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Superpotentials and higher order derivations

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ABSTRACT

We consider algebras defined from quivers with relations that are kth order derivations of a superpotential, generalizing results of Dubois-Violette to the quiver case. We give a construction compatible with Morita equivalence, and show that many important algebras arise in this way, including McKay correspondence algebras for GL_n for all n, and four-dimensional Sklyanin algebras. More generally, we show that any N-Koszul, (twisted) Calabi–Yau algebra must have a (twisted) superpotential, and construct its minimal resolution in terms of derivations of the (twisted) superpotential. This yields an equivalence between N-Koszul twisted Calabi–Yau algebras A and algebras defined by a superpotential ω such that an associated complex is a bimodule resolution of A. Finally, we apply these results to give a description of the moduli space of four-dimensional Sklyanin algebras using the Weil representation of an extension of $\mathsf{SL}_2(\mathbb{Z}/4)$.

1. Introduction

Let Q be a quiver and $\mathbb{C}Q$ its path algebra. This means that Q is an oriented graph, and $\mathbb{C}Q$ is the algebra of $\mathbb{C}$-linear combinations of paths in the quiver, with multiplication given by concatenation of paths (setting $p \cdot q = 0$ if p and q cannot be concatenated). This algebra is graded by path length, which we denote by $|p|$.

If p and q are the paths we define the partial derivative of q with respect to p as

$$\partial_p q := \begin{cases} r & \text{if } q = pr \text{ with } r \text{ a path,} \\ 0 & \text{otherwise.} \end{cases}$$

We can extend this operation linearly to get a map $\partial_p : \mathbb{C}Q \to \mathbb{C}Q$. Note that if $p = e$ is a trivial path (i.e. a vertex) then taking the derivative is the same as multiplication on the left: $\partial_e q = eq$.

Similarly to [1,2], we define the *derivation-quotient algebra* of $\omega \in \mathbb{C}Q$ of order k as the path algebra modulo the derivatives of ω by paths with length k:

$$\mathcal{D}(\omega, k) := \mathbb{C}Q/\langle \partial_p \omega; |p| = k\rangle. \tag{1.1}$$

We are particularly interested in ω which are super-cyclically symmetric, i.e., are sums of elements of the form

$$\sum_{i=1}^{n} (-1)^{(i-1)(n+1)} a_i a_{i+1} \cdots a_n a_1 a_2 \cdots a_{i-1}, \quad a_i \in Q. \tag{1.2}$$

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Such ω will be called *superpotentials*.[3] We will also consider *twisted superpotentials*. These are elements in $(\mathbb{C}Q)_n$, the vector space spanned by paths of length n, that are invariant under the $\mathbb{C}$-linear map

$$a_1 \cdots a_n \mapsto (-1)^{n+1}\sigma(a_n)a_1 \cdots a_{n-1}. \tag{1.3}$$

where σ is a graded $\mathbb{C}$-algebra automorphism of $\mathbb{C}Q$ which fixes the trace function $\mathbb{C}Q \xrightarrow{\mathrm{Tr}} \mathbb{C}$ such that $\mathrm{Tr}(p) = 1$ if p is a vertex, and $\mathrm{Tr}(p) = 0$ for all other paths p. (If σ is trivial one recovers the notion of a superpotential. We need twisted superpotentials to address the McKay correspondence for GL_n, as opposed to SL_n.)

For the case $k = 1$ in (1.1), algebras defined by superpotentials have been greatly studied. Examples include three-dimensional Sklyanin algebras [3], algebras coming from the three-dimensional McKay correspondence [4,5], cluster algebras and mutations [6], and algebras derived from exceptional collections on Calabi–Yau varieties [7]. The fact that all these algebras have a superpotential can be traced back to a common homological property: the Calabi–Yau property. In one of its forms, this property states that an algebra A is CY-n if A has a projective A-bimodule resolution $\mathcal{P}^\bullet$ that is self-dual:

$$\mathrm{Hom}_{A-A}(\mathcal{P}^\bullet, A \otimes A) \cong \mathcal{P}^{n-\bullet}. \tag{1.4}$$

Similarly, one has the twisted Calabi–Yau property, where the resolution is self-dual to a twist of itself by an automorphism σ of A:

$$\mathrm{Hom}_{A-A}(\mathcal{P}^\bullet, A \otimes A) \cong \mathcal{P}^{n-\bullet} \otimes_A A^\sigma,$$

where A^σ is the bimodule obtained from A by twisting the right multiplication by σ ($a \cdot x \cdot b = ax\sigma(b)$, for $x \in A^\sigma$, $a, b \in A$).

It is known that graded three-dimensional Calabi–Yau algebras always derive from a superpotential [8], i.e., are of the form (1.1) with $k = 1$. Also, in [4, Theorem 3.6.4], a wide class of Calabi–Yau algebras of any dimension are shown to arise from a much more general type of superpotential.

In [1,2], in the one-vertex case (working over a field), these results were generalized to higher order derivations. In particular, [2, Theorem 11] implies that any AS-Gorenstein algebra over a field which is also Koszul is equal to $\mathcal{D}(\omega, k)$ for some ω, k (more generally, this is shown replacing Koszul with N-Koszul, a generalization to the case of algebras presented by homogeneous relations of degree N rather than two [9]). We recall that a graded algebra over a field k is AS-Gorenstein if

$$\mathrm{Ext}^i_A(k, k) \cong \begin{cases} 0 & i \neq n \\ k & i = n. \end{cases}$$

It is clear by extending the proof of Proposition 4.3 in [10] that graded twisted Calabi–Yau algebras over a field are AS-Gorenstein. In the other direction, the result of Dubois-Violette implies that, in the Koszul case, AS-Gorenstein algebras are twisted Calabi–Yau. If we work over a general semisimple algebra S instead of k, then the same relation holds between twisted Calabi–Yau and the AS-Gorenstein property with k replaced by S. In this paper, we will therefore use the twisted Calabi–Yau condition.

One of the main goals of this paper is to generalize [2] to the several-vertex case. Precisely, we give a Morita-invariant construction of algebras $\mathcal{D}(\omega, k)$ over any semisimple $\mathbb{C}$-algebra (Section 2. Using this, we show that algebras which occur in the higher-dimensional McKay correspondence also derive from a superpotential (Section 3). We give a method to compute the superpotential for the path algebra with relations which is Morita equivalent to $\mathbb{C}[V]\#G$ and illustrate this with some examples. These results generalize those of Crawley-Boevey and Holland [11,12] and Ginzburg [4] in the cases $G = \mathsf{SL}_2, \mathsf{SL}_3$. We then prove (Section 6) that any N-Koszul, (twisted) Calabi–Yau algebra over a semisimple algebra is of the form $\mathcal{D}(\omega, k)$, where ω is a (twisted) superpotential. This last theorem generalizes [2, Theorem 11] to the quiver case, and gives another proof of the fact that McKay correspondence algebras are given by a (twisted) superpotential. It also generalizes [8] to arbitrary Calabi–Yau dimension. More generally, we show that N-Koszul twisted Calabi–Yau algebras are equivalent to algebras $A = \mathcal{D}(\omega, k)$ such that an associated complex (6.1) yields a bimodule resolution of A.

We end by illustrating this theorem in the case of Sklyanin algebras of dimension 4 (Section 7), which was the main motivating example behind Section 6. We give a formula for the superpotential (which was done in [2, Section 6.4], see also [1], in different language and over $\mathbb{R}$). In Section 7.0.1, we describe the twisted superpotentials associated to the algebras from [13] related to the Sklyanin algebras. We explain that the Sklyanin McKay correspondence algebras involve subgroups of the Heisenberg group over $\mathbb{Z}/4$.

As an application of our results, we give a simple representation-theoretic computation of the moduli space of Sklyanin algebras of dimension 4 (Theorem 7.9). This description involves considering the projective space of superpotentials. Since the automorphism group of a generic Sklyanin is a form of the Heisenberg group over $\mathbb{Z}/4$ equipped with the Heisenberg representation (which is uniquely determined by the action of its center), we are able to find a version of the Weil representation acting on superpotentials. Pulling this back, we obtain a description of the moduli space in terms of the original parameters for the Sklyanin algebras. We remark that, while it is probably possible to obtain this result using the geometry associated to Sklyanin algebras (an elliptic curve and a point of that curve), and our result can also be deduced from, e.g., [27] and [14, Proposition 3.1, Section 9], it is interesting that the theorem follows purely from representation-theoretic consequences of the action of the Heisenberg group by automorphisms on the Sklyanin algebra.

[3] In the N-Koszul case for $N > 2$, i.e., where n is greater than the expected Hochschild dimension of $\mathcal{D}(\omega, k)$, we will need to consider superpotentials which are cyclically rather than super-cyclically symmetric: i.e., the sign is eliminated. See Section 6.

2. Coordinate-free potentials

In this section we formulate potentials, derivations, and $\mathcal{D}(\omega, k)$ in a categorical way for a tensor algebra over a semisimple algebra.

2.1. Duals, duals, duals…

Let S be a finite-dimensional semisimple algebra over $\mathbb{C}$ and let V be an S-bimodule. We will denote the ordinary tensor product over $\mathbb{C}$ by $\otimes$ and the tensor product over S by $\otimes_S$.

There are at least 4 distinct way to construct a dual bimodule to V:

- The space of linear morphisms to $\mathbb{C}$: $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with bimodule action $(s_1\psi s_2)(w) = \psi(s_2 w s_1)$.
- The space of right-module morphisms to S: $V^{*R} := \mathrm{Hom}_{\text{mod-}S}(V, S)$ with bimodule action $(s_1\psi s_2)(w) = s_1\psi(s_2 w)$.
- The space of left-module morphisms to S: $V^{*L} := \mathrm{Hom}_{S\text{-mod}}(V, S)$ with bimodule action $(s_1\psi s_2)(w) = \psi(w s_1)s_2$.
- The space of bimodule morphisms to $S \otimes_{\mathbb{C}} S$: $V^{*B} := \mathrm{Hom}_{S\text{-bimod}}(V, S \otimes S)$. Using Sweedler notation, we write $\psi \in V^{*B}$ as $\psi_1 \otimes_C \psi_2$, with bimodule action $(s_1\psi s_2)_1(w) \otimes (s_1\psi s_2)_2(w) = \psi_1(w)s_2 \otimes s_1\psi_2(w)$.

These duals extend to 4 contravariant functors $*, *R, *L, *B : S\text{-bimod} \to S\text{-bimod}$. All these different constructions are not canonically isomorphic in the category of S-bimodules, so in order to identify them we need an extra datum. This extra datum is a nondegenerate trace function $\mathrm{Tr} : S \to \mathbb{C}$ which defines natural isomorphisms L, R, B from the complex dual to the the 3 other duals by demanding that, for $\psi \in V^*$,

$$\forall w \in V : \psi(w) = \mathrm{Tr} R\psi(w) = \mathrm{Tr} L\psi(w) = \mathrm{Tr}((B\psi)_1(w))\mathrm{Tr}((B\psi)_2(w)).$$

Moreover, these identifications are compatible with Morita equivalence: if $e \in S$ is an idempotent such that $SeS = S$, then the trace on S restricts to a nondegenerate trace on eSe. The images of the identification maps under the Morita equivalence $\mathcal{M} : S\text{-bimod} \to eSe\text{-bimod}$ are precisely the identification maps of the restricted trace. From now on we will fix a trace on S and omit the functors.

For $\psi \in V^*$ and $w \in V$, we will denote the pairings to $\mathbb{C}$ and S by

$$\langle \psi, x\rangle = \langle x, \psi\rangle = \psi(x) \quad \text{while } [\psi x] := R\psi(x) \quad \text{and} \quad [x\psi] := L\psi(x).$$

The resulting S-bimodule morphisms, $[\,] : V^* \otimes_S V \to S$ and $[\,] : V \otimes_S V^* \to S$ called the evaluation maps. The duals of the evaluation maps are called the coevaluation maps:

$$\mathsf{coev}_R : S \to V \otimes_S V^* \quad \text{and} \quad \mathsf{coev}_L : S \to V^* \otimes_S V.$$

We will write the image of 1 under the coevaluation as

$$\mathsf{coev}_R(1) = \sum_{Rx} x \otimes_S x^* \quad \text{and} \quad \mathsf{coev}_L(1) = \sum_{Lx} x^* \otimes_S x.$$

In these expressions Rx and Lx stand for summation over right and left S-module bases of V, respectively. These elements satisfy the following standard evaluation–coevaluation identities:

$$\forall \zeta \in V^* : \zeta = \sum_{Rx} [\zeta x]x^* = \sum_{Lx} x^*[x\zeta]$$
$$\forall u \in V : u = \sum_{Rx} x[x^* u] = \sum_{Lx} [ux^*]x.$$

The bracket notation can be extended to tensor products to obtain maps $[\,] : (V^*)^{\otimes_S k} \times V^{\otimes_S l} \to V^{\otimes_S (l-k)}$ (for $l \geq k$) such that

$$[\phi_1 \otimes_S \cdots \otimes_S \phi_k, w_1 \otimes_S \cdots \otimes_S w_l] = [\phi_1[\phi_2 \cdots [\phi_k w_1] \cdots w_{k-1}]w_k] \cdot w_{k+1} \otimes_S \cdots \otimes_S w_l,$$

and similarly $[\,] : V^{\otimes_S l} \times (V^*)^{\otimes_S k} \to V^{\otimes_S (l-k)}$. If $k = l$ we end up with an element in S. In this case, we define also the pairing $\langle , \rangle : (V^*)^{\otimes_S k} \otimes_S V^{\otimes_S k} \to \mathbb{C}$ by $\langle x, y\rangle := \mathrm{Tr}[xy]$. For $k > l$, we may replace the image $V^{\otimes_S (l-k)}$ by $(V^*)^{\otimes_S (k-l)}$. These satisfy associativity identities, e.g., $[(\phi \otimes_S \psi)x] = [\phi[\psi x]]$ and $[[\phi x]\psi] = [\phi[x\psi]]$ if $\psi \in (V^*)^{\otimes_S k}$, $\phi \in (V^*)^{\otimes_S l}$ and $x \in V^{\otimes_S n}$ with $n \geq k + l$.

2.2. Potentials

A *weak potential* of degree n is an element of degree n in the tensor algebra $T_S V$ that commutes with the S-action:

$$\omega \in V^{\otimes_S n} \quad \text{such that } \forall s \in S : s\omega = \omega s.$$

A weak potential is called a *superpotential* if

$$\forall \psi \in V^* : [\psi\omega] = (-1)^{n-1}[\omega\psi].$$

Let τ be a graded $\mathbb{C}$-algebra automorphism of $T_S V$ that preserves the trace. This gives us an automorphism of S as a $\mathbb{C}$-algebra, and we can define for any bimodule M the left twist ${}_\tau M$ to be the vector space M equipped with the bimodule action $s_1 \cdot x \cdot s_2 := s_1^\tau x s_2$. The right twist M_τ is defined analogously. We obtain isomorphisms ${}_{\tau^{-1}}S \cong S_\tau$, ${}_{\tau^{-1}} V \cong V_\tau$ using τ, and ${}_{\tau^{-1}}V^* \cong (V^*)_\tau$ using τ^*.

We then define a *twisted weak potential* of degree n to be an element

$$\omega \in V^{\otimes_S n} \quad \text{such that } \forall s \in S : s^\tau \omega = \omega s.$$

A twisted superpotential is a twisted weak potential ω satisfying

$$\forall \psi \in V^* : [\psi\omega] = (-1)^{n-1}[\omega\psi^{\tau^*}].$$

For every (twisted) weak potential ω of degree n and every nonnegative $k \le n$, we can define a bimodule morphism

$$\Delta_k^\omega : (V^{\otimes_S k})^* \otimes_{S_{\tau^{-1}}} S \to V^{\otimes_S (n-k)} : \psi \otimes_S x \to [\psi\omega x].$$

We will denote the image of Δ_k^ω by $W_{n-k} \subset V^{\otimes_S (n-k)}$.

Definition 2.1. We define the derivation-quotient algebra of ω of order k to be the path algebra modulo the ideal generated by the S-bimodule W_{n-k}:

$$\mathcal{D}(\omega, k) := \mathbb{C}Q/\langle \mathrm{Im}\Delta_k^\omega\rangle = \mathbb{C}Q/\langle W_{n-k}\rangle.$$

Here, $\langle M\rangle$ stands for the smallest two-sided ideal containing M.

