagram above. The fact that $\operatorname{im}(\partial^{s*})$ maps to $\operatorname{im}(\partial')$ is a little less clear.

It suffices to show that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z}[H]^E \arrow[r, "\sim"] & \mathbb{Z}^{E\times H} \\
\mathbb{Z}[H]^V \arrow[r, "\sim"] \arrow[u, "\partial^{s*}"] & \mathbb{Z}^{V\times H} \arrow[u, "\partial'^t"']
\end{tikzcd}$$

Again, we chase the diagram clockwise and counterclockwise starting at $\mathbb{Z}[H]^V$. It suffices to show that $\iota_v(e_h)$ is mapped to the same element in both ways. We first chase the diagram in the counterclockwise direction.

First, $\iota_v(e_h)$ maps to $e_{v,h} \in \mathbb{Z}^{V\times H}$ under the isomorphism. To find the image under ∂', we need to the edges k_h that are directed into and out of v_h. By definition of a derived graph, the edges that are directed out of v_h are the edges k_h for all edges k directed out of v in G. Let the indices corresponding to these vectors be S_1. The edges that are directed into v_h are the edges $k_{s(k)^{-1}h}$ where for all edges k directed into v. Let the indices corresponding to these vectors be S_1. The image of $e_{v,h}$ in $\mathbb{Z}^{V\times E}$ is the sum of the basis elements corresponding to these edges, $\sum_{(k,h)\in S_1} e_{k,h} - \sum_{(k,h')\in S_2} e_{k,h'}$.

Now, we chase the diagram clockwise. By the definition of ∂^{s*}, $\partial^{s*}\iota_v(e_h)$ maps to $\sum_{(k,h)\in S_1} \iota_k(e_h) - \sum_{(k,h')\in S_2} \iota_k(e_{h'})$, where S_1 and S_2 are the same as above. This maps to precisely $\sum_{(k,h)\in S_1} e_{k,h} - \sum_{(k,h')\in S_2} e_{k,h'}$ in $\mathbb{Z}^{E\times H}$, so the diagram commutes.

Finally, for the diagram chasing, we have the following induced isomorphisms

$$\begin{tikzcd}
\mathbb{Z}[H]^E/(\operatorname{im}(\partial^{s*}) + \ker(\partial^s)) \arrow[r, "\sim"] \arrow[d, "\overline{\partial^s}"] & \mathbb{Z}^{E\times H}(\operatorname{im}(\partial') + \ker(\partial')) \arrow[d, "\overline{\partial'}"] \\
\operatorname{im}(\partial^s)/\operatorname{im}(\partial^s\partial^{s*}) \arrow[r, "\sim"] & \operatorname{im}(\partial')/\operatorname{im}(\partial'\partial'^t)
\end{tikzcd}$$

where all the arrows are isomorphisms. □

Now, we want to find the cokernel of the induced injection of critical groups $\hat{\kappa} : K(G) \to K(G^s)$.

Definition 14. Let I be the two-sided ideal $\mathbb{Z}\sum_{h\in H} e_h$ of the group algebra $\mathbb{Z}[H]$. Define the *reduced directed incidence matrix* of ∂^s to be the result after applying the projection map from $\mathbb{Z}[H] \to \mathbb{Z}[H]/I$ to each entry. We will denote this as $\overline{\partial^s}$. We define the operator $*$ on the reduced directed incidence matrix to take $\overline{\partial^s}$ to $\overline{\partial^{s*}}$, the result after applying the projection $\mathbb{Z}[H] \to \mathbb{Z}[H]/I$ to each entry of ∂^{s*}.

Proposition 6.2. *Let I be the two-sided ideal $\mathbb{Z}\sum_{h\in H} e_h$ of the group algebra $\mathbb{Z}[H]$.*

Then, the cokernel of the induced injection $\hat{\kappa}$ from $K(G)$ to $K(G^s)$ is of the form

$$(\mathbb{Z}[H]/I)^E/(\operatorname{im}(\overline{\partial^{s*}}) + \ker(\overline{\partial^s})) = \operatorname{im}(\overline{\partial^s})/\operatorname{im}(\overline{\partial^s}\,\overline{\partial^{s*}}).$$

Proof. Recall from Remark 3, the induced injection in terms of the edge presentation is the map induced by $\hat{\psi}$ in the following diagram

$$\begin{tikzcd}
\mathbb{Z}^E \arrow[r, hook, "\hat{\psi}"] \arrow[d, "\kappa_1"] & \mathbb{Z}^{E\times H} \arrow[d, "\kappa_2"] \\
\mathbb{Z}^E/(\operatorname{im}(\partial^t)\oplus\ker(\partial)) \arrow[r, hook, "\overline{\hat{\phi}}"] & \mathbb{Z}^{E\times H}/(\operatorname{im}(\partial'^t)\oplus\ker(\partial'))
\end{tikzcd}$$

where we define the map $\hat{\psi}$ from $\mathbb{Z}^E$ to $\mathbb{Z}^{E'}$ that sends e_k for k an edge of G to $\sum_{k'\in p^{-1}(k)} e_k$, where we defined $p^{-1}(k)$ to be all the edges in G'_p that map to k under the covering map from G^s to G. From the isomorphism between $\mathbb{Z}^{E\times H}$ with $\mathbb{Z}[H]^E$ and the discussion above, this yields

$$\begin{tikzcd}
\mathbb{Z}^E \arrow[r, hook, "\hat{\psi}"] \arrow[d, "\kappa_1"] & \mathbb{Z}[H]^E \arrow[d, "\kappa_2"] \\
\mathbb{Z}^E/(\operatorname{im}(\partial^t)\oplus\ker(\partial)) \arrow[r, hook, "\hat{\kappa}"] & \mathbb{Z}[H]^E/(\operatorname{im}(\partial^{s*})+\ker(\partial^s))
\end{tikzcd}$$

Since the image of $\hat{\psi}$ is precisely, $(\mathbb{Z}\sum_{h\in H} e_h)^E$, if we let I be the two-sided ideal $\mathbb{Z}\sum_{h\in H} e_h$ of the group algebra $\mathbb{Z}[H]$, the cokernel of $\hat{\psi}$ is precisely $(\mathbb{Z}[H]/I)^E$. This yields the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^E \arrow[r, "\hat{\psi}"] \arrow[d, "\kappa_1"] & \mathbb{Z}[H]^E \arrow[r] \arrow[d, "\kappa_2"] & (\mathbb{Z}[H]/I)^E \arrow[r] \arrow[d, "\kappa_3"] & 0 \\
0 \arrow[r] & \mathbb{Z}^E/(\operatorname{im}(\partial^t)\oplus\ker(\partial)) \arrow[r, "\hat{\kappa}"] & \mathbb{Z}[H]^E/(\operatorname{im}(\partial^{s*})+\ker(\partial^s)) \arrow[r] & \operatorname{coker}(\hat{\kappa}) \arrow[r] & 0
\end{tikzcd}$$

where the map κ_3 is induced as the image of $\hat{\psi}$ goes to zero when we go from $\mathbb{Z}[H]^E$ to $\mathbb{Z}[H]^E/(\operatorname{im}(\partial^{s*})+\ker(\partial^s))$ and then to $\operatorname{coker}(\hat{\kappa})$ (this is because the image of $\mathbb{Z}^E$ is zero when we go from $\mathbb{Z}^E$ to $\mathbb{Z}^E/(\operatorname{im}(\partial^t)\oplus\ker(\partial))$ to $\mathbb{Z}[H]^E/(\operatorname{im}(\partial^{s*})+\ker(\partial^s))$ to $\operatorname{coker}(\hat{\kappa})$ and that the diagram is commutative). Therefore, if $\overline{\operatorname{im}(\partial^{s*})}$ and $\overline{\ker(\partial^s)}$ are the images of $\operatorname{im}(\partial^{s*}), \ker(\partial^s) \subset \mathbb{Z}[H]^E$ in $(\mathbb{Z}[H]/I)^E$, then $\operatorname{coker}(\hat{\kappa}) = (\mathbb{Z}[H]/I)^E/(\overline{\operatorname{im}(\partial^{s*})}+\overline{\ker(\partial^s)})$.

We claim that $\overline{\operatorname{im}(\partial^{s*})}$ and $\overline{\ker(\partial^s)}$ are equal to $\operatorname{im}(\overline{\partial^{s*}})$ and $\ker(\overline{\partial^s})$. We claim that both of these facts come from the commutativity of the following diagram (the diagram commutes because the projection $\mathbb{Z}[H] \to \mathbb{Z}[H]/I$ is a homomorphism):

$$\begin{tikzcd}
\mathbb{Z}[H]^E \arrow[r, "\partial^s"] \arrow[d, "\kappa_1"] & \mathbb{Z}[H]^V \arrow[d, "\kappa_2"] \\
(\mathbb{Z}[H]/I)^E \arrow[r, "\overline{\partial}^s"] & (\mathbb{Z}[H]/I)^E
\end{tikzcd}$$

The fact that $\overline{\operatorname{im}(\partial^{s*})} = \operatorname{im}(\overline{\partial^{s*}})$ is a result of following the image of $\mathbb{Z}[H]^E$ both clockwise and counter-clockwise, respectively. The fact that $\overline{\ker(\partial^s)} = \ker(\overline{\partial^s})$ follows the fact that $\kappa_1((\partial^s)^{-1}(0)) = (\overline{\partial^s})^{-1}(\kappa_2(0))$. Therefore, we have shown the cokernel is

$$(\mathbb{Z}[H]/I)^E/(\operatorname{im}(\overline{\partial^{s*}})+\ker(\overline{\partial^s})).$$

The isomorphism from $(\mathbb{Z}[H]/I)^E/(\operatorname{im}(\overline{\partial^{s*}})+\ker(\overline{\partial^s}))$ to $\operatorname{im}(\overline{\partial^s})/\operatorname{im}(\overline{\partial^s\partial^{s*}})$ can be seen by considering the composite of the map $\overline{\partial^s} : (\mathbb{Z}[H]/I)^E \to \operatorname{im}(\overline{\partial^s})$ and the projection map from $\operatorname{im}(\overline{\partial^s})$ to $\operatorname{im}(\overline{\partial^s})/\operatorname{im}(\overline{\partial^s\partial^{s*}})$ and noting that the kernel is exactly $\operatorname{im}(\overline{\partial^{s*}})+\ker(\overline{\partial^s})$. □

Example 10. Suppose $G = (V, E)$ is a graph and G^s is derived from the assignment of the trivial element of a group H to each edge. Then, G^s is $|H|$ disjoint copoies of G, so we would expect $\operatorname{coker}(\hat{\kappa})$, the cokernel of the induced injection of critical groups, to be $K(G)^{|H|-1}$.

