Dimer models and the special McKay correspondence

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1 Introduction

Dimer models are introduced by string theorists (see e.g. [11, 12, 13, 16, 17, 18]) to study supersymmetric quiver gauge theories in four dimensions. A dimer model is a bicolored graph on a 2-torus encoding the information of a quiver with relations. If a dimer model is non-degenerate, then the moduli space $\mathcal{M}_\theta$ of stable representations of the quiver with dimension vector $(1, \ldots, 1)$ with respect to a generic stability parameter θ in the sense of King [23] is a smooth toric Calabi-Yau 3-fold [20]. The convex hull of one-dimensional cones of the fan describing this toric manifold is a lattice polygon described in purely combinatorial way using perfect matchings. Although the structure of the fan is not determined by this lattice polygon, any fan structure gives an equivalent derived categories of coherent sheaves [1, 3].

In this paper, we study the behavior of the dimer model under the operation of removing a vertex from the lattice polygon and taking the convex hull of the rest. This generalizes the work of Gulotta [15] where he studies the operation of removing a triangle from the lattice polygon, and the special McKay correspondence plays an essential role in this generalization. The main result in this paper is the following:

Theorem 1.1. *For any lattice polygon, there is a dimer model such that the derived category $D^b \operatorname{mod} \mathbb{C}\Gamma$ of finitely-generated modules over the path algebra of the resulting quiver Γ with relations is equivalent to the derived category of coherent sheaves on a toric Calabi-Yau 3-fold determined by the lattice polygon.*

The idea for the proof is to show that the derived equivalence after the removal of a vertex is equivalent to that before the removal. This allows us to to construct a dimer model for any lattice polygon starting from that for a sufficiently large triangle by successively removing vertices.

The organization of this paper is as follows: We recall the special McKay correspondence for a finite subgroup of $GL_2(\mathbb{C})$ in section 2, and the relation with continued fractions in the case of an abelian subgroup in section 3. We collect basic definitions on dimer models in section 4, and discuss consistency conditions in section 5. In section 6, we introduce the concept of large hexagons, which will be our main technical tool. The operation on the dimer model corresponding to the removal of a vertex from a lattice polygon will be described in section 7. This operation will be shown to preserve the consistency condition in section 8. In sections from 9 to 13, we show that the following property is preserved under this operation; the tautological bundle on the moduli space is a tilting object, whose endomorphism algebra is isomorphic to the path algebra with

relations. The proof of Theorem 1.1 is given in section 14. In section 15, we generalize the main result of [8].

2 The special McKay correspondence

Let G be a finite small subgroup of $GL_2(\mathbb{C})$ and $Y = G$-$\mathrm{Hilb}(\mathbb{C}^2)$ be the Hilbert scheme of G-orbit in $\mathbb{C}^2$ [25]. Then the Hilbert-Chow morphism

$$\pi : Y \to X = \mathrm{Spec}\,\mathbb{C}[x,y]^G$$

gives the minimal resolution of the quotient singularity [19].

Definition-Lemma 2.1 (Esnault [10]). *Let $\mathcal{M}$ be a sheaf on Y and $\mathcal{M}^\vee$ be its dual sheaf. Then there exists a reflexive module M on X such that $\mathcal{M} \cong \widetilde{\mathcal{M}} := \pi^* M/\text{torsion}$ if and only if the following three conditions are satisfied:*

1. $\mathcal{M}$ is locally-free.

2. $\mathcal{M}$ is generated by global sections.

3. $H^1((\mathcal{M})^\vee \otimes \omega_Y) = 0$.

In this case $\mathcal{M}$ is said to be full.

Since any reflexive module on X can be written as $(\rho^\vee \otimes \mathbb{C}[x,y])^G$ for some representation ρ of G, indecomposable full sheaves on Y are in one-to-one correspondence with irreducible representations of G.

Definition 2.2. An object $\mathcal{E}$ in a triangulated category $\mathcal{T}$ is *acyclic* if

$$\mathrm{Ext}^k(\mathcal{E}, \mathcal{E}) = 0, \qquad k \neq 0.$$

It is a *generator* if for any other object $\mathcal{F}$,

$$\mathrm{Ext}^k(\mathcal{E}, \mathcal{F}) = 0$$

for any $k \in \mathbb{Z}$ implies $\mathcal{F} \cong 0$. An acyclic generator is said to be a *tilting object*.

A tilting object induces a derived equivalence:

Theorem 2.3 (Rickard [26], Bondal [2]). *Let $\mathcal{E}$ be a tilting object in the derived category $D^b \operatorname{coh} X$ of coherent sheaves on a smooth quasi-projective variety X. Then $D^b \operatorname{coh} X$ is equivalent to the derived category of finitely-generated modules over the endomorphism algebra $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$.*

The following theorem is the McKay correspondence as a derived equivalence for a finite subgroup of $SL_2(\mathbb{C})$:

Theorem 2.4 (Kapranov and Vasserot [21], see also Bridgeland, King and Reid [4]). *When G is a finite subgroup of $SL_2(\mathbb{C})$, the direct sum of indecomposable full sheaves is a tilting object whose endomorphism ring is Morita equivalent to the crossed product algebra $G \ltimes \mathbb{C}[x, y]$.*

This is no longer true when $G \not\subset SL_2(\mathbb{C})$, and one has to restrict the class of full sheaves. The following theorem is due to Wunram:

Theorem 2.5 (Wunram [31, Main Result]). *Let $E = \bigcup_{i=1}^{r} E_i$ be the decomposition into irreducible components of the exceptional set E. Then for every curve E_i there exists exactly one indecomposable reflexive module M_i such that the corresponding full sheaf $\widetilde{M_i} = \pi^* M_i / \text{torsion}$ satisfies the conditions $H^1((\widetilde{M})^\vee) = 0$ and*

$$c_1(\widetilde{M_i}) \cdot E_j = \delta_{ij}$$

A full sheaf is said to be *special* if there is an index $1 \leq i \leq r$ such that $\mathcal{M} = \mathcal{M}_i$ or it is isomorphic to the structure sheaf $\mathcal{O}_Y$. The special full sheaf $\mathcal{O}_Y$ corresponds to the trivial representation and is denoted by $\mathcal{M}_0$. Special full sheaves are characterized as follows:

Theorem 2.6 (Wunram [31, Theorem 1.2]). *An indecomposable full sheaf $\mathcal{M}$ is special if and only if $H^1(\mathcal{M}^\vee) = 0$.*

An irreducible representation ρ of G is said to be special if the corresponding full sheaf $\mathcal{M}_\rho = \pi^* \left((\rho^\vee \otimes \mathbb{C}[x, y])^G \right) / (\text{torsion})$ is special. Note that $\mathcal{M}_\omega$ is isomorphic to the dualizing sheaf of Y, where $\omega = \det(\rho_{\text{Nat}}^\vee)$ is the determinant of the dual of the natural representation $\rho_{\text{Nat}} : G \hookrightarrow GL_2(\mathbb{C})$.

Theorem 2.7 (Wunram [31, Theorem 1.2]). *An irreducible representation ρ of G is special if and only if the natural inclusion map $\mathcal{M}_\rho \otimes \mathcal{M}_\omega \to \mathcal{M}_{\rho \otimes \omega}$ is an isomorphism.*

Special full sheaves generate the derived category of coherent sheaves on Y:

Theorem 2.8 (Van den Bergh [28, Theorem B]). *The direct sum of indecomposable special full sheaves is a tilting object.*

The special McKay correspondence as a derived equivalence is studied by Craw [6] and Wemyss [29].

3   Specials and continued fractions

For relatively prime integers $0 < q < n$, consider the cyclic small subgroup $G = \langle \frac{1}{n}(1, q) \rangle$ of $GL_2(\mathbb{C})$ generated by

$$\frac{1}{n}(1, q) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^q \end{pmatrix},$$

where ζ is a primitive n-th root of unity. We label the irreducible representations of G by elements $a \in \mathbb{Z}/n\mathbb{Z}$ so that a sends the above generator to ζ^a. We recall how the geometry of the minimal resolution of $\mathbb{C}^2/G$ is described by the continued fraction expansion of n/q in this section, and collect lemmas which will later be useful.

Define integers r, $b_1, \ldots, b_r$ and $i_0, \ldots, i_{r+1}$ as follows: Put $i_0 := n$, $i_1 := q$ and define i_{t+2}, b_{t+1} inductively by

$$i_t = b_{t+1}i_{t+1} - i_{t+2} \quad (0 < i_{t+2} < i_{t+1})$$

until we finally obtain $i_r = 1$ and $i_{r+1} = 0$. This gives a continued fraction expansion

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}$$

and $-b_t$ is the self intersection number of the t-th irreducible exceptional curve C_t.

For a general representation d, the degrees of the full sheaf $\mathcal{L}_d$ are given in the following way:

Theorem 3.1 (Wunram [30, Theorem]). *For an integer d with $0 \le d < n$, there is a unique expression*

$$d = d_1 i_1 + d_2 i_2 + \cdots + d_r i_r$$

where $d_i \in \mathbb{Z}_{\ge 0}$ are non-negative integers satisfying

$$0 \le \sum_{t > t_0} d_t i_t < i_{t_0}$$

for any t_0. Then, we have

$$\deg \mathcal{M}_d|_{C_t} = d_t.$$

for any t.

Remark 3.2. Non-negative integers d_i in Theorem 3.1 can be computed by $e_0 = d$ and

$$e_t = d_{t+1}i_{t+1} + e_{t+1}, \qquad 0 \le e_{t+1} < i_{t+1}$$

for $0 \le t \le r - 1$.

Corollary 3.3. *Special representations are given by $i_0 \equiv i_{r+1}, i_1, \ldots, i_r$, and the labeling of specials and irreducible components are related by*

$$\deg \mathcal{M}_{i_s}|_{C_t} = \delta_{st}.$$

Lemma 3.4 (Wunram [30, Lemma 1]). *A sequence $(d_1, \ldots, d_r) \in (\mathbb{Z}_{\ge 0})^r$ is obtained from an integer $d \in [0, n-1]$ as in the previous theorem if and only if the following hold:*

- $0 \le d_t \le b_t - 1$ *for any t.*

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- If $d_s = b_s - 1$ and $d_t = b_t - 1$ for $s < t$, then there is l with $s < l < t$ and $d_l \leq b_l - 3$.

Let $q' \in [0, n-1]$ be the integer with $qq' \equiv 1 \mod n$. Then $\langle \frac{1}{n}(1, q) \rangle$ coincides with $\langle \frac{1}{n}(q', 1) \rangle$ as a subgroup of $GL_2(\mathbb{C})$. Introduce the dual sequence $j_0, \ldots, j_{r+1}$ by $j_{r+1} = n$, $j_r = q'$, and

$$j_t = j_{t-1}b_{t-1} - j_{t-2}, \qquad 0 \leq j_{t-2} < j_{t-1}.$$

Then one has $j_1 = 1$ and $j_0 = 0$.

Lemma 3.5 (Wunram [30, Lemma 2]). *Let $d = d_1 i_1 + \cdots + d_r i_r$ be as in Theorem 3.1 and put $f = d_1 j_1 + \cdots + d_r j_r$. Then one has $qf \equiv d \mod n$.*

In particular, special representations are given by

$$i_0 \equiv q j_0, \ i_1 \equiv q j_1, \ \ldots, \ i_r \equiv q j_r. \tag{1}$$

Note that $(i_t)_{t=0}^r$ is decreasing and $(j_t)_{t=0}^r$ is increasing.

4 Dimer models and quivers

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a real two-torus equipped with an orientation. A *bicolored graph* on T consists of

- a set $B \subset T$ of black nodes,

- a set $W \subset T$ of white nodes, and

- a set E of edges, consisting of embedded closed intervals e on T such that one boundary of e belongs to B and the other boundary belongs to W. We assume that two edges intersect only at the boundaries.

A bicolored graph on T is called a *dimer model* if the set of edges divide T into simply-connected polygons.

A *quiver* consists of

- a set V of vertices,

- a set A of arrows, and

- two maps $s, t : A \to V$ from A to V.

For an arrow $a \in A$, $s(a)$ and $t(a)$ are said to be the *source* and the *target* of a respectively. A *path* on a quiver is an ordered set of arrows $(a_n, a_{n-1}, \ldots, a_1)$ such that $s(a_{i+1}) = t(a_i)$ for $i = 1, \ldots, n-1$. We also allow for a path of length zero, starting and ending at the same vertex. The *path algebra* $\mathbb{C}Q$ of a quiver $Q = (V, A, s, t)$ is the algebra spanned by the set of paths as a vector space, and the multiplication is defined by the concatenation of paths;

$$(b_m, \ldots, b_1) \cdot (a_n, \ldots, a_1) = \begin{cases} (b_m, \ldots, b_1, a_n, \ldots, a_1) & s(b_1) = t(a_n), \\ 0 & \text{otherwise.} \end{cases}$$

A *quiver with relations* is a pair of a quiver and a two-sided ideal $\mathcal{I}$ of its path algebra. For a quiver $\Gamma = (Q, \mathcal{I})$ with relations, its path algebra $\mathbb{C}\Gamma$ is defined as the quotient algebra $\mathbb{C}Q/\mathcal{I}$.

A dimer model (B, W, E) encodes the information of a quiver $\Gamma = (V, A, s, t, \mathcal{I})$ with relations in the following way: The set V of vertices is the set of connected components of the complement $T \setminus (\bigcup_{e \in E} e)$, and the set A of arrows is the set E of edges of the graph. The directions of the arrows are determined by the colors of the nodes of the graph, so that the white node $w \in W$ is on the right of the arrow. In other words, the quiver is the dual graph of the dimer model equipped with an orientation given by rotating the white-to-black flow on the edges of the dimer model by minus 90 degrees.