2.3. Path algebras and quivers

Let us look at all these concepts in the case of a path algebra of a quiver. A quiver Q consists of a set of vertices, Q_0, a set of arrows, Q_1, and two maps, $h, t : Q_1 \to Q_0$, assigning to every arrow its head and tail. We set $S = \mathbb{C}^{Q_0}$, where the vertices form a basis of idempotents, we equip it with a trace Tr such that all vertices have trace 1. We construct the S-bimodule $V := \mathbb{C}^{Q_1}$ such that, for every arrow $a \in Q_1$, we have the identity $a = h(a)at(a)$. The path algebra can now be viewed as $\mathbb{C}Q := T_S V$. Note that, with this notation, the composition of arrows is given by

$$ab = \xleftarrow{a}\xleftarrow{b} .$$

The set Q_1 of arrows is a basis for V. Define $a^* \in V^*$ so that $\{a^* \mid a \in Q_1\}$ is a dual basis to Q_1. Tensoring these bases together, we obtain the dual bases for $V^{\otimes_S k} = \mathbb{C}Q_k$, $(V^*)^{\otimes_S k} = \mathbb{C}Q_k^*$ which are nothing but the paths and dual paths of length k. The brackets have the following form for paths $p, q \in \mathbb{C}Q_k$:

$$[p^*q] = \delta_{pq}t(q), \qquad [qp^*] = \delta_{pq}h(q), \quad \text{and} \quad \langle p^*, q\rangle = \delta_{pq}.$$

More generally, if p, q are any paths, then we obtain that bracketing corresponds to taking partial derivatives:

$$\partial_p q = [p^*q].$$

A weak potential is an element $\omega \in \mathbb{C}Q_k$ that consists only of closed paths (i.e. ω is a linear combination of elements p such that $h(p) = t(p)$), and Δ_k^ω corresponds to the map $(\mathbb{C}Q_k)^* \to \mathbb{C}Q_{d-k} : p^* \to \partial_p\omega$. The weak potential ω is a superpotential if and only if $[a^*\omega] = (-1)^{n-1}[\omega a^*]$ for all $a \in Q_1$, which is the same as saying that $\vec{\omega} = (-1)^{n-1}\omega$, where $\vec{\omega}$ denotes the cyclic shift: $\overrightarrow{a_1 \ldots a_n} = a_n a_1 \ldots a_{n-1}$, where $a_i \in Q_1$ for all i.

If τ is a graded trace-preserving automorphism of $\mathbb{C}Q$ then a twisted weak potential consists of a linear combination of paths p that satisfy $h(p) = \tau(t(p))$ (note that τ must induce a permutation on the set of paths of length k for all k). It is a twisted superpotential if and only if $[(a^\tau)^*\omega] = (-1)^{n-1}[\omega a^*]$ for all $a \in Q$, i.e, $\vec{\omega}^\tau = (-1)^{n-1}\omega$, where $\vec{\omega}^\tau$ is the twisted cyclic shift $\mathbb{C}$-linearly defined by $\overrightarrow{a_1 \ldots a_n}^\tau = a_n^\tau a_1 \ldots a_{n-1}$ for $a_i \in Q_1$.

2.4. Morita equivalence

The new formulation has the advantage that it is compatible with standard Morita equivalence:

Lemma 2.2. *Let $e \in S$ be an idempotent such that $SeS = S$. If $M \subset T_S V$ is an S-bimodule then there is a Morita equivalence between $A = T_S V/\langle M\rangle$ and*

$$T_{eSe}(eVe)/\langle eMe\rangle.$$

If ω is a (twisted) weak potential and $e^\tau = e$, then we have that

$$e\mathcal{D}(\omega, k)e = \mathcal{D}(e\omega e, k).$$

Proof. By the standard Morita equivalence between S and eSe, we have a functor

$$\mathcal{F} : S\text{-bimod} \to eSe\text{-bimod}$$

which maps M to eMe. This functor commutes with tensor products $\mathcal{F}(M \otimes_S N) \cong \mathcal{F}(M) \otimes_{eSe} \mathcal{F}(N)$, where $e(m \otimes_S n)e \mapsto eme \otimes_{eSe} ene$ is the natural isomorphism. The same holds for duals and direct sums. This implies that $\mathcal{F}(T_S V) = e(T_S V)e \cong T_{eSe}(eVe)$, and for any S-subbimodule $M \subset T_S V$, that $\mathcal{F}(M) \subset \mathcal{F}(T_S V)$ and $\mathcal{F}(\langle M\rangle) = \langle \mathcal{F}(M)\rangle$. We obtain an isomorphism between $T_{eSe}(eVe)/\langle eMe\rangle$ and $e(T_S V/\langle M\rangle)e$, which is Morita equivalent to $T_S V/\langle M\rangle$.

Finally, if $\omega \in V^{\otimes_S n}$ is a (twisted) weak potential of degree n and $W_{n-k} := \mathrm{Im}\Delta_k^{\omega} \in V^{\otimes_S (n-k)}$, we get that $\mathcal{F}(W_{n-k}) = \mathcal{F}(\mathrm{Im}\Delta_k^{\omega}) = \mathrm{Im}(\mathcal{F}(\Delta_k^{\omega}))$. Since

$$\mathcal{F}(\Delta_k^{\omega})(e\phi e \otimes_{eSe} exe) = [e\phi e\omega exe] = [(e\phi e)(e\omega e)(exe)] = \Delta_k^{e\omega e}(e\phi e \otimes_{eSe} exe),$$

we deduce that $\mathcal{F}(\Delta_k^{\omega}) = \Delta_k^{e\omega e}$, and hence $e\mathcal{D}(\omega, k)e = \mathcal{D}(e\omega e, k)$. □

3. McKay correspondence and potentials

Let G be any finite group, and let V be an arbitrary finite-dimensional representation. We can look at the tensor algebra $T_{\mathbb{C}}V^*$ and the ring $\mathbb{C}[V] \cong \mathrm{Sym}_{\mathbb{C}}V^*$ of polynomial functions on V. This last ring can be seen as the $(n-2)$nd-derived algebra coming from the superpotential:

$$\omega = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\sigma} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \in T_{\mathbb{C}}V^*.$$

where $x_1 \dots x_n$ form a basis for V^*. Indeed, for every path $p = x_{i_1} \dots x_{i_{n-2}} \in (T_{\mathbb{C}}V)_{n-2}$, we get that $\partial_p \omega$ is zero if some of the x_{i_j} are identical, and otherwise it is equal to the commutator between the two basis elements that do not occur in p (in some order). We conclude that

$$\mathbb{C}[V] \cong \mathcal{D}(\omega, n-2).$$

If R is a ring with G acting as automorphisms we can construct the smash product $R\#G$. As a vector space, this ring is isomorphic to $R \otimes \mathbb{C}G$, and the product is given by

$$(r_1 \otimes g_1) \cdot (r_2 \otimes g_2) = r_1(g_1 \cdot r_2) \otimes g_1 g_2.$$

For the tensor algebra TV^* we can rewrite the smash product as a tensor algebra over the group algebra $\mathbb{C}G$. Let us define $U = V^* \otimes \mathbb{C}G$. The $\mathbb{C}G$-bimodule action on it is given by

$$g(v \otimes x)h := gv \otimes gxh.$$

It is easy to see that for every k we have

$$\begin{aligned}(T_{\mathbb{C}}V^*\#G)_k &\cong V^* \otimes \dots \otimes V^* \otimes \mathbb{C}G \\ &\cong (V^* \otimes_{\mathbb{C}} \mathbb{C}G) \otimes_{\mathbb{C}G} \dots \otimes_{\mathbb{C}G} (V^* \otimes_{\mathbb{C}} \mathbb{C}G) = (T_{\mathbb{C}G}U)_k.\end{aligned}$$

The special bimodule action on U makes the identifications also compatible with the product, so that $T_{\mathbb{C}}V^*\#G \cong T_{\mathbb{C}G}U$. So the smash of the tensor algebra is again a tensor algebra but now over the semisimple algebra $\mathbb{C}G$. This algebra is isomorphic to

$$\bigoplus_{S_i} \mathtt{Mat}_{dim\, S_i \times dim\, S_i}(\mathbb{C}),$$

where we sum over all irreducible representations of G (up to isomorphism). The standard traces of these matrix algebras provide us a trace on $\mathbb{C}G$.

Lemma 3.1. *If $R \cong T_{\mathbb{C}}V^*/\langle M\rangle$, where M is a vector space of relations which is invariant under the G-action on $T_{\mathbb{C}}V^*$, then*

$$R\#G \cong T_{\mathbb{C}G}U/\langle M \otimes \mathbb{C}G\rangle.$$

Proof. If M is a G-invariant vector space in $T_{\mathbb{C}}V^*$, then $M \otimes \mathbb{C}G$ can be considered as a $\mathbb{C}G$-subbimodule of $T_{\mathbb{C}}V^*\#G$. This means that if $\mathfrak{i} \lhd T_{\mathbb{C}}V^*$ is a G-invariant ideal, then $\mathfrak{i} \otimes \mathbb{C}G$ is an ideal of $T_{\mathbb{C}}V^*\#G$. Moreover, if $\mathfrak{i} = \langle M\rangle$ with M a G-invariant subspace of $T_{\mathbb{C}}V^*$, then $\mathfrak{i} \otimes \mathbb{C}G = \langle M \otimes \mathbb{C}G\rangle$. So

$$\frac{T_{\mathbb{C}G}V^* \otimes \mathbb{C}G}{\langle M \otimes \mathbb{C}G\rangle} = \frac{(T_{\mathbb{C}}V^*) \otimes \mathbb{C}G}{\langle M\rangle \otimes \mathbb{C}G} = \frac{T_{\mathbb{C}}V^*}{\langle M\rangle} \otimes \mathbb{C}G = R\#G. \quad \square$$

Suppose $R = \mathbb{C}[V]$ with its action of G. Now $\mathbb{C}\omega \cong \bigwedge^n V^*$ is a one-dimensional G-representation. This means that $\bigwedge^n V^* \otimes \mathbb{C}G$ is a bimodule of the form $\mathbb{C}G_\tau$ where $\tau(g) = (\det g)g$ and hence the element $\omega \otimes 1$ is a twisted weak potential. It is easy to check that

$$(\mathrm{Im}\Delta_k^{\omega}) \otimes \mathbb{C}G = \mathrm{Im}((\Delta_k^{\omega}) \otimes \mathrm{id}_{\mathbb{C}G}) = \mathrm{Im}(\Delta_k^{(\omega\otimes 1)}).$$

Furthermore we see that τ permutes the summands of $\mathbb{C}G$ corresponding to the irreducible representations: the summand corresponding to S_i gets sent to the one corresponding to $\bigwedge^n V^* \otimes S_i$. One may find an $e = \sum e_i$ such that $e^\tau = e$, where the e_i are primitive idempotents such that $\mathbb{C}Ge_i \cong S_i$ for all i, with one e_i for each S_i. To find this, first note that, for any two idempotents $e_1, e_2 \in \mathbb{C}G$ and any $m \geq 0$, we have

$$e_1(T_{\mathbb{C}}V\#G)e_2 \supset e_1(\mathbb{C}G \otimes V^{\otimes m})e_2 \cong (e_1\mathbb{C}G) \otimes_{\mathbb{C}G} (V^{\otimes m} \otimes \mathbb{C}Ge_2) \cong \mathrm{Hom}_G(e_1\mathbb{C}G, V^{\otimes m} \otimes (\mathbb{C}Ge_2)). \tag{3.1}$$

Now, for any irreducible representation W of G, let $m \geq 1$ be the smallest positive integer such that $(\bigwedge^n V^*)^{\otimes m} \otimes W \cong W$ as representations of G. Let $\psi : W \xrightarrow{\sim} (\bigwedge^n V^*)^{\otimes m} \otimes W$ be an intertwiner, and let $w \in W$ be a nonzero eigenvector, up to scaling, of ψ: that is, $\psi(w) = x \otimes w$ for some nonzero $x \in (\bigwedge^n V^*)^{\otimes m}$. Then, let $e_1 \in \mathbb{C}G$ be a primitive idempotent such that there is an isomorphism $\mathbb{C}Ge_1 \xrightarrow{\sim} W$ which sends e_1 to w. Then, it follows that $e_1xe_1 \in e_1(T_{\mathbb{C}}V\#G)e_1$ represents the intertwiner $\psi' \in \mathrm{Hom}(\mathbb{C}Ge_1, (\bigwedge^n V^*)^{\otimes m} \otimes \mathbb{C}Ge_1)$ sending e_1 to $x \otimes e_1$, and is hence nonzero. Thus, $e_1x = xe_1$ as elements of $T_{\mathbb{C}}V\#G$. Next, let $e_2, e_3, \ldots, e_m$ be the primitive idempotents such that $e_i\omega = \omega e_{i+1}$ in $T_{\mathbb{C}}V\#G$ for all $1 \leq i \leq m-1$, where $\omega \in \bigwedge^n V^* \subset T_{\mathbb{C}}V$ is the superpotential from before. Then, it follows that $e_m\omega = \omega e_1$, and thus the set $\{e_1, e_2, \ldots, e_m\}$ is closed under the twist, and is a set of primitive idempotents for distinct irreducible representations. Performing this construction once on each orbit of isomorphism classes of irreducible representations under tensoring by $\bigwedge^n V^*$, we obtain the desired full collection of primitive idempotents e_i and the element $e = \sum_i e_i$ such that $e = e^\tau$. We will identify the irreducible representations S_i with $\mathbb{C}Ge_i$, and consider therefore $e_i \in \mathbb{C}Ge_i = S_i$ as a specified nonzero vector.

Note that $\mathrm{Tr}(e_i) = 1$, just as we want it to be for a path algebra. The element $e\omega e$ is a twisted superpotential because the original ω is, and the property of being a twisted superpotential is preserved by taking ω to $\omega \otimes 1$.

We deduce the following result:

Theorem 3.2. *The algebra $\mathbb{C}[V]\#G$ is a derivation-quotient algebra of order $n-2$ with a (twisted if $G \not\subset \mathsf{SL}_n$) superpotential of degree n. The same is true for the corresponding Morita equivalent path algebra with relations.*

How do we work out the potential in terms of paths in the path algebra? If G is a finite group acting on V then the quiver underlying $e(\mathbb{C}[V]\#G)e$ is called the McKay quiver of G and V. Its vertices e_i are in one to one correspondence to the irreducible representations S_i of G. The trace function on $\mathbb{C}G$ then allows us to identify $\mathbb{C}G^*$ with $\mathbb{C}G$ as $\mathbb{C}G$-bimodules: $\mathbb{C}G \to \mathbb{C}G^* : g \mapsto \mathrm{Tr}(g \cdot -)$. Therefore S_i^* is isomorphic to $e_i\mathbb{C}G$ as a right module.

The number of arrows from e_i to e_j is equal to the dimension of

$$e_j(V^* \otimes \mathbb{C}G)e_i = \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}Ge_j, (V^* \otimes \mathbb{C}G)e_i) = \mathrm{Hom}_{\mathbb{C}G}(S_j, V^* \otimes S_i).$$

This means that we can (and do) identify each arrow $a : e_i \to e_j$ with a certain intertwiner morphism $\psi_a : S_{h(a)} \to V^* \otimes S_{t(a)}$, such that the set of arrows gives a basis of these intertwiner maps. Here, we use the notation $S_{e_i} := S_i$. We have a dual basis a^*, which can be interpreted as a collection of maps

$$\psi_{a^*} : S_{t(a)} \to V \otimes S_{h(a)},$$

using the natural pairing between $\mathrm{Hom}_{\mathbb{C}G}(S_j, V^* \otimes S_i)$ and $\mathrm{Hom}_{\mathbb{C}G}(S_i, V \otimes S_j)$.

By construction, the set of vertices $e_i \in Q_0 \subset \mathbb{C}Q$ is closed under the twist τ. We can choose intertwiners so that $Q_1 \subset \mathbb{C}Q$ is also closed under the twist, as follows. First, $\omega \in \bigwedge^n V^* \subset T^nV^*$ gives an intertwiner $\psi_{\tau(e_i)\omega e_i} : S_{\tau(e_i)} \xrightarrow{\sim} \bigwedge^n V^* \otimes S_{e_i}$, given by $\tau(e_i) \mapsto \omega \otimes e_i$, for all vertices e_i. We will denote this simply by $\psi_\omega := \psi_{\tau(e_i)\omega e_i}$.