To show this is indeed true, fix $h_0 \in H$ and consider the isomorphism of $\mathbb{Z}$-modules that sends an element x of $\mathbb{Z}^{E\times(H\setminus h_0)}$ to the element $y \in (\mathbb{Z}[H]/I)^E$, where the component of y corresponding to an edge k of E is the equivalence class of $\sum_{h\in H\setminus h_0} x_{k,h}e_h$ in $\mathbb{Z}[H]/I$. This map is in fact bijective, as there is a *unique* element in each equivalence class of $\mathbb{Z}[H]/I$ with the coefficient in front of e_{h_0} zero.

We also claim that $\mathrm{im}(\partial^t)^{H\setminus h_0}$ and $\ker(\partial)^{H\setminus h_0}$ map isomorphically unto $\mathrm{im}(\overline{\partial^{s*}})$ and $\ker(\overline{\partial^s})$ under this map. We claim that this follows from the fact that in our special case, $\mathrm{im}(\partial'^t) = \mathrm{im}(\partial^t)^H$, $\ker(\partial') = \ker(\partial)^H$ and that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z}^{E\times H} \arrow[r, "\sim"] \arrow[d, "\kappa_1"] & \mathbb{Z}[H]^E \arrow[d, "\kappa_2"] \\
\mathbb{Z}^{E\times(H\setminus h_0)} \arrow[r, "\sim"] & (\mathbb{Z}[H]/I)^E
\end{tikzcd}$$

We have shown above that $\mathrm{im}(\overline{\partial^{s*}})$ and $\ker(\overline{\partial^s})$ are the images of $\mathrm{im}(\partial^{s*})$ and $\ker(\partial^s)$ under κ_2. The commutativity of the diagram shows that this is exactly the image of $\mathrm{im}(\partial'^t) = \mathrm{im}(\partial^t)^H$, $\ker(\partial') = \ker(\partial)^H$ in $(\mathbb{Z}[H]/I)^E$ when we follow the maps counterclockwise. Therefore, we have an induced isomorphism between $(\mathbb{Z}^E/(\mathrm{im}(\partial^t)+\ker(\partial)))^{H\setminus\{h_0\}}$ and $(\mathbb{Z}[H]/I)^E/(\mathrm{im}(\overline{\partial^{s*}})+\ker(\overline{\partial^s}))$, as desired.

6.2. **Special case when $H = \mathbb{Z}/p\mathbb{Z}$ for p a prime.** If H is the cyclic group $H = \mathbb{Z}/p\mathbb{Z}$, and I is the two-sided ideal $\sum_{h\in H} e_h$ of $\mathbb{Z}[H]$, then the quotient $\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]/I$ is isomorphic to $\mathbb{Z}[\omega_p]$ where ω_p is a primitive p^{th} root of unity. This is because the cyclotomic polynomial of ω_p is $1+x+\cdots+x^{p-1}$, so the kernel of the map α from $\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$ to $\mathbb{Z}[\omega_p]$ that sends e_{a^k} to ω_p^k, where a is a generator of $H = \mathbb{Z}/p\mathbb{Z}$, is precisely I.

Therefore, we can interpret Proposition 6.2 in the special case in terms of $\mathbb{Z}[\omega_p]$

Proposition 6.3. *If $H = \mathbb{Z}/p\mathbb{Z}$, the cokernel of the induced injection $\hat{\kappa}$ from $K(G)$ to $K(G^s)$ is of the form*

$$\mathbb{Z}[\omega_p]^E/(\mathrm{im}(\overline{\partial^{s*}})+\ker(\overline{\partial^s})) = \mathrm{im}(\overline{\partial^s})/\mathrm{im}(\overline{\partial^s\partial^{s*}}),$$

where $\overline{\partial^{s*}}$ and $\overline{\partial^s}$ are understood to be the result after applying the map α from $\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$ to $\mathbb{Z}[\omega_p]$ that sends e_{a^k} to ω_p^k for each entry of the matrices.

6.3. **Special case when $H = \mathbb{Z}/2\mathbb{Z}$.** When $p = 2$ in the preceeding subsection, then $\omega_p = -1$ and $\mathbb{Z}[\omega_p]$ is just $\mathbb{Z}$. We would like to understand what $\overline{\partial^{s*}}$ and $\overline{\partial^s}$ are in Proposition 6.3 in this case.

Recall that

$$\partial^s_{v,k} = \begin{cases} 1(=e_1) & \text{if } k \text{ is not a loop and is directed away from } v \\ -e_{s(k)} & \text{if } k \text{ is not a loop and is directed into } v \\ 1-e_{s(k)} & \text{if } k \text{ is a loop at } v \\ 0 & \text{else.} \end{cases}$$

If we let the $s(e)$ being the identity denote a positive edge and $s(e)$ being the nontrivial group element denote a negative edge, then the image of this matrix after we apply the map $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \to \mathbb{Z}$ to each entry is

$$\overline{\partial^s_{v,k}} \begin{cases} 1 & \text{if } k \text{ is not a loop and is directed away from } v \\ -1 & \text{if } k \text{ is not a loop and is directed into } v \text{ and } k \text{ has positive sign} \\ 1 & \text{if } k \text{ is not a loop and is directed into } v \text{ and } k \text{ has negative sign} \\ 0 & \text{if } k \text{ is a loop at } v \text{ and } k \text{ has positive sign} \\ 2 & \text{if } k \text{ is a loop at } v \text{ and } k \text{ has negative sign} \\ 0 & \text{else.} \end{cases}$$

This is exactly the directed incidence matrix of a signed graph where we assign the edge a positive sign if $s(e)$ is trivial and assign the edge a negative sign otherwise. Therefore, Proposition 6.3 specializes to the following.

Proposition 6.4. *Then, the cokernel of the induced injection $\hat{\kappa}$ from $K(G)$ to $K(G^s)$ is of the form*

$$\mathbb{Z}^E/(\mathrm{im}(\overline{\partial^{s*}})+\ker(\overline{\partial^s})) = \mathrm{im}(\overline{\partial^s})/\mathrm{im}(\overline{\partial^s\partial^{s*}}),$$

where $\overline{\partial^s}$ *and* $\overline{\partial^s}$ *are the directed incidence matrices of the signed graph obtained from* G *from the map* $s : E \to \mathbb{Z}/2\mathbb{Z}$.

7. 2-coverings of signed graphs and the closure property

We discuss induced injection and surjections of signed graphs. The arguments given at the beginning generalize easily to this case. The main reason for consider coverings of signed graphs is that the cokernel of an induced injection for the case of 2-covers is the critical group of another signed graph.

For this reason, we will restrict ourselves to 2-coverings, though there is no reason the results that are not related to the closure property would not generalize.

7.1. **Covering spaces of signed graphs.** Everything in Section 3.1 generalizes naturally to signed graphs. We will reproduce the section below with a few small changes.

Definition 15. Define a signed graph G to have four types of edges:

(1) positive edges (possibly loops)
(2) negative edges (possibly loops)
(3) negative half-loops (these will be denoted in diagrams a directed loops labeled with negative sign).
(4) positive half-loops (these will be denoted in diagrams a directed loops labeled with positive sign).

Remark 9. For our purposes, one (positive) negative loop and two (positive) negative half-loops will be indistinguishable in terms of the critical groups of both the original graph and the derived graph.

Definition 16. Let S_2 denote the symmetric group on $\{1, 2\}$. A permutation voltage assignment in S_2 for a directed, signed graph $G = (V, E)$ is a function s that assigns to each edge of G a permutation in S_2. The pair (G, s) is called a 2-permutation voltage graph.

Definition 17. Given a 2-permutation voltage graph (G, s), a derived graph G^s is constructed as follows:

(1) The vertex set is the cartesian product $V \times \{1, 2\}$. For convenience, the vertices will be denoted as v_i instead of (v, i).
(2) For each edge e:
   (a) If edge e is not a half loop assigned the nontrivial permutation, then, suppose e runs from u to v in G. Then, we make two copies of e in G^s. These will be denoted as e_1 and e_2. For $i = 1$ and $i = 2$, e_i runs from u_i and $v_{\pi(i)}$, where π is the permutation $s(e)$ associated to e by s.
   (b) If e is a half loop at v assigned the nontrivial permutation, then there is an edge e' directed from v_1 to v_2.

Remark 10. Here, one edge is viewed to cover a half loop twice.

Also, note that our choice to direct an edge that covers a half loop from v_1 to v_2 is arbitrary.

An example of a derived covering space is in Figure 8.

Definition 18. Define the *directed incidence matrix* $\partial \in \mathbb{Z}^{V\times E}$ of a signed graph to be such that

$$\partial_{v,k} = \begin{cases} 1 (= e_1) & \text{if } k \text{ is not a loop and is directed away from } v \\ -1 & \text{if } k \text{ has positive sign, is not a loop and is directed into } v \\ 1 & \text{if } k \text{ has negative sign, is not a loop and is directed into } v \\ 2 & \text{if } k \text{ is a negative (half) loop at } v \\ 0 & \text{else (in particular } k \text{ is a positive (half) loop at } v). \end{cases}$$

Definition 19. Define the *transpose directed incidence matrix* $\partial^T \in \mathbb{Z}^{E\times V}$ of a signed graph to be such that

$$\partial^T_{k,v} = \begin{cases} 1 (= e_1) & \text{if } k \text{ is not a loop and is directed away from } v \\ -1 & \text{if } k \text{ has positive sign, is not a loop and is directed into } v \\ 1 & \text{if } k \text{ has negative sign, is not a loop and is directed into } v \\ 2 & \text{if } k \text{ is a negative loop at } v \\ 1 & \text{if } k \text{ is a negative half loop at } v \\ 0 & \text{else (in particular } k \text{ is a positive (half) loop at } v). \end{cases}$$