The relations of the quiver are described as follows: For an arrow $a \in A$, there exist two paths $p_+(a)$ and $p_-(a)$ from $t(a)$ to $s(a)$, the former going around the white node connected to $a \in E = A$ clockwise and the latter going around the black node connected to a counterclockwise. Then the ideal $\mathcal{I}$ of the path algebra is generated by $p_+(a) - p_-(a)$ for all $a \in A$.

A *representation* of Γ is a module over the path algebra $\mathbb{C}\Gamma$ with relations. In other words, a representation is a collection $((V_v)_{v \in V}, (\psi(a))_{a \in A})$ of vector spaces V_v for $v \in V$ and linear maps $\psi(a) : V_{s(a)} \to V_{t(a)}$ for $a \in A$ satisfying relations in $\mathcal{I}$. The *dimension vector* of a representation $((V_v)_{v \in V}, (\psi(a))_{a \in A})$ is given by $(\dim V_v)_{v \in V} \in \mathbb{Z}^V$. We regard $\mathbb{Z}^V$ as the quotient of the Grothendieck group of the abelian category $\mathbb{C}\Gamma$-mod of finite dimensional representations of $\mathbb{C}\Gamma$

For $\theta \in \mathrm{Hom}(\mathbb{Z}^V, \mathbb{Z})$ such that $\theta(1, \ldots, 1) = 0$, a $\mathbb{C}\Gamma$-module M with dimension vector $(1, \ldots, 1)$ is said to be θ-*stable* if for any non-trivial submodule $N \subsetneq M$, one has $\theta(N) > 0$. M is θ-*semistable* if $\theta(N) \geq 0$ holds instead of $\theta(N) > 0$. This stability condition is introduced by King [23] to construct the moduli space $\mathcal{M}_\theta$ (resp. $\overline{\mathcal{M}_\theta}$) of θ-stable (resp. θ-semistable) representations. A stability parameter θ is said to be *generic* if semistability implies stability

A *small loop* on a quiver coming from a dimer model is the product of arrows surrounding a node of the dimer model. A path p is said to be *minimal* if it is not equivalent to a path containing a small loop. A path p is said to be *minimum* if any path from $s(p)$ to $t(p)$ is equivalent to the product of p and a power of a small loop. For a pair of vertices of the quiver, a minimum path from one vertex to another may not exist, and will always be minimal when it exists.

Small loops starting from a fixed vertex are equivalent to each other. Hence the sum ω of small loops over the set of vertices is a well-defined element of the path algebra. ω belongs to the center of the path algebra and there is the universal map

$$\mathbb{C}\Gamma \to \mathbb{C}\Gamma[\omega^{-1}]$$

into the localization of the path algebra by the multiplicative subset generated by ω. Two paths are called *weakly equivalent* if they are mapped to the same element in $\mathbb{C}\Gamma[\omega^{-1}]$.

A *perfect matching* (or a *dimer configuration*) on a dimer model $G = (B, W, E)$ is a subset D of E such that for any node $v \in B \cup W$, there is a unique edge $e \in D$ connected to v. A dimer model is said to be *non-degenerate* if for any edge $e \in E$, there is a perfect matching D such that $e \in D$.

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Consider the bicolored graph $\widetilde{G}$ on $\mathbb{R}^2$ obtained from G by pulling-back by the natural projection $\mathbb{R}^2 \to T$, and identify the set of perfect matchings of G with the set of periodic perfect matchings of $\widetilde{G}$. Fix a reference perfect matching D_0. Then for any perfect matching D, the union $D \cup D_0$ divides $\mathbb{R}^2$ into connected components. The height function h_{D,D_0} is a locally-constant function on $\mathbb{R}^2 \setminus (D \cup D_0)$ which increases (resp. decreases) by 1 when one crosses an edge $e \in D$ with the black (resp. white) node on his right or an edge $e \in D_0$ with the white (resp. black) node on his right. This rule determines the height function up to additions of constants. The height function may not be periodic even if D and D_0 are periodic, and the *height change* $h(D, D_0) = (h_x(D, D_0), h_y(D, D_0)) \in \mathbb{Z}^2$ of D with respect to D_0 is defined as the difference

$$h_x(D, D_0) = h_{D,D_0}(p + (1, 0)) - h_{D,D_0}(p),$$
$$h_y(D, D_0) = h_{D,D_0}(p + (0, 1)) - h_{D,D_0}(p)$$

of the height function, which does not depend on the choice of $p \in \mathbb{R}^2 \setminus (D \cup D_0)$. More invariantly, height changes can be considered as an element of $H^1(T, \mathbb{Z})$. The dependence of the height change on the choice of the reference matching is given by

$$h(D, D_1) = h(D, D_0) - h(D_1, D_0)$$

for any three perfect matchings D, D_0 and D_1. We will often suppress the dependence of the height difference on the reference matching and just write $h(D) = h(D, D_0)$.

For a fixed reference matching D_0, the characteristic polynomial of G is defined by

$$Z(x, y) = \sum_{D \in \mathrm{Perf}(G)} x^{h_x(D)} y^{h_y(D)}.$$

It is a Laurent polynomial in two variables, whose Newton polygon coincides with the convex hull of the set

$$\{(h_x(D), h_y(D)) \in \mathbb{Z}^2 \mid D \text{ is a perfect mathing}\}$$

consisting of height changes of perfect matchings of the dimer model. A perfect matching D is said to be a *corner perfect matching* if the height change $h(D)$ is on the vertex of the Newton polygon of the characteristic polynomial. The *multiplicity* of a perfect matching D is the number of perfect matchings whose height change is the same as D.

A perfect matching can be considered as a set of walls which block some of the arrows; for a perfect matching D, let Q_D be the subquiver of Q whose set of vertices is the same as Q and whose set of arrows consists of $A \setminus D$ (recall that $A = E$). The path algebra $\mathbb{C}Q_D$ of Q_D is a subalgebra of $\mathbb{C}Q$, and the ideal $\mathcal{I}$ of $\mathbb{C}Q$ defines an ideal $\mathcal{I}_D = \mathcal{I} \cap \mathbb{C}Q_D$ of $\mathbb{C}Q_D$. A path $p \in \mathbb{C}Q$ is said to be an allowed path with respect to D if $p \in \mathbb{C}Q_D$.

To a perfect matching, one can associate a representation of the quiver with dimension vector $(1, \ldots, 1)$ by sending any allowed path to 1 and other paths to 0. A perfect matching is said to be *simple* if this quiver representation is simple, i.e., has no non-trivial subrepresentation. This is equivalent to the condition that there is an allowed path starting and ending at any given pair of vertices.

A *zig-zag path* is a path on a dimer model which makes a maximum turn to the right on a white node and to the left on a black node. Note that it is not a path on a quiver. We assume that a zig-zag path does not have an endpoint, so that it is either periodic or infinite in both directions. Figure 1 shows an example of a part of a dimer model and a zig-zag path on it.
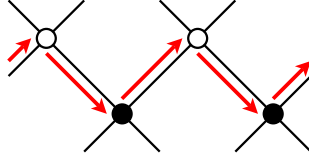

Figure 1: A zig-zag path

For a given zig-zag path z, assume that there is a perfect matching D_0 which intersect half of the edges constituting z (i.e., every other edge of z belongs to D_0). Then the height change of any other perfect matching D with respect to D_0 in the direction of z is positive:

$$\langle h(D, D_0), [z] \rangle \geq 0. \tag{2}$$

Here, $[z] \in H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$ is the homology class of $[z]$, which is paired with the height change considered as an element of $H^1(T, \mathbb{Z})$. This follows from the fact that as one goes around T along z, one crosses no edge in D_0 and every edge one crosses has a white node on your right. In this way, every zig-zag path gives an inequality which bound the Newton polygon of the characteristic polynomial. The homology class $[z]$ of a zig-zag path considered as an element of $\mathbb{Z}^2$ will be called its *slope*.

The main theorem of [20] states that when a dimer model is non-degenerate, then the moduli space $\mathcal{M}_\theta$ is a smooth Calabi-Yau toric 3-fold for general θ. A toric divisor in $\mathcal{M}_\theta$ gives a perfect matching so that the stabilizer group of the divisor is determined by the height change of the perfect matching.

As an example, consider the dimer model in Figure 2. The corresponding quiver is shown in Figure 3, whose relations are given by

$$\mathcal{I} = (dbc - cbd, dac - cad, adb - bda, acb - bca).$$

This dimer model is non-degenerate, and has four perfect matchings $D_0, \ldots, D_3$ shown in Figure 4. Their height changes with respect to D_0 are given in Figure 5, so that the
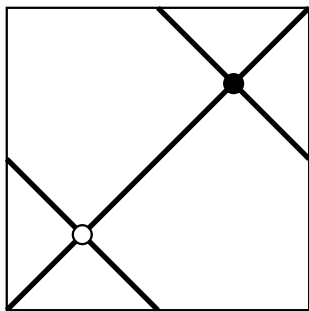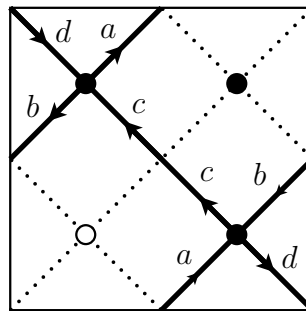

Figure 2: A dimer model

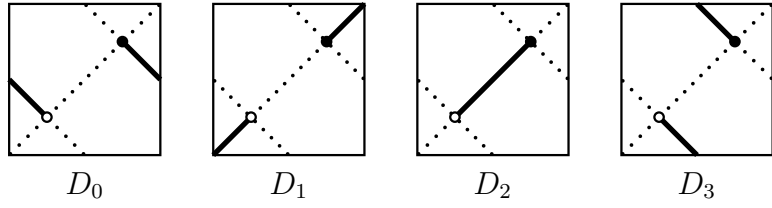Figure 3: The corresponding quiver
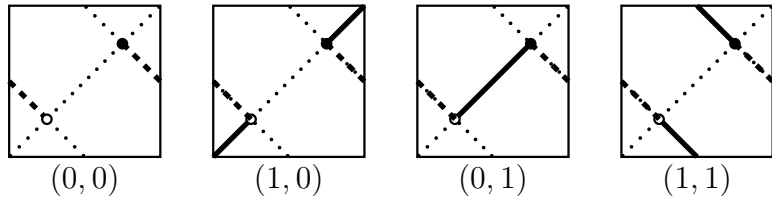
Figure 4: Four perfect matchings
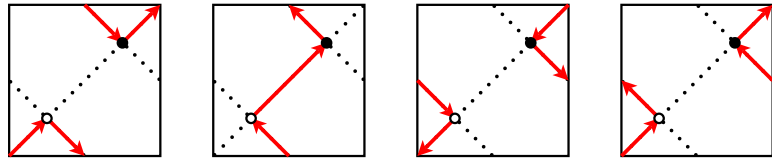

Figure 5: Height changes
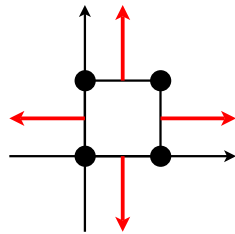

Figure 6: Four zig-zag paths

Figure 7: The lattice polygon

characteristic polynomial is

$$Z(x, y) = 1 + x + y + xy.$$

This dimer model has four zig-zag paths as shown in Figure 6. Note that the homology class of these four paths are normal to the edges of the Newton polygon the characteristic polynomial as shown in Figure 7.

5 Consistency conditions for dimer models

To define the consistency, we consider intersections of zigzag paths. Here we regard a zigzag path on the universal cover as a sequence (e_i) of edges e_i parameterized by $i \in \mathbb{Z}$, up to translations of i.

Definition 5.1. Let $z = (e_i)$ and $w = (f_i)$ be two zigzag paths on the universal cover. We say that z and w intersect if there are $i, j \in \mathbb{Z}$ with $e_i = f_j$ such that if u, v are the maximum and the minimum of t with $e_{i+t} = f_{j-t}$ respectively, then $u - v \in 2\mathbb{Z}$. In this case, the sequence $(e_{i+v} = f_{j-v}, e_{i+v+1} = f_{j-v-1}, \ldots e_{i+u} = f_{j-u})$ of intersections is counted as a single intersection. We say that z has a self-intersection if there is a pair $i \neq j$ with $e_i = e_j$ such that the directions of z at e_i and e_j are opposite, and $u - v \in 2\mathbb{Z}$ for u and v defined similarly as above. We say that z is homologically trivial if the map $i \mapsto e_i$ is periodic.

Note that if $u - v > 0$ in the above definition, then the nodes between e_v and e_u are divalent. According to this definition, there are cases where z and w have a common nodes or common edges, but they do not intersect as shown in Figure 8.
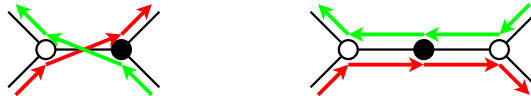

Figure 8: Examples of an intersection (left) and a non-intersection (right)

The consistency condition that we use in this paper is the following:

Definition 5.2. A dimer model is said to be *consistent* if

- there is no homologically trivial zig-zag path,

- no zig-zag path has a self-intersection on the universal cover, and

- no pair of zig-zag paths intersect each other on the universal cover in the same direction more than once.