Next, we choose the intertwiners ψ_a so that the following diagram commutes (for all $a \in Q_1$):

$$\begin{array}{ccc} S_{\tau(h(a))} & \xrightarrow{\quad\psi_a\quad} & V^* \otimes S_{\tau(t(a))} \\ \downarrow{\scriptstyle \psi_\omega} & & \downarrow{\scriptstyle \mathrm{id}_{V^*} \otimes \psi_\omega} \\ \bigwedge^n V^* \otimes S_{h(a)} & \xrightarrow{\mathrm{id}_{\bigwedge^n V^*} \otimes \psi_{a^\tau}} & \bigwedge^n V^* \otimes V^* \otimes S_{t(a)}. \end{array} \tag{3.2}$$

Then, the corresponding elements $Q_1 \subset \mathbb{C}Q$ will be closed under the twist τ. Furthermore, assign to every path p of length k the composite maps $\psi_p : S_{h(p)} \to (V^*)^{\otimes k} \otimes S_{t(p)}$ and $\psi_{p^*} : S_{t(p)} \to V^{\otimes k} \otimes S_{h(p)}$. These also fit into the commutative diagram above, replacing a by p.

For every $k \leq n$ we have an antisymmetrizer: $\alpha_k : V^{\otimes k} \to \bigwedge^k V : v_1 \otimes \cdots \otimes v_k \mapsto v_1 \wedge \ldots \wedge v_k$. If p is a path of length n, consider the composite

$$S_{t(p)} \xrightarrow{\psi_{p^*}} V^{\otimes n} \otimes S_{h(p)} \xrightarrow{\alpha_n \otimes \mathrm{id}_{S_{h(p)}}} \bigwedge^n V \otimes S_{h(p)} \xrightarrow{\mathrm{id}_{\bigwedge^n V} \otimes \psi_\omega} \bigwedge^n V \otimes \bigwedge^n V^* \otimes S_{t(p)} \longrightarrow S_{t(p)}, \tag{3.3}$$

where the last arrow is given by the pairing between V and V^*. Since this is an intertwiner, by Schur's lemma it is a scalar, which we denote by c_p. In other words, c_p is such that $(\alpha'_n \otimes \mathrm{id}_{S_{t(p)}}) \circ \psi_p(e_{h(p)}) = c_p^{-1}(\omega \otimes e_{t(p)})$, where $\alpha'_k : (V^*)^{\otimes k} \to \bigwedge^k V^*$ is the antisymmetrizer.

These scalars allow us to write down an explicit form of the superpotential. The weak potential $\omega \otimes 1$ in $T_{\mathbb{C}G}(V^* \otimes \mathbb{C}G)$ acts as a linear function on $(\mathbb{C}G^* \otimes V)^{\otimes_{\mathbb{C}G} n} \cong V^{\otimes n} \otimes \mathbb{C}G$, given by the last three arrows of (3.3) followed by the trace Tr. Hence, $(\omega \otimes 1)(p^*) = \mathrm{tr}((\omega \otimes 1)(\psi_{p^*})) = \mathrm{tr}((\omega \otimes 1)((\alpha_n \otimes \mathbf{1}_{S_{h(p)}})(\psi_{p^*}))) = \mathrm{tr}(c_p) = c_p \mathit{dim}\, t(p)$. Here we used tr for the canonical trace $\mathrm{End}(S_{t(p)}) \to \mathbb{C}$, to distinguish it from Tr. Because the Morita equivalence between $\mathbb{C}G$ and $e\mathbb{C}Ge$ is compatible with taking the dual, we see that

$$e(\omega \otimes 1)e = \sum_{|p|=n} \big(e(\omega \otimes 1)e\big)(p^*)p = \sum_{|p|=n} c_p \mathit{dim}\, t(p)p =: \Phi$$

and so $\mathbb{C}[V]\#G$ is Morita equivalent to

$$\frac{T_{eSe}(eVe)}{\langle \mathrm{Im}(\Delta_{n-2}^{e(\omega\otimes 1)e})\rangle} \cong \frac{\mathbb{C}Q}{\langle \partial_q \Phi : q \text{ is a path of length } n-2\rangle}.$$

4. Corollaries and remarks

In this section we show how the main result of the last section recovers several known results in the literature. In particular we show that for a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$, we recover the preprojective algebra; for a finite small[4] subgroup of $\mathsf{GL}_2(\mathbb{C})$ we recover the mesh relations; and for a finite subgroup of $\mathsf{SL}_3(\mathbb{C})$ we recover the superpotential found by Ginzburg [4]. Furthermore, if the group is abelian in $\mathsf{GL}_n(\mathbb{C})$, we can also recover the toric result.

Recall our convention that, when referring to quivers, xy means y followed by x.

We start with the toric case: suppose G is a finite abelian subgroup of $\mathsf{GL}_n(\mathbb{C})$. Being abelian we may choose a basis $e_1, \dots, e_n$ of V that diagonalizes the action of G and thus we get n characters $\rho_1, \dots, \rho_n$ defined by setting $\rho_i(g)$ to be the ith diagonal element of g. It is clear that e_i is a basis for the representation ρ_i.

In what follows it is convenient to suppress tensor product signs as much as possible, so we write $\rho_{i,j}\rho$ for $\rho_i \otimes \rho_j \otimes \rho$. In this notation $\mathit{det}_V = \rho_{1,\dots,n}$. Denote the set of irreducible representations by $\mathit{Irr}(G)$.

Corollary 4.1 ([15]). *Let G be a finite abelian subgroup of $\mathsf{GL}_n(\mathbb{C})$. Then the McKay quiver is the directed graph with a vertex for each irreducible representation ρ and an arrow*

$$\rho_i\rho \xrightarrow{x_i^\rho} \rho$$

for all $1 \le i \le n$ and $\rho \in \mathit{Irr}(G)$. Furthermore, the path algebra of the McKay quiver modulo the relations

$$\{x_j^\rho x_i^{\rho_j\rho} = x_i^\rho x_j^{\rho_i\rho} : \rho \in \mathit{Irr}(G), 1 \le i, j \le n\}$$

is isomorphic to the skew group ring $\mathbb{C}[V]\#G$.

Proof. The first statement regarding the McKay quiver is trivial since $V = \rho_1 \oplus \dots \oplus \rho_n$. Furthermore since G is abelian the idempotent e in Section 3 is the identity and so we really are describing the skew group ring up to isomorphism, not just Morita equivalence.

For the relations, we build a potential as follows: first recall we have a basis $e_1, \dots, e_n$ of V (from which e_i is a basis for each ρ_i). Since the ρ_i generate the group of characters this gives a basis for every representation. Now if we view the map x_i^ρ as an intertwiner $\rho_i \otimes \rho \to V \otimes \rho$ it is clear that it can be represented as $e_i \otimes v_\rho \mapsto e_i \otimes v_\rho$ where v_ρ is the basis element of ρ.

This means that if a path $p : \mathit{det}_V \otimes \rho \to \rho$ of length n contains two x's with the same subscript then $c_p = 0$. Consequently, for any given $\rho \in \mathit{Irr}(G)$, the only nonzero contributions to the potential from paths $\mathit{det}_V \otimes \rho \to \rho$ of length n come from

$$\mathit{det}_V \otimes \rho \xrightarrow{x_{\sigma(1)}^{\rho_{\sigma(2),\dots,\sigma(n)}\rho}} \rho_{\sigma(2),\dots,\sigma(n)}\rho \xrightarrow{x_{\sigma(2)}^{\rho_{\sigma(3),\dots,\sigma(n)}\rho}} \rho_{\sigma(3),\dots,\sigma(n)}\rho \longrightarrow \cdots \longrightarrow \rho_{\sigma(n)}\rho \xrightarrow{x_{\sigma(n)}^{\rho}} \rho$$

where $\sigma \in \mathfrak{S}_n$. Thus for each $\rho \in \mathit{Irr}(G)$ we obtain a contribution to the potential

$$\Phi_\rho := \sum_{\sigma \in \mathfrak{S}_n} \omega(e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(n)}) x_{\sigma(n)}^\rho x_{\sigma(n-1)}^{\rho_{\sigma(n)}\rho} \dots x_{\sigma(2)}^{\rho_{\sigma(3),\dots,\sigma(n)}\rho} x_{\sigma(1)}^{\rho_{\sigma(2),\dots,\sigma(n)}\rho}.$$

Adding these contributions one obtains the potential $\Phi = \sum_{\rho \in \mathit{Irr}(G)} \Phi_\rho$. It is easy to see that differentiating Φ with respect to paths of length $n-2$ gives the required relations. □

4 Here, "small" means that the group has no pseudoreflections, i.e., no elements whose eigenspace of eigenvalue one has codimension one.

As another corollary to Theorem 3.2 we have

Corollary 4.2 (*Reiten–Van den Bergh [16]*). *Suppose G is a finite small subgroup of* $\mathrm{GL}_2(\mathbb{C})$. *Then the relations on the McKay quiver which give an algebra Morita equivalent to* $\mathbb{C}[x,y]\#G$ *are precisely the mesh relations from AR theory on* $\mathbb{C}[[x,y]]^G$ *and the superpotential is exactly the sum of all mesh relations with the appropriate sign.*

More precisely, by AR theory for every arrow $a : i \to j$ *there is an arrow* $\tau(j) \to i$ *which we denote by* a^π. *The twist on the vertices equals the Auslander–Reiten translation, and the mesh relation associated to a vertex j is* $\sum_{h(a)=j} aa^\pi$.

In particular for a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, *the preprojective algebra of the corresponding extended Dynkin diagram is Morita equivalent to* $\mathbb{C}[x,y]\#G$.

Proof. We will work out the proof in the completed case and then go back by taking the associated graded ring. Denote by $R = \mathbb{C}[[x,y]]$ the ring of formal power series in two variables and consider the Koszul complex over R

$$0 \longrightarrow R\otimes det_V \longrightarrow R\otimes V \longrightarrow R \longrightarrow \mathbb{C} \longrightarrow 0\ .$$

We know that this comes from a superpotential (if we consider, for all m, $(V^*)^{\otimes m} \subset \mathbb{C}\langle\langle x,y\rangle\rangle =$ the ring of noncommutative power series in x, y, and define derivation quotients $\mathcal{D}(\omega,k)$ for $\omega \in (V^*)^{\otimes m}$ of $\mathbb{C}\langle\langle x,y\rangle\rangle$ just as in the uncompleted case). We proved that the algebra obtained by smashing with a group G also comes from a (possibly twisted) superpotential, so

$$0 \longrightarrow R\otimes det_V \otimes \mathbb{C}G \longrightarrow R\otimes V\otimes \mathbb{C}G \longrightarrow R\otimes \mathbb{C}G \longrightarrow \mathbb{C}G \longrightarrow 0$$

(which is the minimal projective resolution of the $R\#G$ module $\mathbb{C}G$) arises from a superpotential, i.e. the relations on $R\#G$ can be read off from the fact that the composition of the first two nontrivial maps is zero.

For convenience label the elements of $Irr(G)$ by $\sigma_0, \sigma_1, \dots, \sigma_n$ where σ_0 corresponds to the trivial representation. Since $\mathbb{C}G = \oplus_{i=0}^n \sigma_i^{\oplus dim(\sigma_i)}$ the above exact sequence decomposes into

$$\oplus_{i=0}^n(\ 0 \longrightarrow R\otimes det_V\otimes\sigma_i \longrightarrow R\otimes V\otimes \sigma_i \longrightarrow R\otimes\sigma_i \longrightarrow \sigma_i \longrightarrow 0\ )^{\oplus dim(\sigma_i)}$$

so the relations on $R\#G$ can be read off from the fact that the composition of the first two nontrivial maps in each summand is zero. But now, by [17, 10.9],

$$\begin{aligned} \mathfrak{proj} R\#G &\approx \mathfrak{CM}R^G \\ M &\mapsto M^G \end{aligned}$$

is an equivalence of categories, where $\mathfrak{proj}R\#G$ is the category of finitely generated projective $R\#G$ modules, and $\mathfrak{CM}R^G$ is the category of maximal Cohen–Macaulay modules for R^G. Thus taking G-invariants of the above exact sequence, the relations on $R\#G$ can be read off from the fact that the composition of the first two nontrivial maps in each summand of

$$\oplus_{i=0}^n(\ 0 \to (R\otimes det_V\otimes\sigma_i)^G \to (R\otimes V\otimes\sigma_i)^G \to (R\otimes\sigma_i)^G \to \sigma_i^G \to 0\ )^{\oplus dim(\sigma_i)}$$

is zero. It is clear that $\sigma_i^G = 0$ for $i \neq 0$ whilst $\sigma_0^G = \mathbb{C}$. By [17, 10.13], for $i \neq 0$ the summands above are precisely the AR short exact sequences, and for $i = 0$ the sequence has the appropriate AR property. Thus the relations on $eR\#Ge$ are precisely the mesh relations.

Because the mesh relations are graded and taking the associated graded is compatible with the Morita equivalence we can conclude that the relations of $e\mathbb{C}[x,y]\#Ge$ are also given by the mesh relations and the superpotential will be the sum of all mesh relations. □

Because we work with superpotentials there is a redundancy in the coefficients of the potential:

Lemma 4.3. *Choose a basis of arrows in* $\mathbb{C}Q_1 = e(T_{\mathbb{C}}V\#G)e$ *that is closed under the application of the twist* τ. *Then the coefficients of* $e(\omega\otimes 1)e = \sum_{|p|=n} c_p p$ *have the following property: if* $p = p_1 \dots p_n$ *is a path of length n then*

$$c_{p_1\dots p_n} = (-1)^{n-1} c_{p_n^\tau p_1 p_2 \dots p_{n-1}}.$$

Proof. This follows immediately from Theorem 3.2 and the discussion in Section 2.3. □

Note that if $G \le \mathrm{SL}_n(\mathbb{C})$, the twist is trivial so we can work with any basis for the arrows. In this case, not only does the above lemma simplify the calculation of the c_p, but it also tells us that we can write our superpotential up to cyclic permutation (if one defines the notation carefully). This generalizes a result of Ginzburg [4] for $\mathrm{SL}_3(\mathbb{C})$.

The superpotential highly depends on the representatives we chose for the arrows in Q. From the point of view of the quiver we have an action of the graded automorphism group,

$$Aut_{\mathbb{C}Q_0}\mathbb{C}Q = \prod_{e_i,e_j\in Q_0} \mathrm{GL}(e_i(\mathbb{C}Q)_1 e_j),$$

on the space $(\mathbb{C}Q)_n$, and all potentials that give an isomorphic derivation-quotient algebra are in the same orbit. An interesting problem is to find (and define the notion of) nice representatives for the arrows, and hence for the superpotential.

5. Examples of McKay correspondence superpotentials

In this section we illustrate Theorem 3.2 by computing examples. We first illustrate that our theorem does not depend on whether or not G has pseudoreflections by computing an example of a non-abelian group $G \leq \mathsf{GL}_2(\mathbb{C})$ where $\mathbb{C}^2/G := \mathsf{Spec}\,\mathbb{C}[x, y]^G$ is smooth:

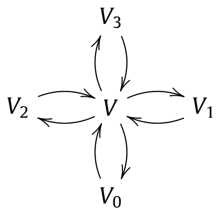
Example 5.1. Consider the dihedral group $D_8 = \langle g, h : g^4 = h^2 = 1, h^{-1}gh = g^{-1}\rangle$ viewed inside $\mathsf{GL}_2(\mathbb{C})$ as

$$g = \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^{-1} \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

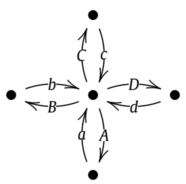
It is clear that the invariant ring is $\mathbb{C}[xy, x^4 + y^4]$ and so is smooth. Denoting the natural representation by V, the character table for this group is

	1	g^2	g	h	gh
V_0	1	1	1	1	1
V_1	1	1	1	−1	−1
V_2	1	1	−1	1	−1
V_3	1	1	−1	−1	1
V	2	−2	0	0	0

and so the McKay quiver has the shape



We shall show that the algebra Morita equivalent to the skew group ring is:



$$\begin{array}{cc} Da = 0 & Cb = 0 \\ Ad = 0 & Bc = 0 \\ \multicolumn{2}{c}{aA + dD = bB + cC} \end{array}$$

Note that $\tau(a) = d$, $\tau(d) = a$, $\tau(A) = D$, $\tau(D) = A$ and likewise with the c's and D's. Notice also that there are five relations, which coincides with the number of paths of length zero (i.e. the number of vertices). We now check that the relations above are correct.