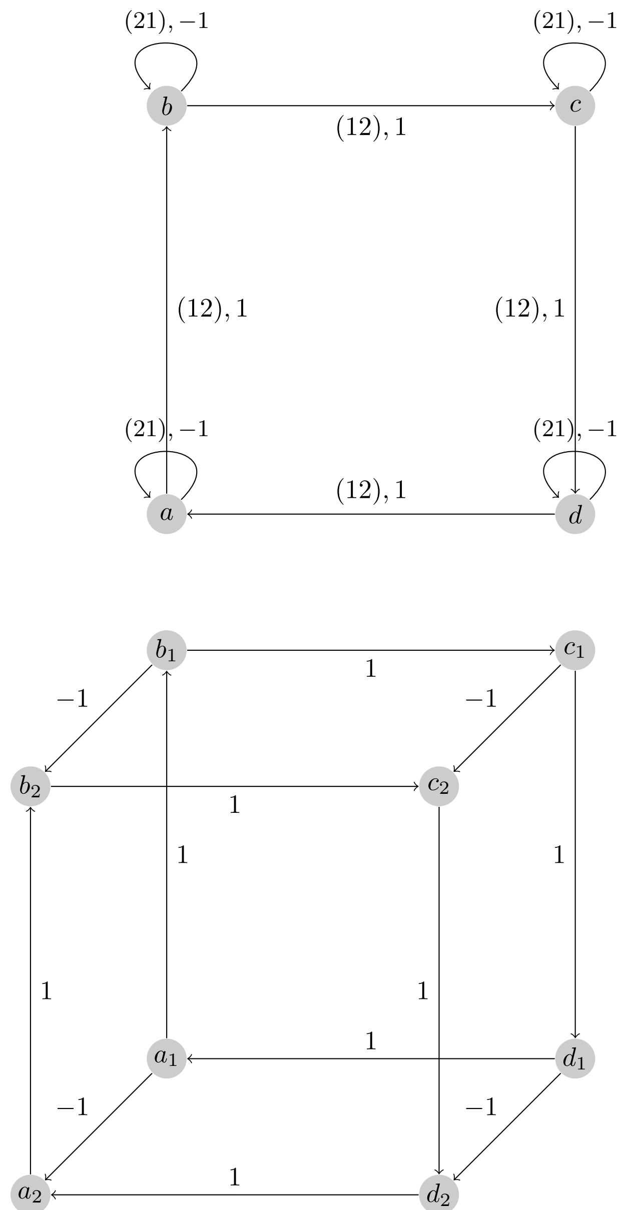

FIGURE 8. Ordinary Voltage graph and derived covering. Note that if we changed all the half loops to positive half loops in the ordinary voltage graph, we would have gotten the cube as the cover.

In particular, ∂^T is the transpose of ∂ except for the columns of ∂ that correspond to negative half loops.

It can be verified by direct matrix multiplication that

$$\partial\partial^T_{v,u} = \begin{cases} |\text{Number of negative edges between } v \text{ and } u| - \\ |\text{Number of positive edges between } v \text{ and } u| & \text{if } v \neq u \\ \\ |\text{Number of non-loops adjacent to } v| + \\ 4|\text{Number of negative loops at } v| + \\ 2|\text{Number of half loops at } v| & \text{if } v = u. \end{cases}$$

In particular, $\partial\partial^T$ is symmetric.

Definition 20. Given a signed graph G, we define the critical group $K(G)$ to be

$$\frac{\mathbb{Z}^E}{\operatorname{im}(\partial^T)+\ker(\partial)} = \operatorname{im}(\partial)/\operatorname{im}(\partial\partial^T).$$

Example 11. Consider signed graph in Figure 8 with vertices a, b, c, d. Then,

$$\partial = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array}, \partial^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\partial\partial^T = \begin{pmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{pmatrix}.$$

7.2. **Induced Surjections.** Let $G_1 = (V_1, E_1)$ be a signed graph. Assign a permutation to each edge of G_1 and construct the derived graph $G' = (V', E')$ (a 2-covering of G_1). Define $G_2 = (V_2, E_2)$ to be the signed graph obtained by reversing the sign of all the edges of G_1 assigned a nontrivial permutation.

We will index V_1 with $\{1, \dots, m\}$. Since G_2 is the same graph as G_1 if we forget about sign, there will be no harm in indexing V_2 also as $\{1, \dots, m\}$ (the same vertex set). Since there are two vertices of G' for each vertex of G_1, we can index G' with $\{\pm 1, \dots, \pm m\}$ such that vertices $\pm k$ of G' map to vertex k of G_1 under the covering map.

Let ∂_1, ∂_2, and ∂' be the directed incidence matrices of G_1, G_2, and G' respectively. We claim there are induced surjection of the critical group of G' unto G_1 and G_2 as follows:

Proposition 7.1. *Consider the surjection $\phi : \mathbb{Z}^{V'} \to \mathbb{Z}^{V_1}$ of lattices that sends $e_{\pm i}$ to e_i. Then,*

(1) ϕ restricts to a surjection from $\operatorname{im}(\partial')$ to $\operatorname{im}(\partial_1)$.
(2) The surjection induces a surjection from $K(G')$ to $K(G_1)$ as in the following commutative diagram:

$$\begin{tikzcd} \operatorname{im}(\partial') \arrow[r, two heads, "\phi"] \arrow[d, two heads, "\kappa'"] & \operatorname{im}(\partial_1) \arrow[d, two heads, "\kappa_1"] \\ K(G') := \operatorname{im}(\partial')/\operatorname{im}(\partial'\partial'^T) \arrow[r, "\kappa"] & \operatorname{im}(\partial_1)/\operatorname{im}(\partial_1\partial_1^T) \end{tikzcd}$$

.

Proof. To see that ϕ restricts to a surjection from $\operatorname{im}(\partial')$ to $\operatorname{im}(\partial_1)$, we first define a map $\psi : \mathbb{Z}^{E'} \to \mathbb{Z}^E$ that sends each edge k' of E' to the edge that it covers under the covering map $p : G' \to G_1$. (In particular, if the edge that k' covers happens to be a half loop, we still only send it to $e_{p(k')}$, not $2e_{p(k')}$. This is because the directed incidence matrix ∂_1 makes no distinction between half loops and loops.

If $e \in E$ is not a half loop, then e will be hit by two edges under the covering map p. If $e \in E$ is a half loop, then e will be hit by one edge under the covering map p. Showing ϕ is surjective reduces to verifying the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z}^{E'} \arrow[r, two heads, "\psi"] \arrow[d, two heads, "\partial'"] & \mathbb{Z}^E \arrow[d, two heads, "\partial_1"] \\ \operatorname{im}(\partial') \arrow[r, "\phi"] & \operatorname{im}(\partial_1) \end{tikzcd}$$

To verify this, it suffices to show that $e_{k'}$ maps to the same element both clockwise and counterclockwise, where k' is an edge in E'. Suppose k' is directed from v' to u' (where v' and u' might be the same). Then, if we go clockwise, $p(k')$ is an edge directed from $p(v')$ to $p(u')$. This maps to a $e_{p(u')} \pm e_{p(v')}$, where the sign depends on the sign of k'.

If we go counterclockwise, k' maps to $e_{u'} \pm e_{v'}$, where the sign depends on the sign of k'. Then, $e_{u'} \pm e_{v'}$ maps to $e_{p(u')} \pm e_{p(v')}$, as desired.

Finally, showing that the surjection $\phi : \mathrm{im}(\partial') \to \mathrm{im}(\partial_1)$ induces a surjection reduces to showing that $\mathrm{im}(\partial'\partial'^T)$ goes to zero under $\kappa_1 \circ \phi$. To show this, I claim that it can be checked that the column of $\partial'\partial'^T$ corresponding to the vertex indexed by $\pm i$ maps to the column of $\partial_1\partial_1^T$ corresponding to the vertex indexed by i. □

7.3. **Sign flips and voltage flips.** Preserving the same setup as the previous subsection, we describe two operations on G_1.

Definition 21. Define a *sign flip* at a vertex v of G_1 (or G' or G_2) to reverse all the signs of all the edges incident to v.

A self loop is understood to not be affected. We can interpret this has being flipped twice.

Remark 11. It can be checked that a sign flip of the vertex of G_1 indexed by i corresponds to sign flips at the vertices indexed by $+i$ and $-i$ of G' and a sign flip at the vertex indexed by i of G_2.

Remark 12. In the presentation of the group G_1 (or G' or G_2) as $\mathrm{im}(\partial_1)/\mathrm{im}(\partial_1\partial_1^T)$, we can interpret a sign flip at the vertex indexed by i as replacing the standard basis vector e_i in $\mathbb{Z}^{V_1}$ with $-e_i$. If we reexpress $\partial\partial^T$ after replacing e_i with $-e_i$, this corresponds to flipping the sign of the column of $\partial\partial^T$ corresponding to i and then doing the same to the row corresponding to i. In particular, performing a sign flip does not change the critical group.

However, if we blindly reexpress ∂ (or ∂^T) by just replacing e_i with $-e_i$, we might get a column with two -1's (flipping the head of a positively signed edge) or "change the direction" of a negative edge (flipping the head of a negatively signed edge). To avoid this, to compute ∂, we flip vertex i of G_1 (preserving directions) and then read off ∂ as usual.

Definition 22. Define a *voltage flip* at a vertex v of G_1 to reverse all the permutation assignments of all the edges incident to v.

A self loop is understood to not be affected. We can interpret this has being flipped twice.

Remark 13. It can be checked that a voltage flip at the vertex of G_1 indexed by i corresponds to having the vertices of G' indexed by $+i$ and $-i$ switch places (so in particular does not alter G') and a sign flip of G_2 at the vertex i.

In particular, this gives a surjection from $K(G')$ to $K(G_2)$.

Proposition 7.2. *Consider the surjection $\phi : \mathbb{Z}^{V'} \to \mathbb{Z}^{V_2}$ of lattices that sends e_{+i} to e_i amd e_{-i} to $-e_i$. Then,*

(1) ϕ restricts to a surjection from $\mathrm{im}(\partial')$ to $\mathrm{im}(\partial_2)$.
(2) The surjection induces a surjection from $K(G')$ to $K(G_2)$ as in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{im}(\partial') \arrow[r, two heads, "\phi"] \arrow[d, two heads, "\kappa'"] & \mathrm{im}(\partial_2) \arrow[d, two heads, "\kappa_2"] \\
K(G') := \mathrm{im}(\partial')/\mathrm{im}(\partial'\partial'^T) \arrow[r, "\kappa"] & \mathrm{im}(\partial_2)/\mathrm{im}(\partial_2\partial_2^T)
\end{tikzcd}$$

.

Proof. Consider the result after we perform a sign flip at each vertex in $\{-1, -2, \ldots, -m\}$ of G'. If we consider the cut of the vertices of G' into $\{+1, +2, \ldots, +m\}$ and $\{-1, -2, \ldots, -m\}$, then an edge changes sign if and only if it crosses the cut.

This is precisely what it means for the lift of any edge k in G_1 to change sign if and only if k is assigned the nontrivial permutation. Therefore, after performing the flips, we find that the new graph $G'' = (V', E')$

is now a cover of G_2 and we can apply Proposition 7.1 to find an induced surjection of $K(G'')$ to $K(G_2)$ that is induced by the surjection of lattices ϕ' that sends e_{+i} to e_i and $-e_{-i}$ (of the original basis of $\mathbb{Z}^{V'}$) to e_i.

If we interpret what this means in terms of the original graph G', this is the same as an induced surjection from $K(G')$ to $K(G_2)$ that sends e_{+i} to e_i and e_{-i} to $-e_i$, which is what Proposition 7.2 claims. □

7.4. **Closure Property.** Preserving the setup in Subsection 7.2, we recall that Propositions 7.1 and 7.2 give us induced surjections from $K(G')$ to $K(G_1)$ and $K(G_2)$.