Here, the third condition means that if a pair (z, w) of zig-zag paths has two intersections a and b and the zig-zag path z points from a to b, then the other zig-zag path w must point from b to a. We assume that a consistent dimer model is non-degenerate in this section. We prove this assumption in Proposition 6.2 in section 6 using Lemma 5.10 and Lemma 5.11, neither of which needs this assumption for the proof.

Figure 9: A homologically trivial zig-zag path

Figure 10: An inconsistent dimer model

Figure 11: A pair of zig-zag paths in the same direction intersecting twice

Figure 9 shows a part of an inconsistent dimer model which contains a homologically trivial zig-zag path. Figure 10 shows an inconsistent dimer model, which contains a pair of zig-zag paths intersecting in the same direction twice as in Figure 11.

On the other hand, a pair of zig-zag paths going in the opposite direction may intersect twice in a consistent dimer model. Figure 13 shows a pair of such zig-zag paths on a consistent dimer model in Figure 12.

Figure 12: A consistent non-isoradial dimer model

Figure 13: A pair of zig-zag paths in the opposite direction intersecting twice

For a node in a dimer model, the set of zig-zag paths going through the edges adjacent to the node has a natural cyclic ordering.

Definition 5.3 (Gulotta [15, section 3.1])**.** A dimer model is *properly ordered* if

- there is no homologically trivial zig-zag path,

- no pair of zig-zag paths in the same homology class have a common node, and

- the cyclic order of the zig-zag paths around any node of the dimer model is compatible with the cyclic order determined by their slopes.

A consistent dimer model is properly ordered:

Lemma 5.4. *In a consistent dimer model, the cyclic order of the zig-zag paths around any node of the dimer model is compatible with the cyclic order determined by their slopes.*

Proof. Let z_1, z_2 and z_3 be a triple of zig-zag paths passing through a node of the dimer model along neighboring edges at the node whose cyclic order around the node does not respect the cyclic order determined by their slopes. Then two of them must intersect more than once in the same direction on the universal cover. $\square$

Remark 5.5. Although a pair of zig-zag paths in the opposite direction intersecting more than once is allowed in a consistent dimer model, such a pair will make an *isoradial embedding* of the dimer model impossible: An isoradial embedding of a graph is an embedding into a torus (or a plane) so that each face is inscribed in a circle of unit length. It is known by Kenyon and Schlenker [22] that the existence of an isoradial embedding of a bicolored graph is equivalent to the absence of a pair of zig-zag paths with more than one common edges.

We first show that under the existence of a perfect matching, the consistency condition is equivalent to the following condition introduced by Mozgovoy and Reineke [24, Condition 4.12]:

Definition 5.6. A dimer model is said to satisfy the *first consistency condition* of Mozgovoy and Reineke if weakly equivalent paths are equivalent.

Mozgovoy and Reineke proved that the path algebra of the quiver with relation coming from a dimer model is a Calabi-Yau 3 algebra in the sense of Ginzburg [14] assuming the consistency condition and one extra condition which they call the second consistency condition. The latter condition has been shown to follow from the former by Davison [9]. Broomhead has proved the Calabi-Yau 3 property of the path algebra with relations associated with an isoradial dimer model [5]. Calabi-Yau 3 condition implies the derived equivalence by Bridgeland, King and Reid [4] and Van den Bergh [27].

To obtain a criterion for the minimality of a path, we discuss the intersection of a path of the quiver and a zigzag path. Note that paths of the quiver and zigzag paths are both regarded as sequences of arrows of the quiver, where the former are finite and the latter are infinite.

Definition 5.7. Let $p = a_1 a_2 \ldots$ be a path of the quiver and $z = (b_i)_{i \in \mathbb{Z}}$ be a zigzag path. We say p intersects with z at an arrow a if there are i, j with $a = a_i = b_j$, satisfying the following condition: If u, v denote the maximum and the minimum of t with $a_{i+t} = b_{j-t}$ respectively, then $u - v$ is even. In this case, the sequence $(a_{i+v} = b_{j-v}, \ldots, a_{i+u} = b_{j-u})$ is counted as a single intersection.

Figure 14 shows an example of a non-intersection; the path shown in red does not intersect with the zig-zag path shown in blue. Note that the red path is equivalent to the green path, which does not have a common edge (or an arrow) with the blue zig-zag path.

The following lemma is obvious from the definition of the equivalence relations of paths:

Lemma 5.8. *Let z be a zigzag path on the universal cover. Suppose that a path p' is obtained from another path p by replacing $p_+(a) \subset p$ with $p_-(a)$ or the other way around for a single arrow a, as in the definition of the equivalence relations of paths.*
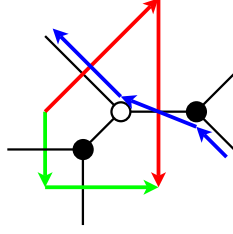
Figure 14: An example of a non-intersection

If neither $ap_+(a)$ nor $p_+(a)a$ is a part of p, then there is a natural bijection between the intersections of z and p and those of z and p'. If a is not a part of p, then this bijection preserves the order of intersections given by the orientation of the path.

The first half of Lemma 5.8 immediately gives the following:

Corollary 5.9. *A minimal path which does not intersect with a zig-zag path z cannot be equivalent to a path intersecting z.*

Lemma 5.8 also gives the following:

Corollary 5.10. *Let p be a path of the quiver. If there is no zigzag path that inetersects p more than once in the same direction on the universal cover, then p is minimal.*

Proof. Note that the condition implies that if p contains $p_+(a)$ or $p_-(a)$ for an arrow a, then p dones not contain a. Suppose p' is a path related to p as in Lemma 5.8. Since p does not contain small loops $ap_+(a)$ or $p_+(a)a$, Lemma 5.8 implies that p' also satisfies the condition and therefore does not contain a small loop. By repeating this argument, we can see that if a path is equivalent to p, then it does not contain a small loop. $\square$

The following lemma shows that the consistency condition implies the first consistency condition of Mozgovoy and Reineke:

Lemma 5.11. *If weak equivalence does not imply equivalence, then the dimer model is not consistent.*

Proof. Assume that a consistent dimer model has a pair of weakly equivalent paths which are not strictly equivalent. Then there is a pair (a, b) of paths such that

- There is an integer $i \geq 0$ such that either $(a, b\omega^i)$ or $(a\omega^i, b)$ is weakly equivalent but not strictly equivalent.

- If one of a and b contains loops, then it is a loop and the other one is a trivial path.

- a and b meet only at the endpoints.

Choose one of such pairs so that the area of the region bounded by a and b is minimal among such pairs.

Figure 15 shows a pair (a, b) of such paths. We may assume that a is a non-trivial path. Let v_1 and v_2 be be the source and the target of a respectively. To show the
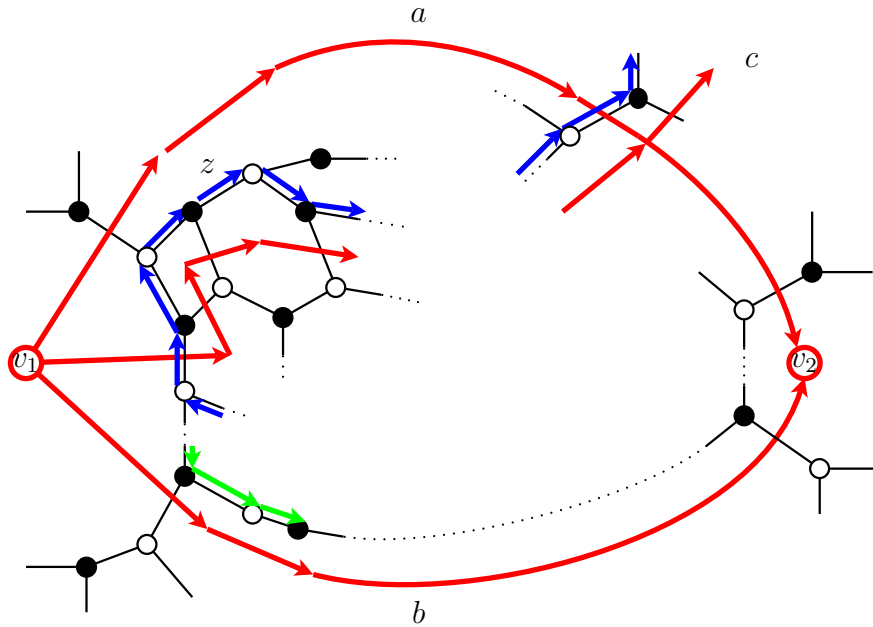
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Figure 15: A pair of inequivalent paths which are weakly equivalent

inconsistency of the dimer model, consider the zig-zag path z which starts from the white node just on the right of the first arrow in the path a as shown in blue in Figure 15. We show that if z crosses a, then it contradicts the minimality of the area. Assume that z crosses a, and consider the path c which goes along z as in Figure 15. Since z crosses a, the path c also crosses a. Let v_3 be the vertex where a and c intersects, and a' and c' be the parts of a and c from v_1 to v_3 respectively. The part of a from v_3 to v_2 will be denoted by d as in Figure 16. The path c' is minimal by Corollary 5.10.

Suppose c' is different from b. Then by the minimality of the area and the minimality of c', there are non-negative integers i and j such that a' is equivalent to $c'\omega^i$ and either $(dc'\omega^j, b)$ or $(dc', b\omega^j)$ are equivalent pairs. Then one of $(a, b\omega^{i-j})$, $(a\omega^{j-i}, b)$ and $(a, b\omega^{i+j})$ is an equivalent pair, which contradicts the assumption. If dc' coincides with b, then b is equivalent to a path that goes along the opposite side of z as in Figure 17, which contradicts the minimality of the area.

Hence the zig-zag path z cannot cross the path a. In the same way, the zig-zag path shown in green in Figure 15 cannot cross the path b. It follows that if we extend these two zig-zag paths in both directions, then they will intersect in the same direction more than once or have a self-intersection. □

Lemma 5.12. *In a consistent dimer model, a path p is minimal if and only if there is no zigzag path that intersects with p more than once in the same direction on the universal cover.*

Proof. The if part is in Corollary 5.10 and we show the only if part. Suppose there is a zigzag path z as above. Let a_1 and a_2 be arrows on the intersection of z and p such that the directions are from a_1 to a_2 on both z and p, and thier parts between a_1 and a_2 do not meet each other. Let p' be the part of p from $s(a_1)$ to $t(a_2)$. There is a
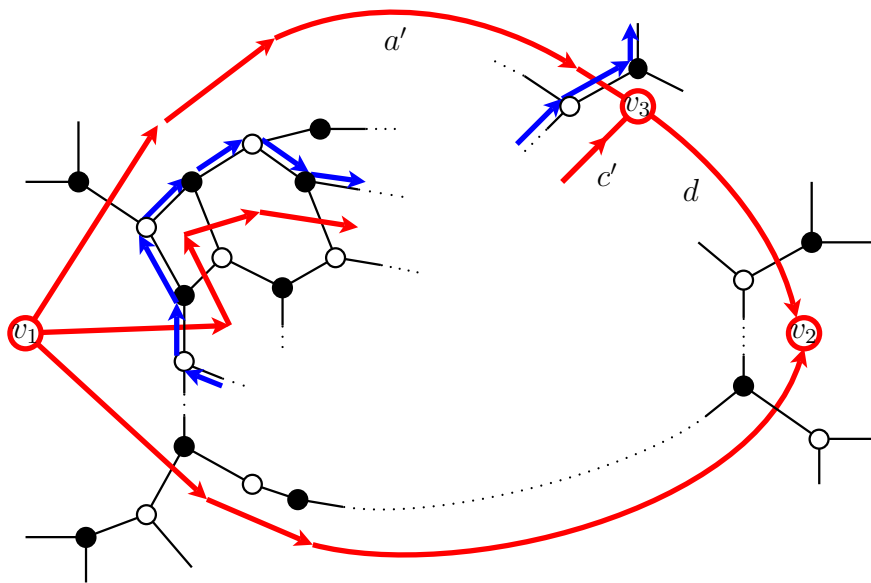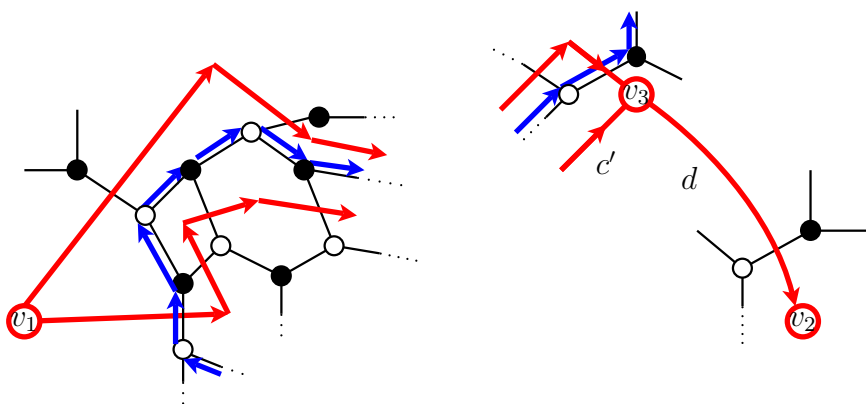
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Figure 16: The paths a' and c'

Figure 17: A path equivalent to dc'

path q from $s(a_1)$ to $t(a_2)$ which is pararell to z. Since q is minimal by Corollary 5.10 and the consistency, Lemma 5.11 implies that there is an integer $i \geq 0$ such that p' is equivalent to $q\omega^i$. If p' is also minimal, i must be zero and therefore p' is equivalent to q. This contradicts Lemma 5.8 and thus p is not minimal. □