For each $i \in \{0, 1, 2, 3\}$, let v_i denote a basis vector for V_i. Let $\{e_1, e_2\}$ be the (standard) basis for V in which the above matrices are written, and let $\{e_1^*, e_2^*\}$ be the dual basis of V^*. Assume that $\omega = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$. For our idempotent e, we have a choice of the component in V, since V is two-dimensional. Let us pick the component in V to be the one corresponding to a multiple of e_1 (rather than of e_2). Take the following τ-closed basis of arrows (i.e., intertwiners) ψ_{x^*} for $\mathbb{C}Q_1$, as x ranges over the edges of Q_1. The right side of the line indicates the edge that the (summand of the) map corresponds to:

$$\begin{array}{rlr}
V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 & \mapsto \mathbb{C}(v_0 \otimes e_1) \oplus \mathbb{C}(v_0 \otimes e_2) = V_0 \otimes V & (A) \\
V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 & \mapsto \mathbb{C}(v_1 \otimes e_1) \oplus \mathbb{C}(-v_1 \otimes e_2) = V_1 \otimes V & (D) \\
V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 & \mapsto \mathbb{C}(v_2 \otimes e_2) \oplus \mathbb{C}(v_2 \otimes e_1) = V_2 \otimes V & (B) \\
V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 & \mapsto \mathbb{C}(v_3 \otimes e_2) \oplus \mathbb{C}(-v_3 \otimes e_1) = V_3 \otimes V & (C) \\
V_0 \oplus V_1 \oplus V_2 \oplus V_3 = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_3 & \mapsto \mathbb{C}\,(e_1 \otimes e_2 + e_2 \otimes e_1) & (a) \\
& \oplus\mathbb{C}\,(e_1 \otimes e_2 - e_2 \otimes e_1) & (d) \\
& \oplus\mathbb{C}\,(e_1 \otimes e_1 + e_2 \otimes e_2) & (b) \\
& \oplus\mathbb{C}\,(e_1 \otimes e_1 - e_2 \otimes e_2) & (c) \\
& = V \otimes V. &
\end{array}$$

Since the determinant representation is V_1, if we consider paths of length 2 ending at a given vertex ρ, the only possible ones with nonzero c_p must start at $\rho \otimes V_1$.

Now, the coefficients c_p are easy to compute using (3.3). For example, with $p = Ad$, we have

$$V_1 \xrightarrow{d} V \otimes V \xrightarrow{A\otimes 1} V_0 \otimes V \otimes V \xrightarrow{\mathrm{id}_{V_0} \otimes \alpha_2} V_0 \otimes \bigwedge^2 V \xrightarrow{\psi_\omega \otimes \mathrm{id}_{\bigwedge^2 V}} V_1 \otimes \bigwedge^2 V^* \otimes \bigwedge^2 V \longrightarrow V_1$$

takes

$$v_1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1 \mapsto v_0 \otimes e_1 \otimes e_2 - v_0 \otimes e_2 \otimes e_1 \mapsto 2v_0 \otimes \omega^* \mapsto 2v_1 \otimes \omega \otimes \omega^* \mapsto 2v_1,$$

and so $c_{Ad} = 2$. For another example, take $p = aA$. Then, we have

$$e_1 \mapsto v_0 \otimes e_1 \mapsto e_1 \otimes (e_2 \otimes e_1) + e_2 \otimes (e_1 \otimes e_1) \mapsto e_1 \otimes -\omega^* \mapsto e_1 \otimes \omega \otimes -\omega^* \mapsto -e_1, \tag{5.1}$$

so $c_{aA} = 1$. Continuing in this fashion our potential (after dividing through by 2) is

$$(Da - aA + Ad - dD) + (-Cb + bB - Bc + cC),$$

which in compact form may be written as $[Da]_\tau - [Cb]_\tau$, where $[x]_\tau$ is the sum of all signed τ-cyclic permutations of x that give rise to distinct monomials (cf. Lemma 4.3). Since $n-2 = 0$ we do not differentiate, and these are precisely the relations.

Note that we needed here the fact that, in the superpotential, the coefficient of each path p is $c_p \dim t(p)$: this was necessary for the coefficient of Ad to be negative the coefficient of aA by our computation above. Without this, the superpotential would not be τ-cyclically symmetric.

Remark 5.2. Taking a different basis of arrows for $\mathbb{C}Q_1$ (which is not closed under τ) may lead to a potential which is not invariant under twisted cyclic permutation.

Remark 5.3. In the above example if we change h slightly and so our group is now the binary dihedral group $\mathbb{D}_{3,2}$ generated by

$$a = \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^{-1} \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

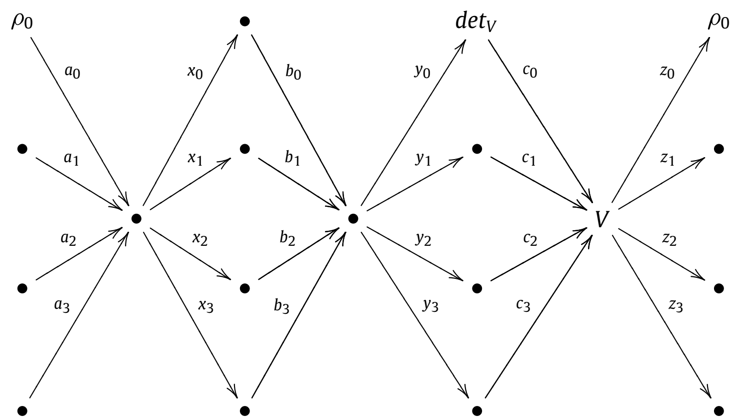
inside $\mathsf{SL}_2(\mathbb{C})$, then although the character table and so shape of the McKay quiver is the same, the relations differ. Indeed, by Corollary 4.2 the relations are now the preprojective relations. This can also be verified directly by choosing a basis of arrows for $\mathbb{C}Q_1$ closed under τ.

We now illustrate Corollary 4.2 with an example of a finite small subgroup of $\mathsf{GL}_2(\mathbb{C})$:

Example 5.4. Take $G = \mathbb{D}_{5,2}$, i.e. the group inside $\mathsf{GL}_2(\mathbb{C})$ generated by

$$G = \left\langle \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_4 \\ \varepsilon_4 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_6 & 0 \\ 0 & \varepsilon_6 \end{pmatrix} \right\rangle.$$

The McKay quiver is



where the trivial, determinant and natural representations are illustrated, and the ends of the two sides are identified. Note that the permutation τ induced by tensoring with the determinant representation rotates this picture to the left, and so the fact that the permutation coincides with the AR translate is implicit. The mesh relations are

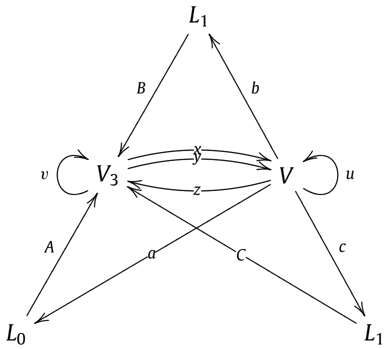
$$\begin{array}{lll} x_0a_0 = 0 & y_0b_0 = 0 & z_0c_0 = 0 \\ x_1a_1 = 0 & y_1b_1 = 0 & z_1c_1 = 0 \\ x_2a_2 = 0 & y_2b_2 = 0 & z_2c_2 = 0 \\ x_3a_3 = 0 & y_3b_3 = 0 & z_3c_3 = 0 \end{array} \qquad \begin{array}{l} b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0 \\ c_0y_0 + c_1y_1 + c_2y_2 + c_3y_3 = 0 \\ a_0z_0 + a_1z_1 + a_2z_2 + a_3z_3 = 0 \end{array}$$

and so we have 15 relations, matching the number of paths of length 0 (i.e. the number of vertices).

Example 5.5. Take $G = \frac{1}{7}(1, 2, 4) \ltimes \langle\tau\rangle$, i.e. the group inside $\mathtt{SL}_3(\mathbb{C})$ generated by

$$G = \left\langle \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$$

where $\varepsilon^7 = 1$. The McKay quiver is



Denote the basis of L_i by l_i for $1 \le i \le 3$, the basis V by e_1, e_2, e_3, and the basis of V_3 by j_1, j_2, j_3. Take the basis of arrows (=intertwiners) for $\mathbb{C}Q_1$ corresponding to the following images of the above bases (with the edge as listed in the right side of the row), where γ is a cube root of unity:

$$\begin{array}{rll} V \mapsto L_0 \otimes V & = \mathbb{C}(l_0 \otimes e_1) \oplus \mathbb{C}(l_0 \otimes e_2) \oplus \mathbb{C}(l_0 \otimes e_3) & (a) \\ V \mapsto L_1 \otimes V & = \mathbb{C}(l_1 \otimes \gamma e_1) \oplus \mathbb{C}(l_1 \otimes \gamma^2 e_2) \oplus \mathbb{C}(l_1 \otimes e_3) & (b) \\ V \mapsto L_2 \otimes V & = \mathbb{C}(l_2 \otimes \gamma^2 e_1) \oplus \mathbb{C}(l_2 \otimes \gamma e_2) \oplus \mathbb{C}(l_2 \otimes e_3) & (c) \\ (V \oplus V_3^{\oplus 2}) \mapsto V \otimes V & = \mathbb{C}(e_3 \otimes e_3) \oplus \mathbb{C}(e_1 \otimes e_1) \oplus \mathbb{C}(e_2 \otimes e_2) & (u) \\ & \oplus \mathbb{C}(e_1 \otimes e_2) \oplus \mathbb{C}(e_2 \otimes e_3) \oplus \mathbb{C}(e_3 \otimes e_1) & (x) \\ & \oplus \mathbb{C}(e_2 \otimes e_1) \oplus \mathbb{C}(e_3 \otimes e_2) \oplus \mathbb{C}(e_1 \otimes e_3) & (y) \\ (L_0 \oplus L_1 \oplus L_2 \oplus V \oplus V_3) \mapsto V_3 \otimes V & = \mathbb{C}\,(j_1 \otimes e_3 + j_2 \otimes e_1 + j_3 \otimes e_2) & (A) \\ & \oplus \mathbb{C}\left(j_1 \otimes \gamma^2 e_3 + j_2 \otimes \gamma e_1 + j_3 \otimes e_2\right) & (B) \\ & \oplus \mathbb{C}\left(j_1 \otimes \gamma e_3 + j_2 \otimes \gamma^2 e_1 + j_3 \otimes e_2\right) & (C) \\ & \oplus \mathbb{C}(j_2 \otimes e_2) \oplus \mathbb{C}(j_3 \otimes e_3) \oplus \mathbb{C}(j_1 \otimes e_1) & (z) \\ & \oplus \mathbb{C}(j_2 \otimes e_3) \oplus \mathbb{C}(j_3 \otimes e_1) \oplus \mathbb{C}(j_1 \otimes e_2) & (v) \end{array}$$

A calculation shows that the superpotential can be written as

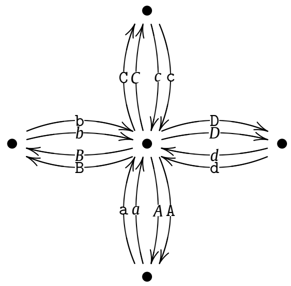
$$\circlearrowleft \quad a(x - y)A + b(x - \gamma y)B + c(x - \gamma^2 y)C - zux + vzy + \frac{1}{3}uuu - \frac{1}{3}vvv.$$

Differentiating with respect to the paths of length $3 - 2 = 1$ gives the relations

$$\begin{array}{lc} \partial_A & ax = ay \\ \partial_B & bx = \gamma by \\ \partial_C & cx = \gamma^2 cy \\ \partial_a & xA = yA \\ \partial_b & xB = \gamma yB \\ \partial_c & xC = \gamma^2 yC \\ \partial_x & Aa + Bb + Cc = zu \\ \partial_y & Aa + \gamma Bb + \gamma^2 Cb = vz \\ \partial_u & xz = u^2 \\ \partial_v & zy = v^2. \end{array}$$

Example 5.6. As in Example 5.1 consider the group D_8, but now acting on $(V \oplus V^*) \cong (V \oplus V)$. Since D_8 is generated inside V by pseudoreflections it follows that inside $V \oplus V$ it is generated by symplectic reflections, thus in this case $\mathbb{C}[V \oplus V]\#G$

is the undeformed symplectic reflection algebra. The McKay quiver is now



The superpotential is then given by the sum of the following terms and all distinct terms obtained by signed cyclic permutations:

$$\begin{array}{llllllll}
(\mathtt{A}a\mathtt{A}a) & -(A\mathtt{a}Aa) & -2(\mathtt{Ad}Da) & (\mathtt{A}d\mathtt{D}a) & (A\mathtt{dD}a) & (\mathtt{A}b\mathtt{B}a) & -(A\mathtt{bB}a) & -(\mathtt{A}c\mathtt{C}a) \\
(A\mathtt{cC}a) & (A\mathtt{a}A\mathtt{a}) & (\mathtt{A}dD\mathtt{a}) & (A\mathtt{d}D\mathtt{a}) & -2(Ad\mathtt{Da}) & -(\mathtt{A}bB\mathtt{a}) & (A\mathtt{b}B\mathtt{a}) & (\mathtt{A}cC\mathtt{a}) \\
-(A\mathtt{c}C\mathtt{a}) & (\mathtt{D}d\mathtt{D}d) & -(D\mathtt{dD}d) & -(\mathtt{D}b\mathtt{B}d) & (D\mathtt{bB}d) & (\mathtt{D}c\mathtt{C}d) & -(D\mathtt{cC}d) & (D\mathtt{d}Dd) \\
(\mathtt{D}bB\mathtt{d}) & -(D\mathtt{b}B\mathtt{d}) & -(\mathtt{D}cC\mathtt{d}) & (D\mathtt{c}C\mathtt{d}) & (\mathtt{B}b\mathtt{B}b) & -(B\mathtt{bB}b) & -2(\mathtt{Bc}Cb) & (\mathtt{B}c\mathtt{C}b) \\
(B\mathtt{cC}b) & (B\mathtt{b}B\mathtt{b}) & (\mathtt{B}cC\mathtt{b}) & (B\mathtt{c}C\mathtt{b}) & -2(Bc\mathtt{Cb}) & (\mathtt{C}c\mathtt{C}c) & -(C\mathtt{cC}c) & (C\mathtt{c}C\mathtt{c}).
\end{array}$$

(Recall, since we are inside $\mathsf{SL}_4(\mathbb{C})$, a negative sign is introduced with cyclic permutation.) Differentiating appropriately gives the relations

$$\begin{array}{cccc}
Da = 0 & Ad = 0 & Cb = 0 & Bc = 0 \\
\mathtt{Da} = 0 & \mathtt{Ad} = 0 & \mathtt{Cb} = 0 & \mathtt{Bc} = 0 \\
\mathtt{D}a = -D\mathtt{a} & \mathtt{A}a = A\mathtt{a} & \mathtt{C}a = C\mathtt{a} & \mathtt{B}a = B\mathtt{a} \\
\mathtt{D}b = D\mathtt{b} & \mathtt{A}b = A\mathtt{b} & \mathtt{C}b = -C\mathtt{b} & \mathtt{B}b = B\mathtt{b} \\
\mathtt{D}c = D\mathtt{c} & \mathtt{A}c = A\mathtt{c} & \mathtt{C}c = C\mathtt{c} & \mathtt{B}c = -B\mathtt{c} \\
\mathtt{D}d = D\mathtt{d} & \mathtt{A}d = -A\mathtt{d} & \mathtt{C}d = C\mathtt{d} & \mathtt{B}d = B\mathtt{d}
\end{array}$$

$$aA + bB = cC + dD$$

$$\mathtt{aA} + \mathtt{bB} = \mathtt{cC} + \mathtt{dD}$$

$$\mathtt{a}A + b\mathtt{B} = a\mathtt{A} + \mathtt{b}B = \mathtt{c}C + d\mathtt{D} = c\mathtt{C} + \mathtt{d}D = \Sigma$$

where $\Sigma = \frac{1}{2}(a\mathtt{A} + b\mathtt{B} + c\mathtt{C} + d\mathtt{D}) = \frac{1}{2}(\mathtt{a}A + \mathtt{b}B + \mathtt{c}C + \mathtt{d}D)$.

