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Wolfgang Ebeling

Poincaré series and monodromy of a two-dimensional quasihomogeneous hypersurface singularity

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Abstract. A relation is proved between the Poincaré series of the coordinate algebra of a two-dimensional quasihomogeneous isolated hypersurface singularity and the characteristic polynomial of its monodromy operator. For a Kleinian singularity not of type A_{2n}, this amounts to the statement that the Poincaré series is the quotient of the characteristic polynomial of the Coxeter element by the characteristic polynomial of the affine Coxeter element of the corresponding root system. We show that this result also follows from the McKay correspondence.

Introduction

S. M. Gusein-Zade, F. Delgado, and A. Campillo [GDC] have shown that for an irreducible plane curve singularity the Poincaré series of the ring of functions on the curve coincides with the zeta function of its monodromy transformation.

In this paper we show that there is also a relation between the Poincaré series of the coordinate algebra of a two-dimensional quasihomogeneous isolated hypersurface singularity and the characteristic polynomial of its monodromy operator.

Let (X, x) be a normal surface singularity with good $\mathbb{C}^*$-action. The coordinate algebra A is a graded algebra. We consider the Poincaré series $p_A(t)$ of A. Let $\{g; b; (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)\}$ be the orbit invariants of (X, x). We define

$$\psi_A(t) := (1-t)^{2-r} \prod_{i=1}^{r} (1 - t^{\alpha_i}),$$
$$\phi_A(t) := p_A(t)\psi_A(t).$$

Let (X, x) be a hypersurface singularity. Then $\phi_A(t)$ is a product of cyclotomic polynomials.

K. Saito [Sa3, Sa4] has introduced a duality between polynomials which are products of cyclotomic polynomials. He has shown that V. I. Arnold's strange duality

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W. Ebeling: Institut für Mathematik, Universität Hannover, Postfach 6009, 30060 Hannover, Germany. e-mail: ebeling@math.uni-hannover.de

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between the 14 exceptional unimodal hypersurface singularities is related to such a duality between the characteristic polynomials of the monodromy operators of the singularities. It is now well-known that Arnold's strange duality is related to the mirror symmetry of $K3$ surfaces (see e.g. [D5]).

The main results of the paper are the following. We show that the dual (in Saito's sense) of the rational function $\tilde{\phi}_A(t) := \phi_A(t)/(1-t)^{2g}$ is the characteristic polynomial of the monodromy operator of (X, x) (Theorem 1). Similar results can be proved for isolated complete intersection singularities (abbreviated ICIS in the sequel) of certain types (see Theorem 3 and 4).

If (X, x) is a Kleinian singularity not of type A_{2n}, then $\psi_A(t)$ is the characteristic polynomial of the affine Coxeter element of the corresponding root system and the above result implies that $\phi_A(t)$ is the characteristic polynomial of the Coxeter element. Hence the Poincaré series of a polyhedral group is the quotient of these two polynomials. We derive this result also directly from the McKay correspondence using ideas of the paper [Kos]. There are various formulas for Poincaré series of binary polyhedral groups in [Kn, Kos, M2, Sp], but this relation seems to be new.

1. Main results

Let (X, x) be a normal surface singularity with a good $\mathbb{C}^*$-action. So X is a normal two-dimensional affine algebraic variety over $\mathbb{C}$ which is smooth outside its *vertex* x. Its coordinate ring A has the structure of a graded $\mathbb{C}$-algebra $A = \bigoplus_{k=0}^{\infty} A_k$, $A_0 = \mathbb{C}$, and x is defined by the maximal ideal $\mathfrak{m} = \bigoplus_{k=1}^{\infty} A_k$.

According to I. Dolgachev [D2], there exist a simply connected Riemann surface $\mathcal{D}$, a discrete cocompact subgroup Γ of $\mathrm{Aut}(\mathcal{D})$ and a line bundle $\mathcal{L}$ on $\mathcal{D}$ to which the action of Γ lifts such that

$$A_k = H^0(\mathcal{D}, \mathcal{L}^k)^\Gamma.$$

Let $Z := \mathcal{D}/\Gamma$. By [P, Theorem 5.1] (see also [W2, Theorem 5.4.1]), there exist a divisor D_0 on Z, $p_1, \ldots, p_r \in Z$, and integers α_i, β_i with $0 < \beta_i < \alpha_i$ and $(\alpha_i, \beta_i) = 1$ for $i = 1, \ldots, r$ such that

$$A_k = L\left(kD_0 + \sum_{i=1}^{r} \left[k\frac{\alpha_i - \beta_i}{\alpha_i}\right] p_i\right).$$

Here $[x]$ denotes the largest integer $\leq x$, and $L(D)$ for a divisor D on Z denotes the linear space of meromorphic functions f on Z such that $(f) \geq -D$. We number the points p_i so that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_r$. Let g be the genus of Z and define $b :=$ degree $D_0 + r$. Then $\{g; b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}$ are called the *orbit invariants* of (X, x), cf. e.g. [W3]. Define $\mathrm{vdeg}(\mathcal{L}) := -b + \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i}$.