The following lemmas show that the first consistency condition of Mozgovoy and Reineke together with the existence of a perfect matching implies the consistency condition:

Lemma 5.13. *Assume that a dimer model has a perfect matching and a pair of zig-zag paths intersecting in the same direction twice on the universal cover, none of which has a self-intersection. Then there is a pair of inequivalent paths which are weakly equivalent.*

Proof. For a pair (z, w) of zig-zag paths intersecting in the same direction twice, consider the pair (a, b) of paths as shown in red in Figure 18. Then there is a minimal path
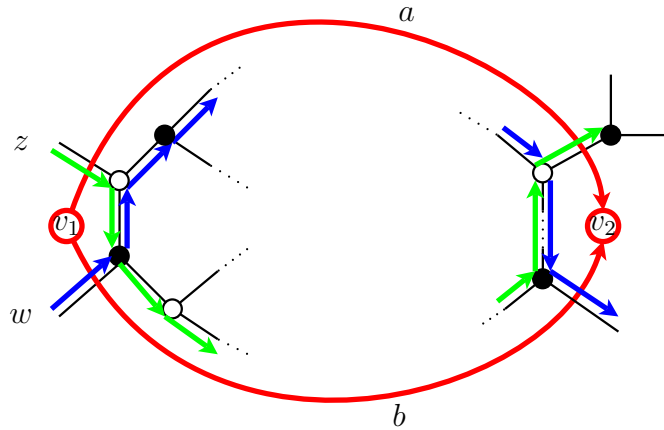

Figure 18: A pair of inequivalent paths which are weakly equivalent

a' which does not intersect with w such that $a = a'\omega^k$ for some $k \in \mathbb{N}$. The existence of such a' and k follows from Corollary 5.9 and the existence of a perfect matching: A perfect matching intersects with a in a finite number of points, and the number of intersection decreases by one as one factors out a small loop. Hence the process of deforming the path without letting it intersect with w and factoring out a small loop if any must terminate in a finite steps, and the resulting path cannot be equivalent to a path intersecting with w by Corollary 5.9. Similary, there is a minimal path b' from v_1 to v_2 which does not intersect with z, and the path a' cannot be equivalent to b'. □

Lemma 5.14. *Assume that a dimer model has a perfect matching and a zigzag path with a self-intersection. Then there is a pair of inequivalent paths which are weakly equivalent.*

Proof. Let z be a zigzag path with a self-intersection and $e_0 e_1 e_2 \ldots e_n e_0$ be a loop in z, where e_i's are mutually distinct edges. The union of $e_1, \ldots, e_n$ forms a circle denoted by C.
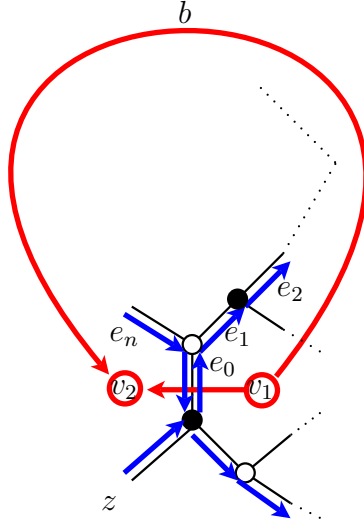

Figure 19: A pair of inequivalent paths which are weakly equivalent

Regarding e_0 as an arrow, we put $v_1 = s(e_0)$ and $v_2 = t(e_0)$. There is a path b from v_1 to v_2 which goes along z. The edge e_0 as an arrow of the quiver also forms a path from v_1 to v_2, which is obviously minimal. We show that there is a minimal path b' from v_1 to v_2 which does not intersect C and which is not equivalent to e_0. We first remove as many small loops from b as possible without making it intersect with C. The resulting path may not be minimal yet since it might allow a deformation first to a path intersecting C and then to a path containing small loops. Assume that a path p' from v_1 to v_2 intersecting C is obtained from another path p which does not intersect with C by replacing $p_-(a) \subset p$ with $p_+(a)$ (or the other way around) for a single arrow a. Since C is a part of a zigzag path, if p does not contain a small loop, then the arrow a must be e_0. Thus p contains $p_-(e_0)$ (or $p_+(e_0)$)and is written as $p = p_1 p_-(e_0) p_2$ (or $p = p_1 p_+(e_0) p_2$), where p_1 and p_2 are paths form v_1 to v_2. Both of them are not equivalent to e_0 and we can replace p with p_i that is not homotopic to e_0 in $\mathbb{R} \setminus C$. By repeating this process, one can find a minimal path b' inequivalent to e_0. $\square$

The following lemma can be shown in completely analogous way:

Lemma 5.15. *Assume that a dimer model has a perfect matching and a zig-zag path with the trivial homology class, then there is a closed path on the quiver which is weakly equivalent to some power of a small loop but not equivalent.*

Indeed, consider the path which goes around the zig-zag path, and factor out all the possible small loops. Then one ends up with a path weakly equivalent to a power of a small loop but not strongly equivalent to it.

For example, the path on the quiver shown in Figure 20 is weakly equivalent to a small loop as shown in Figure 21, although it is not strongly equivalent.
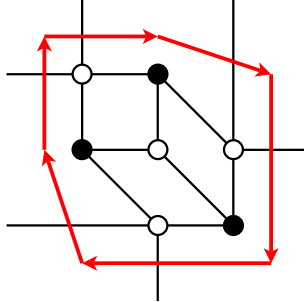

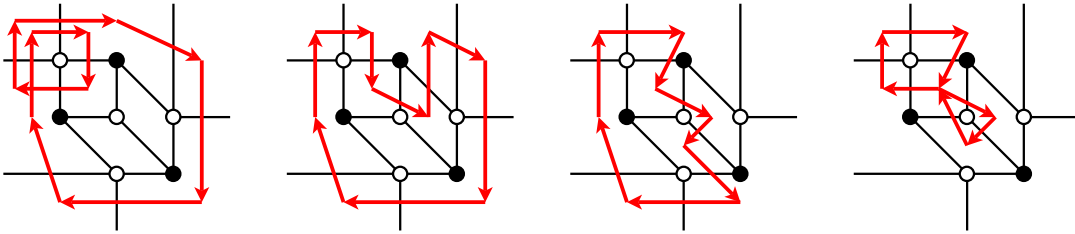Figure 20: Homologically tirival zig-zag path and a closed path on the quiver

Figure 21: Deforming a path on the quiver

We end this section with the following lemma:

Lemma 5.16. *The algebra homomorphism from the path algebra with relations of the quiver associated with a consistent dimer model to the endomorphism algebra of the tautological bundle on the moduli space of quiver representations is injective.*

Proof. A consistent dimer model is non-degenerate by Proposition 6.2. Therefore, the moduli space contains a three-dimensional algebraic torus $\mathbb{T}$ as an open set. We can fix trivializations of the restrictions of the tautological bundles on $\mathbb{T}$ so that $\mathbb{T}$ acts on these line bundles. If two paths p and q from u to v are not equivalent, they are not weakly equivalent by the first consistency and the associated maps form $\mathcal{E}_u|_{\mathbb{T}}$ to $\mathcal{E}_v|_{\mathbb{T}}$ have different weights with respect to $\mathbb{T}$-action. Thus the homomorphism is injective. $\square$

6 Large hexagons and corner perfect matchings

In this section, we introduce the concept of *large hexagons* and discuss the relation between the following:
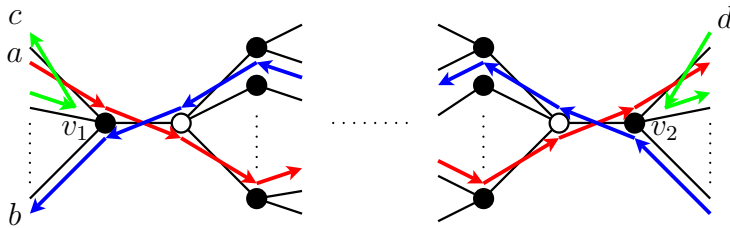
1. The choice of a vertex in the toric diagram.

Figure 22: A pair of zig-zag paths intersecting twice

2. The choice of a surjection from the quiver to the McKay quiver of some abelian subgroup G in $GL_3(\mathbb{C})$.

3. The choice of an open subscheme isomorphic to G-Hilb($\mathbb{C}^3$) in the moduli space $\mathcal{M}_\theta$ for some θ.

4. The division of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ into large hexagons.

The division into large hexagons will be done by drawing two zigzag path with adjacent slopes. Recall that the *slope* of a zig-zag path on a dimer model is its homology class, considered as an element in $\mathbb{Z}^2$ by identifying $H_1(T, \mathbb{Z})$ with $\mathbb{Z}^2$ in the standard way. Note that the slope in a consistent dimer model is always primitive, and there may be several zig-zag paths with a given slope. The set of slopes naturally has a cyclic order, which allows one to talk about adjacency of slopes. A pair of zig-zag paths with adjacent slopes in a consistent dimer models are allowed to intersect more than once only in a sequence of adjacent edges connected by divalent vertices:

Lemma 6.1. *Assume that either a pair of zig-zag paths in a consistent dimer model intersect each other more than once on the universal cover, or they have common points other than their intersection. Here, an intersection of two zigzag paths is defined in Definition 5.1. Then the slopes of this pair of zig-zag paths are not adjacent.*

Proof. Assume that there is a pair (a, b) of zig-zag paths intersecting twice in the opposite direction as in Figure 22. Let v_1 and v_2 be the vertices adjacent to the first and the last edge where a and b intersect. Then there are two other zig-zag paths c and d such that c intersects with a at the edge adjacent to the vertex v_1 and d intersects with a at the edge adjacent to the vertex v_2. Then the slopes of c and d must come in between a and b by Lemma 5.4, preventing them to be adjacent.

If they have common points other than their intersection, then Lemma 5.4 implies they are not adjacent. $\qquad\square$

It is obvious that any pair of lines on a torus will divide the torus into parallelograms. Since two adjacent zig-zag paths in a consistent dimer model intersect along a connected union of edges instead of a point, they divide the torus into hexagons. Such hexagons will be called *large hexagons*, and will be the main technical tool in this paper. Figure 23 shows a part of a large square tiling, and an example of a collection of zig-zag paths with adjacent slopes is shown in Figure 24. One can see that these zig-zag path divides the torus into large hexagons as shown in Figure 25.
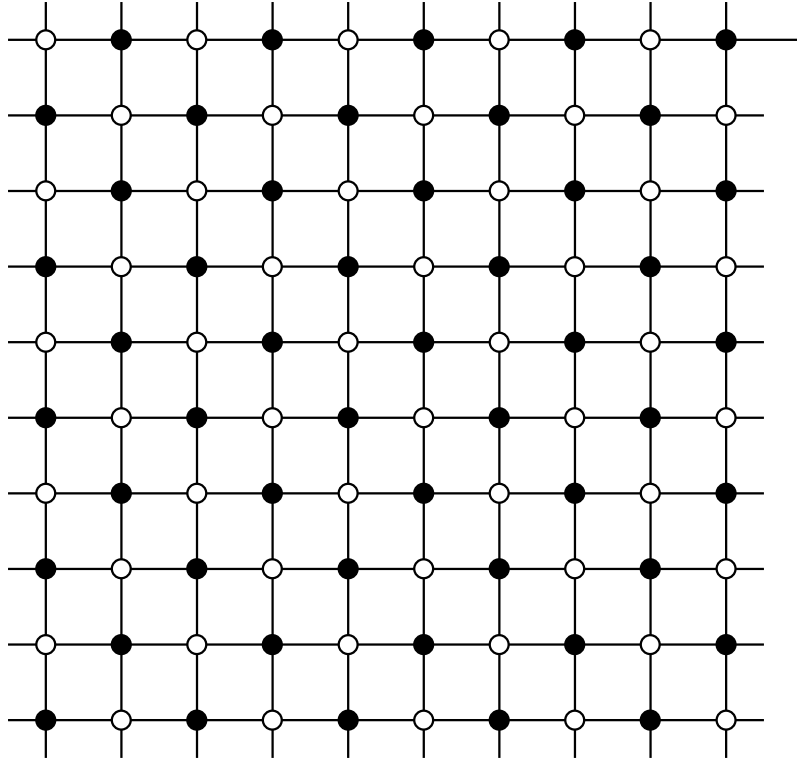
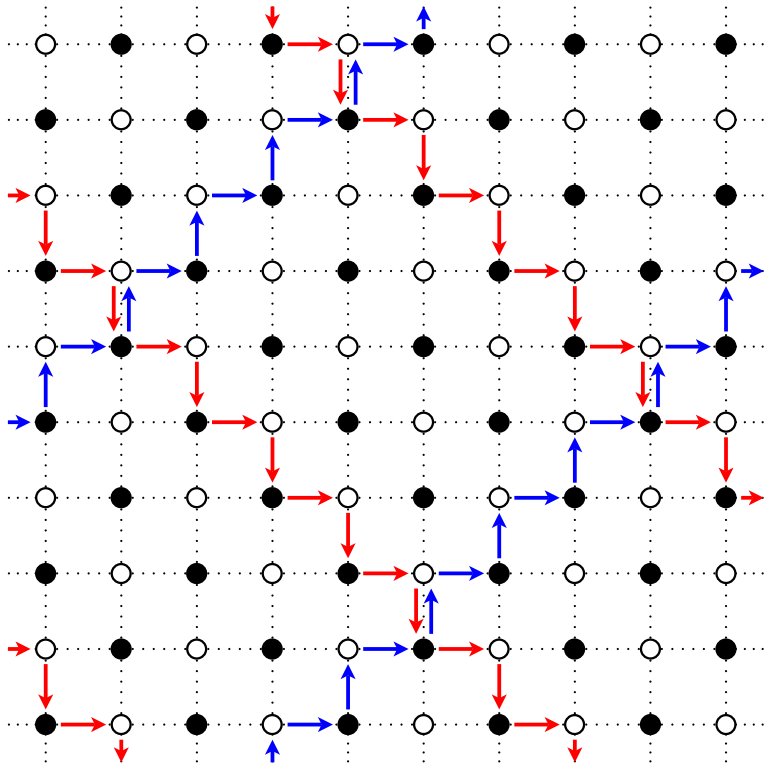Figure 23: A part of a large square tiling

