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D. Stanton 9/5/14 — based on work with P. Hersh & J. Shareshian "The $q=-1$ phenomenon via homology concentration"

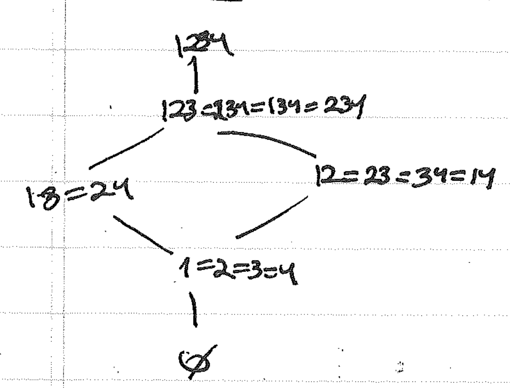
Let S_n act on $2^{[n]}$

S_n = symmetric group

What is the orbit generating function

$$X_n(q) = \sum_{\text{orbits } \mathcal{O}} q^{|\mathcal{O}|} \quad ?$$

EXAMPLE: $n=4$ replacing S_n by C_n



$$q^4 + q^3 + 2q^2 + q + 1$$

For a general $G \le S_n$

$$X_G(q) = \frac{1}{|G|}\sum_{g\in G} P_{\text{class}(g)}(q) \quad = \text{pattern inventory for 2-colorings}$$

$G = C_4$

$$= \frac{1}{4}\Big(\underbrace{(1+q)^4}_{\text{id}} + \underbrace{2(1+q^4)}_{2\ 4\text{-cycles}} + \underbrace{(1+q^2)^2}_{(13)(24)} \Big)$$

THM: (de Bruijn) $X_G(\overset{q}{-1}) = \#$ of self-complementary orbits

e.g. $\left(1+q+2q^2+q^3+q^4\right)_{q=-1} = 2 = \#\{13=24,\ 12=23=34=14\}$

IDEA: Interpret $X_G(-1)$ as the Euler characteristic of an algebraic complex.

$$C_n \xrightarrow{D_n} C_{n-1} \xrightarrow{D_{n-1}} \cdots \longrightarrow C_2 \xrightarrow{D_2} C_1 \xrightarrow{D_1} C_0$$

C_i = $\mathbb{F}_2$-vectorspace spanned by basis elements e_S indexed by i-element subsets S

$$D_i(e_S) = \sum_{i \in S} e_{S \setminus \{i\}}$$ (check $D^2 = 0$ because we're working over $\mathbb{F}_2$)

Same thing on $2^{[n]}/G$

e.g. $G = C_4$

$$\begin{tikzcd}[column sep=small]
& G1234 \arrow[d, dash] & \\
& G123 \arrow[dl, dash] \arrow[dr, dash] & \\
G13 \arrow[dr, dash] & & G12 \arrow[dl, dash] \\
& G1 \arrow[d, dash] & \\
& G\phi &
\end{tikzcd}$$

$$\begin{aligned} D(G1234) &= G123 + G134 + G234 + G134 \\ &= 4 \cdot G123 \\ &= 0 \end{aligned}$$

$$0 \le X_G(-1) = \sum_{i=0}^{n} \dim C_i \cdot (-1)^i$$

$$= \sum_{i=0}^{n} \dim H_i \cdot (-1)^i \quad \text{where } H_i = \ker D_i / \operatorname{im} D_{i+1}$$

We can <u>hope</u> $H_{2i+1} = 0 \ \forall i$ showing RHS ≥ 0.

What happens for $G = C_n$?

For n odd, $X_G(-1) = 0$ and one can show $H_i = 0 \ \forall i$.

What about for n even?

C_n-orbits = 2-colored necklaces

FACTS:

$$X_n(q) = \frac{1}{n} \sum_{d \mid n} \varphi(d)\left(1+q^d\right)^{n/d}$$

$$X_n(-1) = \frac{1}{n} \sum_{\substack{d \mid n \\ d \text{ even}}} \varphi(d)\, 2^{n/d}$$

$$X_{2n}(-1) = \frac{X_n(-1) + X_n(1)}{2}$$

DATA:

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}
$n=2$	1	0	0								
$n=4$	1	0	1	0	0						
$n=6$	1	0	0	0	1	0					
$n=8$	1	0	1	0	1	0	1	0	0		
$n=10$	1	0	0	0	2	0	0	0	1	0	0

Let $A_n(q) = \sum_{i=0}^{n} \dim H_i \cdot q^i$

PROBLEM 1A:

CONJ (in HSS paper): $A_{2n}(q) = \dfrac{q^2 A_n(q^2) + X_n(q^2)}{1+q^2}$

KNOWN: • n odd it's proven
• $n \leq 18$ checked via computer
• H_0, H_1 are OK

ALSO: $(1+q^2) \mid A_{4n}(q)$ has been checked for $n \leq 25$.

PROBLEM 1B: $G \leq S_n$ acting on $2^{[n]}$
$\chi_G(-1) = \#$ self-complementary orbits

$G \leq GL_n(\mathbb{F}_q)$ acting on $L_n(q) =$ lattice of all subspaces of $\mathbb{F}_q^n$

Let $\chi_G(t) = \sum_{G\text{-orbits } \mathcal{O}} t^{\dim(\mathcal{O})}$

dimension of any subspace in the orbits

e.g. $G = \langle e \rangle$ has $\chi_G(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k$

Q: Does $\chi_G(-1)$ have meaning?

For $G = \langle e \rangle$, $\chi_G(-1) = \begin{cases} (1-q)(1-q^3)\cdots(1-q^{2M-1}) & \text{if } n = 2M \\ 0 & \text{if } n \text{ odd} \end{cases}$