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10/3/14

Fixing $1\le k\le l$ ~~there is~~ a famous graded ring $R = H^\bullet(Gr(k,\mathbb{C}^{k+l}),\mathbb{Q})$

$$\cong \mathbb{Q}[e_1,e_2,\ldots,e_k,h_1,h_2,\ldots,h_l] \Big/ \begin{pmatrix} e_1-h_1, \\ e_2-e_1h_1+h_2, \\ e_3-e_2h_1-e_1h_2+h_3, \\ \vdots \\ e_kh_{l-1}-e_{k-1}h_l, \\ e_kh_l \end{pmatrix}$$

$\deg(e_i)=\deg(h_i)=i$

i.e.
$$\sum_{\substack{i+j=d:\\ 1\le i\le k\\ 1\le j\le l}} (-1)^i e_i h_j = 0 \quad \text{for} \quad d=1,2,\ldots,k+l$$

with a famous Hilbert series

$$\mathrm{Hilb}(R,q) := \sum_{d=0}^{kl} \dim_{\mathbb{Q}}(R_d)\cdot q^d$$

$$= \underbrace{\begin{bmatrix} k+l \\ k\end{bmatrix}_q}_{\overset{q=1}{\rightsquigarrow}\binom{k+l}{k}} := \frac{(1-q^{l+1})(1-q^{l+2})\cdots(1-q^{l+k})}{(1-q)(1-q^2)\cdots(1-q^k)}$$

CONJ: (Tudose-R. 2003 arXiv: 0309281) If we let $R^{(m)}$ = the subalgebra of R gen'd by its degrees $R_0,R_1,\ldots,R_m$ (= subalgebra gen'd by $e_1,\ldots,e_m$ or by $h_1,\ldots,h_m$)

so $\mathbb{Q}=R^{(0)}\subset \underset{\mathbb{Q}[e_1]}{R^{(1)}} \subset R^{(2)}\subset\cdots\subset R^{(k)}=R$,

then, for $m=1,2,\ldots,k$

$$\mathrm{Hilb}(\underbrace{R^{(m)}/R^{(m-1)}}_{\text{quotient } \mathbb{Q}\text{-vector space}},q) = q^m \underbrace{\begin{bmatrix} l\\ m\end{bmatrix}_q}_{\overset{q=1}{\rightsquigarrow}\binom{l}{m}} \cdot \underbrace{\left(\sum_{j=0}^{k-m} q^{j(l-m+1)} \begin{bmatrix} m+j-1 \\ j\end{bmatrix}_q\right)}_{\overset{q=1}{\rightsquigarrow}\binom{k}{m}}$$

consistent with $\binom{k+l}{k} = \sum_m \binom{l}{m}\binom{k}{m}$

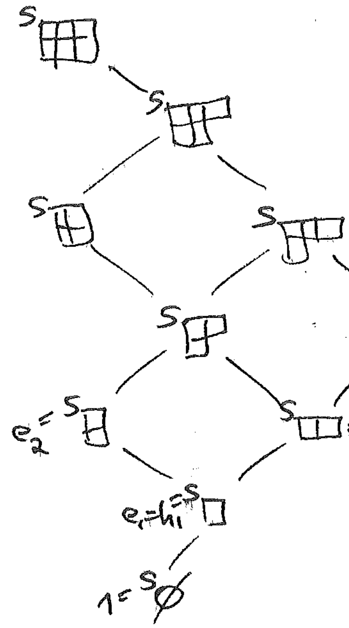
$k=2$
$l=3$ $\quad R = \mathbb{Q}[e_1, \underset{\overset{\|}{k}}{e_2}, h_1, h_2, \underset{\overset{\|}{l}}{h_3}] \Big/ (e_1 - h_1,$

$e_2 - e_1h_1 + h_2,$

$e_2h_1 - e_1h_2 + h_3$

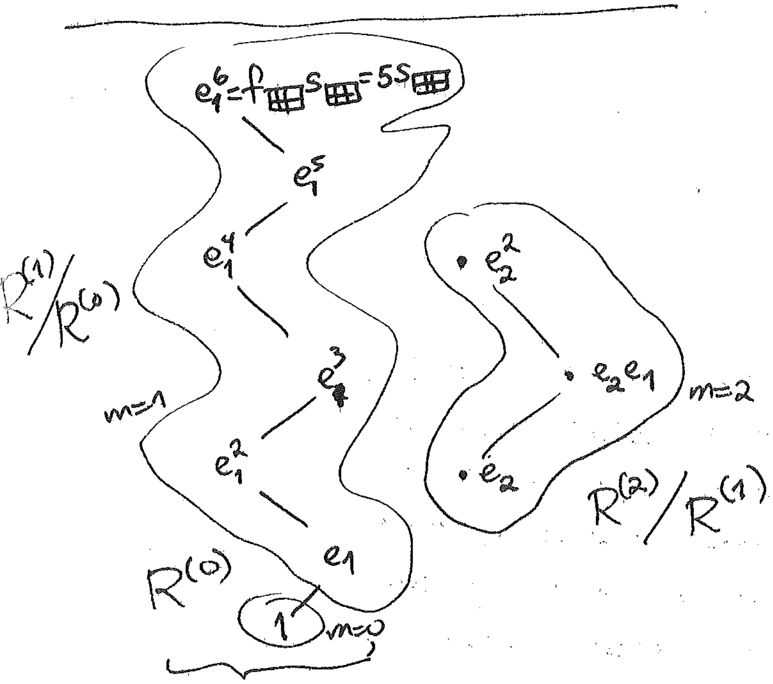
$-e_1h_3 + e_2h_2,$

$e_2h_3)$



$s_{\square\square\square} = h_3 = \underset{-e_2h_1}{e_1h_2} = e_1(e_1^2 - e_2) \underset{-e_2e_1}{} = e_1^3 - 2e_1e_2 = \det \begin{bmatrix} e_1 & e_2 & 0 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{bmatrix}$
$\quad \| \quad s_{\square\square\square}$

$s_{\square\square} = h_2 = e_1h_1 - e_2 = e_1^2 - e_2$



$$\underset{m=2}{\mathrm{Hilb}(R^{(2)}/R^{(1)}, q)} = q^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \underbrace{\left(\sum_{j=0}^{0} q^{j(3-2+1)} \begin{bmatrix} 2+j-1 \\ j \end{bmatrix}_q \right)}_{q^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q}$$

$$= q^2 [3]_q$$

$R^{(1)}$

$$\underset{m=1}{\mathrm{Hilb}(R^{(1)}/R^{(0)}, q)} = q^1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \underbrace{\underbrace{\left(\sum_{j=0}^{2-1} q^{j\overbrace{(3-1+1)}^{3}} \begin{bmatrix} j \\ j \end{bmatrix}_q \right)}_{q^0 + q^3}}_{[\underset{l}{3} \cdot \underset{k}{2}]_q}$$

$$= q^1 [6]_q \checkmark$$