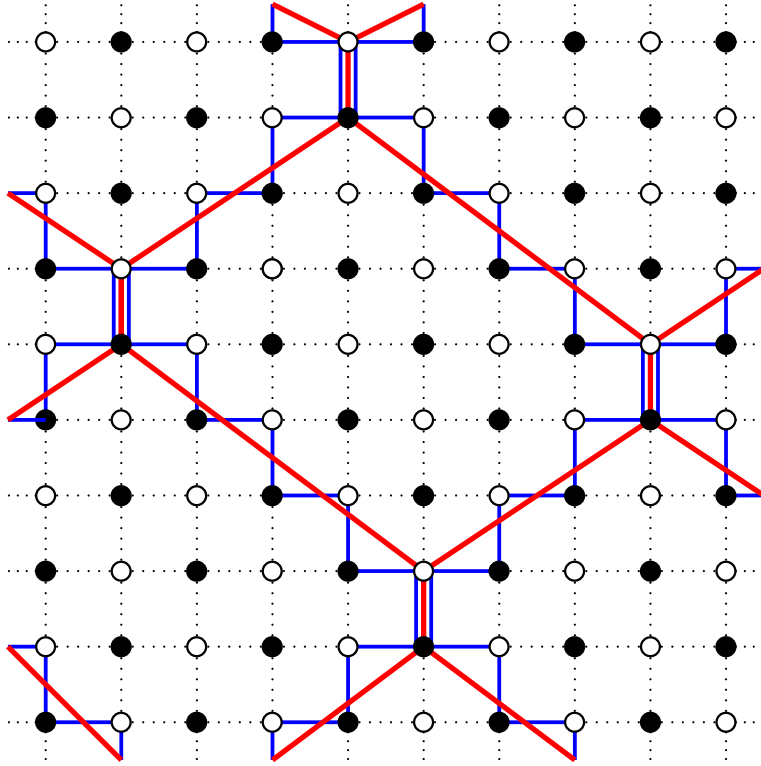Figure 24: A pair of zig-zag paths with adjacent slopes

Figure 25: Large hexagons

Inside each large hexagon is a pair of distinguished vertices of the quiver adjacent to the edges where two zig-zag paths intersects. Note that an (connected component of the) intersection of the two zigzag paths consists of odd number of edges. The source of the arrow corresponding to the terminal edges of an intersection is called the *source* in the large hexagon and the other vertex is the *sink* in it.

Proposition 6.2. *A consistent dimer model is non-degenerate.*

Proof. For an edge in a consistent dimer model, choose a zig-zag path z containing the edge and another zig-zag path w whose slope is adjacent to that of z. Then z and w divide the torus into large hexagons, and the paths from the source to the sink along z and w are minimal by Lemma 5.10 since the slopes of z and w are adjacent. Choose a perfect matching on the union of z and w such that the edge we have started with is contained, and take the union with the set of edges in the interiors of the large hexagons which are not crossed by any minimal path from the source to the sink. To show that it is really a perfect matching, fix a large hexagon and suppose that the union of edges inside the large hexagon which are not crossed by any minimal path from the source to sink has a connected component consisting of two or more edges. Then the component does not have an endpoint since otherwise the edge connected to the endpoint must be isolated. Since a minimal path does not meet the component, the two mimimal path from the source to the sink along the boundary of the hexagon cannot be equivalent. This contradicts Lemma 5.11. □

For a pair (z, w) of zig-zag paths with adjacent slopes, we can choose a perfect matching on the union of z and w which contains arrows in the intersections of z and
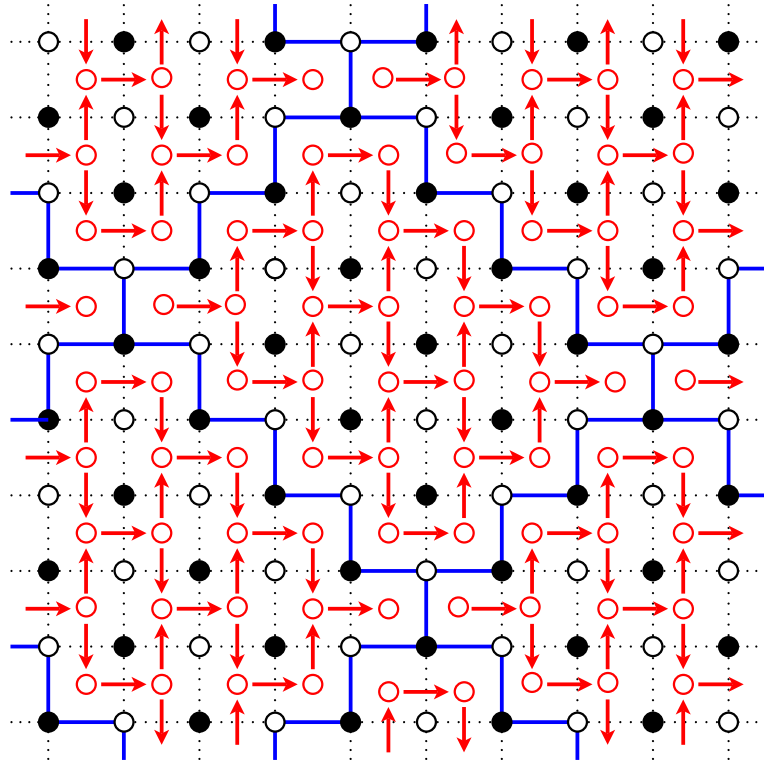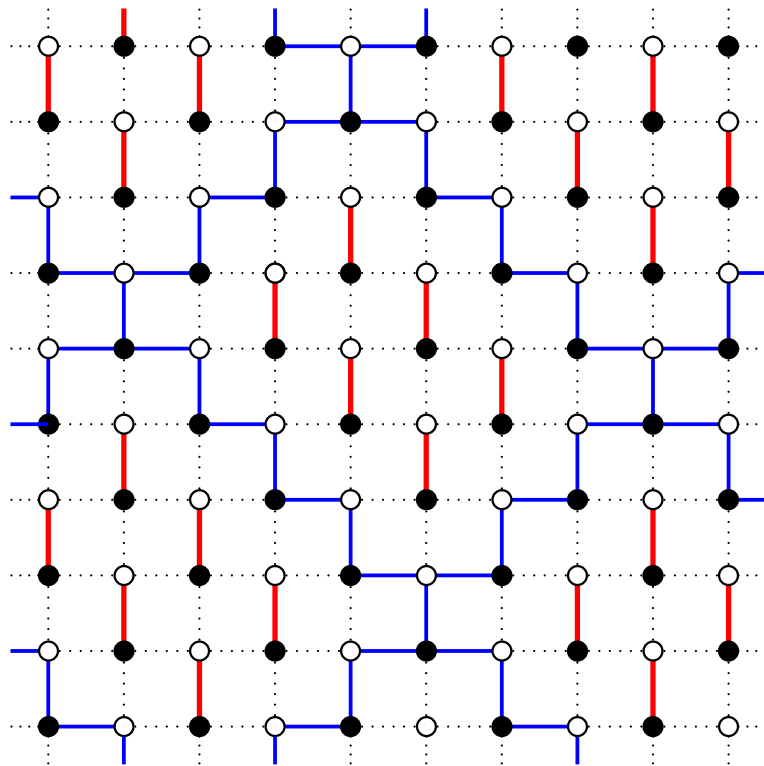
22

Figure 26: Sources and sinks

Figure 27: A perfect matching

w starting from the sources of large hexagons. There is a perfect matching obtained by taking the union with the set of edges in the interiors of large hexagons as in the above proof, which is said to *come from a pair of zig-zag paths with adjacent slopes.* We assume the consistency condition through the rest of this section.

The proof of Proposition 6.2 also shows the following:

Lemma 6.3. *There exists a path from the source to any vertex inside the large hexagon wihch is allowed by the perfect matching construced in Proposition 6.2. Similarly, there exists an allowed path to the sink from any vertex inside the large hexagon.*

The tesselation by large hexagons forms a new dimer model, and the resulting quiver Λ with relations can be identified with the McKay quiver for a suitable $G \subset SL(3, \mathbb{C})$ acting on $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$ in the following way:

- Choose any vertex of Λ and identify it with the trivial represenation.

- The arrow from the source of one large hexagon to the sink of the adjacent large hexagon is identified with "multiplication by z".

- The cyclic order of three arrows starting from a vertex of Λ coming from the orientation of the torus is given by (x, y, z).

Now we discuss the embedding of G-Hilb$(\mathbb{C}^3)$ into $\mathcal{M}_\theta$ for a suitable choice of θ. Let D be the perfect matching coming from a pair of zig-zag paths with adjacent slopes. We can define a linear map

$$h : \mathbb{C}\Gamma \to \mathbb{C}\Lambda$$

as follows:

- An arrow inside a large hexagon that is not contained in D goes to the idempotent of the large hexagon.

- An arrow inside a large hexagon that is contained in D goes to the small loop.

- Suppose an arrow a of Γ is on the boundary of two large hexagons. Let b be the arrow of Λ connecting the two large hexagons. If a is in the same direction as b, then a goes to b. If a is in the opposite direction, then a goes to the path of length two that connects the two large hexagons in the same direction as a.

It is easy to see that the above map is well-defined. Although it is not an algebra homomorphism, it has the following property: If p and q are paths of Γ with $pq \neq 0$, then $h(pq) = h(p)h(q)$.

Let H be the set of large hexagons and A' be the set of arrows of Λ. To a representaion $((V_h)_{h \in H}, (\psi(a'))_{a' \in A'})$ of Λ, we can associate a representaion $(((V_{h(v)})_{v \in V}), (\psi(h(a))_{a \in A}))$ of Γ. This gives rise to a functor

$$\mathbb{C}\Lambda\text{-mod} \to \mathbb{C}\Gamma\text{-mod}.$$

By this functor, G-clusters are sent to representations of Γ with dimension vector $(1, \ldots, 1)$.

Choose a vertex $h_0 \in H$ and fix it. We consider it as the trivial representation in the McKay quiver. We first choose a parameter $\eta \in \operatorname{Hom}(\mathbb{Z}^V, \mathbb{R})$ such that

- If a vertex v is not the source, then $\eta(v) = 1$.

- The sum of $\eta(v)$ inside a fixed large hexagon is 0.

Then for a sufficiently small $\epsilon > 0$, we define θ so that

- If v is the sorce of a large hexagon other than h_0, then $\theta(v) = \eta(v) + \epsilon$.

- If v is the source of h_0, then $\theta(v) = \eta(v) - (\#G - 1)\epsilon$.

- For the other vertices v, put $\theta(v) = \eta(v)$.

Then it is easy to see the following:

Lemma 6.4. *Every G-cluster goes to a θ-stable representation of Γ. This gives rise to an open immersion G-Hilb$(\mathbb{C}^3) \to \mathcal{M}_\theta$.*

Now we can prove the following:

Proposition 6.5. *The following are equivalent for a perfect matching D in a consistent dimer model:*

1. *D is simple.*

2. *D is multiplicity free.*

3. *D is a corner perfect matching.*

4. *D comes from a pair of zig-zag paths with adjacent slopes.*

The proof is divided into several steps:

Step 1. *A perfect matching is a corner perfect matching if and only if it comes from a pair of zig-zag paths with adjacent slopes.*

Proof. The if part follows from the fact that the height change of a perfect matching coming from a pair of zig-zag paths with adjacent slopes satisfies equality in (2) coming from both of these zig-zag paths. Now consider the convex hull of the set of height changes of corner perfect matchings coming from pairs of zig-zag paths with adjacent slopes. Then inequality (2) shows that the height change of any other perfect matching is contained in it, and the only if part follows. □

Step 2. *A perfect matching coming from a pair of zig-zag paths with adjacent slopes is simple.*

Proof. We have to show that the corresponding quiver representation M is simple, i.e., has no non-trivial submodule. This follows from the fact that in a perfect matching coming from a pair of zig-zag paths with adjacent slopes, one can find an allowed path from any vertex to any other vertex in the quiver. Indeed, starting from any vertex, one can first go to the sink of the large hexagon h_1 where the vertex belongs, and then to the source of adjacent large hexagon h_2 in the z-direction by the path corresponding to xy. Recall that one can go from the source of a large hexagon to any other vertex in the same large hexagon only through an allowed path. Note also that one can go from the source of one large hexagon to the source of another large hexagon adjacent in the x- and y-direction. Since one can go from one large hexagon to any other large hexagon by multiplying sufficiently many x and y, the lemma follows. □

Step 3. *A perfect matching is multiplicity-free if and only if it is simple.*

Proof. Let us first prove the only if part: Assume M has a non-trivial submodule. Then one can find a stability parameter θ such that M is not θ-semistable. Since M is 0-semistable and the map $\mathcal{M}_\theta \to \overline{\mathcal{M}}_0$ is projective, there is another θ-semistable representation N with the same height change.