Now assume that (X, x) is Gorenstein. By [D4], there exists an integer R such that $\mathcal{L}^{-R}$ and the tangent bundle $T_{\mathcal{D}}$ of $\mathcal{D}$ are isomorphic as Γ-bundles and

$$R \cdot \mathrm{vdeg}(\mathcal{L}) = 2 - 2g - r + \sum_{i=1}^{r} \frac{1}{\alpha_i},$$
$$R\beta_i \equiv 1 \bmod \alpha_i, \quad i = 1, \ldots, r.$$

Following [D3, 3.3.15] we call R the *exponent* of (X, x). Since b and the β_i are determined by the α_i and the number R, we write the orbit invariants also as $g; \alpha_1, \ldots, \alpha_r$.

The Gorenstein surface singularities with good $\mathbb{C}^*$-action fall into three classes [D4,W3]:

(1) $\mathcal{D} = \mathbb{P}^1(\mathbb{C})$: Then $R = -2$ or $R = -1$, $g = 0$, and (X, x) is a Kleinian singularity.
(2) $\mathcal{D} = \mathbb{C}$: Then $R = 0$, $r = 0$, and $g = 1$. Hence (X, x) is a simply elliptic singularity [Sa1].
(3) $\mathcal{D} = \mathbb{H}$, the upper half plane: The remaining Gorenstein surface singularities with good $\mathbb{C}^*$-action belong to this class. We have $R \geq 1$.

We consider the *Poincaré series* of the algebra A

$$p_A(t) = \sum_{k=0}^{\infty} a_k t^k,$$

where $a_k = \dim A_k$. It is well known that $p_A(t)$ is a rational function and the order of the pole of $p_A(t)$ at $t = 1$ is equal to the dimension of A, hence equal to 2. Moreover, $p_A(t)$ has simple poles at the α_i-th roots of unity different from 1.

We define

$$\begin{aligned}
\psi_A(t) &:= (1-t)^{2-r} \prod_{i=1}^{r} (1 - t^{\alpha_i}), \\
\phi_A(t) &:= p_A(t)\psi_A(t), \\
\tilde{\phi}_A(t) &:= \frac{\phi_A(t)}{(1-t)^{2g}}.
\end{aligned}$$

Then $\phi_A(t)$ is a polynomial.

Now let (X, x) be an ICIS with weights $q_1, \ldots, q_n$ and degrees $d_1, \ldots, d_{n-2}$. Then its Poincaré series is given by (see e.g. [W3, Proposition (2.2.2)])

$$p_A(t) = \frac{\prod_{i=1}^{n-2}(1 - t^{d_i})}{\prod_{j=1}^{n}(1 - t^{q_j})}.$$

Hence $p_A(t)$, $\psi_A(t)$, $\phi_A(t)$, and $\tilde{\phi}_A(t)$ are rational functions of the form

$$\phi(t) = \prod_{m|h} (1 - t^m)^{\chi_m} \quad \text{for } \chi_m \in \mathbb{Z} \text{ and for some } h \in \mathbb{N}.$$

Given a rational function

$$\phi(t) = \prod_{m|h} (1 - t^m)^{\chi_m},$$

K. Saito [Sa3] has defined a dual rational function

$$\phi^*(t) = \prod_{k|h} (1 - t^k)^{-\chi_{h/k}}.$$

We are now able to state the main results of the paper.

Theorem 1. *Let (X, x) be a quasihomogeneous hypersurface singularity in $\mathbb{C}^3$. Then $\tilde{\phi}_A^*(t)$ is the characteristic polynomial of the classical monodromy operator of (X, x).*

The proof of Theorem 1 will be given in Sect. 2.

