C.P.S. 11/14/2014 L. Billera

(from Veit Elser)

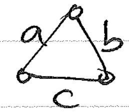
G a simple connected graph

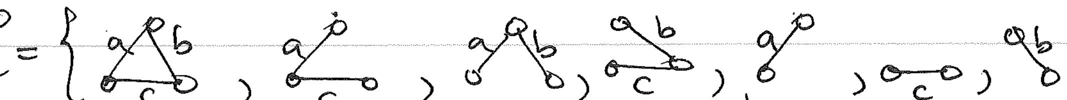
$\mathcal{C}$ = set of connected subgraphs $H \subset G$

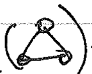
st. $V(H)$ is a vertex cover of G, i.e. it meets all edges of G

Define $f_k(G) := (-1)^{|V(G)|+1} \sum_{H \in \mathcal{C}} (-1)^{|E(H)|} |V(H)|^k$

for $k \geq 1$

EXAMPLE: $G =$ 

$\mathcal{C} = \{$  $\}$

$f_k($$) = (-1)^4 (-3^k + 3 \cdot 3^k - 3 \cdot 2^k)$

$= 6 (3^{k-1} - 2^{k-1}) = \begin{cases} 0 & k=1 \\ \text{positive for } k \geq 2 \end{cases}$

THM (Elser, "Cluster models of percolation ..." J. Phys. A Math Gen. 17 (1984), 1515–1523)

$f_1(\circ) = 1$ else $f_1(G) = 0$

$f_2(G) = 0$

a tree on $n \geq 2$, not a path

$f_2(\text{path}) = 2$

$f_2(n\text{-cycle}) = n(n-1)$

$f_2(K_n) = n!$

CONJ: $f_k(G) \geq 0$ $\forall$ connected simple G

LEMMA: If $|V(G)| \geq 2$ then given $v \in V(G)$

$$\sum_{\substack{H \in \mathcal{C} \\ v \in V(H)}} (-1)^{|E(H)|} = 0 .$$

Note: Given $H \in \mathcal{C}$ and $e \in E(G) \setminus E(H)$ ~~then~~ $H \cup \{e\} \in \mathcal{C}$.

So the complements of $\{E(H) : H \in \mathcal{C}\}$ form a simplicial complex