Now we prove the if part: Assume that M is simple and take any module N with the same height change as M. Choose a stability parameter θ such that semistability implies stability and N is θ-stable. Since M is also θ-stable with the same height change as N, N and M must belong to the same $\mathbb{T}$-orbit, so that the corresponding perfect matching is identical. $\square$

Step 4. *A simple perfect matching is a corner perfect matching.*

Proof. Since simple modules are θ-stable for any θ, the divisor corresponding to a simple perfect matching is not contracted in the affine quotient $\overline{\mathcal{M}}_0$. Hence it must be a corner perfect matching. $\square$

7   Description of the algorithm

Let $G = (B, W, E)$ be a consistent dimer model on the 2-torus T and D be a corner perfect matching. The algorithm to remove the vertex of the lattice polygon corresponding to D is the following:

1. Choose a pair of zig-zag paths with adjacent slopes, corresponding to the edges of the Newton polygon of the characteristic polynomial of G, incident to the height change of D.

2. The pair of zig-zag paths chosen above divides T into large hexagons. Choose an identification of the resulting honeycomb lattice with the McKay quiver for a finite small group $G \subset GL_2(\mathbb{C}) \subset SL_3(\mathbb{C})$ by choosing the large hexagon corresponding to the trivial representation.

3. Remove the edges of the dimer corresponding to the arrows of the quiver going from the sources of the large hexagons corresponding to special representations to the sinks of the adjacent large hexagons related by "multiplication by z".

In terms of the quiver, the above operation corresponds either to merging sources of special large hexagons with the sinks of adjacent large hexagons along the arrows corresponding to "multiplication by z", or to adding inverse to such arrows. The former point of view will be used in section 8 to prove the preservation of the consistency under this operation, and the latter point of view will be used in sections 9, 10, 12, and 13 to prove the derived equivalence inductively.

Note that there are several choices in general in the above algorithm. As an example, consider the construction of dimer models for the hexagon in Figure 30 starting from the dimer model in Figure 31 corresponding to the square lattice polygon in Figure 28 by removing two vertices.
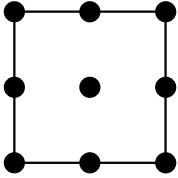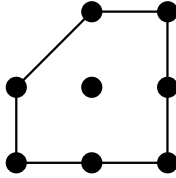
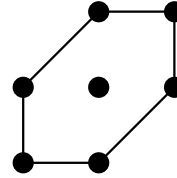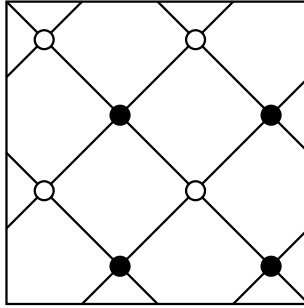Figure 28: A square   Figure 29: A pentagon   Figure 30: A hexagon

Figure 31: A dimer model for the square lattice polygon

To remove the top left vertex from the square lattice polygon in Figure 28, we have to choose a pair of zig-zag paths, one from each of those with homology classes $(-1, 0)$ (shown in red in Figure 32) and $(0, 1)$ (shown in blue in Figure 32). There are four choices in Step 1, which actually do not matter for symmetry reasons. There is no choice in Step 2 and Figure 33 shows the resulting dimer model.
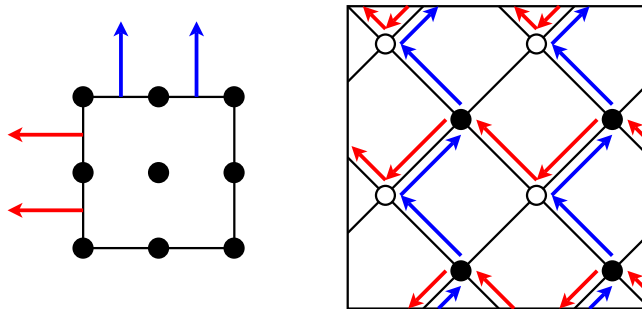

Figure 32: Zig-zag paths

Now consider the the removal of the lower-right vertex from the pentagonal lattice polygon in Figure 29. In this case there are four choices in Step 1, which lead to the dimer models shown in Figure 35. Note that models 2 and 4 are obtained from models 1 and 3 respectively by changing the colors of the nodes, so that the corresponding quivers are related by the reversal of arrows.

Model 1 has a divalent white node, and one obtains the dimer model in Figure 36 by contracting it. Model 3 is equivalent to the dimer model shown in Figure 37.

The zig-zag paths on the dimer model in Figure 36 is shown in Figure 38.

From the dimer model in Figure 36, one can construct the dimer model for $\mathbb{P}^2$ by removing three vertices from the lattice polygon as in Figure 39.
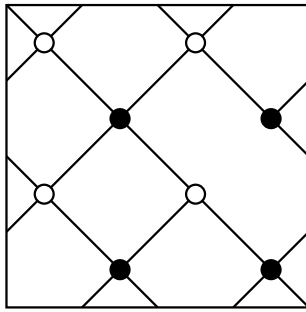
27

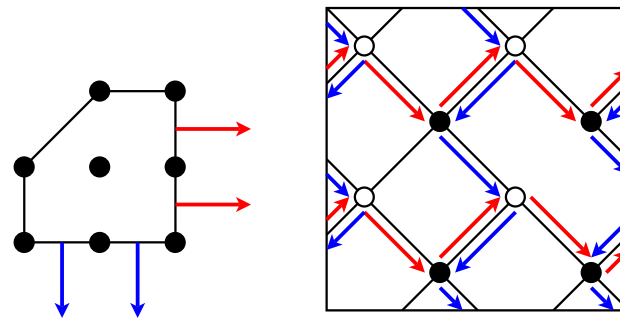Figure 33: The dimer model for the pentagonal lattice polygon
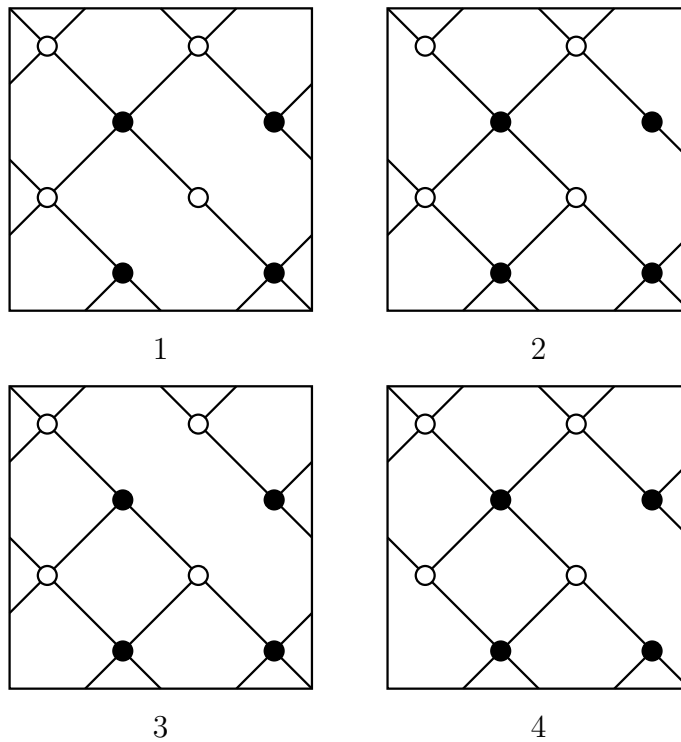

Figure 34: Zig-zag paths

1

2

3

4

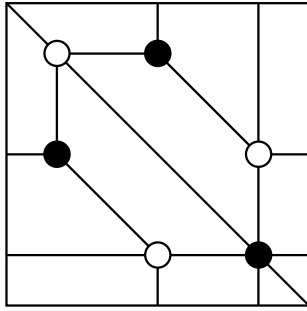Figure 35: Dimer models for the hexagonal lattice polygon

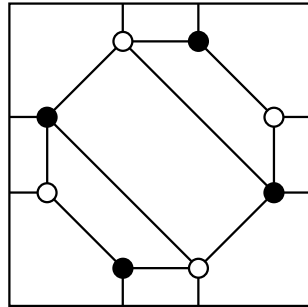Figure 36: A dimer model equivalent to Model 1

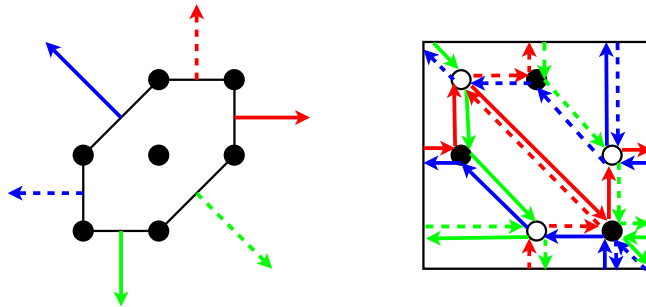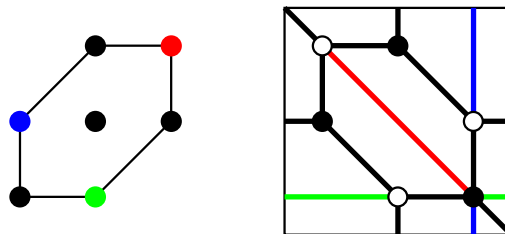Figure 37: A dimer model equivalent to Model 3

Figure 38: Zig-zag paths

Figure 39: From a hexagon to a triangle

From the dimer model in Figure 36, one can construct the dimer model for $\mathbb{P}^1 \times \mathbb{P}^1$ by removing two vertices from the lattice polygon as in Figure 40.
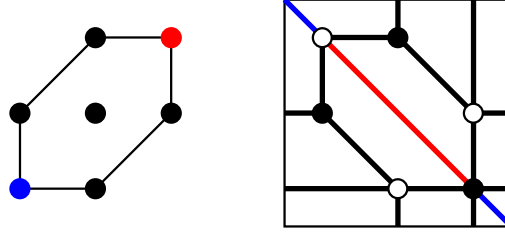

Figure 40: From a hexagon to a square

8　Preservation of the consistency

We use the same notation as in section 3. We prove the following in this section:

Proposition 8.1. *A consistent dimer model remains consistent after the operation described in section 7.*

We need the following lemma to prove Proposition 8.1:

Lemma 8.2. *Let $t \in [1, r+1]$, $a \in [0, i_{t-1} - i_t)$ and $b \in [0, j_t - j_{t-1})$ be integers. Then $i_{t-1} + a + bq$ is special if and only if $a = b = 0$.*

Proof. We may assume $t \in [2, r]$ since otherwise there is no such a or b. Write

$$a = d_t i_t + d_{t+1} i_{t+1} + \cdots + d_r i_r$$

as in Theorem 3.1. By using $b_t i_t = i_{t-1} + i_{t+1}$ and the assumption $a < i_{t-1} - i_t$, we obtain

$$(d_t + 1 - b_t) i_t + (d_{t+1} + 1) i_{t+1} + d_{t+2} i_{t+2} + \cdots + d_r i_r < 0$$

and hence $d_t \leq b_t - 2$. Moreover, if the equality $d_t = b_t - 2$ holds, then we have

$$d_{t+1} i_{t+1} + d_{t+2} i_{t+2} + \cdots + d_r i_r < i_t - i_{t+1},$$

which is of the same form as $a < i_{t-1} - i_t$ with t increased by 1. Thus we can inductively show: $d_t \leq b_t - 2$ and if $d_k = b_k - 1$ for some $k > t$, then there is an integer $l \in (t, k)$ with $d_l \leq b_l - 3$.

Using the dual sequence, we can write

$$b = d_{t-1} j_{t-1} + d_{t-2} j_{t-2} + \cdots + d_1 j_1$$

and argue in the same way to conclude: $d_{t-1} \leq b_{t-1} - 2$ and if $d_k = b_k - 1$ for some $k < t - 1$, then there is an integer $l \in (k, t-1)$ with $d_l \leq b_l - 3$.

Thus the sequence $(d_1, \ldots, d_{t-2}, d_{t-1} + 1, d_t, \ldots, d_r)$ satisfies the condition in Lemma 3.4 and represents the degrees of the tautological bundle corresponding to the representation $i_{t-1} + a + bq$. By the definition of special full sheaves, it is special if and only if $d_1 = \cdots = d_r = 0$. □

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Now we prove Proposition 8.1:

Step 1. The case G-$Hilb(\mathbb{C}^3) \setminus G$-$Hilb(\mathbb{C}^2)$ for finite small $G \subset GL_2(\mathbb{C})$: Let Λ be the hexagonal dimer model for G-Hilb($\mathbb{C}^3$) and Λ' be the dimer model for G-Hilb($\mathbb{C}^3$) $\setminus$ G-Hilb($\mathbb{C}^2$) obtained from Λ by the operation in section 7 (i.e., by removing edges corresponding to "multiplication by z" from special representations).

Of three homology classes of zig-zag paths on Λ, only the ones consisting of edges corresponding to multiplications by x, y survives in Λ'. Other two zigzag paths will be transformed into new zigzag paths on Λ', indexed by t with $t \in \mathbb{Z}/r\mathbb{Z}$ as follows: Start with the edge corresponding to the multiplication by y into the special hexagon i_{t-1} and go along the old zigzag path consisting of multiplications by y, z until one arrives at the next special hexagon i_t, where one is blocked by a removed edge. Then one changes the direction and go along the old zigzag path consisting of multiplications by x, z. By virtue of (1), one comes back to the starting point without meeting any other removed edges.