We claim that the kernel of the induced surjection from $K(G')$ into $K(G_1)$ can be described in terms of $K(G_2)$ and vice versa. First, we can assume that G is connected as the general case can be described in terms of each connected component.

Now, we will split our work into three cases:

(1) G does not have a cycle such that the number of edges assigned transpositions in the cycle is odd.
(2) G does have a cycle such that the number of edges assigned tranpositions in the cycle is odd and either ∂_1 has full rank or ∂_2 has full rank.
(3) G does have a cycle such that the number of edges assigned tranpositions in the cycle is odd and ∂_1 and ∂_2 have full rank.

It can be shown that it is impossible to not be in Case 1 and yet have both ∂_1 and ∂_2 not of full rank. To see this, we can apply sign flips to G_1 so that G_1 has all positive signs. Then, G_2 has a cycle with an odd number of negative edges, so that ∂_2 is full rank. Therefore, our three cases does contain all the possibilities.

Also, note that in Cases 2 and 3, G' is a connected graph.

7.5. **Case 1.** For case 1, it can be shown that a sequence of voltage flips can be applied so that all the permutations assigned to G_1 are the trivial permutations so that $G_1 = G_2$ and G' is two disjoint copies of G_1. Intuitively, this can be seen as "untwisting" G' into two separate pieces. Then, we see that $K(G') = K(G_1) \oplus K(G_2)$.

Remark 14. If we actually want to recover the injection of $K(G_2)$ as the kernel, we would first consider the case where all the permutations assigned to G_1 are trivial and consider the injection of $K(G_2)$ to $K(G')$ induced by the lattice map $\hat{\phi} : \mathbb{Z}^{V_2} \to \mathbb{Z}^{V'}$ that sends e_i to $e_i - e_{-i}$.

Then, in general, we would have to keep track of the voltage flips we performed and track how to recover the original injection of $K(G_2)$ into $K(G')$. If ∂_2 has full rank, we don't have to perform this exercise, but if ∂_2 does not have full rank, there could be issues about mapping into $\mathrm{im}(\partial')$ or hitting the entire kernel if we just blindly use the map $\hat{\phi} : \mathbb{Z}^{V_2} \to \mathbb{Z}^{V'}$ that sends e_i to $e_i - e_{-i}$.

7.6. **Case 2.** Without loss of generalize, we can assume that ∂_1 is not of full rank and ∂_2 is of full rank, since, as noted in the proof of Proposition 7.2, flipping signs at $\{-1, \ldots, -m\}$ of G' reverses the roles of G_1 and G_2.

Since ∂_1 is not of full rank, we can perform sign flips at the vertices of G_1 so that all of the edges of G_1 become positively signed. In particular, this means G' also has all positive edges. What happens in this case can be described in Proposition 7.3

Definition 23. Define the *sign preserving involution* ι on $\mathbb{Z}^{V'}$ be the endomorphism that sends e_{+i} to e_{-i} and e_{-i} to e_{+i}.

Remark 15. It can be shown that $\partial\partial^T\iota(x) = \iota(\partial\partial^T x)$. This will be the main application of the sign preserving involution.

Proposition 7.3. *If G_1 is connected and has all positive edges (so G' has all positive edges), then we have the following two exact sequences:*

$$0 \leftarrow K(G_1) \leftarrow K(G') \leftarrow K(G_2) \leftarrow 0$$
$$0 \rightarrow K(G_1) \rightarrow K(G') \rightarrow K(G_2) \rightarrow 0,$$

where the surjections are the surjections from Propositions 7.1 and 7.2.

Proof. For the first exact sequence, we first note that if ϕ is the map of lattices from $\mathbb{Z}^{V'}$ to $\mathbb{Z}^{V_1}$ that sends $e_{\pm i}$ to e_i, then $\phi(\mathrm{im}(\partial'\partial'^T))$ is exactly $\mathrm{im}(\partial_1\partial_1^T)$. To see this, it suffices to recall that the column of $\partial'\partial'^T$ corresponding to vertex $+i$ or $-i$ maps to the column of $\partial_1\partial_1^T$ corresponding to vertex i.

This means in particular that if the image of a vector $x \in \mathbb{Z}^{V'}$ is equivalent to 0 after quotienting out by $\mathrm{im}(\partial_1\partial_1^T)$, then there is a vector $x' \in \mathbb{Z}^{V'}$ equivalent to x under $\mathrm{im}(\partial'\partial'^T)$ that maps exactly to zero under ϕ.

Therefore, the kernel of the map $K(G') \to K(G_1)$ is the subgroup $\mathbb{Z}^{V'}_{\text{skew}}/\mathbb{Z}^{V'}_{\text{skew}} \cap \mathrm{im}(\partial'\partial'^T)$, where $\mathbb{Z}^{V'}_{\text{skew}}$ is sublattice of $\mathbb{Z}^{V'}$ containing all vectors x such that $x + \iota(x) = 0$, where ι is the sign preserving involution on $\mathbb{Z}^{V'}$. Note that since G' is a connected unsigned graph $\mathbb{Z}^{V'}_{\text{skew}} \subset \mathrm{im}(\partial')$

It suffices to find $\mathbb{Z}^{V'}_{\text{skew}} \cap \mathrm{im}(\partial'\partial'^T)$. To do so, suppose $x \in \mathbb{Z}^{V'}$ such that $\partial'\partial'^T x \in \mathbb{Z}^{V'}_{\text{skew}}$. Then, we know that $\partial'\partial'^T x + \iota(\partial'\partial'^T x) = 0$, or $x + \iota(x) \in \ker(\partial'^T)$. Since $\ker(\partial'^T)$ is generated by the all 1's vector (recall that G' is unsigned), we can add $\frac{c}{2}$ times the all 1's vector to x so that $x = \iota(x)$, where c is an integer. Equivalently, x now lies in the lattice $\mathbb{Z}^{V'}_{\text{skew}} + (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2})^t$, where the additional vector has a $\frac{1}{2}$ corresponding to each vertex in $\{+1, \dots, +m\}$ and a $-\frac{1}{2}$ corresponding to each vertex in $\{-1, \dots, -m\}$.

Conversely, it can be shown that for any vector $x \in \mathbb{Z}^{V'}_{\text{skew}} + (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2})$, $\partial'\partial'^T x$ is in $\mathbb{Z}^{V'}_{\text{skew}}$.

Consider the injection $\hat{\phi} : \mathbb{R}^{V_2} \to \mathbb{R}^{V'}$ of lattices that sends e_i to $e_i - e_{-i}$. It can be checked that $\hat{\phi}(\partial_2\partial_2^T x) = \partial'\partial'^T(\hat{\phi}(x))$.

Using the fact that $\hat{\phi} : \mathbb{Z}^{V_2} \to \mathbb{Z}^{V'}_{\text{skew}}$ is an isomorphism, it can be shown that

$$\mathbb{Z}^{V'}_{\text{skew}}/\mathbb{Z}^{V'}_{\text{skew}} \cap \mathrm{im}(\partial'\partial'^T) = \mathbb{Z}^{V_2}/\partial_2\partial_2^T(\mathbb{Z}^{V_2} + (\frac{1}{2}, \dots, \frac{1}{2})^t)$$

From Example 2, we know that there is an isomorphism between $\mathbb{Z}^{V_2}/\partial_2\partial_2^T(\mathbb{Z}^{V_2} + (\frac{1}{2}, \dots, \frac{1}{2})^t)$ and $\widehat{\mathrm{im}(\partial_2)/\mathrm{im}(}\partial_2\partial_2^T)$, which means the kernel of the surjection from $K(G')$ to $K(G_1)$ is isomorphic to $K(G_2)$ as desired.

The proof of the other exact sequence is similar. Let ϕ be the map of lattices from $\mathbb{Z}^{V'}$ to $\mathbb{Z}^{V_2}$ that sends e_{+i} to e_i and e_{-i} to $-e_i$, then $\phi(\mathrm{im}(\partial'\partial'^T))$ is exactly $\mathrm{im}(\partial_2\partial_2^T)$. To see this, it suffices to recall that the column of $\partial'\partial'^T$ corresponding to vertex $+i$ to the column of $\partial_2\partial_2^T$ corresponding to vertex i and the column of $\partial'\partial'^T$ corresponding to $-i$ maps to the negative of the column of $\partial_2\partial_2^T$ corresponding to vertex i.

This means in particular that if the image of a vector $x \in \mathbb{Z}^{V'}$ is equivalent to 0 after quotienting out by $\mathrm{im}(\partial_2\partial_2^T)$, then there is a vector $x' \in \mathbb{Z}^{V'}$ equivalent to x under $\mathrm{im}(\partial'\partial'^T)$ that maps exactly to zero under ϕ.

Therefore, the kernel of the map $K(G') \to K(G_2)$ is the subgroup $\mathrm{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}}/\mathrm{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}} \cap \mathrm{im}(\partial'\partial'^T)$, where $\mathbb{Z}^{V'}_{\text{sym}}$ is sublattice of $\mathbb{Z}^{V'}$ containing all vectors x such that $x = \iota(x)$, where ι is the sign preserving involution on $\mathbb{Z}^{V'}$.

It suffices to find $\mathrm{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}} \cap \mathrm{im}(\partial'\partial'^T)$. To do so, suppose $x \in \mathbb{Z}^{V'}$ such that $\partial'\partial'^T x \in \mathrm{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}}$. Then, we know that $\partial'\partial'^T x - \iota(\partial'\partial'^T x) = 0$, or $x - \iota(x) \in \ker(\partial'^T)$. Since $\ker(\partial'^T)$ is generated by the all 1's vector (recall that G' is unsigned), $x - \iota(x)$ must be a multiple of the all 1's vector. However, the dot product of $x - \iota(x)$ with the all 1's must be zero since the sum of its components is zero, so $x - \iota(x)$ must in fact equal 0.

Conversely, it can be shown that for any vector $x \in \mathbb{Z}^{V'}_{\text{sym}}$, $\partial'\partial'^T x \in \mathrm{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}}$.

Consider the injection $\hat{\phi} : \mathbb{Z}^{V_1} \to \mathbb{Z}^{V'}$ of lattices that sends e_i to $e_i + e_{-i}$. It can be checked that $\hat{\phi}(\partial_1\partial_1^T x) = \partial'\partial'^T(\hat{\phi}(x))$.