The calculations involving this example were done using a computer program written in GAP [18]. The source code of this program can be downloaded at http://www.algebra.ua.ac.be/research/mckay.gap.

6. Koszul algebras

Thus far, we have explained that, for $G < \mathsf{GL}(V)$, $\mathbb{C}[V]\#G$, and hence the quiver algebras Morita equivalent to it, are twisted Calabi–Yau and derived from a twisted superpotential (in the case $G < \mathsf{SL}(V)$, we may remove the word "twisted"). Here we explain that this is part of a more general phenomenon: *any* $(N\text{-})$*Koszul, (twisted) Calabi–Yau algebra is of the form* $\mathcal{D}(\omega, k)$. This was proved in [2] for algebras over a field, so our result generalizes this to the quiver case. We also prove a converse: any algebra of the form $A = \mathcal{D}(\omega, k)$ is $(N\text{-})$Koszul and (twisted) Calabi–Yau if and only if a natural complex attached to ω is a bimodule resolution of A.

Recall that a graded algebra A is Koszul if A has a bimodule resolution with all maps of degree 1. This is clearly invariant under a Morita equivalence $A \sim eAe$, using the functor described in Section 2. Then, McKay correspondence algebras are Koszul, by the following well-known lemma:

Lemma 6.1. *If $G \subset \mathsf{SL}(V) \cong \mathsf{SL}_n$ is finite, then $\mathbb{C}[V]\#G$ is CY-n and Koszul.*

Proof. The standard Koszul bimodule resolution $\mathcal{K}^\bullet$ for $\mathbb{C}[V]$ is self-dual, so $\mathbb{C}[V]$ is CY-n. The kth term of this resolution is $\mathcal{K}^k = \mathbb{C}[V] \otimes \bigwedge^k V^* \otimes \mathbb{C}[V]$ and it is isomorphic to the $(\mathcal{K}^{n-k})^*$ dual term because of the pairing

$$\bigwedge^k V^* \times \bigwedge^{n-k} V^* \to \mathbb{C} : (v_1, v_2) \to a \iff \phi_1 \wedge \phi_2 = a x_1 \wedge \ldots \wedge x_n.$$

Since $G \subset \mathsf{SL}(V)$, this pairing is a pairing of left $\mathbb{C}G$-modules.

All of the terms M above are G-modules compatibly with the $\mathbb{C}[V]$-bimodule structure (precisely, they are $(\mathbb{C}[V] \otimes \mathbb{C}[V])\#G$-modules). Now we apply the exact functor $M \mapsto M \otimes \mathbb{C}G$. This turns the above resolution into a projective $\mathbb{C}[V]\#G$-bimodule resolution of $\mathbb{C}[V] \otimes \mathbb{C}G \cong \mathbb{C}[V]\#G$, where for each term M in the resolution, the bimodule action is now given by $(xg)(m \otimes h)(x'g') = (x \cdot g(m) \cdot gh(x') \otimes ghg')$. The Koszul property follows from the fact that tensoring by $\mathbb{C}G$ preserves the grading, and the self-duality property carries over. □

In order to formulate the theorem, we need to introduce a natural complex $\mathcal{W}^\bullet$ attached to any superpotential ω, for $A = \mathcal{D}(\omega, k)$ (which may be viewed as a bimodule version of [2, (5.3)]). For simplicity, we will assume for now that $|\omega| = k + 2$, so that A is quadratic.

Recall the spaces W_i defined above Definition 2.1. Consider the complex

$$\mathcal{W}^\bullet := 0 \to A \otimes_S W_{|\omega|} \otimes_S A \xrightarrow{d_{|\omega|}} A \otimes_S W_{|\omega|-1} \otimes_S A \to \cdots \to A \otimes_S W_1 \otimes_S A \xrightarrow{d_1} A \otimes_S W_0 \otimes_S A \to 0, \tag{6.1}$$

where, for $v_1, \ldots, v_i \in W$ and $a, a' \in A$,

$$d_i = \varepsilon_i(\mathrm{split}_L + (-1)^i \mathrm{split}_R)|_{A \otimes_S W_i \otimes_S A},$$
$$\mathrm{split}_L(a \otimes_S v_1 v_2 \cdots v_i \otimes_S a') = a v_1 \otimes_S v_2 \cdots v_i \otimes_S a',$$
$$\mathrm{split}_R(a \otimes_S v_1 v_2 \cdots v_i \otimes_S a') = a \otimes_S v_1 \cdots v_{i-1} \otimes_S v_i a',$$
$$\varepsilon_i := \begin{cases} (-1)^{i(|\omega|-i)}, & \text{if } i < (|\omega|+1)/2, \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to check that the above yields a complex, i.e., $d_i \circ d_{i+1} = 0$. Moreover, the terms, aside from A itself, are projective bimodules, and the maps are A-bimodule maps. We will see that it is exact if and only if A is Koszul and Calabi–Yau. More precisely, we will prove:

Theorem 6.2. *An algebra $T_S W/\langle R \rangle$ is Koszul and Calabi–Yau if and only if it is of the form $\mathcal{D}(\omega, k)$ and the corresponding complex (6.1) is exact in positive degree and $H^0(\mathcal{W}^\bullet) = A$. In this case, (6.1) is the Koszul resolution of A, and it is self-dual.*

Remark 6.3. The condition that $\mathcal{W}^\bullet$ is a resolution of A is very subtle and hard to check for a given potential. In the one-vertex case it is shown in [1,2] that this implies some special regularity conditions on ω.

We begin with the

Lemma 6.4. *For any superpotential ω, the complex (6.1) is self-dual.*

Proof. First, note that ω induces perfect pairings

$$\langle\, , \rangle : W^*_{|\omega|-i} \otimes_S W^*_i \to \mathbb{C}, \qquad \langle \alpha, \beta \rangle := [(\alpha \otimes_S \beta)\omega],$$

satisfying the supersymmetry property,

$$\langle \alpha, \beta \rangle = (-1)^{|\alpha||\beta|} \langle \beta, \alpha \rangle.$$

This yields an isomorphism $\eta : W^*_{|\omega|-i} \xrightarrow{\sim} W_i$, and hence a duality pairing of bimodules

$$\langle\, , \rangle : (A \otimes_S W_i \otimes_S A) \otimes_S (A \otimes_S W_{|\omega|-i} \otimes_S A) \to A \otimes_S A, \qquad \langle a \otimes_S x \otimes_S a', b \otimes_S y \otimes_S b' \rangle := a'b \otimes_S [\eta^{-1}(x)y] \otimes_S b'a.$$

This explains why the terms in the above complex are in duality.

It remains to check that the differentials satisfy the self-duality property: $d_i = d^*_{|\omega|+1-i}$. It suffices to show that $\mathrm{split}_L|_{A \otimes_S W_i \otimes_S A}$ is identified with $\varepsilon_i \varepsilon_{|\omega|-i} \mathrm{split}^*_R|_{A \otimes_S W^*_{|\omega|-i} \otimes_S A}$ under the above duality. That is, for all $x \in W_i$, $y \in W_{|\omega|+1-i}$, we need to check that

$$\langle 1 \otimes_S y \otimes_S 1, \mathrm{split}_L(1 \otimes_S x \otimes_S 1) \rangle = \varepsilon_i \varepsilon_{|\omega|-i} \langle \mathrm{split}_R(1 \otimes_S y \otimes_S 1), 1 \otimes_S x \otimes_S 1 \rangle.$$

This amounts to checking that, for all $\alpha \in W_1^* = W^*$, we have

$$[\eta^{-1}([\alpha x])y] = \varepsilon_i \varepsilon_{|\omega|-i} [x \eta^{-1}[y\alpha]].$$

By associativity identities and the definition of η, the LHS of is $[(\eta^{-1}(x) \otimes_S \xi)y]$. The RHS may be rewritten as

$$\varepsilon_i \varepsilon_{|\omega|-i} (-1)^{|x|(|y|-1)} [\eta^{-1}(x)[y\xi]] = \varepsilon_i \varepsilon_{|\omega|-i} (-1)^{i(|\omega|-i)} [(\eta^{-1}(x) \otimes_S \xi)y].$$

Thus, setting ε_i as specified above yields the desired self-duality. □

Lemma 6.5. *The complex (6.1) is a subcomplex of the Koszul complex for $\mathcal{D}(\omega, |\omega| - 2)$.*

Proof. The Koszul complex can be defined as follows. If $A = T_S W/\langle R\rangle$ where R is an S-subbimodule of $W \otimes_S W$, then we denote by $R^\perp$ the submodule of $W^* \otimes_S W^*$ that annihilates R. The Koszul dual of A is $A^! := T_S W^*/\langle R^\perp\rangle$ and it is again a graded algebra. For each k we have a projection $(W^*)^{\otimes_S k} \to A^!_k$, and, dually, this gives us injections $(A^!_k)^* \hookrightarrow W^{\otimes_S k}$. The Koszul complex $\mathcal{K}^\bullet$ is defined by the maps $d : A \otimes_S (A^!_k)^* \otimes_S A \to A \otimes_S (A^!_{k-1})^* \otimes_S A$ which are constructed analogously to the maps in (6.1). To prove the lemma we only have to show that $W_k \subset (A^!_k)^*$.

What does $(A^!_k)^*$ look like? Because $A^!_k = (W^*)^{\otimes_S k}/(\sum_l (W^*)^{\otimes_S l} \otimes_S R^\perp \otimes_S (W^*)^{\otimes_S (k-l-2)})$ one has that $w \in (A^!_k)^*$ if and only if $\langle w, \phi\rangle = 0$ for all $\phi \in (\sum_l (W^*)^{\otimes_S l} \otimes_S R^\perp \otimes_S (W^*)^{\otimes_S (k-l-2)})$. This is the same as to say that

$$w \in \bigcap_l W^{\otimes_S l} \otimes_S R \otimes_S W^{\otimes_S (k-l-2)} = \bigcap_l W^{\otimes_S l} \otimes_S W_2 \otimes_S W^{\otimes_S (k-l-2)}.$$

We conclude immediately that $W_k \subset (A^!_k)^*$. $\square$

Remark 6.6. According to [19, Theorem 5.4], the pairing between the W_k can be extended to a pairing between the $(A^!_k)^*$, but this pairing is not necessarily perfect. Only in the AS-Gorenstein (Koszul twisted Calabi–Yau) case this pairing becomes perfect because then $(A^!_k)^*$ and W_k coincide.

Proof of Theorem 6.2. If (6.1) is a resolution of A then since all of the differentials have degree $+1$ with respect to the grading of A, this would imply (by one definition of Koszulity) that A is Koszul, and that (6.1) is a Koszul resolution of A (more generally, for any graded algebra, any free bimodule resolution of A with differentials of positive degree must be minimal and unique). Then, by Lemma 6.4, A is Calabi–Yau as well.

Conversely, suppose that A is CY of dimension n and Koszul. Using the CY property, [8, Theorem A.5.2] shows that there is a trace function $\mathrm{Tr} : \mathrm{Ext}^n_A(S, S) \to \mathbb{C}$ such that

$$\mathrm{Tr}(\alpha * \beta) = (-1)^{k(n-k)}\mathrm{Tr}(\beta * \alpha), \quad \alpha \in \mathrm{Ext}^k(S, S), \beta \in \mathrm{Ext}^{n-k}(S, S) \tag{6.2}$$

induces a perfect pairing, where $*$ denotes the Yoneda cup product. Using the Koszul property, we may identify $\mathrm{Ext}^n(S, S)$ with a quotient of $(W^*)^{\otimes_S n}$, so that a trace function becomes canonically an element $\omega \in W^{\otimes_S n}$. Then, (6.2) says precisely that ω is a superpotential.

By nondegeneracy, the trace pairing induces an isomorphism $\mathrm{Ext}^2(S, S) \cong \mathrm{Ext}^{n-2}(S, S)^*$. Furthermore, $\mathrm{Ext}^2(S, S) \cong R$, so this isomorphism translates into the statement that $W_2 \cong R$, and hence $W_2 = R$ by Lemma 6.5. Thus, $A \cong \mathcal{D}(\omega, n-2)$.

Similarly, nondegeneracy of the trace pairing provides an isomorphism between $\mathrm{Ext}^i(S, S)$ and a subspace of W_i for all i, and hence (applying Lemma 6.5 again) (6.1) must be the Koszul resolution of A. $\square$

Remark 6.7. This theorem is a generalization of Theorem 3.2 (at least in the nontwisted case): to obtain Theorem 3.2, we combine Theorem 6.2 and Lemma 6.1. See below for the twisted case.

Finally, we explain briefly how to generalize to N-Koszul and twisted Calabi–Yau algebras. First, for the σ-twisted setting, all that changes is that the twisted superpotential property proves a σ-twisted self-duality in Lemma 6.4, and conversely in the proof of Theorem 6.2. For the converse, one needs to use the modified Serre duality $f * g = (-1)^{k(n-k)}F'(g) * f$, where $F' : \mathrm{Ext}^1(S, S) \xrightarrow{\sim} \mathrm{Ext}^1({}_{\sigma^{-1}}S, {}_{\sigma^{-1}}S)$ is the map coming from the functor on A-modules, $M \mapsto {}_{\sigma^{-1}}M$, which is the composition of the Serre functor with the shift by $-n$. In this case, if we compose with the canonical isomorphisms $V^* \cong \mathrm{Ext}^1(S, S)$ and $V^* \cong \mathrm{Ext}^1({}_{\sigma^{-1}}S, {}_{\sigma^{-1}}S)$, then we obtain the map $\sigma : V^* \xrightarrow{\sim} V^*$. As a result, the superpotential ω becomes a σ-twisted superpotential.

Next, for the N-Koszul setting, first recall [9] that an N-Koszul algebra is an algebra A presented by homogeneous relations of degree N so that there is a free resolution of A with differentials of degrees alternating between $N-1$ and 1:

$$\cdots \to A \otimes_S Y_2 \otimes_S A \xrightarrow{d_2} A \otimes_S Y_1 \otimes_S A \xrightarrow{d_1} A \otimes_S A \xrightarrow{m} A \to 0,$$

where d_i has degree 1 if i is odd, and $N-1$ if i is even.

In this setting, it is natural to define N-complexes of bimodules instead of complexes [20]. These are sequences of maps $\mathcal{K}_i \xrightarrow{d} \mathcal{K}_{i-1}$ such that $d^N = 0$ (instead of $d^2 = 0$). As in the ordinary Koszul case one can define a self-dual bimodule N-complex

$$\mathcal{W}^\bullet := 0 \to A \otimes_S W_{|\omega|} \otimes_S A \xrightarrow{d_{|\omega|}} A \otimes_S W_{|\omega|-1} \otimes_S A \to \cdots \to A \otimes_S W_1 \otimes_S A \xrightarrow{d_1} A \otimes_S W_0 \otimes_S A \to 0, \tag{6.3}$$

where $d_i = \varepsilon_i(\mathrm{split}_L + (q)^i \mathrm{split}_R)|_{A \otimes_S W_i \otimes_S A}$, and q is a primitive Nth root of 1. This N-complex is a subcomplex of the Koszul N-complex as defined in [19].

From now on, assume $N > 2$. We can contract N-complexes to obtain complexes. In the case of our self-dual N-complex, we get

$$\begin{aligned} &0 \to A \otimes_S W_{mN+1} \otimes_S A \to A \otimes_S W_{mN} \otimes_S A \to A \otimes_S W_{(m-1)N+1} \otimes_S A \\ &\quad \to \cdots \to A \otimes_S W_N \otimes_S A \to A \otimes_S W_1 \otimes_S A \to A \otimes_S W_0 \otimes_S A \to 0, \end{aligned} \tag{6.4}$$

where the differentials alternate between $\pm(\mathsf{split}_L^{N-1}+\mathsf{split}_L^{N-2}\mathsf{split}_R+\cdots+\mathsf{split}_L\mathsf{split}_R^{N-2}+\mathsf{split}_R^{N-1})$ and $\pm(\mathsf{split}_L-\mathsf{split}_R)$. This is not self-dual anymore unless we are in the case that $|\omega| = mN + 1$ for some $m \in N$.