Let (X, x) be a Kleinian singularity. Then $g = 0$, and therefore $\tilde{\phi}_A(t) = \phi_A(t)$. It is well known that the Kleinian singularities correspond to root systems of type A_l, D_l, E_6, E_7, or E_8. The classical monodromy operator of (X, x) is the Coxeter element of the corresponding root system. The polynomial $\psi_A(t)$ is the characteristic polynomial of the affine Coxeter element of the corresponding root system (see [St, p. 591] or [Sp, 6.2], if (X, x) is not of type A_l). In the case A_{2n}, we have

$$\phi_A(t) = \frac{1 - t^{4n+2}}{1 - t^2}.$$

In this case, $\phi_A^*(t) \neq \phi_A(t)$. In all other cases one can verify that $\phi_A^*(t) = \phi_A(t)$ (cf. Table 1). Therefore we obtain from Theorem 1:

Theorem 2. *Let (X, x) be a Kleinian singularity not of type A_{2n}. Then $\phi_A(t)$ and $\psi_A(t)$ are the characteristic polynomials of the Coxeter element and the affine Coxeter element respectively of the corresponding root system. Hence $p_A(t)$ is the quotient of these polynomials.*

In Sect. 3 we shall give a direct proof of Theorem 2 using the McKay correspondence.

2. Poincaré series and monodromy

In this section we shall prove Theorem 1.

Let (X, x) be a Gorenstein surface singularity with a good $\mathbb{C}^*$-action. The residue of the Poincaré series $p_A(t)$ at a primitive α_i-th root of unity can be computed as follows.

Proposition 1. *Let (X, x) be Gorenstein and R be the exponent of (X, x). Let $\xi_i = \exp(2\pi\sqrt{-1}/\alpha_i)$. Then the residue of $p_A(t)$ at $t = \xi_i$ is equal to*

$$\sum_{\alpha_i | \alpha_j} \frac{\xi_i \cdot \xi_i^R}{\alpha_j (1 - \xi_i^R)}.$$

From Proposition 1 we can derive the following proposition generalizing [W1, Proposition (2.8)]). For integers $a_1, \ldots, a_r$ we denote by $\langle a_1, \ldots, a_r \rangle$ their least common multiple and by $(a_1, \ldots, a_r)$ their greatest common divisor.

Proposition 2. *Let A be the coordinate algebra of a Gorenstein surface singularity with good $\mathbb{C}^*$-action. Suppose that A is generated by 3 elements of weights q_1, q_2, q_3. For each $i = 1, \ldots, r$ let $\xi_i = \exp(2\pi\sqrt{-1}/\alpha_i)$. Then the Poincaré series is given by*

$$p_A(t) = \frac{(1 - t^d)}{(1 - t^{q_1})(1 - t^{q_2})(1 - t^{q_3})}$$

if and only if the following conditions hold:

(a) $(q_1, q_2, q_3) = 1$.
(b) $2g - 2 + r - \sum_{i=1}^{r} \frac{1}{\alpha_i} = \frac{Rd}{q_1 q_2 q_3}$.
(c) *For each i*

$$\sum_{\alpha_i | \alpha_j} \frac{\xi_i \cdot \xi_i^R}{\alpha_j (1 - \xi_i^R)} = \begin{cases} \frac{-\xi_i (1 - \xi_i^d)}{q_{i_1} (1 - \xi_i^{q_{i_2}})(1 - \xi_i^{q_{i_3}})} & \text{if } \alpha_i | q_{i_1} \text{ and } \alpha_i \not| q_{i_2}, q_{i_3}, \\ \frac{-d\xi_i}{q_{i_1} q_{i_2} (1 - \xi_i^{q_{i_3}})} & \text{if } \alpha_i | q_{i_1}, \alpha_i | q_{i_2}, \alpha_i \not| q_{i_3}. \end{cases}$$

(d) *For all i and j so that* $i \neq j$, $(q_i, q_j) | d$.
(e) $d = q_1 + q_2 + q_3 + R$.

Let (X, x) be an isolated hypersurface singularity in $\mathbb{C}^3$ given by a quasihomogeneous equation $f(z_1, z_2, z_3) = 0$ of degree d and weights q_1, q_2, q_3. Then the Poincaré series is given by

$$p_A(t) = \frac{(1 - t^d)}{(1 - t^{q_1})(1 - t^{q_2})(1 - t^{q_3})}.$$

From Proposition 2 one can easily derive the following proposition which was proven by P. Orlik and P. Wagreich [OW, 3.6 Proposition 1] using another method.

Proposition 3 (Orlik,Wagreich). *Let (X, x) be a quasihomogeneous isolated hypersurface singularity. Let $w_i = d/q_i = u_i/v_i$ where $(u_i, v_i) = 1$ und $u_i, v_i \geq 1$. Assume $1 \leq v_1 \leq v_2 \leq v_3$. Then the table below indicates the number of orbit invariants α of each type:*