Using the fact that $\hat{\phi} : \mathrm{im}(\partial_1) \to \mathrm{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}}$ is an isomorphism, it can be shown that

$$\mathrm{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}}/\mathrm{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}} \cap \mathrm{im}(\partial'\partial'^T) = \mathrm{im}(\partial_1)/\mathrm{im}(\partial_1\partial_1^T),$$

which is exactly $K(G_1)$, as desired. $\square$

Corollary 7.4. *The exact sequence*

$$0 \leftarrow K(G_1) \leftarrow K(G') \leftarrow K(G_2) \leftarrow 0$$

from Proposition 7.3 splits at all odd primes.

Proof. This can be proved using exactly the same as the argument used in Section 4.3. Recall that the main fact used is that in the following commutative diagram (where the middle vertical map is the identity)

$$\begin{tikzcd} 0 \arrow[r] & K(G_1) \arrow[r] & K(G') \arrow[r] \arrow[d, "\wr"] & K(G_2) \arrow[r] & 0 \\ 0 & K(G_1) \arrow[l] & K(G') \arrow[l] & K(G_2) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

following the arrows from $K(G_1)$ to $K(G')$ and then back to $K(G_1)$ results in $2K(G_1)$, which has an index in $K(G_1)$ that is a power of 2. $\square$

7.7. **Case 3.** Finally, we have to deal with the case where ∂_1 and ∂_2 are full rank and there is a cycle in G_1 with an odd number of edges assigned the transposition. We first isolate the fundamental difference of this case from Case 2.

Proposition 7.5. *Suppose G_1 is connected, ∂_1 and ∂_2 have full rank, and G_1 contains a cycle with an odd number of edges assigned the tranposition. Then, ∂' has full rank.*

Proof. First note that G_1 containing a cycle with an odd number of edges assigned the tranposition implies that G' is connected. Also, recall that ∂' having full rank is equivlant to there existing a cycle with an odd number of negative edges.

This is equivalent to finding a cycle in G_1 with an odd number of negative edges and an even number of transpositions. We claim that given our conditions this is always possible. We show this by contradiction. Suppose there is no cycle in G_1 with an odd number of negative edges and an even number of transpositions.

Then, every cycle in G_1 with an odd number of negative edges must necessarily have an odd number of edges assigned the transposition. We will show that in addition, every cycle in G_1 with an even number of negative edges must have an even number of edges assigned the transposition.

To see this, suppose not. Since ∂_1 has full rank, there exists a cycle C_1 such that C_1 has an odd number of negative edges. By our assumption, C_1 has an odd number of edges assigned the transposition. Suppose there is a cycle C_2 with an even number of negative edges with an odd number of edges assigned the transposition.

Then, we can create a larger cycle C_3 from C_1 and C_2 by taking any path joining a vertex in C_1 and C_2 (such a path must exist since G_1 is connected). Suppose this path joins v_1 and v_2. Then, our cycle C_3 will start at v_1, cycle back to v_1 through C_1, go to v_2 in C_2 through the path, cycle back to v_2 through C_2 and then go back to V_1 through the path. Then, our cycle C_3 has a odd number of negative edges and an even number of edges assigned the transposition, which is not allowed.

Finally, we see that this is a contradiction, as, if every cycle in G_1 with an odd number of negative edges has an odd number of edges assigned a transposition and every cycle with an even number of negative edges has an even number of edges assigned the transposition, then every cycle in G_2 has an even number of negative edges. But we assumed that ∂_2 is also full rank, therefore we have a contradiction and ∂' must have full rank. $\square$

What happens in this case is summarized in the following proposition

Proposition 7.6. *Suppose G_1 is connected, ∂_1 and ∂_2 have full rank, and G_1 contains a cycle with an odd number of edges assigned the tranposition.*

Then, we have the following sequences of maps

$$0 \leftarrow K(G_1) \leftarrow K(G') \leftarrow K(G_2) \leftarrow 0$$
$$0 \rightarrow K(G_1) \rightarrow K(G') \rightarrow K(G_2) \rightarrow 0,$$

where the surjections are the surjections from Propositions 7.1 and 7.2. The sequences are exact except $K(G_2)$ hits a subgroup of index 2 in the kernel of the surjection $K(G') \to K(G_1)$ and similarly, $K(G_1)$ hits a subgroup of index 2 in the kernel of the surjection $K(G')$ rightarrow $K(G_2)$.

Proof. The proof of this will be extremely similar to the proof of Proposition 7.3. The fundamental difference is that the kernel of ∂' is empty.

For the first exact sequence, we first note that if ϕ is the map of lattices from $\mathbb{Z}^{V'}$ to $\mathbb{Z}^{V_1}$ that sends $e_{\pm i}$ to e_i, then $\phi(\mathrm{im}(\partial'\partial'^T))$ is exactly $\mathrm{im}(\partial_1\partial_1^T)$. To see this, it suffices to recall that the column of $\partial'\partial'^T$ corresponding to vertex $+i$ or $-i$ maps to the column of $\partial_1\partial_1^T$ corresponding to vertex i.

This means in particular that if the image of a vector $x \in \mathbb{Z}^{V'}$ is equivalent to 0 after quotienting out by $\operatorname{im}(\partial_1\partial_1^T)$, then there is a vector $x' \in \mathbb{Z}^{V'}$ equivalent to x under $\operatorname{im}(\partial'\partial'^T)$ that maps exactly to zero under ϕ.

Therefore, the kernel of the map $K(G') \to K(G_1)$ is the subgroup $\mathbb{Z}^{V'}_{\text{skew}}/\mathbb{Z}^{V'}_{\text{skew}} \cap \operatorname{im}(\partial'\partial'^T)$, where $\mathbb{Z}^{V'}_{\text{skew}}$ is sublattice of $\mathbb{Z}^{V'}$ containing all vectors x such that $x + \iota(x) = 0$, where ι is the sign preserving involution on $\mathbb{Z}^{V'}$. Note that since ∂' has full rank, $\operatorname{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{skew}}$ is all of $\mathbb{Z}^{V'}_{\text{skew}}$

It suffices to find $\mathbb{Z}^{V'}_{\text{skew}} \cap \operatorname{im}(\partial'\partial'^T)$. To do so, suppose $x \in \mathbb{Z}^{V'}$ such that $\partial'\partial'^T x \in \mathbb{Z}^{V'}_{\text{skew}}$. Then, we know that $\partial'\partial'^T x + \iota(\partial'\partial'^T x) = 0$, or $x + \iota(x) \in \ker(\partial'^T)$. Since $\ker(\partial'^T)$ is trivial, $x = -\iota(x)$ and $x \in \mathbb{Z}^{V'}_{\text{skew}}$.

Conversely, it can be shown that for any vector $x \in \mathbb{Z}^{V'}_{\text{skew}}$, $\partial'\partial'^T x$ is in $\mathbb{Z}^{V'}_{\text{skew}}$.

Consider the injection $\hat{\phi} : \mathbb{R}^{V_2} \to \mathbb{R}^{V'}$ of lattices that sends e_i to $e_i - e_{-i}$. It can be checked that $\hat{\phi}(\partial_2\partial_2^T x) = \partial'\partial'^T(\hat{\phi}(x))$.

Using the fact that $\hat{\phi} : \mathbb{Z}^{V_2} \to \mathbb{Z}^{V'}_{\text{skew}}$ is an isomorphism, it can be shown that

$$\mathbb{Z}^{V'}_{\text{skew}}/\mathbb{Z}^{V'}_{\text{skew}} \cap \operatorname{im}(\partial'\partial'^T) = \mathbb{Z}^{V_2}/\operatorname{im}(\partial_2\partial_2^T)$$

Since $\operatorname{im}(\partial_2)$ is a sublattice of $\mathbb{Z}^{V_2}$ of index 2 and $\operatorname{im}(\partial_2\partial_2^T) \subset \operatorname{im}(\partial_2)$, $K(G_2)$ is of index 2 in $\mathbb{Z}^{V'}_{\text{skew}}/\mathbb{Z}^{V'}_{\text{skew}} \cap \operatorname{im}(\partial'\partial'^T) = \mathbb{Z}^{V_2}/\operatorname{im}(\partial_2\partial_2^T)$, as desired.

The proof of the other exact sequence is similar. Let ϕ be the map of lattices from $\mathbb{Z}^{V'}$ to $\mathbb{Z}^{V_2}$ that sends e_{+i} to e_i and e_{-i} to $-e_i$, then $\phi(\operatorname{im}(\partial'\partial'^T))$ is exactly $\operatorname{im}(\partial_2\partial_2^T)$. To see this, it suffices to recall that the column of $\partial'\partial'^T$ corresponding to vertex $+i$ to the column of $\partial_2\partial_2^T$ corresponding to vertex i and the column of $\partial'\partial'^T$ corresponding to $-i$ maps to the negative of the column of $\partial_2\partial_2^T$ corresponding to vertex i.

This means in particular that if the image of a vector $x \in \mathbb{Z}^{V'}$ is equivalent to 0 after quotienting out by $\operatorname{im}(\partial_2\partial_2^T)$, then there is a vector $x' \in \mathbb{Z}^{V'}$ equivalent to x under $\operatorname{im}(\partial'\partial'^T)$ that maps exactly to zero under ϕ.

Therefore, the kernel of the map $K(G') \to K(G_2)$ is the subgroup $\mathbb{Z}^{V'}_{\text{sym}}/\mathbb{Z}^{V'}_{\text{sym}} \cap \operatorname{im}(\partial'\partial'^T)$, where $\mathbb{Z}^{V'}_{\text{sym}}$ is sublattice of $\mathbb{Z}^{V'}$ containing all vectors x such that $x = \iota(x)$, where ι is the sign preserving involution on $\mathbb{Z}^{V'}$. Note that since ∂' has full rank, $\operatorname{im}(\partial') \cap \mathbb{Z}^{V'}_{\text{sym}}$ is all of $\mathbb{Z}^{V'}_{\text{sym}}$

It suffices to find $\mathbb{Z}^{V'}_{\text{sym}} \cap \operatorname{im}(\partial'\partial'^T)$. To do so, suppose $x \in \mathbb{Z}^{V'}$ such that $\partial'\partial'^T x \in \mathbb{Z}^{V'}_{\text{sym}}$. Then, we know that $\partial'\partial'^T x - \iota(\partial'\partial'^T x) = 0$, or $x - \iota(x) \in \ker(\partial'^T)$. Since $\ker(\partial'^T)$ is trivial, $x = \iota(x)$.

Conversely, it can be shown that for any vector $x \in \mathbb{Z}^{V'}_{\text{sym}}$, $\partial'\partial'^T x \in \mathbb{Z}^{V'}_{\text{sym}}$.

Consider the injection $\hat{\phi} : \mathbb{Z}^{V_1} \to \mathbb{Z}^{V'}$ of lattices that sends e_i to $e_i + e_{-i}$. It can be checked that $\hat{\phi}(\partial_1\partial_1^T x) = \partial'\partial'^T(\hat{\phi}(x))$.