We see in (6.4) that the length of the complex is even, i.e., if it is exact, then A should have odd Hochschild dimension, $2m+1$. In this case, since the desired perfect pairing $\mathsf{Ext}^1(S,S)\times\mathsf{Ext}^{2m}(S,S)\to\mathbb{C}$ will be symmetric, we should work with cyclically symmetric potentials rather than super-cyclically symmetric potentials.

For $N > 2$, the N-Koszul generalization of Theorem 6.2 then becomes

Theorem 6.8. *An algebra $A = T_S W/\langle R\rangle$ is N-Koszul (for $N > 2$) and σ-twisted Calabi–Yau if and only if $A = \mathcal{D}(\omega, (m-1)N+1)$ for a σ-twisted cyclically symmetric potential ω of degree $mN+1$ for some $m \geq 1$, and the corresponding complex (6.4) is exact in positive degree and $H^0(\mathcal{W}^\bullet) = A$. In this case, (6.4) is the N-Koszul resolution of A, it is twisted self-dual, and A has twisted CY dimension $2m+1$.*

The proof of the theorem uses basically the same arguments as in the proof of Theorem 6.2 but adapted to the N-Koszul situation in accordance with the results of [9,19].

Remark 6.9. As also remarked in [10], a graded three-dimensional CY algebra is automatically N-Koszul, since then the graded self-dual resolution has maps of degrees 1, $N-1$, and 1. Hence, we recover the twisted generalization of the main result of [8], that the algebra comes from a twisted superpotential (and moreover that the associated complex is the minimal bimodule resolution of A).

7. Sklyanin algebras

In this section we consider the four-dimensional Sklyanin algebras as introduced by Sklyanin in [21,22]. These algebras may be thought of as "elliptic deformations" of the polynomial algebra in four variables, and they are in particular Koszul and have the same Hilbert series $\frac{1}{(1-t)^4}$ as the polynomial ring.

Following [23, Section 0], fix values α, β, and γ satisfying[5]

$$\alpha+\beta+\gamma+\alpha\beta\gamma = 0. \tag{7.1}$$

Then, the algebra A is defined by

$$A := \mathbb{C}\langle x_0, x_1, x_2, x_3\rangle / I,$$

where I is the two-sided ideal generated by the relations r_i, s_i[6]:

$$\begin{aligned} r_1 &:= x_0x_1 - x_1x_0 - \alpha(x_2x_3+x_3x_2), & s_1 &:= x_0x_1 + x_1x_0 - (x_2x_3 - x_3x_2),\\ r_2 &:= x_0x_2 - x_2x_0 - \beta(x_3x_1+x_1x_3), & s_2 &:= x_0x_2 + x_2x_0 - (x_3x_1 - x_1x_3),\\ r_3 &:= x_0x_3 - x_3x_0 - \gamma(x_1x_2+x_2x_1), & s_3 &:= x_0x_3 + x_3x_0 - (x_1x_2 - x_2x_1).\end{aligned}$$

We would like to find a superpotential for A. This must be a supercyclic element which is homogeneous of degree four. The superpotential also sits in I because for every homogeneous element u and every $n \leq |u|$, we have the identity $u = \sum_{|p|=n} p\partial_p u$ (the summation is over the monomials of degree n in $\mathbb{C}\langle x_0,x_1,x_2,x_3\rangle$). It is easy to compute that, under the assumption

$$(\alpha,\beta,\gamma)\notin\{(\alpha,-1,1),(1,\beta,-1),(-1,1,\gamma)\}, \tag{7.2}$$

the space of such elements is one-dimensional and spanned by the following element:

$$\omega := \kappa_1(r_1s_1+s_1r_1)+\kappa_2(r_2s_2+s_2r_2)+\kappa_3(r_3s_3+s_3r_3), \tag{7.3}$$

where $(\kappa_1,\kappa_2,\kappa_3)\neq(0,0,0)$ is determined up to a nonzero multiple by

$$\kappa_1(1+\alpha)=\kappa_3(1-\gamma), \qquad \kappa_1(1-\alpha)=\kappa_2(1+\beta), \qquad \kappa_2(1-\beta)=\kappa_3(1+\gamma). \tag{7.4}$$

Proposition 7.1. *The element ω is a superpotential. Moreover, for any α,β,γ satisfying (7.2), $A\cong\mathcal{D}(\omega,2)$, and in this case, the resolution (6.1) is a self-dual resolution of A, making A Calabi–Yau.*

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5 In the original form [21,22], see also e.g. [24,25], not all values α,β,γ satisfying this equation are considered—only those that arise from an elliptic curve and a point of that curve. By, e.g., [23], these are the values where (7.2) holds and $\alpha,\beta,\gamma\neq 0$; cf. Theorem 7.4.

6 Our notation r_i is for the relation involving $x_0x_i - x_ix_0$, and s_i is the relation involving $x_0x_i + x_ix_0$.

Proof. It is easy to verify that by the precise choice of the κ_i, ω is a superpotential (in fact, it makes sense and is cyclically supersymmetric even if (7.2) is not satisfied). Next, suppose (7.2) holds. Then, κ_1, κ_2, and κ_3 are nonzero. The elements r_i, s_i are linearly independent and we can complete this set with elements $\{u_1, \ldots, u_{10}\}$ to obtain a basis for $\mathbb{C}\langle x_0, x_1, x_2, x_3\rangle_2$. This gives us a dual basis r_i^*, s_i^*, u_i^* and

$$[r_i^*\omega] = \kappa_i s_i, \qquad [s_i^*\omega] = \kappa_i r_i, \qquad [u_i^*\omega] = 0.$$

This shows that $A \cong \mathcal{D}(\omega, 2)$.

To deduce that (6.1) is a resolution of A, we make use of

Theorem 7.2 ([23]). *Assuming (7.2), A is Koszul. Moreover,* $H(A^!, t) = (1+t)^4$.

In the above theorem, $H(V, t)$ denotes the Hilbert series of a graded vector space V, i.e., $H(V, t) = \sum_{m\geq 0} \dim V(m)t^m$. The hard part of the above theorem is the Koszulity.

Now, by Lemma 6.5 and the formula for the Koszul complex (see the proof of Lemma 6.5), it suffices only to show that $\dim W_i = \binom{n}{i}$ for all i. For $i = 2$, this follows from the above observations; then, it follows by applying partial derivatives to the relations r_j, s_j that this is true for $i = 1$. Since $i = 0$ is obvious, we get $\dim W_i = \binom{n}{i}$ for all i by the self-duality of $\mathcal{W}^\bullet$. Thus, A is Calabi–Yau with self-dual resolution $\mathcal{W}^\bullet$. □

Remark 7.3. It is also easy to derive that A is Calabi–Yau directly from [23]: in particular, in [23] it is shown that $A^!$ is Frobenius, and one may easily show that $A^!$ is in fact symmetric. Our contribution here is in producing a superpotential and showing that the minimal (Koszul) resolution of A is produced in this way.

7.0.1. *Modified Sklyanin algebras from [13]*

In [13], some new algebras related to the above are defined and shown to be Koszul, and have the same Hilbert series $\frac{1}{(1-t)^4}$ as the polynomial ring in four variables. Here, we explain that these algebras are not Calabi–Yau, but rather twisted Calabi–Yau, with twisted superpotential described below. We omit the proofs, which are the same as for the Sklyanin algebra. Following [13], let us assume in this subsection that $\{\alpha, \beta, \gamma\} \cap \{0, 1, -1\} = \emptyset$.

Heuristically, these algebras are "elliptic deformations" of the algebra $\mathbb{C}\langle x_0, x_1, x_2, x_3\rangle/(-x_0^2 + x_1^2 + x_2^2 + x_3^2, x_ix_j - x_jx_i \mid \{i, j\} \neq \{2, 3\})$ in the same way that the Sklyanin algebras are deformations of $\mathbb{C}[x_0, x_1, x_2, x_3]$.

Precisely, the relations are given by using any five of the relations r_i, s_j, and replacing the sixth with the new relation $q := d_1\Omega_1 + d_2\Omega_2$, where

$$\Omega_1 := -x_0^2 + x_1^2 + x_2^2 + x_3^2, \qquad \Omega_2 := x_1^2 + \frac{1+\alpha}{1-\beta}x_2^2 + \frac{1-\alpha}{1+\gamma}x_3^2.$$

We obtain the algebra $A' = \mathbb{C}\langle x_0, x_1, x_2, x_3\rangle/I'$, where I' is the ideal generated by q and five of the r_i, s_j. (The geometric motivation for studying A' is that it and the Sklyanin algebra A both surject to the same ring $B := A/(\Omega_1, \Omega_2)$ of geometric relevance.)

First, suppose that the relations are $q, r_2, r_3, s_1, s_2, s_3$ (so r_1 is not a relation). We claim that A is twisted Calabi–Yau with twisting $\sigma(x_0) = -x_0, \sigma(x_1) = -x_1, \sigma(x_2) = x_2, \sigma(x_3) = x_3$, and with unique twisted superpotential (up to scaling) given by

$$\lambda_1(qs_1 + s_1q) + \lambda_2(r_2r_3 - r_3r_2) + \lambda_3(s_2s_3 - s_3s_2),$$

with $(\lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^2$ determined by

$$d_2\lambda_1 = \lambda_2(\beta\gamma + 1), \qquad d_1\lambda_1 = -\lambda_2 + \lambda_3,$$

provided that any nonzero solution to the above has all of $\lambda_1, \lambda_2, \lambda_3$ nonzero, i.e., (d_1, d_2) is not a multiple of $(1, 0)$ or $(1, -1-\beta\gamma)$.

Next, suppose that the relations are $q, r_1, r_2, r_3, s_2, s_3$ (so s_1 is not a relation). Then, A is twisted Calabi–Yau with the same twisting as above, and the unique twisted superpotential (up to scaling) is given by

$$\lambda_1(qr_1 + r_1q) + \lambda_2(r_2s_3 - s_3r_2) + \lambda_3(s_2r_3 - r_3s_2),$$

with $(\lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^2$ determined by

$$\alpha d_1\lambda_1 = \lambda_2 - \lambda_3, \qquad \alpha(d_1 + d_2)\lambda_1 = \beta\lambda_2 + \gamma\lambda_3,$$

again provided all of $\lambda_1, \lambda_2, \lambda_3$ can be nonzero (i.e., (d_1, d_2) is not a multiple of $(1, \beta - 1)$ or $(1, -1-\gamma)$). Any other A' can be obtained from this or the previous paragraph by a cyclic permutation of the parameters and relations.

Finally, in [13], also the algebra $A'_\infty = \mathbb{C}\langle x_0, x_1, x_2, x_3\rangle/(r_2, s_2, r_3, s_3, \Omega_1, \Omega_2)$ is studied, and shown to be Koszul and have the same Hilbert series as the polynomial ring in four variables (just as in all the other examples). We claim that this algebra is twisted Calabi–Yau with twisting $\sigma(x_i) = -x_i$ for all i. In other words, the twisted superpotential ω (which is unique up to scaling) is actually cyclically symmetric. We omit the formula for the twisted superpotential.

7.1. McKay correspondence for four-dimensional Sklyanin algebras

It makes sense to think of the potential (7.3) as a deformed version of the volume form in the case of the polynomial algebra in four variables: precisely, the potential ω for a Koszul Calabi–Yau algebra always spans the top Hochschild homology group (here, $HH_4(A, A)$) as a free module over the center $HH^0(A, A)$, viewing ω as a standard Hochschild cycle. This may be deduced by applying $M \mapsto M \otimes_{A \otimes_S A^{op}} A$ to (6.1) and comparing with the standard Hochschild complex; cf. [2, Proposition 10]. We are interested in automorphisms of A preserving the potential. It is well known (e.g., [24,25]) that the finite Heisenberg group acts on the Sklyanin algebra A by automorphisms. In fact:

Theorem 7.4 *([26, Section 2]). Assume that $\alpha, \beta, \gamma \neq 0$ and (7.2) holds. If α, β or $\gamma \neq \pm\sqrt{-3}$ then the group of graded automorphisms of A ($\subset \mathrm{Aut}(V)$) is isomorphic to a group $\widetilde{H}$ given by a central extension,*

$$1 \to \mathbb{C}^{\times} \to \widetilde{H} \to (\mathbb{Z}/4 \oplus \mathbb{Z}/4) \to 1, \tag{7.5}$$

such that for lifts X, Y of $(1, 0), (0, 1)$ to $\widetilde{H}$, $[X, Y] = \sqrt{-1} \in \mathbb{C}^{\times}$. In the case $\alpha = \beta = \gamma = \pm\sqrt{-3}$, the group has the form $\widetilde{H} \ltimes \mathbb{Z}/3$.

Explicit matrix generators for this group are given in [26, Section 2] and are also given below, at the end of Section 7.3.

Notation. Call the subgroup of $\mathrm{Aut}(V)$ preserving $\omega \in V^{\otimes 4}$ the automorphism group of ω, and denote it by $\mathrm{Aut}(\omega)$.

The only elements of $\mathbb{C}^{\times}$ that act trivially on $V^{\otimes 4}$ are fourth roots of unity. As a result, in the situation above where $\mathrm{Aut}(A) = \widetilde{H}$, the automorphism group of the superpotential ω will be finite, of order 64. It turns out this is one of the "$\mathbb{Z}/4$-Heisenberg groups," which we describe as follows. Let $X, Y \in \widetilde{H}$ be elements as in the theorem, chosen to have the property $X^4 = Y^4 = -1$. Then, H is the group generated by X and Y. It is a central extension

$$1 \to \mu_4 \to H \to \mathbb{Z}/4 \oplus \mathbb{Z}/4 \to 1, \tag{7.6}$$

where $\mu_4 \subset \mathbb{C}^{\times}$ is the subgroup of elements of order four. A presentation for H is given by

$$H \cong \langle X, Y, Z \mid XZ = ZX, YZ = ZY, Z^4 = 1, X^4 = Y^4 = Z^2, [X, Y] = Z \rangle. \tag{7.7}$$

We deduce the following:

Proposition 7.5. *For any α, β, γ as in Theorem 7.4, $\mathrm{Aut}(\omega) \cong H$, unless $\alpha = \beta = \gamma = \pm\sqrt{-3}$, in which case this group is $H \rtimes \mathbb{Z}/3$, where $\mathbb{Z}/3$ acts nontrivially on H.*

As a consequence, we see that, under the assumptions of Theorem 7.4, $\omega \otimes 1 \in A\#H$ is still a superpotential, and hence also gives a superpotential for any algebra Morita equivalent to $A\#H$. Letting $f_1, \ldots, f_m$ be a full set of primitive idempotents (one for each irreducible representation of H), and $f := f_1 + \cdots + f_m$, we then have

Proposition 7.6. *The algebra $f(A\#H)f$ is Calabi–Yau. For any finite subgroup $G < \widetilde{H}$, $f'(A\#G)f'$ is twisted Calabi–Yau, where f' is the sum of a full set of primitive idempotents for G.*

These algebras may be considered the elliptic McKay correspondence algebras in dimension four, and $f(A\#H)f$ is the maximal Calabi–Yau one, in the sense that H is maximal (and so the McKay quiver is also the largest possible).

7.2. The case $\alpha = 0$

The Theorem 7.4 did not apply to the case that one of α, β, γ is zero. Since we only need (7.2) to obtain a Calabi–Yau algebra and a potential, it is worth proving the analogue of Theorem 7.4 in the degenerate cases $(\alpha, \beta, \gamma) \in \{(0, \beta, -\beta), (\alpha, 0, -\alpha), (\alpha, -\alpha, 0), (0, 0, 0)\}$ (and we will use this in the next subsection). By symmetry, we restrict ourselves to the case $\alpha = 0$.

It is likely that this result is known, but we did not find it in the literature. We remark that, in [23, Section 1], it is shown that these degenerate cases are iterated Ore extensions.