$\alpha =$	(q_2, q_3)	(q_1, q_3)	(q_1, q_2)	q_3	q_2	q_1
$1 = v_1 = v_2 = v_3$	$\frac{d}{\langle q_2, q_3 \rangle}$	$\frac{d}{\langle q_1, q_3 \rangle}$	$\frac{d}{\langle q_1, q_2 \rangle}$			
$1 = v_1 = v_2 < v_3$	$\frac{d - q_2}{\langle q_2, q_3 \rangle}$	$\frac{d - q_1}{\langle q_1, q_3 \rangle}$	$\frac{d}{\langle q_1, q_2 \rangle}$	1		
$1 = v_1 < v_2 \leq v_3$	$\frac{d - q_2 - q_3}{\langle q_2, q_3 \rangle}$	$\frac{d - q_1}{\langle q_1, q_3 \rangle}$	$\frac{d - q_1}{\langle q_1, q_2 \rangle}$	1	1	
$1 < v_1 \leq v_2 \leq v_3$	$\frac{d - q_2 - q_3}{\langle q_2, q_3 \rangle}$	$\frac{d - q_1 - q_3}{\langle q_1, q_3 \rangle}$	$\frac{d - q_1 - q_2}{\langle q_1, q_2 \rangle}$	1	1	1

The blank entries are zero if q_i does not divide q_j for $j \neq i$. If $q_i | q_j$, then $(q_i, q_j) = q_i$ and we list those orbit invariants under the column headed (q_i, q_j).

On the other hand, we consider the characteristic polynomial of the monodromy operator of (X, x). Let $\mathcal{X}_t$ be a Milnor fibre of the singularity (X, x) and denote by $M := H_2(\mathcal{X}_t, \mathbb{Z})$ the corresponding Milnor lattice. Let $c : M \to M$ be the classical monodromy operator of the singularity (X, x). It is well-known that c is quasi-unipotent and therefore the eigenvalues of c are roots of unity. We write the characteristic polynomial $\phi_M(t) = \det(t I - c)$ of c as

$$\phi_M(t) = \prod_{m | h} (t^m - 1)^{\chi_m} \quad \text{for } \chi_m \in \mathbb{Z} \text{ and for some } h \in \mathbb{N}.$$

The characteristic polynomial $\phi_M(t)$ can be computed as follows [MO] (see also [Sa4]). Consider the rational function

$$\Phi(T) := T^{-d}\frac{(T^d - T^{q_1})(T^d - T^{q_2})(T^d - T^{q_3})}{(T^{q_1} - 1)(T^{q_2} - 1)(T^{q_3} - 1)}.$$

By [Sa2, (1.3) Theorem] there exist finitely many integers $m_1, \ldots, m_\mu$ such that

$$\Phi(T) = T^{m_1} + \ldots + T^{m_\mu}.$$

Then $\omega_i := \exp(2\pi\sqrt{-1}m_i/d)$, $i = 1, \ldots, \mu$, are the zeros of $\phi_M(t)$. Put $\Lambda_k := \omega_1^k + \ldots + \omega_\mu^k$ for $k \in \mathbb{N}$ and let $\omega := \exp(2\pi\sqrt{-1}/d)$. Then one has $\Lambda_k = \Phi(\omega^k)$. From this one can derive that

$$\Lambda_k = \left(\delta(kq_1 \bmod d)\frac{d}{q_1} - 1\right)\left(\delta(kq_2 \bmod d)\frac{d}{q_2} - 1\right)\left(\delta(kq_3 \bmod d)\frac{d}{q_3} - 1\right)$$

where δ is the delta function, i.e., $\delta(0) := 1$ and $\delta(x) := 0$ for $x \neq 0$. The numbers Λ_k and χ_m are related by the formula

$$\Lambda_k = \sum_{m|k} m\chi_m.$$

Proof of Theorem 1. We have

$$\tilde{\phi}_A(t) = \frac{(1 - t^d)(1 - t^{\alpha_1})\cdots(1 - t^{\alpha_r})}{(1 - t)^{2g-2+r}(1 - t^{q_1})(1 - t^{q_2})(1 - t^{q_3})}.$$

From Proposition 3 we conclude that $q_i|d$ or $q_i = \alpha_j$ for some j, $1 \le j \le r$, and that $\alpha_i|d$ or $\alpha_i = q_j$ for some j, $1 \le j \le 3$. Therefore we may assume that

$$\tilde{\phi}_A(t) = \frac{(1 - t^d)\prod_{\alpha_i|d}(1 - t^{\alpha_i})}{(1 - t)^{2g-2+r}\prod_{q_j|d}(1 - t^{q_j})}.$$

Hence

$$\tilde{\phi}_A^*(t) = \frac{(1 - t^d)^{2g-2+r}\prod_{q_j|d}(1 - t^{d/q_j})}{(1 - t)\prod_{\alpha_i|d}(1 - t^{d/\alpha_i})}.$$

Denote by $\tilde{\Lambda}_k$ the sum of the k-th powers of the roots of $\tilde{\phi}_A^*(t) = 0$. For the proof of Theorem 1 we must show that $\tilde{\Lambda}_k = \Lambda_k$ for all $k \in \mathbb{N}$.