Now let us check the consistency of the new dimer model. It is obvious that a new zigzag path has no self-intersection on the universal cover. Choose two zigzag paths on Λ'. If they are both old, then they do not meet at all. If one is old and the other is new, then they meet more than once in general but always in the opposite direction. If they are both new, then Lemma 8.2 ensures that they meet at most once on the universal cover.

Step 2. The general case : Old zigzag paths except the chosen two survive, and new zigzag paths are described in the same way as above using the large hexagons. Let us analyze intersections of two zig-zag paths in the new dimer model. If two zig-zag paths are both old or new, then the same reasoning as step 1 shows that they do not intersect in the same direction twice. Take one new zigzag path and one survivor from the old one, and suppose they meet twice in the same direction. Since we have chosen two zigzag paths with adjacent slopes to perform the operation, the slope of the survivor cannot be in between the slopes of these two zig-zag paths. This implies that the survivor must meet either of the two zig-zag paths twice in the same direction, thus contradicting the consistency of the old dimer model. $\square$

Let $([z_i])_{i=1}^k$ be the set of slopes of zig-zag paths ordered cyclically starting from any zig-zag path. Here k is the number of zig-zag paths and some of the slopes may coincide in general. Define another sequence $(w_i)_{i=1}^r$ in $\mathbb{Z}^2$ by $w_0 = 0$ and

$$w_{i+1} = w_i + [z_{i+1}]', \qquad i = 0, \ldots, k-1$$

where $[z_{i+1}]'$ is obtained from $[z_{i+1}]$ by rotating 90 degrees. Note that one has $w_r = 0$ since every edge is contained in exactly two zigzag paths with differnt directions and therefore the homology classes of the zig-zag paths add up to zero. One can define a lattice polytope by taking the convex hull of $(w_i)_{i=1}^r$. The following theorem is proved by Gulotta for properly ordered dimer models. The same result follows for consistent dimer models as a corollary to Proposition 8.1:

Corollary 8.3 (Gulotta [15, Theorem 3.3]). *For a consistent dimer model, the lattice polygons obtained from hight changes and from zig-zag paths coincide up to translation.*

Indeed, any corner perfect matching in a consistent dimer model comes from a pair of adjacent edges, and one can perform the operation of removing the corner perfect matching to reduce to the case of the unit triangle, which is obvious.

9 Preservation of the tilting condition: the case of G-**Hilb**$(\mathbb{C}^3)$ versus G-**Hilb**$(\mathbb{C}^3) \setminus G$-**Hilb**$(\mathbb{C}^2)$

Let G be a finite small subgroup of $GL_2(\mathbb{C})$ and put $Y = G\text{-Hilb}(\mathbb{C}^2)$, $U = G\text{-Hilb}(\mathbb{C}^3)$ and $U^0 = U \setminus Y$. The tautological bundle on U corresponding to an irreducible representation ρ of G and its restriction to U^0 will be denoted by $\mathcal{R}_\rho$ and $\mathcal{L}_\rho$ respectively. We prove the following in this section:

Proposition 9.1. *The direct sum $\bigoplus_{\rho:\ irreducible} \mathcal{L}_\rho$ over the set of irreducible representations of G is a tilting object in $D^b \operatorname{coh} U^0$. If G is not contained in $SL_2(\mathbb{C})$, then this tilting object contains isomorphic summands and the direct sum $\bigoplus_{\rho:\ non\text{-}special} \mathcal{L}_\rho$ over the set of non-special representations of G is the tilting object obtained by removing redundant summands.*

Here we do not have to assume that G is abelian. In the case when G is abelian, Alastair Craw has pointed out that Proposition 9.1 follows from [7]. We first prove the following two lemmas:

Lemma 9.2. *Let $\mathcal{E}$ be a tilting object in $D^b \operatorname{coh} U$. Then the pull-back of $\mathcal{E}$ by $\iota : U^0 \to U$ is a generator in $D^b \operatorname{coh} U^0$.*

Proof. For any coherent sheaf $\mathcal{F}$ on U^0, there is a coherent sheaf $\widetilde{\mathcal{F}}$ on U such that $\iota^* \widetilde{\mathcal{F}} = \mathcal{F}$. Since $\mathcal{E}$ is a tilting object, $\mathcal{F}$ is a direct summand of an object in $D^b \operatorname{coh} U$ obtained from $\mathcal{E}$ by taking mapping cones. Since derived restriction commutes with the operation of taking mapping cones, this shows that $\mathcal{F}$ is obtained from $\iota^* \mathcal{E}$ by taking direct summands and mapping cones. This implies that $\iota^* \mathcal{E}$ is a generator in $D^b \operatorname{coh} U^0$. $\square$

Lemma 9.3. *The local cohomology $H_Y^i(U, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau)$ vanishes for $i \geq 2$.*

Proof. We use
$$H_Y^i(U, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \cong \varinjlim_n \operatorname{Ext}_U^i(\mathcal{O}_{nY}, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau)$$
to compute the local cohomology. Replacing $\mathcal{O}_{nY}$ by a complex $[\mathcal{O}_U(-nY) \to \mathcal{O}_U]$, we obtain
$$\operatorname{Ext}_U^i(\mathcal{O}_{nY}, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \cong H^{i-1}(\mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau \otimes \mathcal{O}_U(nY)|_{nY})$$
where $\mathcal{O}_U(nY)|_{nY}$ is isomorphic to the dualizing sheaf ω_{nY} by virtue of the crepantness of U. When $n = 1$, this vanishes since Y is a resolution of an affine surface, $H^1(\mathcal{R}_\rho^\vee \otimes \omega_Y) = 0$ and $\mathcal{R}_\tau|_Y$ is generated by global sections by the definition of a full sheaf. For $n > 1$, we use the exact sequence
$$0 \to \omega_{(n-1)Y} \to \omega_{nY} \to \omega_Y^{\otimes n} \to 0$$
and note that ω_Y is generated by global sections to obtain the desired vanishing. $\square$

The direct sum of tautological bundle restricts to a tilting object:

Lemma 9.4. *The direct sum $\bigoplus_\rho \mathcal{L}_\rho$ over the set of irreducible representations of G is a tilting object.*

Proof. The restriction $\bigoplus_\rho \mathcal{L}_\rho$ is a generator by Lemma 9.2. The vanishing of $H^i(\mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau|_{U^0})$ for $i \geq 1$ follows from the long exact sequence

$$\cdots \to H_Y^i(U, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \to H^i(U, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \to H^i(U^0, \mathcal{L}_\rho^\vee \otimes \mathcal{L}_\tau) \to \cdots \qquad (3)$$

and Lemma 9.3. $\qquad \square$

Note that the inclusion $G \subset SL_3(\mathbb{C})$ is given by $\rho_{\mathrm{Nat}} \oplus \omega$ where $\omega = \det(\rho_{\mathrm{Nat}}^\vee)$ is the determinant of the dual of the natural representation $\rho_{\mathrm{Nat}} : G \hookrightarrow GL_2(\mathbb{C})$. For every tautological bundle $\mathcal{R}_\rho$, "multiplication by z" gives rise to a map

$$z_\rho : \mathcal{R}_\rho \to \mathcal{R}_{\rho \otimes \omega}$$

which defines an effective divisor

$$Z_\rho = \{P \in U | \wedge^r z_\rho(P) = 0\}$$

where $r = \mathrm{rank}\,\mathcal{R}_\rho = \dim \rho$. Note that $\mathcal{R}_\omega|_Y \cong \omega_Y \cong \mathcal{O}_Y(Y)$.

Proposition 9.5.

  1. Z_ρ is a connected divisor which is a sum of rY and compact divisors.

  2. ρ is special if and only if $Z_\rho = rY$.

Proof. Put $\mathcal{F} = \mathrm{Coker}\, z_\rho$. Then by the Fourier-Mukai transform, $\mathcal{F}$ is sent to

$$[\rho \otimes \mathcal{O}_{\mathbb{C}^3} \xrightarrow{\cdot z} \rho \otimes \omega \otimes \mathcal{O}_{\mathbb{C}^3}] \cong \rho \otimes \omega \otimes \mathcal{O}_{\mathbb{C}^2}.$$

Since $\rho \otimes \omega \otimes \mathcal{O}_{\mathbb{C}^2}$ is indecomposable, so is $\mathcal{F}$. Therefore the support of $\mathcal{F}$, which coincides with Z_ρ as a set, is connected. Moreover, pushing forward to $\mathbb{C}^3/G$ yields $H^0(\mathcal{F}) \cong (\rho \otimes \omega \otimes \mathcal{O}_{\mathbb{C}^2})^G$, which is a rank r reflexive sheaf on $\mathbb{C}^2/G$. This implies that the coefficient of Y in Z_ρ is r and the other components are compact.

Write $Z_\rho = rY + Z'$ with Z' compact. Then Z' represents $c_1(\mathcal{R}_{\rho \otimes \omega}(-Y)) - c_1(\mathcal{R}_\rho)$. Hence Theorem 2.7 shows that ρ is special if and only if the intersection $Z' \cap Y$ is empty. Since $Z_\rho = rY + Z'$ is connected, this is equivalent to the condition $Z_\rho = rY$. $\qquad \square$

This immediately implies the following:

Corollary 9.6. *The map z_ρ induces an isomorphism $\mathcal{L}_\rho \cong \mathcal{L}_{\rho \otimes \omega}$ if and only if ρ is special.*

The following lemma ensures that one can remove special representations from $\bigoplus_\rho \mathcal{L}_\rho$ and concludes the proof of Proposition 9.1:

Lemma 9.7. *Assume that G is not contained in $SL_2(\mathbb{C})$. Then for any special ρ, there is an integer $l > 0$ such that $\rho \otimes \omega^l$ is non-special.*

Proof. If not, then Theorem 2.7 implies that $\mathcal{R}_\rho|_Y \otimes \omega_Y^l$ are all full sheaves whose first Chern classes are mutually distinct. This contradicts the finiteness of indecomposable full sheaves. $\qquad \square$

10 Preservation of surjectivity: the case of G-**Hilb**$(\mathbb{C}^3)$ versus G-**Hilb**$(\mathbb{C}^3) \setminus G$-**Hilb**$(\mathbb{C}^2)$

Let G be a finite small abelian subgroup of $GL_2(\mathbb{C})$ and put $U = G\text{-Hilb}(\mathbb{C}^3)$, $Y = G\text{-Hilb}(\mathbb{C}^2)$, $U^0 = U \setminus Y$ and $\mathcal{L}_d = \mathcal{R}_d|_{U^0}$ as in the previous section. The McKay quiver of G will be denoted by Λ and let Λ' be the quiver obtained from Λ by adding inverse arrows to the arrows starting from special representations corresponding to "multiplication by z". We prove the following in this section:

Proposition 10.1. *The natural map from $\mathbb{C}\Lambda'$ to the endomorphism algebra of $\bigoplus_i \mathcal{L}_i$ is surjective.*

The "N-lattice" corresponding to the toric varieties $\mathbb{C}^3/G$ and U is

$$N = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{n}(1, q, n - (1 + q))$$

and $\mathbb{C}^3/G$ corresponds to the cone $(\mathbb{R}_{\geq 0})^3 \cap N$.

The following fact is well-known.

Lemma 10.2. *The cone in $N_\mathbb{R}$ corresponding to $\mathrm{Spec}(H^0(\mathcal{O}_{U^0}))$ is generated by the vectors $\frac{1}{n}(j_t, i_t, n - (i_t + j_t))$ for $0 \leq t \leq r + 1$.*

Let x, y, z be the coordinate of $\mathbb{C}^3$ such that the action of $G = \langle \frac{1}{n}(1, q, n - (1 + q)) \rangle$ is diagonalized. Then the rational sections of $\mathcal{R}_d$ form a vector space with a basis consisting of Laurent monomials $x^a y^b z^c$ with $a + bq - (1 + q)c \equiv d \mod n$. Thus we can embed the line bundle $\mathcal{R}_d^{\otimes n}$ into $\mathcal{O}_U$ in a natural way and it defines an effective exceptional divisor E_d on U with $\mathcal{R}_c^{\otimes n} = \mathcal{O}_U(-E_d)$.

Let $C = (c_{st})$ be the negative of the intersection matrix of the resolution $Y \to \mathbb{C}^2/G$, namely, it is an $r \times r$ matrix with entries

$$c_{st} = \begin{cases} b_i & s = t, \\ -1 & |s - t| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

whose determinant is $n = |G|$. Let η_{st} be the (s, t) entry of the integer matrix nC^{-1}. It is easy to see $i_t = \eta_{t1}$ and $j_t = \eta_{tr}$ for $1 \leq t \leq r$.

Let D_t be the divisor on U corresponding to the ray $\mathbb{R}_{\geq 0}(j_t, i_t, n - (i_t + j_t))$ in $N_\mathbb{R}$. Then, since a line bundle on Y is determined by the degrees of the restrictions to the exceptional curves, the fact that $\mathcal{O}(-E_{i_s})|_{(Y \cap D_t)}$ is of degree $n\delta_{st}$ implies the following.

Lemma 10.3. *The coefficient of D_t in E_{i_s} is η_{st}.*

For integers $f, g \in [0, n - 1]$, write $f = \sum_t f_t i_t$ and $g = \sum_t g_t i_t$ as in Theorem 3.1.