Theorem 7.7. *(i) Assume $(\alpha, \beta, \gamma) = (0, \beta, -\beta)$ with $\beta \neq 0$. Then, the graded automorphism group of A is generated by $\mathbb{C}^{\times}$, the group $SO(2, \mathbb{C})$ acting on $\mathrm{Span}\{x_2, x_3\}$, i.e., $\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix} \middle| \, a^2 + b^2 = 1 \right\}$, and the elements $\left\{ \begin{pmatrix} 0 & \frac{1}{\pm\sqrt{\beta}} & 0 & 0 \\ \pm\sqrt{\beta} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \right\}$, where i denotes a square root of -1.*

(ii) If $\alpha = \beta = \gamma = 0$, then the automorphism group is $\mathbb{C}^{\times} \cdot SO(3, \mathbb{C})$, with $SO(3, \mathbb{C})$ the automorphism group of $\mathrm{Span}\{x_1, x_2, x_3\}$ together with its standard symmetric bilinear form $(x_i, x_j) = \delta_{ij}$.

Proof. (i) The vector $r_1 = x_0 \wedge x_1$ must be preserved up to scalar by any automorphism, so the span of x_0, x_1 is preserved; then the only element of $\mathrm{Sym}^2\, \mathsf{Span}\{x_0, x_1\}$ in the symmetrization of the relations is $x_0x_1 + x_1x_0$. Hence, any automorphism must send (x_0, x_1) to $(\lambda x_0, \mu x_1)$ or $(\mu x_1, \lambda x_0)$. Up to the automorphism $x_0 \mapsto \sqrt{\beta}x_1, x_1 \mapsto \frac{1}{\sqrt{\beta}}x_0, x_2 \mapsto x_2, x_3 \mapsto x_3$ and scaling, we may assume that our automorphism ψ satisfies $\psi(x_0) = \lambda x_0, \psi(x_1) = \lambda^{-1}x_1$. Since then $x_0x_1 + x_1x_0$ is preserved, looking at s_1, we see that $x_2x_3 - x_3x_2$ is preserved, and hence the span of x_2, x_3 is preserved.

Next, note that the relations project isomorphically to Λ^2V. Let $[x, y] := xy - yx$ denote the commutator and $\{x, y\} := xy + yx$ the anticommutator. We have $\psi(r_2) = [\lambda x_0, \psi(x_2)] - \frac{\beta}{\lambda}\{\psi(x_3), x_1\}$. If we write $\lambda\psi(x_2) = ax_2 + bx_3$, then we must have $\psi(r_2) = ar_2 + br_3$. This implies that, restricted to $\mathsf{Span}\{x_2, x_3\}$, ψ must have the form

$$\psi = \begin{pmatrix} \frac{a}{\lambda} & -b\lambda \\ \frac{b}{\lambda} & a\lambda \end{pmatrix}. \tag{7.8}$$

Applying the same reasoning to $\psi(x_3)$, we deduce furthermore that $\lambda^4 = 1$. This yields the claimed description.

(ii) Let R be the vector space spanned by the relations. In the case $\alpha = \beta = \gamma = 0$, the intersection $\Lambda^2V \cap R$ is $\mathsf{Span}\{x_0\} \wedge V$, and hence any automorphism ψ must send x_0 to a multiple of itself. Up to scaling, let us assume that $\psi(x_0) = x_0$. Then, the fact that the relations project isomorphically to Λ^2V yields a canonical isomorphism $\Lambda^2V/(\Lambda^2V \cap R) \xrightarrow{\sim} \mathsf{Span}\{x_1, x_2, x_3\}$, sending $w \in \Lambda^2V$ to the unique element v such that $w - (x_0v + vx_0) \in R$. Any automorphism ψ as above preserves not only $U := \mathsf{Span}\{x_1, x_2, x_3\}$, but also the isomorphism $\Lambda^2U \xrightarrow{\sim} U$ obtained from the above. This isomorphism can also be expressed as the composition $\Lambda^2U \xrightarrow{\sim} \Lambda^2U^* \xrightarrow{\sim} U$ where the first map uses the standard symmetric bilinear form $(x_i, x_j) = \delta_{ij}$, and the second map uses the standard volume form. It easily follows that automorphisms preserving this map are exactly $SO(3, \mathbb{C})$. Conversely, any element preserving the map $\Lambda^2U \xrightarrow{\sim} U$ is easily shown to be an automorphism of A. This yields the result. □

Corollary 7.8. *In either case of the theorem, the automorphism group of the potential* $\mathsf{Aut}(\omega)$ *is generated by the elements listed, except that* $\mathbb{C}^\times$ *is replaced by the group* $\mu_4 \subset \mathbb{C}^\times$ *of fourth roots of unity.*

Proof. It suffices to observe that all of the elements listed, aside from the scalars $\mathbb{C}^\times$, preserve the potential μ_4. This is straightforward in (i); for part (ii), by looking at the coefficient of $x_0x_1x_2x_3$ under the action of $SO(3, \mathbb{C})$ on ω, one sees that the character $SO(3, \mathbb{C}) \to \mathbb{C}^\times, g \mapsto \frac{g(\omega)}{\omega}$ is the same as the determinant, and hence is trivial. □

As a consequence, we may again consider $A\#G$ for any finite subgroup $G \subset \mathsf{Aut}(\omega)$, which will be a Calabi–Yau algebra, and in the case that G is in $\mathsf{Aut}(A)$ but not in $\mathsf{Aut}(\omega)$, we get a twisted Calabi–Yau algebra. As before, one may consider the Morita equivalent algebras and write down their potentials.

7.3. Moduli space of four-dimensional Sklyanin algebras

In this subsection we will use the theory of the Weil representation over $\mathbb{Z}/4$ and the preceding results to give a simple computation of the moduli space of Sklyanin algebras in dimension four. Throughout, when we say "isomorphism" or "automorphism" of Sklyanin algebras, we mean a graded isomorphism or automorphism.

First, we note that, given any (α, β, γ), the algebras associated to this triple and any cyclic permutation are isomorphic: the permutation $x_0 \mapsto x_0, x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1$ sends the relations for (α, β, γ) to the relations for (γ, α, β). Similarly, the map $x_0 \mapsto x_0, x_1 \mapsto x_2, x_2 \mapsto -x_1, x_3 \mapsto x_3$ sends the relations for (α, β, γ) to the relations for $(-\beta, -\alpha, -\gamma)$.

Hence, if we consider the $\mathfrak{S}_3$ action on the surface $\mathcal{S}$ given by $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$, given by multiplying the standard permutation action by the sign representation, we get a map

$$\mathcal{S}/\mathfrak{S}_3 \twoheadrightarrow \{\text{Isomorphism classes of four-dimensional Sklyanin algebras}\}. \tag{7.9}$$

Theorem 7.9. *The map (7.9) is a bijection.*

The rest of the subsection will be devoted to the proof of the theorem. The main case of the theorem concerns those parameters satisfying the conditions of Theorem 7.4, and we will prove the result by finding a description of the moduli space of potentials in terms of the Heisenberg and Weil representations.

Remark 7.10. Note that, in the locus of elements satisfying Theorem 7.4, the $\mathfrak{S}_3$ action is free except at the two points $\alpha = \beta = \gamma = \pm\sqrt{-3}$. Here, these two points form a two-element orbit, and the isotropy $\mathbb{Z}/3$ is picked up by the automorphism group at these points (cf. Theorem 7.4).

First, let us handle the degenerate cases when one of α, β, γ is zero. Suppose only one is zero, and without loss of generality, say it is α. Then $(\alpha, \beta, \gamma) = (0, \beta, -\beta)$. Note that, in this case, the automorphism group of A is independent of the value of β. In particular, any $\psi : V \xrightarrow{\sim} V$ inducing an isomorphism $A \xrightarrow{\sim} A'$ with A' of the same form must normalize the connected component of the identity of the common automorphism group, i.e., $\mathbb{C}^\times \cdot SO(2)$. Since ψ must therefore preserve the trivial weight spaces of $SO(2)$ and either preserve or interchange the nontrivial weight spaces, ψ must have the form $\psi = \psi' \oplus \psi''$, where $\psi' = \psi|_{\mathrm{Span}\{x_0,x_1\}}$ and $\psi'' = \psi|_{\mathrm{Span}\{x_2,x_3\}}$, and $\psi'' \in O(\mathrm{Span}\{x_2, x_3\})$ (the orthogonal group). Up to an automorphism of A, we may assume that $\psi'' = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$, with $\varepsilon \in \{1, -1\}$. By the same argument as in the proof of Theorem 7.7, we must have that ψ' is either diagonal or strictly off-diagonal, and using the automorphism $\begin{pmatrix} 0 & \pm\frac{1}{\sqrt{\beta}} \\ \pm\sqrt{\beta} & 0 \end{pmatrix}$, we may assume ψ' is diagonal, say $\psi' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Using that the relations for A and A' both contain s_1, s_2, s_3, it follows that $\mu\lambda = \varepsilon$, $\mu = \lambda$, and $\mu = \varepsilon\lambda$. Put together, this says that $\varepsilon = 1$ and $\mu = \lambda = \pm 1$. This is already an automorphism of A, so $\psi \in \mathrm{Aut}(A)$. That is, A and A' already had the same relations. So (7.9) is injective when restricted to parameters (α, β, γ) with exactly one parameter equal to zero.

In the case $\alpha = \beta = \gamma = 0$, it is clear that no other triple (α, β, γ) yields an isomorphic algebra.

We did not restrict ourselves to the Calabi–Yau condition (7.2), so let us also explain the contrary cases. First assume $(\alpha, \beta, \gamma) = (\alpha, -1, 1)$ with $\alpha \neq \pm 1$. Call R the span of the relations. We quickly compute the automorphism group of A as follows. We see that R contains the rank-two tensors

$$x_0x_2 + x_1x_3, \qquad x_2x_0 - x_3x_1, \qquad x_0x_3 - x_1x_2, \qquad x_3x_0 + x_2x_1. \tag{7.10}$$

Set $U := \mathrm{Span}\{x_0, x_1\}$ and $U' := \mathrm{Span}\{x_2, x_3\}$. Then, the rank-two tensors form a union of an open subvariety of $R \cap (U \otimes U')$ and an open subvariety of $R \cap (U' \otimes U)$. Thus, any automorphism of A must either preserve or interchange U and U'. Moreover, equip U and U' each with their standard symmetric bilinear forms. We see that, given nonzero vectors $w_1, w_2 \in U$, the subspace of relations $\{w_1w_1' - w_2w_2' \mid w_1', w_2' \in U'\} \cap R$ is two-dimensional if and only if $(w_1, w_2) = 0$. Hence, any automorphism of A which preserves U, U' must also preserve their standard symmetric bilinear forms. Thus, $\mathrm{Aut}(A)$ must be a subgroup of $(\mathbb{C}^\times SO(U)) \oplus (\mathbb{C}^\times SO(U')) \rtimes \mathbb{Z}/2$, where $1 \in \mathbb{Z}/2$ interchanges U and U', e.g., it may be the element $x_0 \mapsto \sqrt[4]{-\alpha} \cdot x_2 \mapsto x_0, x_1 \mapsto \sqrt{-1} \cdot \sqrt[4]{-\alpha} \cdot x_3 \mapsto x_1$. We claim that the automorphism group is $\mathbb{C}^\times (SO(U) \oplus SO(U')) \rtimes \mathbb{Z}/2$. To prove this it suffices to show that any automorphism of A in $\mathbb{C}^\times \oplus \mathbb{C}^\times$ is diagonal, i.e., if $\psi \in \mathrm{Aut}(A)$ has the property that $\psi|_U$ and $\psi|'_U$ are scalar, then the two scalars are equal. Such an element must preserve the relation s_1, which implies the needed result.

If we had any two parameters $(\alpha, -1, 1), (\alpha', -1, 1)$, with $\alpha, \alpha' \notin \{\pm 1\}$, then any intertwiner $A \xrightarrow{\sim} A'$ between the two, by the above analysis, must also either preserve U, U' and their standard symmetric bilinear forms (hence $\alpha = \alpha'$), or else must interchange U and U'; up to $SO(U) \times SO(U')$, we may assume that the map sends x_0 to a multiple of x_2 and vice versa, and sends x_1 to a multiple of x_3 and vice versa. It easily follows that $\alpha = \alpha'$.

The case where $\alpha, \beta, \gamma \in \{\pm 1\}$ is trivial since all of these cases are under the same orbit of $\mathfrak{S}_3$ (and they cannot be equivalent to any other example because their relations have the largest subvarieties of rank-two tensors, or alternatively, because we show in all other examples that this case is not equivalent).

Thus, we have reduced the theorem to the nondegenerate case when α, β, and γ are all nonzero and (7.2) is satisfied. We may also assume that α, β, γ are not all equal, since these two cases are isomorphic and they are the only ones where the automorphism group has size 192. We will not make further mention of these assumptions.

Recall the Heisenberg group $H \cong \mathrm{Aut}(\omega)$ from the previous section. We will need the Stone–von Neumann theorem in our context (we omit the proof, which is easy):

Lemma 7.11 (*Stone–von Neumann Theorem*). *There is a unique irreducible representation of H which sends elements $\zeta \in \mu_4$ to the corresponding scalar matrix $\zeta \cdot \mathrm{id}$.*

Call this the **Heisenberg representation**. Note that our given representation V of H is of this form.

Notation. Let $\mathrm{Aut}(H, \mu_4)$ denote the subgroup of the automorphism group of H which acts trivially on the center $\mu_4 < H$. Similarly, let $\mathrm{Inn}(H, \mu_4) = \mathrm{Inn}(H)$ be the inner automorphisms, and $\mathrm{Out}(H, \mu_4)$ be $\mathrm{Aut}(H, \mu_4)$ modulo inner automorphisms.

We know that a Sklyanin algebra is specified by a potential $\omega \in V^{\otimes 4}$, up to a scalar multiple. Now, let us fix one such algebra A_0 with potential ω_0. Then, V naturally has the structure of the unique irreducible Heisenberg representation of Lemma 7.11, given by any fixed isomorphism $H \cong \mathrm{Aut}(\omega_0) \subset \mathrm{Aut}(V)$. Let $\rho_0 : H \to \mathrm{Aut}(V)$ be such a representation. Further, once and for all identifying μ_4 with the center of H, we will assume that ρ_0 restricts to the tautological inclusion $\mu_4 \hookrightarrow \mathbb{C}^\times \cdot \mathrm{id}$ on the center.

So, we have fixed the data (A_0, ω_0, ρ_0). Now, given any other algebra A with potential $\omega \in V^{\otimes 4}$, it is equipped with a Heisenberg representation $\rho : H \to \mathrm{Aut}(V)$ which is unique up to precomposition with an element of $\mathrm{Aut}(H, \mu_4)$. By Lemma 7.11 and Schur's Lemma, there must be a unique up to scalar intertwiner $\psi : V \xrightarrow{\sim} V$ such that $\psi\rho_0(h)\psi^{-1} = \rho(g)$ for all $h \in H$. Hence, we obtain the vector $\psi^{-1}(\omega) \in V^{\otimes 4}$. This vector is uniquely determined by (A, ω, ρ) up to scaling.

If we had picked a different potential ω, this could also only affect the vector $\psi^{-1}(\omega)$ by scaling.

If, instead of ρ, we had chosen $\rho' = \rho \circ \phi$ for some element $\phi \in \mathsf{Aut}(H, \mu_4)$, then instead of $\psi^{-1}(\omega) \in V^{\otimes 4}$, we would have obtained $\psi_\phi^{-1}\psi^{-1}(\omega)$, where $\psi_\phi^{-1} : V \xrightarrow{\sim} V$ is any intertwiner (unique up to scaling) between ρ_0 and $\rho_0 \circ \phi$, i.e., such that $\psi_\phi \rho_0(h)\psi_\phi^{-1} = \rho_0(\phi(h))$.