(a) We first have

$$\tilde{\Lambda}_1 = -1 = \Lambda_1.$$

(b) Now suppose that $kq_{i_1} \equiv 0 \bmod d$ but $kq_{i_2} \not\equiv 0 \bmod d$, $kq_{i_3} \not\equiv 0 \bmod d$. Then we claim that for all i with $\alpha_i|d$ the number $\frac{d}{\alpha_i}$ does not divide k. For suppose the contrary. By Proposition 3 we have $\alpha_i = (q_{j_1}, q_{j_2})$ or $\alpha_i = q_{j_1}$. Now $\frac{d}{(q_{j_1}, q_{j_2})}|k$ implies $d|kq_{j_1}$ and $d|kq_{j_2}$, which contradicts our assumption. But $\alpha_i = q_{j_1}$ is only possible if $\alpha_i \not| d$. Therefore we have shown

$$\tilde{\Lambda}_k = -1 + \frac{d}{q_{i_1}} = \Lambda_k.$$

(c) We now consider the case that $kq_{i_1} \equiv 0 \bmod d$, $kq_{i_2} \equiv 0 \bmod d$, $kq_{i_3} \not\equiv 0 \bmod d$. By Proposition 3 and the same arguments as in (b), we see that $\alpha_i | d$ and $\frac{d}{\alpha_i} | k$ only if $\alpha_i = (q_{i_1}, q_{i_2})$. By Proposition 3 we therefore get

$$\begin{aligned}\tilde{\Lambda}_k &= -1 + \delta(d \bmod q_{i_1})\frac{d}{q_{i_1}} + \delta(d \bmod q_{i_2})\frac{d}{q_{i_2}} - \sum_{\alpha_j = (q_{i_1}, q_{i_2})} \frac{d}{\alpha_j} \\ &= -1 + \frac{d}{q_{i_1}} + \frac{d}{q_{i_2}} - \frac{d^2}{q_{i_1} q_{i_2}} \\ &= -\left(\frac{d}{q_{i_1}} - 1\right)\left(\frac{d}{q_{i_2}} - 1\right) \\ &= \Lambda_k.\end{aligned}$$

(d) Finally, assume that $kq_1 \equiv 0 \bmod d$, $kq_2 \equiv 0 \bmod d$, and $kq_3 \equiv 0 \bmod d$. Since the greatest common divisor of q_1, q_2, q_3 is 1, it follows that $d|k$. Then we have

$$\begin{aligned}\tilde{\Lambda}_k &= (2g - 2 + r)d + \sum_{j=1}^{3} \delta(d \bmod q_j)\frac{d}{q_j} - \sum_{i=1}^{r} \delta(d \bmod \alpha_i)\frac{d}{\alpha_i} - 1 \\ &= (2g - 2 + r)d + \sum_{j=1}^{3} \frac{d}{q_j} - \sum_{i=1}^{r} \frac{d}{\alpha_i} - 1.\end{aligned}$$

By Proposition 2(b) and (e) we get

$$\begin{aligned}\tilde{\Lambda}_k &= \frac{Rd^2}{q_1 q_2 q_3} + \sum_{j=1}^{3} \frac{d}{q_j} - 1 \\ &= \frac{d^2(d - q_1 - q_2 - q_3)}{q_1 q_2 q_3} + \sum_{j=1}^{3} \frac{d}{q_j} - 1 \\ &= \left(\frac{d}{q_1} - 1\right)\left(\frac{d}{q_2} - 1\right)\left(\frac{d}{q_3} - 1\right) \\ &= \Lambda_k.\end{aligned}$$

This completes the proof of Theorem 1. □

3. The McKay correspondence

In this section we shall derive Theorem 2 from the McKay correspondence.

Let (X, x) be a Kleinian singularity. Then $\mathcal{D} = \mathbb{P}^1(\mathbb{C})$ and Γ is a finite subgroup of $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) = PGL(2, \mathbb{C})$. We may assume that $\Gamma \subset PSU(2) \cong SO(3)$. Up to conjugacy, there are five classes of such groups: (1) $\mathcal{C}_n$, the cyclic group of order n, $n \geq 1$, (2) $\mathcal{D}_{l-2}$, the dihedral group of order $2(l-2)$, $l \geq 4$, (3) $\mathcal{T}$, the tetrahedral