Using the fact that $\hat{\phi} : \mathbb{Z}^{V_1} \to \mathbb{Z}^{V'}_{\text{sym}}$ is an isomorphism, it can be shown that

$$\mathbb{Z}^{V'}_{\text{sym}}/\mathbb{Z}^{V'}_{\text{sym}} \cap \operatorname{im}(\partial'\partial'^T) = \mathbb{Z}^{V_1}/\operatorname{im}(\partial_1\partial_1^T).$$

Since $\operatorname{im}(\partial_1)$ is a sublattice of $\mathbb{Z}^{V_1}$ of index 2 and $\operatorname{im}(\partial_1\partial_1^T) \subset \operatorname{im}(\partial_2)$, $K(G_1)$ is of index 2 in $\mathbb{Z}^{V'}_{\text{sym}}/\mathbb{Z}^{V'}_{\text{sym}} \cap \operatorname{im}(\partial'\partial'^T) = \mathbb{Z}^{V_1}/\operatorname{im}(\partial_1\partial_1^T)$, as desired. $\square$

Corollary 7.7. *The sequence*

$$0 \leftarrow K(G_1) \leftarrow K(G') \leftarrow K(G_2) \leftarrow 0$$

from Proposition 7.6 splits at all odd primes.

Proof. This can be proved using exactly the same as the argument used in Section 4.3. Recall that the main fact used is that in the following commutative diagram (where the middle vertical map is the identity)

$$\begin{tikzcd}
0 \arrow[r] & K(G_1) \arrow[r] & K(G') \arrow[r] \arrow[d, "\wr"] & K(G_2) \arrow[r] & 0 \\
0 & K(G_1) \arrow[l] & K(G') \arrow[l] & K(G_2) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

following the arrows from $K(G_1)$ to $K(G')$ and then back to $K(G_1)$ results in $2K(G_1)$, which has an index in $K(G_1)$ that is a power of 2. It is true that $K(G_2)$ only hits a subgroup of index 2 of the kernel of the surjection of $K(G')$ onto $K(G_1)$, but replacing $K(G_2)$ with the kernel will not change the Sylow $p-$group of $K(G_2)$ for p odd. □

7.8. **Critical Group of n-cube.** Using the results in this section, we will develop an alternative interpretation for Bai's proof of the Sylow $p-$ component of the critical group of the n-cube for p odd [2].

Following the notation of Bai [2], let $L_{n,k}$ be the result after taking the $n-$cube and adding k negative half loops at each vertex. Consider the permutation voltage assignment on $L_{n,k+1}$, where we have all the edges assigned the identity permutation except for exactly one of the half loops at each vertex.

Then, the covering derived from this assignment of permutations is $L_{n+1,k}$. From Corollary 7.7, we know that the sequence

$$0 \leftarrow K(L_{n,k}) \leftarrow K(L_{n+1,k}) \leftarrow K(L_{n,k+1}) \leftarrow 0$$

splits at all odd primes by letting $G_1 = L_{n,k}$, $G' = L_{n+1,k}$ and $G_2 = L_{n,k+1}$. To see this, we can take $L_{n,k}$, add a positive half loop at each vertex, and then assign the trivial permutation to each edge of $L_{n,k}$, except for the positive half loops which are assigned the nontrivial permutation. Then, the derived covering is $L_{n+1,k}$ and G_2 is $L_{n,k+1}$. Thus, following the argument used in Theorem 1.2 of [2], the Sylow p-group of p odd of $Q_n = L_{n,0}$ is

$$\begin{aligned}\mathrm{Syl}_p(K(L_{n,0})) &= \mathrm{Syl}_p(K(L_{n-1,0})) \oplus \mathrm{Syl}_p(K(L_{n-1,1}))\\ &= \mathrm{Syl}_p(K(L_{n-2,0})) \oplus \mathrm{Syl}_p(K(L_{n-2,1}))^2 \oplus \mathrm{Syl}_p(K(L_{n-2,2}))\\ &= \vdots\\ &= \prod_{k=0}^{n} (\mathrm{Syl}_p(K(L_{0,k}))^{\binom{n}{k}}.\end{aligned}$$

Finally, $L_{0,k}$ is just the graph with one vertex and k negative half loops. The critical group is $2\mathbb{Z}/2k\mathbb{Z} = \mathbb{Z}/k\mathbb{Z}$. Therefore, we recover Bai's result:

$$\prod_{k=0}^{n} (\mathrm{Syl}_p(\mathbb{Z}/k\mathbb{Z}))^{\binom{n}{k}}$$

is the Sylow $p-$group of $K(Q_n)$ for p odd.

8. Examples

Here we will have specific examples of critical groups that can be computed using what we have developed.

8.1. **Complete Bipartite Graph $K_{n,n}$ plus a number of perfect matchings.** Let G_1 be the complete graph on n vertices with k positive half loops at each vertex (so G_1 is a signed graph). Assign every edge of G_1 the nontrivial permutation. Then, the derived graph covering G' is a complete bipartite graph $K_{n,n}$ plus $k-1$ times the perfect matching where $+1$ is matched to -1, $+2$ is matched to -2, and so on. If $k = 0$, then G' is the complete bipartite graph without a perfect matching. This is also known as the crown graph.

Let G_2 be the signed graph obtained by reversing the signs of all the edges of G_1. Then, from Proposition 7.3, we know that there is an exact sequence $0 \to K(G_1) \to K(G') \to K(G_2) \to 0$. It is well-known that $K(G_1) = (\mathbb{Z}/n\mathbb{Z})^{n-2}$.

We can also compute $K(G_2)$ directly.

Proposition 8.1. *The critical group of the signed graph G_2 obtained by taking a complete graph on n vertices, adding k half loops to each vertex, and assign negative signs to each edge (including the loops) is $(\mathbb{Z}/(n-2+2k)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/(n-1+k)(n+2k-2)\mathbb{Z}$.*

Proof. First, we note that if ∂_2 is the directed incidence matrix of G_2, then $\partial_2\partial_2^T$ is equal to

$$\begin{pmatrix} n-1+2k & 1 & 1 & \cdots & 1\\ 1 & n-1+2k & 1 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \cdots & n-1+2k \end{pmatrix}.$$

In order to compute the critical group, we will use the presentation of $K(G_2)$ given in Example 2 as $\mathbb{Z}^V/\partial\partial^T(\mathbb{Z}^V + (\frac{1}{2}, \dots, \frac{1}{2})^t)$. This is equivalent to computing the Smith Normal form of

$$\begin{pmatrix} n-1+2k & 1 & 1 & \cdots & 1 & n-1+k \\ 1 & n-1+2k & 1 & \cdots & 1 & n-1+k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & n-1+2k & n-1+k \end{pmatrix}.$$

We proceed by somewhat unmotivated row and column operations. First, we can add all the rows to the first row to get

$$\left(\begin{array}{ccccc|c} 2(n-1+2k) & 2(n-1+2k) & 2(n-1+2k) & \cdots & 2(n-1+2k) & (n-1+k)n \\ \hline 1 & n-1+2k & 1 & \cdots & 1 & n-1+k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & n-1+2k & n-1+k \end{array}\right).$$

Then, we subtract the first column from the next $n-1$ columns and subtract $n+k-2$ times the first row from the last column

$$\left(\begin{array}{c|cccc|c} 2(n-1+2k) & 0 & 0 & \cdots & 0 & -(n-1+k)(n+2k-4) \\ \hline 1 & n-2+2k & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & n-2+2k & 1 \end{array}\right).$$

Then, we eliminate up the first column to get

$$\left(\begin{array}{c|cccc|c|c} 0 & 0 & 0 & \cdots & 0 & -(n-2+2k)(2n-2+2k) & -(n-1+k)(n+2k-2) \\ \hline 0 & n-2+2k & 0 & \cdots & 0 & -(n-2+2k) & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & n-2+2k & -(n-2+2k) & 0 \\ \hline 1 & 0 & 0 & 0 & \cdots & n-2+2k & 1 \end{array}\right).$$

Cleaning up a little yields

$$\left(\begin{array}{c|cccc|c|c} 0 & 0 & 0 & \cdots & 0 & 0 & -(n-1+k)(n+2k-2) \\ \hline 0 & n-2+2k & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & n-2+2k & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right).$$

This means $K(G_2)$ is $(\mathbb{Z}/(n-2+2k)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/(n-1+k)(n+2k-2)\mathbb{Z}$, as desired. □

Corollary 8.2. *If $G(n,k)$ is the complete bipartite graph with $k-1$ times a perfect matching added, then $K(G(n,k))$ satisfies the following exact sequence:*

$$0 \to (\mathbb{Z}/n\mathbb{Z})^{n-2} \to K(G(n,k)) \to (\mathbb{Z}/(n-2+2k)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/(n-1+k)(n+2k-2)\mathbb{Z}$$

In many cases, we can determine $K(G(n,k))$ exactly. I admit I haven't tried very hard to deal with the other cases. First, the answer is simple if n is odd

Proposition 8.3. *Suppose n is odd. The critical group $G(n,k)$ is $(\mathbb{Z}/n\mathbb{Z})^{n-2} \oplus (\mathbb{Z}/(n-2+2k)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/(n-1+k)(n+2k-2)\mathbb{Z}$.*

Proof. We know that the exact sequence from Corollary 8.2 splits at all odd primes. Since n is odd, this means the sequence splits at all primes. □

If n is even, we can get a partial answer. In particular, we have a complete answer when $k=0$ or $k=2$. The case where $k=0$ rederives the result in [7] that computes the critical group of a complete bipartite graph $K_{n,n}$ minus a perfect matching.

Proposition 8.4. *Suppose n is even and n and $k-1$ are relatively prime (so in particular k is even). The critical group of $G(n,k)$ is $\mathbb{Z}/(n-2+2k)\mathbb{Z} \oplus (\mathbb{Z}/n(n-2+2k)\mathbb{Z})^{n-3} \oplus \mathbb{Z}/n(n-1+k)(n-2+2k)\mathbb{Z}$.*

Proof. We first show that the number of generators is at most $n-1$. To do so, we note that the Laplacian of $G(n,k)$ is

$$\left(\begin{array}{cccc|cccc} n+k-1 & 0 & \cdots & 0 & -k & -1 & \cdots & -1 \\ 0 & n+k-1 & \cdots & 0 & -1 & -k & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n+k-1 & -1 & -1 & \cdots & -k \\ \hline -k & -1 & \cdots & -1 & n+k-1 & 0 & \cdots & 0 \\ -1 & -k & \cdots & -1 & 0 & n+k-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -k & 0 & 0 & \cdots & n+k-1 \end{array}\right).$$

The upper left n by n matrix has determinant $(n+k-1)^n$. Now, consider the minor formed by taking the row indices in $\{2, n+1, \ldots, 2n\}$ and column indices in $\{2, 3, \ldots, n+1\}$. Then, we get the n by n minor

$$\left(\begin{array}{ccccc|c} 0 & n+k-1 & 0 & \cdots & 0 & -1 \\ \hline -k & -1 & -1 & \cdots & -1 & 0 \\ -1 & -k & -1 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -k & 0 \end{array}\right).$$

If we expand by minors across the first row, we see that the determinant of taken $\pmod{n-1}$ is ± 1 times the determinant of

$$\begin{pmatrix} -k & -1 & -1 & \cdots & -1 \\ -1 & -k & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -k \end{pmatrix}.$$

Since the eigenvalues of the all 1's matrix is $0, 0, \ldots, 0, n-1$, the eigenvalues of this matrix is $-(k-1), -(k-1), \ldots, -(k-1), -(n+k-2)$, which means the determinant of n by n minor above is $\pm(n+k-2) \pmod{n+k-1}$. Therefore, if p is a prime that divides the n by n minor with determinant $(n+k-1)^n$ above, then it won't divide the second n by n minor we found with determinant $\pm(n+k-2) \pmod{n+k-1}$.