Note that, by Lemma 7.11, we have a projective representation $\mathsf{Aut}(H, \mu_4) \to \mathsf{PGL}(V^{\otimes 4})$. Thus, we have obtained a map from Sklyanin algebras to $\mathbb{P}V^{\otimes 4}/\mathsf{Aut}(H, \mu_4)$. In fact, we can do better: since ω is fixed by the action of $\rho(H)$, $\psi^{-1}(\omega_0)$ is fixed by the action of $\rho_0(H)$, and this is the same as the action of $\mathsf{Inn}(H, \mu_4)$ on $\mathbb{P}V^{\otimes 4}$. Hence, letting $T \subset V^{\otimes 4}$ be the subspace of fixed vectors under $\rho_0(H)$, we have a projective representation of $\mathsf{Out}(H, \mu_4)$ on T, and have a map

$$\text{Four-dimensional Sklyanin algebras} \to \mathbb{P}T/\mathsf{Out}(H, \mu_4). \tag{7.11}$$

Furthermore, suppose we have (A, ω, ρ) as above, and another Sklyanin algebra $A' \cong A$, together with an isomorphism $\theta : V \xrightarrow{\sim} V$ carrying the relations of A to the relations of A'. We may pick $\omega' = \theta(\omega)$ as our potential for A', and $\rho' := \theta\rho\theta^{-1}$ as our Heisenberg representation $H \to \mathsf{Aut}(\omega')$. Thus, using the intertwiner $\psi' = \theta \circ \psi$, we see that the image of A and A' under (7.11) is the same. Conversely, if we are given (A, ω, ρ), (A', ω', ρ'), ψ, ψ' such that $\psi^{-1}(\omega) = (\psi')^{-1}(\omega')$, then $\psi' \circ \psi^{-1} : V \xrightarrow{\sim} V$ is an isomorphism carrying ω to ω', and hence induces an isomorphism between (the relations of) A and A'.

We thus obtain a canonical map (having fixed just A_0 and ρ_0):

$$\text{Isomorphism classes of four-dimensional Sklanin algebras} \hookrightarrow \mathbb{P}T/\mathsf{Out}(H, \mu_4). \tag{7.12}$$

The reader will probably recognize that $\mathsf{Out}(H, \mu_4) \cong \mathsf{SL}_2(\mathbb{Z}/4)$ and its action on $\mathbb{P}T$ is a version of the Weil representation, which we will explain.

Let us define $M := (\mathbb{Z}/4)^{\oplus 2}$ and think of this as a free rank-two $\mathbb{Z}/4$-module.

Lemma 7.12. *The outer automorphism group* $\mathsf{Out}(H, \mu_4)$ *of* H *fixing its center is* $\mathsf{SL}_2(\mathbb{Z}/4)$. *We have the exact sequence*

$$1 \to M \to \mathsf{Aut}(H, \mu_4) \to \mathsf{SL}_2(\mathbb{Z}/4) \to 1. \tag{7.13}$$

Here, $\mathsf{Aut}(H, \mu_4)$ denotes the automorphism group of H which acts trivially on the center μ_4. Note that the size of $\mathsf{SL}_2(\mathbb{Z}/4)$ is 48.

Proof. It is clear that the inner automorphism group is $H/\mu_4 \cong M$. This acts by characters $M \to \mu_4$, fixing $\mu_4 < H$. Furthermore, the action of the outer automorphism group on H fixing the center descends to an action on $H/\mu_4 = M$, and this action must preserve commutators since $[H, H] \subset \mu_4$. If we consider $x \wedge y \mapsto [x, y]$ for $x, y \in M \cong H/\mu_4$ to be a volume form, then we obtain an embedding $\mathsf{Out}(H, \mu_4) \hookrightarrow \mathsf{SL}_2(\mathbb{Z}/4)$. We have to show this is surjective. If X, Y are lifts of generators of M to H, they have order 8, and it follows that the same is true for X^aY^b whenever at least one of a, b is odd. As a result, we see that, for any two elements $X', Y' \in H$ such that $[X', Y'] = [X, Y]$, the map $X \mapsto X'$, $Y \mapsto Y'$ must yield an automorphism of H fixing μ_4. □

As a consequence, the action of $\mathsf{Out}(H, \mu_4)$ on $\mathbb{P}T$ is a projective representation of $\mathsf{SL}_2(\mathbb{Z}/4)$, which we will call the Weil representation on T.

Let $\mathcal{S}_0 \subset \mathcal{S}$ be the subset of tuples satisfying the assumptions of Theorem 7.4. Next, we will describe explicitly the map $\mathcal{S}_0 \to \mathbb{P}T/\mathsf{SL}_2(\mathbb{Z}/4)$ and show that its kernel is $\mathfrak{S}_3$. More precisely, we show that this map factors as follows. Let $K \subset \mathsf{SL}_2(\mathbb{Z}/4)$ be the kernel of the canonical surjection $\mathsf{SL}_2(\mathbb{Z}/4) \twoheadrightarrow \mathsf{SL}_2(\mathbb{Z}/2)$ (note that $K \cong (\mathbb{Z}/2)^{\times 3}$). Then, we prove the following

Claim 1. *The map* $(\alpha, \beta, \gamma) \mapsto \omega$ *given by (7.3) factors as follows:*

$$\mathcal{S}_0 \hookrightarrow \mathbb{P}T/K \twoheadrightarrow \mathbb{P}T/\mathsf{SL}_2(\mathbb{Z}/4). \tag{7.14}$$

Moreover, using a natural isomorphism $\mathfrak{S}_3 \cong \mathsf{SL}_2(\mathbb{Z}/2)$*, the action of* $\mathfrak{S}_3$ *on* $\mathcal{S}_0$ *is identified with the action of* $\mathsf{SL}_2(\mathbb{Z}/2)$ *on* $\mathbb{P}T/K$.

The theorem follows immediately from the claim.

To prove the claim, we recall from [26, Section 2] explicit formulas for $\rho_0(X)$, $\rho_0(Y)$. Let $\theta_0, \theta_1, \theta_2, \theta_3 \in \mathbb{C}^\times$ be numbers such that

$$\alpha_0 = \left(\frac{\theta_0\theta_1}{\theta_2\theta_3}\right)^2, \qquad \beta_0 = -\left(\frac{\theta_0\theta_2}{\theta_1\theta_3}\right)^2, \qquad \gamma_0 = -\left(\frac{\theta_0\theta_3}{\theta_1\theta_2}\right)^2. \tag{7.15}$$

(The numbers θ_i are in fact Jacobi's four theta functions associated with an elliptic curve valued at a point of that curve, which may be used to give a geometric definition of A_0. We will not need this fact.) Fix $i = \sqrt{-1} \in \mathbb{C}$. We have:

$$\rho_0(X) = \begin{pmatrix} 0 & 0 & 0 & i\frac{\theta_3}{\theta_0} \\ 0 & 0 & -i\frac{\theta_2}{\theta_1} & 0 \\ 0 & i\frac{\theta_1}{\theta_2} & 0 & 0 \\ i\frac{\theta_0}{\theta_3} & 0 & 0 & 0 \end{pmatrix}, \qquad \rho_0(Y) = \begin{pmatrix} 0 & 0 & -i\frac{\theta_2}{\theta_0} & 0 \\ 0 & 0 & 0 & -\frac{\theta_3}{\theta_1} \\ i\frac{\theta_0}{\theta_2} & 0 & 0 & 0 \\ 0 & \frac{\theta_1}{\theta_3} & 0 & 0 \end{pmatrix}. \tag{7.16}$$

Then, if $(\alpha, \beta, \gamma) \in \mathcal{S}_0$, for any choice of $\theta_0', \theta_1', \theta_2', \theta_3'$ satisfying the version of (7.15) for (α, β, γ), we may define the representation ρ using (7.16) with primed thetas. It is easy to see that an intertwiner ψ carrying ρ_0 to ρ is given by

$$\psi = \begin{pmatrix} \theta_0/\theta_0' & 0 & 0 & 0 \\ 0 & \theta_1/\theta_1' & 0 & 0 \\ 0 & 0 & \theta_2/\theta_2' & 0 \\ 0 & 0 & 0 & \theta_3/\theta_3' \end{pmatrix}. \tag{7.17}$$

As a consequence, we obtain a vector $\psi^{-1}(\omega)$ in T. However, the construction involved a choice of the θ_j', so it is not yet well defined. First, nothing is affected by multiplying all the θ_j' by the same scalar, since everything only involves ratios of the same number of the thetas. So let us assume that $\theta_0' = 1$. Any other choice of $\theta_1', \theta_2', \theta_3'$ must differ by a transformation

$$\begin{pmatrix} 1 \\ \theta_1' \\ \theta_2' \\ \theta_3' \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_3 \end{pmatrix} \begin{pmatrix} 1 \\ \theta_1' \\ \theta_2' \\ \theta_3' \end{pmatrix}, \tag{7.18}$$

where $\varepsilon_i \in \mu_4$, and $\varepsilon_1\varepsilon_2\varepsilon_3 = \pm 1$. First of all, in the case that $\varepsilon_j \in \{\pm 1\}$ for all j and $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, then the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_3 \end{pmatrix}$ is already in $\rho_0(H)$ (and $\rho(H)$, so it will not affect $\psi^{-1}(\omega)$. Factoring the group of ε-matrices (7.18) by this subgroup leaves a group isomorphic to $(\mathbb{Z}/2)^{\times 3}$. Conjugating ρ_0 by the action of this group is easily verified to send ρ_0 to $\rho_0 \circ K$, where $K \subset \mathsf{SL}_2(\mathbb{Z}/4) \cong \mathsf{Out}(H, \mu_4)$ is the kernel of $\mathsf{SL}_2(\mathbb{Z}/4) \twoheadrightarrow \mathsf{SL}_2(\mathbb{Z}/2)$. After all, given any $h \in H$, the elements $i^k \rho_0(hX^{2\ell}Y^{2m})$ for $k, \ell, m \in \mathbb{Z}$ are exactly those that differ from h by a diagonal matrix. Hence, we obtain a well-defined map from tuples $(\alpha, \beta, \gamma) \in \mathcal{S}_0$ to $\mathbb{P}T/K$.

We claim that the resulting map $\mathcal{S}_0 \to \mathbb{P}T/K$ is injective. To see this, note that, since ψ^{-1} is diagonal, we may recover α from $\psi^{-1}(\omega)$ as follows: Write $\psi^{-1}(\omega)$ as a linear combination of terms of the form

$$[x_i, x_j]\{x_k, x_\ell\}, \qquad [x_i, x_j]\{x_k, x_\ell\}, \qquad \{x_i, x_j\}[x_k, x_\ell], \qquad \{x_i, x_j\}\{x_k, x_\ell\}, \tag{7.19}$$

where, as before, $\{x, y\} := xy + yx$ is the anticommutator. We see that

$$\frac{\text{Coefficient in } \psi^{-1}(\omega) \text{ of } \{x_0, x_1\}\{x_2, x_3\}}{\text{Coefficient in } \psi^{-1}(\omega) \text{ of } [x_0, x_1][x_2, x_3]} = \alpha. \tag{7.20}$$

This does not change if we rescale ω or apply an element of K. Similarly, we may recover β, γ from $\psi^{-1}(\omega)$. This proves injectivity.

It remains only to show that the action of $\mathsf{SL}_2(\mathbb{Z}/2)$ is identified with the action given in the theorem of $\mathfrak{S}_3$ under an isomorphism $\mathsf{SL}_2(\mathbb{Z}/2) \cong \mathfrak{S}_3$. Since $\mathfrak{S}_3$ clearly acts by automorphisms and faithfully so except at two points, this must be true, but we give an explicit identification.

There is a natural isomorphism $\mathsf{SL}_2(\mathbb{Z}/2) \cong \mathfrak{S}_3$ given by the induced permutation action on the nonzero elements of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. This identifies with the image of $\mathsf{Aut}(H, \mu_4)$ acting on the quotient $H/\langle X^2, Y^2, \mu_4\rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Next, note that ρ_0 takes $\mathsf{Aut}(H, \mu_4)$ to permutation matrices times diagonal matrices, and $\langle X^2, Y^2, \mu_4\rangle$ is the subgroup mapping to diagonal matrices under ρ_0. Thus, taking the image under ρ_0 and modding by diagonal matrices, the above permutation action becomes the image $\mathfrak{S}_3 \cong \mathfrak{S}_4/(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$ of the conjugation action of $\mathbb{P}(\rho_0(\mathsf{Aut}(H, \mu_4))) = \mathfrak{S}_4$ on $\mathbb{P}(\rho_0(H)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \{(ab)(cd) \mid \{a, b, c, d\} = \{1, 2, 3, 4\}\} \subset \mathfrak{S}_4$.

This shows that the action of $\mathsf{Aut}(H, \mu_4)$ on the LHS of (7.20) via ρ_0, which descends to $\mathsf{SL}_2(\mathbb{Z}/2) \cong \mathfrak{S}_3$, must apply the same element of $\mathfrak{S}_3$ to (α, β, γ) (up to elements of μ_4, and in fact, ± 1), using the RHS of (7.20). If we are slightly more careful, we can see that the action is the permutation action tensored with the sign representation.

This completes the proof of 7.9.

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References

[1] M. Dubois-Violette, Graded algebras and multilinear forms, C. R. Acad. Sci. Paris, Ser. I 341 (2005) 719–725.
[2] M. Dubois-Violette, Multilinear forms and graded algebras, J. Algebra 317 (1) (2007) 198–225. arXiv:math/0604279.
[3] M. Artin, W. Schelter, Graded algebras of global dimension 3, Adv. Math. 66 (1987) 171–216.
[4] V. Ginzburg, Calabi–Yau algebras, math/0612139.
[5] R. Coquereaux, R. Trinchero, On quantum symmetries of ADE graphs, hep-th/0401140.
[6] H. Derksen, J. Weyman, A. Zelevinsky, Quivers with potentials and their representations I: Mutations, arXiv:0704.0649, Selecta Math., New Series (in press).
[7] P.S. Aspinwall, L.M. Fidkowski, Superpotentials for quiver gauge theories, hep-th/0506041.
[8] R. Bocklandt, Graded Calabi Yau algebras of dimension 3, J. Pure Appl. Algebra 212 (1) (2008) 14–32.
[9] R. Berger, Koszulity for nonquadratic algebras, J. Algebra 239 (2) (2001) 705–734.
[10] R. Berger, R. Taillefer, Poincare–Birkhoff–Witt deformations of Calabi–Yau Algebras, J. Noncommut. Geom. 1 (2007) 241–270.
[11] W. Crawley-Boevey, M.P. Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (3) (1998) 605–635.
[12] W. Crawley-Boevey, DMV lectures on representations of quivers, preprojective algebras and deformations of quotient singularities, http://www.maths.leeds.ac.uk/~pmtwc/dmvlecs.pdf.
[13] J.T. Stafford, Regularity of algebras related to the Sklyanin algebra, Trans. AMS 341 (2) (1994) 895–916.
[14] A. Connes, M. Dubois-Violette, Noncommutative finite-dimensional manifolds. II Moduli space and structure of noncommutative 3-spheres, 2005, arXiv:math/0511337.
[15] A. Craw, D. Maclagan, R.R. Thomas, Moduli of McKay quiver representations. II: Grobner basis techniques, J. Algebra 316 (2) (2007) 514–535.
[16] I. Reiten, M. Van den Bergh, Two-dimensional tame and maximal orders of finite representation type, Mem. Amer. Math. Soc. 408 (1989) vii+72.
[17] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, in: LMS Lecture Note Series, vol. 146, 1990, p. 177.
[18] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.10, 2007.
[19] R. Berger, N. Marconnet, Koszul and Gorenstein properties for homogeneous algebras, Algebras and Representation theory 9 (2006) 67–97.
[20] R. Berger, M. Dubois-Violette, M. Wambst, Homogeneous algebras, J. Algebra 261 (2003) 172–185.
[21] E.K. Sklyanin, Some algebraic structures connected with the Yang–Baxter equation, Funktsional. Anal. i Prilozhen. 16 (4) (1982) 22–34.
[22] E.K. Sklyanin, Some algebraic structures connected with the Yang–Baxter equation. Representations of a quantum algebra, Funktsional. Anal. i Prilozhen. 17 (4) (1983) 34–48.
[23] S.P. Smith, J.T. Stafford, Regularity of the four-dimensional Sklyanin algebra, Compositio Math. 83 (3) (1992) 259–289.
[24] A.V. Odesskii, B.L. Feigin, Sklyanin's elliptic algebras, Funktsional. Anal. i Prilozhen. 23 (3) (1989) 45–54. 96.
[25] S.P. Smith, The four-dimensional Sklyanin algebras, in: Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part I, Antwerp, 1992, vol. 8, 1994, pp. 65–80.
[26] S.P. Smith, J.M. Staniszkis, Irreducible representations of the 4-dimensional Sklyanin algebra at points of infinite order, J. Algebra 160 (1993) 57–86.
[27] A. Connes, M. Dubois-Violette, Noncommutative finite-dimensional manifolds. I Spherical manifolds and related examples, Comm. Math. Phys. 230 (2002) 539–579.