Table 1. Kleinian singularities

	Γ	$g;\alpha_1,\dots,\alpha_r$	Weights	π_A
A_{2n}	$\mathcal{C}_{2n+1}$	$0;\ 2n+1,\ 2n+1$	$2,\ 2n+1,\ 2n+1/4n+2$	$4n+2/2$
A_{2n-1}	$\mathcal{C}_n$	$0;\quad n=1$ $0;\ n,\ n\ n>1$	$1,\ n,\ n/2n$	$2n/1$
D_l	$\mathcal{D}_{l-2}$	$0;\ 2,\ 2,\ l-2$	$2,\ l-2,\ l-1/2(l-1)$	$2\cdot 2(l-1)/1\cdot(l-1)$
E_6	$\mathcal{T}$	$0;\ 2,\ 3,\ 3$	$3,\ 4,\ 6/12$	$2\cdot 3\cdot 12/1\cdot 4\cdot 6$
E_7	$\mathcal{O}$	$0;\ 2,\ 3,\ 4$	$4,\ 6,\ 9/18$	$2\cdot 3\cdot 18/1\cdot 6\cdot 9$
E_8	$\mathcal{I}$	$0;\ 2,\ 3,\ 5$	$6,\ 10,\ 15/30$	$2\cdot 3\cdot 5\cdot 30/1\cdot 6\cdot 10\cdot 15$

group of order 12, (4) $\mathcal{O}$, the octahedral group of order 24, (5) $\mathcal{I}$, the icosahedral group of order 60.

If $R=-2$, then $\mathcal{L}^2\cong T_{\mathcal{D}}$ and Γ is a cyclic group of odd order $2n+1$, $n\geq 1$. The orbit invariants of (X,x) are $\{0;\ 1;\ (2n+1,n),\ (2n+1,n)\}$ and (X,x) is a singularity of type A_{2n}, which was excluded in Theorem 2.

Therefore it suffices to consider th e case $R=-1$. It is well known that there is a correspondence between the Kleinian singularities and the irreducible root systems as indicated in Table 1.

Let $R=-1$. Then we have $\mathcal{L}\cong T_{\mathcal{D}}$ and $A_k=S^{2k}(\mathbb{C}^2)^{\Gamma}$ where $S^{2k}(\mathbb{C}^2)$ denotes the $2k$-th symmetric power of $\mathbb{C}^2$ and the action of Γ on $S^{2k}(\mathbb{C}^2)$ is induced by its action on $\mathbb{P}^1(\mathbb{C})$. We relate the series $p_A(t)$ to another Poincaré series which is considered in [Kos]. Let $SU(2)\to PSU(2)$ be the usual double covering and let $G\subset SU(2)$ be the inverse image of $\Gamma\subset PSU(2)$. Let ρ_m be the representation of G on $S^m(\mathbb{C}^2)$ induced by its action on $\mathbb{C}^2$. Let $\gamma_0,\dots,\gamma_l$ be the equivalence classes of irreducible finite dimensional complex representations of G where γ_0 is the class of the trivial representation. For each integer $m\geq 0$ we have a decomposition $\rho_m=\sum_{i=0}^{l}v_{mi}\gamma_i$ with $v_{mi}\in\mathbb{Z}$. We associate to ρ_m the vector $v_m=(v_{m0},\dots,v_{ml})^t\in\mathbb{Z}^{l+1}$. As in [Kos, p. 211] we define

$$P_G(t):=\sum_{m=0}^{\infty}v_m t^m.$$

This is a formal power series with coefficients in $\mathbb{Z}^{l+1}$. We also put $P_G(t)_i:=\sum_{m=0}^{\infty}v_{mi}t^m$. Note that v_{m0} is the dimension of the G-invariant subspace of $S^m(\mathbb{C}^2)$. Since $-I\in G$, we have $v_{m0}=0$ for m odd. Therefore we get

$$p_A(t^2)=P_G(t)_0.$$

J. McKay [M1] has observed that if $\gamma: G\to SU(2)$ is the given 2-dimensional representation of G then the $(l+1)\times(l+1)$-matrix $B=(b_{ij})$, defined by decomposing the tensor products $\gamma_j\otimes\gamma=\bigoplus_i b_{ij}\gamma_i$ into irreducible components, satisfies $B=2I-C$ where C is the affine Cartan matrix of the corresponding root system. Moreover, the indexing is so that the additional vertex in the extended Coxeter–Dynkin diagram corresponding to the matrix C has index 0.