This means in the Smith Normal form of the Laplacian of $G(n,k)$, there must be n 1's on the diagonal.

Since the sequence

$$0 \to (\mathbb{Z}/n\mathbb{Z})^{n-2} \to K(G(n,k)) \to (\mathbb{Z}/(n-2+2k)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/(n-1+k)(n+2k-2)\mathbb{Z}$$

splits at all odd primes, we only need to consider the induced exact sequence of 2-groups. This is $0 \to (\mathbb{Z}/2^{e_1}\mathbb{Z})^{n-2} \to \mathrm{Syl}_2(K(G(n,k))) \to (\mathbb{Z}/2^{e_2}\mathbb{Z})^{n-1}$ of 2-groups, where e_1 is the largest power such that 2^{e_1} divides n and similarly for e_2 and $n+2k-2$. Since there are at most $n-1$ generators of $\mathrm{Syl}_2(K(G(n,k))) = (\mathbb{Z}/2^{e_1+e_2}\mathbb{Z})^{n-2} \oplus \mathbb{Z}/2^{e_2}\mathbb{Z}$ by applying Littlewood Richardson coefficients.

Since the greatest common divisor of n and $n-2k+2$ is at most 2, the 2-groups combining shows that $K(G(n,0))$ can only be $(\mathbb{Z}/n(n-2+2k)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/(n-2+2k)\mathbb{Z} \oplus \mathbb{Z}/(n-1+k)\mathbb{Z} = \mathbb{Z}/(n-2+2k)\mathbb{Z} \oplus (\mathbb{Z}/n(n-2+2k)\mathbb{Z})^{n-3} \oplus \mathbb{Z}/n(n-1+k)(n-2+2k)\mathbb{Z}$. □

Finally, we can recover the critical group of the complete bipartite graph $K_{n,n}$. This is a special case of Lorenzini's computation that the critical group of K_{n_1,n_2} is $(\mathbb{Z}/n_1\mathbb{Z})^{n_2-2} \oplus (\mathbb{Z}/n_2\mathbb{Z})^{n_1-2} \oplus \mathbb{Z}/n_1n_2\mathbb{Z}$ [6].

Proposition 8.5. *The critical group of $G(n,1)$ is $(\mathbb{Z}/n\mathbb{Z})^{2n-4} \oplus \mathbb{Z}/n^2\mathbb{Z}$.*

Proof. From Corollary 8.2, we have the exact sequence

$$0 \to (\mathbb{Z}/n\mathbb{Z})^{n-2} \to K(G(n,1)) \to (\mathbb{Z}/n\mathbb{Z})^{n-2} \oplus \mathbb{Z}/n^2\mathbb{Z} \tag{1}$$

To get more information, we also apply Proposition 5.4 to a graph with 2 vertices and one edge between them to find that $K(G(n,1))$ also satisfies the exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to K(G(n,1)) \to (\mathbb{Z}/n\mathbb{Z})^{2n-3} \to 0. \tag{2}$$

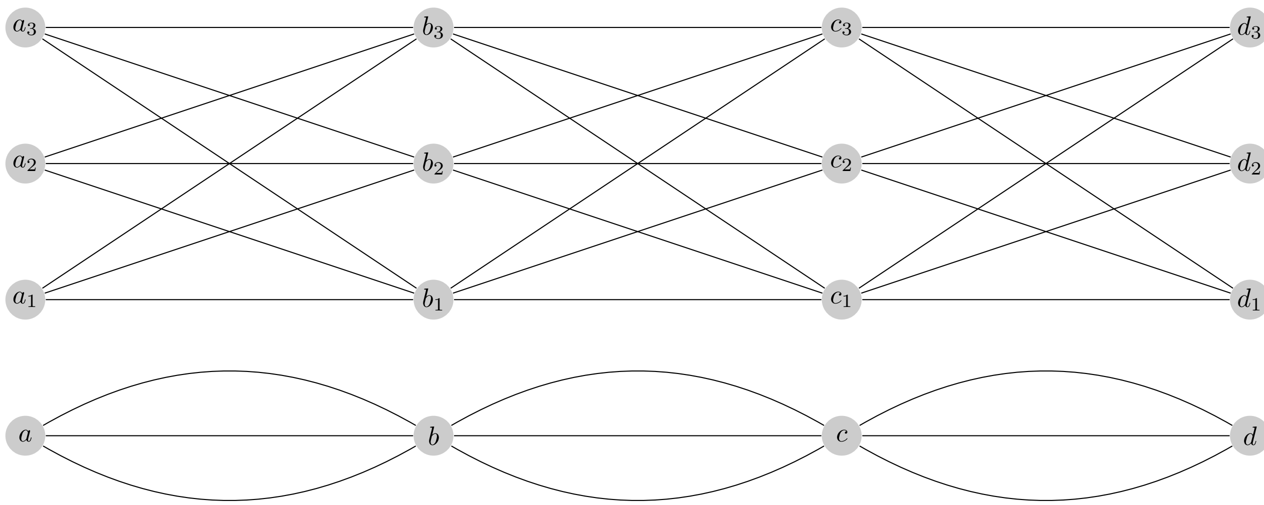

FIGURE 9. G is the path with 4 vertices, $n = 3$, nG is the graph on the bottom, and a covering is the graph on top

.

While (1) splits at all odd primes, we have no information on (2) since we only know that it splits at all primes that do not divide n, which is not very useful. However, if p is a prime and e is the largest power such that p^e divides n, then we know that the induced exact sequence of $p-$groups from (2) is

$$0 \to \mathbb{Z}/p^e\mathbb{Z} \to \mathrm{Syl}_p(K(G(n, 1))) \to (\mathbb{Z}/p^e\mathbb{Z})^{2n-3}.$$

In addition, we know from (1), that $\mathrm{Syl}_p(K(G(n, 1))$ has $\mathbb{Z}/p^{2e}\mathbb{Z}$ as part of its decomposition. From applying Littlewood Richardson, we know the only possibility for $\mathrm{Syl}_p(K(G(n, 1))$ that has $\mathbb{Z}/p^{2e}\mathbb{Z}$ as part of its decomposition is $(Z/p^e\mathbb{Z})^{2n-4} \oplus \mathbb{Z}/p^{2e}\mathbb{Z}$.

Performing this at all primes p yields the critical group of $G(n, 1)$ is $(\mathbb{Z}/n\mathbb{Z})^{2n-4} \oplus \mathbb{Z}/n^2\mathbb{Z}$, as desired. □

8.2. **Complete Split Graphs.** In [3], it is known that the critical group of the complete split graph, where the the clique has size n_1 and the coclique has size n_2, is $(\mathbb{Z}/(n_1+n_2)\mathbb{Z})^{n_1-2} \oplus (\mathbb{Z}/n_1\mathbb{Z})^{n_2-2} \oplus \mathbb{Z}/n_1(n_1+n_2)\mathbb{Z}$.

If we start with a graph on two vertices v_1 and v_2 such that there is an edge between v_1 and v_2 and a self loop at v_1, we can apply Proposition 5.4 to the case where $n_1 = n_2 = n$. Then, we get the following exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to K(G) \to (\mathbb{Z}/2n)^{n-1} \oplus (\mathbb{Z}/n\mathbb{Z})^{n-2},$$

where G is the complete split graph with clique and coclique size n. It can be shown the number of generators is at most $2n - 3$ (by just finding a 2 by 2 matrix in $L(G)$ with determinant ± 1). If n is odd, then applying Littlewood Richardson coefficients yields $(\mathbb{Z}/n\mathbb{Z})^{n-2} \oplus (\mathbb{Z}/2n\mathbb{Z})^{n-2} \oplus \mathbb{Z}/2n^2\mathbb{Z}$. If n is even, then applying Littlewood Richardson coefficients yields only two possibilities: $(\mathbb{Z}/n\mathbb{Z})^{n-2} \oplus (\mathbb{Z}/2n\mathbb{Z})^{n-2} \oplus \mathbb{Z}/2n^2\mathbb{Z}$ and $(\mathbb{Z}/n\mathbb{Z})^{n-3} \oplus (\mathbb{Z}/2n\mathbb{Z})^{n-1} \oplus \mathbb{Z}/n^2\mathbb{Z}$, and the first one agrees with the correct answer.

8.3. **Stacked Complete Bipartite Graphs.**

Proposition 8.6. *Let G' be a graph with kn vertices with $n > 1$. Let the vertex set be $\{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$. Let there be an edge between v_{ij} and $v_{i',j+1}$ for all $1 \leq j < k$ and $1 \leq i, i' \leq n$. See the top graph of Figure 9 for the case where $n = 3$ and $k = 4$.*

Suppose k is even.

(1) If n is odd, then the critical group of G' is $(\mathbb{Z}/2\mathbb{Z})^{(k-2)(n-1)} \oplus (\mathbb{Z}/n\mathbb{Z})^{(n-2)k} \oplus (\mathbb{Z}/n^2\mathbb{Z})^{k-1}$.

(2) If $n = 2$, then the critical group of G' is $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})^{k-2}$.

Proof. Let G be the graph that is a path with k vertices. Then, if G' is $k - 1$ complete bipartite graphs, each with each bipartition of size n stacked on top of each other (see the top graph in Figure 9). Then, Proposition 5.4 yields the exact sequence

$$0 \to (\mathbb{Z}/n\mathbb{Z})^{k-1} \to K(G') \to (\mathbb{Z}/n\mathbb{Z})^{2n-3} \oplus (\mathbb{Z}/2n\mathbb{Z})^{(k-2)(n-1)} \to 0.$$

As a submatrix of the Laplacian of G' it is possible to find an k by k minor that is all 0's except on the all -1's on the superdiagonal and subdiagonal (equivalently, this is the adjacency matrix of the k-path). It is not hard to show that, since k is even, the determinant of this minor is ± 1. This means, there must be at most $(kn-1)-k=(n-1)k-1$ generators of $K(G)$.