Proof of Theorem 2. From the Clebsch–Gordon formula one can derive that

$$Bv_m=v_{m+1}+v_{m-1}$$

for all non-negative integers m where $v_{-1} = 0$ [Kos, (3.3.1)]. This can be reformulated as follows (cf. [Kos, p. 222]). Let V denote the set of all formal power series $x = \sum_{m=0}^{\infty} x_m t^m$ with $x_m \in \mathbb{Z}^{l+1}$. This is a free module of rank $l+1$ over the ring R of formal power series with integer coefficients. Then $x = P_G(t)$ is a solution of the following linear equation in V:

$$((1+t^2)I - tB)x = v_0.$$

Let $M(t)$ be the matrix $((1+t^2)I - tB)$ and $M_0(t)$ be the matrix obtained by replacing the first column of $M(t)$ by $v_0 = (1, 0, \ldots, 0)^t$. Then Cramer's rule yields

$$P_G(t)_0 = \frac{\det M_0(t)}{\det M(t)}.$$

From [B, Ch. V, § 6, Exercice 3] we obtain

$$\det M(t) = \det(t^2 I - c_a), \qquad \det M_0(t) = \det(t^2 I - c),$$

where c is the Coxeter element and c_a is the affine Coxeter element of the corresponding root system. In the case $\Gamma = \mathcal{C}_n$, we have $l+1 = 2n$ and we assume that the numbering of $\gamma_0, \ldots, \gamma_l$ is so that the vertices of the extended Coxeter–Dynkin diagram (which is a cycle) corresponding to $\gamma_0, \ldots, \gamma_{n-1}$ are not connected with each other and the same holds for the vertices corresponding to $\gamma_n, \ldots, \gamma_l$. Note that this differs from the numbering in [B] but agrees with the numbering used for the discussion of the affine Coxeter element in the cases different from A_l in [St]. (The case A_l is excluded in that paper.) This proves Theorem 2. □

For a polynomial

$$\phi(t) = \prod_{m|h} (1 - t^m)^{\chi_m},$$

we use the symbolic notation

$$\pi := \prod_{m|h} m^{\chi_m}.$$

In the theory of finite groups, this symbol is known as a *Frame shape* (cf. [CN]). The Frame shapes π_A corresponding to the polynomials $\phi_A(t)$ are indicated in Table 1.

4. Generalizations to ICIS

In this section we shall consider generalizations of Theorem 1 to certain ICIS.

Let (X, x) be an ICIS in $\mathbb{C}^4$ given by quasihomogeneous equations $g = 0$ and $f = 0$ of degrees d_1 and d_2 respectively. As above, let M be the Milnor lattice and $c : M \to M$ be the monodromy operator of (X, x). By [GH] the characteristic polynomial of the monodromy operator can be computed as above using an appropriate rational function $\Phi(T)$. Similarly to the proof of Theorem 1 one can show:

Theorem 3. *Let (X, x) be a quasihomogeneous ICIS in $\mathbb{C}^4$ with weights q_1, q_2, q_3, q_4 and degrees d_1, d_2. Assume that $g(z_1, z_2, z_3, z_4) = z_1z_4 + z_2z_3$. Define*

$$\tilde{\phi}_A(t) := \frac{\phi_A(t)(1-t^{d_2})}{(1-t)^{2g}(1-t^{d_1})}, \quad \phi_M^\flat(t) := \frac{\phi_M(t)}{(1-t)}.$$

Then we have $\tilde{\phi}_A^*(t) = \phi_M^\flat(t)$.

Theorem 4. *Let (X, x) be a quasihomogeneous ICIS in $\mathbb{C}^4$ with weights q_1, q_2, q_3, q_4 and degrees d_1, d_2. Assume that either*

(A) $g(z_1, z_2, z_3, z_4) = z_1^q + z_2z_3$ and $f(z_1, z_2, z_3, z_4) = f'(z_1, z_2, z_3) + z_4^p$ for some integers $p, q \geq 2$ where $q|d_2$, or

(B) $g(z_1, z_2, z_3, z_4) = z_1^q + (z_2 - z_3)z_4$ and $f(z_1, z_2, z_3, z_4) = az_1^q + z_2(z_3 - z_4)$ for some $a \in \mathbb{C}$, $a \neq 0, 1$, and some integer $q \geq 2$ and $p := 2$.

Define

$$\tilde{\phi}_A(t) := \frac{\phi_A(t)(1-t^{d_2})^{p-1}(1-t^{\frac{d_1}{q}})(1-t^{\frac{d_2}{p}})}{(1-t)^{2g}(1-t^{d_1})(1-t^{\frac{d_2}{q}})^p},$$
$$\phi_M^\flat(t) := \frac{\phi_M(t)(1-t^q)^p}{(1-t)^{p-1}(1-t^{\langle p,q\rangle})^{(p,q)}}.$$

Then we have $\tilde{\phi}_A^*(t) = \phi_M^\flat(t)$.

Note that in the case $p = 2$, $p|q$, the polynomial $\phi_M^\flat(t)$ in Theorem 4 reduces to the corresponding polynomial of Theorem 3.