If n is odd, the cokernel of the exact sequence above can be rewritten as $(\mathbb{Z}/2\mathbb{Z})^{(k-2)(n-1)} \oplus (\mathbb{Z}/n\mathbb{Z})^{(n-1)k-1}$. Applying Littlewood-Richardson coefficients shows that $K(G') = (\mathbb{Z}/2\mathbb{Z})^{(k-2)(n-1)} \oplus (\mathbb{Z}/n\mathbb{Z})^{(n-2)k} \oplus (\mathbb{Z}/n^2\mathbb{Z})^{k-1}$.

If n is 2, applying Littlewood-Richardson coefficients to

$$0 \to (\mathbb{Z}/2\mathbb{Z})^{k-1} \to K(G') \to (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})^{k-2} \to 0 \tag{3}$$

with the knowledge that there are at most $k-1$ generators yields $K(G') = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})^{k-2}$. □

Conjecture 1. Preserving the setup in 8.6, if $n=2$ and k is odd, then $K(G') = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^{k-2}$.

Remark 16. To prove Conjecture 1, it suffices to show that there is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in the 2-group component of $K(G')$, as the exact sequence (3) and Littlewood-Richardson coefficients show that $K(G')$ is $(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^{k-2}$, $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^{k-3}$, or $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})^{k-2}$. Computer evidence suggests that it is the first one. (I spent about a day trying to prove this to give an impression of how hard I tried)

8.4. **A "Circle" of Complete Bipartite Graphs.**

Proposition 8.7. *Let G' be a graph with kn vertices where $n>1$. Let the vertex set be $\{v_{ij} : 1 \le i \le n, 1 \le j \le k\}$. Let there be an edge between v_{ij} and $v_{i',j+1}$ for all $1 \le j < k$ and between $v_{i,k}$ and $v_{i',1}$, where i and i' range over all integers between 1 and n inclusive.*

Suppose k is odd.

1. *If n is odd, then the critical group of G' is $(\mathbb{Z}/2\mathbb{Z})^{(n-1)k} \oplus (\mathbb{Z}/n\mathbb{Z})^{(n-2)k} \oplus (\mathbb{Z}/n^2\mathbb{Z})^{k-2} \oplus \mathbb{Z}/n^2k\mathbb{Z}$.*
2. *If $n=2$, then the critical group of G' is $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/8\mathbb{Z})^{k-1} \oplus \mathbb{Z}/k\mathbb{Z}$.*

Remark 17. If we take the top graph of Figure 9 and glue together verticees a_1 and d_1, a_2 and d_2 and a_3 and d_3, we get the case where $n=k=3$.

Proof. Let G be the graph that is a cycle with k vertices. Then, since $K(G) = \mathbb{Z}/k\mathbb{Z}$, Proposition 5.4 yields the following exact sequence

$$0 \to \mathbb{Z}/nk\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{k-2} \to K(G') \to (\mathbb{Z}/2n\mathbb{Z})^{(n-1)k-1} \oplus \mathbb{Z}/2\mathbb{Z} \to 0.$$

Now, similarly to the proof of Proposition 8.6, it is possible to find a k by k minor that is the adjacency matrix of the k-cycle. It is known that, since k is odd, the determinant of this adjacency matrix is 2 (see [1] Proposition 2.1 for example). This means the number of generators of the 2-group of $K(G')$ is at most $(nk-1)-(k-1)=(n-1)k$ and the number of generators of the p-group of $K(G')$ for any odd prime p is at most $(nk-1)-k=(n-1)k-1$.

If n is odd, the exact sequence can be rewritten as

$$0 \to \mathbb{Z}/nk\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{k-2} \to K(G') \to (\mathbb{Z}/n\mathbb{Z})^{(n-1)k-1} \oplus (\mathbb{Z}/2\mathbb{Z})^{(n-1)k} \to 0.$$

Since 2 does not divide n, the sequence splits at the 2-groups. For the rest, applying Littlewood-Richardson Coefficients shows $K(G') = (\mathbb{Z}/2\mathbb{Z})^{(n-1)k} \oplus (\mathbb{Z}/n\mathbb{Z})^{(n-2)k} \oplus (\mathbb{Z}/n^2\mathbb{Z})^{k-2} \oplus \mathbb{Z}/n^2k\mathbb{Z}$.

If $n=2$, then we have the sequence

$$0 \to \mathbb{Z}/2k\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{k-2} \to K(G') \to (\mathbb{Z}/4\mathbb{Z})^{k-1} \oplus \mathbb{Z}/2\mathbb{Z} \to 0.$$

Since k is odd, the kernel can be rewritten as $\mathbb{Z}/k\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{k-1}$. Applying Littlewood-Richardson Coefficients shows $K(G') = \mathbb{Z}/k\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/8\mathbb{Z})^{k-1}$ or $\mathbb{Z}/k\mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^{k-2}$.

To finish, we will rule out the possibility of $K(G') = \mathbb{Z}/k\mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^{k-2}$. To do so, we will show that $\mathbb{Z}/2\mathbb{Z}$ must be its primary decomposition. We note that G' is an k-sheeted covering of the graph H with two vertices v and u, two edges between v and u, and two self loops at v and at u. This can be seen by mapping $v_{11}, v_{12}, \ldots, v_{1k}$ to v and $v_{21}, v_{22}, \ldots, v_{2k}$ to u. Since k is odd, Corollary 4.2 shows that there is a splitting backmap from $K(H) = \mathbb{Z}/2\mathbb{Z}$ to $K(G')$. This means $\mathbb{Z}/2\mathbb{Z}$ must be in the primary decomposition of $K(G')$, so $K(G') = \mathbb{Z}/k\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/8\mathbb{Z})^{k-1}$ □

8.5. **A Generalization of Sections 8.3 and 8.4.** For what it is worth, it is possible to abstract the conditions used in the case where n is odd in Propositions 8.6 and 8.7 in Proposition 8.8 below.

Proposition 8.8. *Let $G = (V, E)$ be a connected graph. Let G' be the graph created by*

(1) For each vertex $v \in G$, create n copies $v_1, \ldots, v_n$ of G.
(2) For each edge between v and u in G (possible $v = u$), let there be edges between v_i and u_j for all $1 \leq i, j \leq n$.

Let $|V| = k$ and the outegrees of the vertices of G be $d_1, d_2, \ldots, d_k$ (where self-loops count as 1). Suppose

(1) $n > 1$ and n is relatively prime with $d_1, \ldots, d_k$
(2) n is relatively prime with the determinant of the adjacency matrix of G.

Then, if $K(G') = \bigoplus_{i=1}^{k-1} \mathbb{Z}/c_i\mathbb{Z}$ (where many of the c_i might be 1), then

$$K(G') = (\mathbb{Z}/n\mathbb{Z})^{(k-2)n} \oplus \left(\bigoplus_{i=1}^{k-1} \mathbb{Z}/n^2 c_i \mathbb{Z} \right) \oplus \left(\bigoplus_{i=1}^{k} (\mathbb{Z}/d_i\mathbb{Z})^{n-1} \right).$$

Proof. Applying Proposition 5.4 to G yields the following exact sequence:

$$0 \to K(nG) = \bigoplus_{i=1}^{k-1} \mathbb{Z}/nc_i\mathbb{Z} \to K(G') \to \mathbb{Z}/a_1\mathbb{Z} \oplus \left(\bigoplus_{i=2}^{k(n-1)} \mathbb{Z}/na_i\mathbb{Z} \right) \to 0,$$

where $a_1, \ldots, a_{k(n-1)}$ is defined to be the invariant factors of $\bigoplus_{i=1}^{k} (\mathbb{Z}/d_i\mathbb{Z})^{n-1}$ such that a_i divides a_{i+1} for all $1 \leq< k(n-1)$. Since we assumed that n is relatively prime with $d_1, \ldots, d_k$, n is relatively prime with $a_1, \ldots, a_{k(n-1)}$ and we can rewrite the exact sequence above as

$$0 \to \bigoplus_{i=1}^{k-1} \mathbb{Z}/nc_i\mathbb{Z} \to K(G') \to (\mathbb{Z}/n\mathbb{Z})^{k(n-1)-1} \oplus \left(\bigoplus_{i=1}^{k(n-1)} \mathbb{Z}/a_i\mathbb{Z} \right) \to 0.$$

By definition of $a_1, \ldots, a_{k(n-1)}$, we can rewrite the cokernel as

$$0 \to \bigoplus_{i=1}^{k-1} \mathbb{Z}/nc_i\mathbb{Z} \to K(G') \to (\mathbb{Z}/n\mathbb{Z})^{k(n-1)-1} \oplus \left(\bigoplus_{i=1}^{k} (\mathbb{Z}/d_i\mathbb{Z})^{n-1} \right) \to 0.$$

Since we can find the adjacency matrix as an k by k minor of the Laplacian of G', the number of generators of the p-group component of $K(G')$, for any p that divides n, must have at most $(kn - 1) - k = k(n - 1) - 1$ generators by the assumption we made about the adjacency matrix of G having determinant relatively prime to n.

Applying Littlewood-Richardson coefficients to the exact sequence shows that $K(G')$ can only be $(\mathbb{Z}/n\mathbb{Z})^{(k-2)n} \oplus \left(\bigoplus_{i=1}^{k-1} \mathbb{Z}/n^2 c_i\mathbb{Z} \right) \oplus \left(\bigoplus_{i=1}^{k} (\mathbb{Z}/d_i\mathbb{Z})^{n-1} \right)$. □

Example 12. If we let G be the complete graph with k vertices, then if n is relatively prime with $k-1$, then we find the critical group of the complete multipartite graph with k blocks of equal size m, from Proposition 8.8, is $(\mathbb{Z}/n\mathbb{Z})^{(k-2)n} \oplus \mathbb{Z}/n^2\mathbb{Z} \oplus (\mathbb{Z}/n^2k\mathbb{Z})^{k-2} \oplus (\mathbb{Z}/(k-1)\mathbb{Z})^{(n-1)k}$.

Since n is relatively prime with $k-1$, we can combine appropiate generators to yield $(\mathbb{Z}/(k-1)n\mathbb{Z})^{(k-2)n} \oplus \mathbb{Z}/(k-1)n^2\mathbb{Z} \oplus (\mathbb{Z}/(k-1)kn^2\mathbb{Z})^{k-2} \oplus \mathbb{Z}/(k-1)\mathbb{Z}$.

This is the same answer found in Corollary 5 of [5] (though they did not need to make the assumption that n is relatively prime with $k - 1$).

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