In case (A) of Theorem 4, (X, x) is a p-fold suspension and we use the following result which can be derived from [ESt, Theorem 10]. Let $(X, 0)$ be an ICIS in $\mathbb{C}^{n+2}$ of dimension n given by a map germ $F = (g, f) : (\mathbb{C}^{n+2}, 0) \to (\mathbb{C}^2, 0)$. Let $X' = g^{-1}(0)$ and assume that $(X', 0)$ is an isolated singularity. Let $p \in \mathbb{N}$, $p \geq 2$. The *p-fold suspension* of $(X, 0)$ is the ICIS $(\tilde{X}, 0)$ defined by $\tilde{F} = (\tilde{g}, \tilde{f}) : (\mathbb{C}^{n+2} \times \mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ where $\tilde{g}(y, z) = g(y)$ and $\tilde{f}(y, z) = f(y) + z^p$ for $(y, z) \in \mathbb{C}^{n+2} \times \mathbb{C}$. Let ϕ_M, ϕ'_M, and $\tilde{\phi}_M$ be the characteristic polynomials of the monodromy operators of the singularities $(X, 0)$, $(X', 0)$, and $(\tilde{X}, 0)$ respectively. Write

$$\phi_M(t) = \prod_{m|h}(1-t^m)^{\chi_m}, \quad \phi'_M(t) = \prod_{k|h'}(1-t^k)^{\chi'_k}.$$

By [ESt, loc.cit.] we have

$$\tilde{\phi}_M(t) = \prod_{m|h} \frac{(1-t^{\langle m,p\rangle})^{(m,p)\chi_m}}{(1-t^m)^{\chi_m}} \prod_{k|h'}(1-t^{\langle k,p\rangle})^{(k,p)\chi'_k}.$$

In case (B), (X, x) is a Brieskorn-Hamm ICIS. A Brieskorn-Hamm ICIS is a singularity $(V_B(w_1, \ldots, w_n), 0)$ (for integers $w_1, \ldots, w_n$, $w_i \geq 1$, $n \geq 3$), where

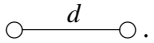
$$V_B(w_1, \ldots, w_n) := \{z \in \mathbb{C}^n \mid b_{i1}z_1^{w_1} + \ldots + b_{in}z_n^{w_n} = 0;\ i = 1, \ldots, n-2\}$$

Table 2. Simply elliptic ICIS

Name	$\{g; b; \}$	Weights	w_i	π_M	π_A
$\tilde{E}_8$	$\{1; 1; \}$	$1, 2, 3/6$	$2, 3, 6$	$2 \cdot 3 \cdot 6/1$	$1 \cdot 6/2 \cdot 3$
$\tilde{E}_7$	$\{1; 2; \}$	$1, 1, 2/4$	$2, 4, 4$	$2 \cdot 4^2/1$	$4/2$
$\tilde{E}_6$	$\{1; 3; \}$	$1, 1, 1/3$	$3, 3, 3$	$3^3/1$	$3/1$
$\tilde{D}_5$	$\{1; 4; \}$	$1, 1, 1, 1/2, 2$	$2, 2, 2, 2$	$2^4/1$	$2^2/1^2$

and $B = (b_{ij})$ is a sufficiently general $(n-2) \times n$-matrix of complex numbers. In our case (X, x) is analytically isomorphic to the singularity $(V_B(q, 2, 2, 2), 0)$ for a suitable matrix B. The orbit invariants of a Brieskorn–Hamm ICIS are given in [NR, Theorem 2.1] and the characteristic polynomial of the monodromy operator is computed in [H].

Example 1. Let (X, x) be a simply elliptic ICIS. Then (X, x) is one of the singularities indicated in Table 2 (cf. [Sa1]). Let $q_1, \ldots, q_n$ be the weights and d be the degree of the equation(s) of (X, x). Put $w_i = \frac{d}{q_{n+1-i}}$, $i = 1, \ldots, n$. Then $(X, x) \cong (V_B(w_1, \ldots, w_n), 0)$. Hence we can apply Theorem 1 or Theorem 4 if $n = 3$ or $n = 4$ respectively. We obtain $\psi_A(t) = (1-t)^2$, $\tilde{\phi}_A(t) = p_A(t)$, and $\phi_M(t) = p_A^*(t)$ if $n = 3$ and $\phi_M^\flat(t) = p_A^*(t)$ otherwise. The polynomial $\phi_A(t) = 1 + (b-2)t + t^2$ is the characteristic polynomial of a rotation of the Euclidian plane by the angle $\frac{2\pi}{d}$ and hence of the Coxeter element corresponding to the Coxeter graph [B]



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