Lemma 10.4. *For integers a, b, c with $a + bq - c(1 + q) \equiv g - f \mod n$, if $x^a y^b z^c$ is a holomorphic section of $\mathcal{L}_f^\vee \otimes \mathcal{L}_g$ on U^0, then $a, b \geq 0$ and*

$$aj_t + bi_t + c(n - (i_t + j_t)) \geq \sum_s (g_s - f_s)\eta_{st} \tag{4}$$

for $1 \leq t \leq r$.

More precisely, the order of zero of the rational section $x^a y^b z^c$ in $\mathcal{L}_f^\vee \otimes \mathcal{L}_g$ along D_t is given by the integer

$$e_t = \frac{1}{n}\left(aj_t + bi_t + c(n - (i_t + j_t)) - \sum_s (g_s - f_s)\eta_{st}\right).$$

By noting $i_t = \eta_{t1}$, $j_t = \eta_{tr}$ and by multiplying the matrix C, we see that (e_t) is determined by the following system of equations:

$$\begin{cases} b - b_1 e_1 + e_2 = g_1 - f_1 - (b_1 - 2)c \\ e_{t-1} - b_t e_t + e_{t+1} = g_t - f_t - (b_t - 2)c \quad (2 \le t \le r - 1) \\ e_{r-1} - b_r e_r + a = g_r - f_r - (b_r - 2)c \end{cases} \tag{5}$$

Thus if $x^a y^b z^c$ is a holomorphic section of $\mathcal{L}_f^\vee \otimes \mathcal{L}_g$ on U^0, then there is a solution $(e_t) \in (\mathbb{Z}_{\ge 0})^r$ to (5). Putting $e_0 := b$ and $e_{r+1} := a$, we consider the second difference

$$e_t'' := e_{t-1} - 2e_t + e_{t+1}$$

for $1 \le t \le r$. Then (5) can be written as

$$e_t'' = g_t - f_t + (b_t - 2)(e_t - c) \quad (1 \le t \le r).$$

Note that $0 \le f_t, g_t \le b_t - 1$. We estimate the right hand side from below. The following is easy:

Lemma 10.5. *Let $e \ge 0$, $b_t \ge 2$, $f_t \le b_t - 1$ and $c < 0$ be integers. Then*

1. *If $-f_t + (b_t - 2)(e - c) < 0$, then we have $-f_t + (b_t - 2)(e - c) = -1$ and $f_t = b_t - 1$.*

2. *If $-f_t + (b_t - 2)(e - c) = 0$, then we have $f_t \ge b_t - 2$.*

Since $(f_1, \ldots, f_r)$ satisfies the condition in Lemma 3.4, this implies:

Corollary 10.6. *Suppose $(e_t) \in (\mathbb{Z}_{\ge 0})^r$ is a solution to (5) for $a, b \ge 0$ and $c < 0$. Then we have the following:*

1. *$e_t'' \ge -1$ for any t.*

2. *If $e_s'' = e_t'' = -1$ for $s < t$, then there is l with $s < l < t$ and $e_l'' \ge 1$.*

3. *There is $t_0 \in [0, r + 1]$ such that*

$$b = e_0 \ge e_1 \ge \cdots \ge e_{t_0} \le \cdots \le e_r \le e_{r+1} = a.$$

Proposition 10.7. *Suppose $x^a y^b z^c$ is a rational section of $\mathcal{L}_f^\vee \otimes \mathcal{L}_g$ satisfying (4). If we assume $c < 0$, then there are a special representation i_s and a rational section $x^{a'} y^{b'} z^c$ of $\mathcal{L}_f^\vee \otimes \mathcal{L}_{i_s}$ satisfying $0 \le a' \le a$, $0 \le b' \le b$ and the inequality (4) where we replace (a, b, g_t) by (a', b', δ_{st}).*

Proof. Let (e_t) be a solution to (5). We may assume that g is not special. It suffices to show that for a suitable choice of s, there is a solution $(h_0, \ldots, h_{r+1}) \in (\mathbb{Z}_{\geq 0})^{r+2}$ to

$$h_t'' = \delta_{ts} - f_t + (b_t - 2)(h_t - c) \quad (1 \leq t \leq r) \tag{6}$$

with $0 \leq h_t \leq e_t$. Note that (h_t) satisfying (6) is determined by any two consecutive values h_p, h_{p+1}. Thus all we have to do is to choose suitable s and values h_p, h_{p+1} for some p such that the solution to (6) satisfies $0 \leq h_t \leq e_t$.

Let $p \leq p'$ be such that

$$e_0 \geq \cdots \geq e_{p-1} > e_p = \cdots = e_{p'} < e_{p'+1} \leq \cdots \leq e_{r+1}.$$

Let q be the largest integer that satisfies

$$(1 \leq q \leq p \text{ and } - f_q + (b_q - 2)(e_p - c) > 0) \text{ or } q = 0.$$

Similarly, let q' be the least integer that satisfies

$$(p' \leq q' \leq r \text{ and } - f_{q'} + (b_{q'} - 2)(e_p - c) > 0) \text{ or } q' = r + 1.$$

We first consider the case where there is an integer $v \in (q, q')$ such that $-f_v + (b_v - 2)(e_p - c) < 0$. In this case, we have $f_v = b_v - 1$ and $-f_v + (b_v - 2)(e_p - c) = -1$ by Lemma 10.5. Such an integer $v \in (q, q')$ is unique by virtue of Lemma 3.4 and Lemma 10.5. We choose s as follows.

1. If $v \in [p, p']$, then put $s = v$.

2. If $v < p$, then $s = p$.

3. If $v > p'$, then $s = p'$.

Note that in each case, we have $g_s > 0$ since $e_s'' > -f_s + (b_s - 2)(e_s - 2)$ holds. We define (h_t) satisfying (6) by the following two consecutive values:

1. If $v \in [p, p']$, then $h_p = h_{p+1} = e_p$.

2. If $v < p$, then $h_p = h_{p+1} = e_p$.

3. If $v > p'$, then $h_{p'-1} = h_{p'} = e_p$.

Then it satisfies

1. $h_{q-1} > h_q = \cdots = h_{q'} < h_{q'+1}.$

2. $h_{p-1} > h_p = \cdots = h_{q'} < h_{q'+1}.$

3. $h_{q-1} > h_q = \cdots = h_{p'} < h_{p'+1}.$

in each case. Again by Lemma 3.4 and Lemma 10.5, we see that $h_t \geq h_p \geq 0$ for any t. To compare h_t and e_t, note that $h_p = e_p$ and $h_{p+1} = e_{p+1}$ (or $h_{p'} = e_{p'}$ and $h_{p'+1} = e_{p'+1}$) hold. Moreover, by our choice of s, we have $\delta_{st} \leq g_t$ for any t. Therefore, we inductively obtain $h_t'' \leq e_t''$ and $h_t \leq e_t$.

The case where there is no such v is easier; we can take any s with $g_s > 0$ and we can define (h_t) by $h_q = h_{q+1} = e_p$. $\qquad\square$

Now we prove Proposition 10.1:

Proof of Proposition 10.1. We have to show that if $x^a y^b z^c$ is a rational section of $\mathcal{L}_f^{-1} \otimes \mathcal{L}_g$ satisfying (4), then there is a path from f to g that is mapped to $x^a y^b z^c$. The assertion is obvious if $c \geq 0$ and hence we assume $c < 0$. Then, we have s, a' and b' as in the proposition. We can regard $x^{a'} y^{b'} z^{c+1}$ as a rational map from $\mathcal{L}_f$ to $\mathcal{L}_{i_s+n-q-1}$, whose orders of zeros along the divisors D_t are the same as those of $x^{a'} y^{b'} z^c$. Therefore, we can represent the map as the product of the map from $\mathcal{L}_f$ to $\mathcal{L}_{i_s+n-q-1}$, "multiplication by z^{-1}", and the path from $\mathcal{L}_{i_s}$ to $\mathcal{L}_g$, and inductively prove the assertion. $\square$

11   Some technical lemmas

This section is devoted to prove technical lemmas on the paths of the quiver associated with a dimer model, which will be needed later. Consider two zigzag paths with adjacent slopes, which associate a corner perfect matching D as in section 6. We have a map

$$h : \mathbb{C}\Gamma \to \mathbb{C}\Lambda$$

where Λ is the path algebra of the McKay quiver whose vertices are large hexagons. There is a corner perfect matching $\bar{D}$ of Λ corresponding to D, which corresponds to the zero locus of "multiplications by z".

Lemma 11.1. *Let v be a vertex of Γ.*

1. *Suppose v is the source of the large hexagon $h(v)$ and a path p of Λ starting from $h(v)$ does not intersect with $\bar{D}$. Then there is a path $\tilde{p}$ of Γ from v to any vertex in the large hexagon $t(p)$ such that $h(\tilde{p}) = p$ and $\tilde{p}$ does not intersect with D.*

2. *Suppose v is the sink of the large hexagon $h(v)$ and a path p of Λ ending at $h(v)$ does not intersect with $\bar{D}$. Then there is a path $\tilde{p}$ of Γ from any vertex in the large hexagon $s(p)$ to v such that $h(\tilde{p}) = p$ and $\tilde{p}$ does not intersect with D.*

The first assertion follows from the following lemma. The second assertion can be obtained similarly.

Lemma 11.2. *Suppose a vertex v of Γ is the source of the large hexagon $h(v)$.*

1. *For any vertex w of Γ in $h(v)$, there is a path q from v to w with $h(q) = e_{h(v)}$ (the idempotent of $h(v)$) which doesn't contain arrows in D.*

2. *If a is an arrow of Λ with $s(a) = h(v)$, then there is a path q' from v to the source of the large hexagon $t(a)$ with $h(q') = a$ which doesn't contain arrows in D.*

Proof. Let w be a vertex in $h(v)$ and take the shortest path q from v to w inside $h(v)$. Then, by the construction of the corner perfect matching D, q doesn't contain arrows in D. For an arrow a as in (2), there is a zigzag path contacting both the sources of $s(a)$ and $t(a)$. Then q' is the path of Γ which is parallel to this zigzag path starting from v and ending at the source of $t(a)$. $\square$

Lemma 11.3. *Suppose a is an arrow of Γ contained in the perfect matching D. Then there is a path q of Γ with the following properties:*

- *q goes from $s(a)$ to the source w of the large hexagon that is adjacent to the sink u of $h(t(a))$ by the arrow b in D with $s(b) = w$ and $t(b) = u$.*

- *$h(bq) \equiv h(a)$*

- *q doesn't contain arrows in D.*

Proof. First assume that a is inside a large hexagon (i.e., $h(s(a)) = h(t(a))$) as in Figure 41. Then there is a shortest path q' from $s(a)$ to u inside $h(t(a))$. In this case, q is obtained by composing q' and the path from u to w that goes around a node. Next consider the case where a is on one of the two zigzag paths determining large hexagons but not on the other one as in Figure 42. In this case, q is the path parallel to the zigzag path on which a is lying. Finally, suppose that a is on the intersection of the two zigzag paths as in Figure 43. In this case, b coincides with a and we can put $q = e_{s(a)}$.
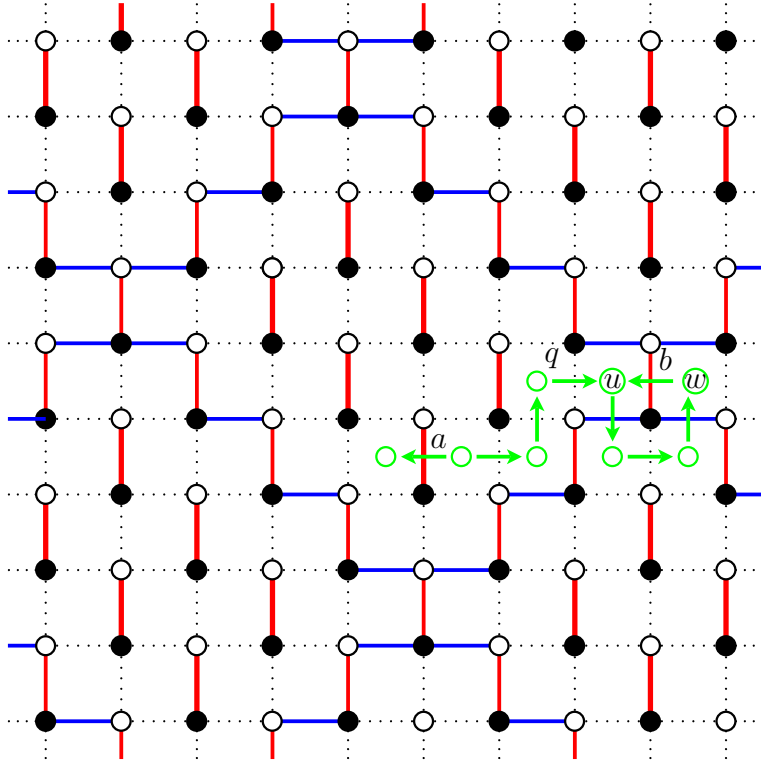

Figure 41: Case 1

Lemma 11.1, 11.3 and its dual yield the following.

Lemma 11.4. *Let a be an arrow of Γ contained in the perfect matching D.*

- *Suppose p is a path from $t(a)$ to the sink u of some large hexagon and p does not contain arrows in D. Let b be the arrow such that $t(b) = u$ and $s(b)$ is the source of the adjacent large hexagon. Then, there is a path p' with $ap \equiv p'b$.*
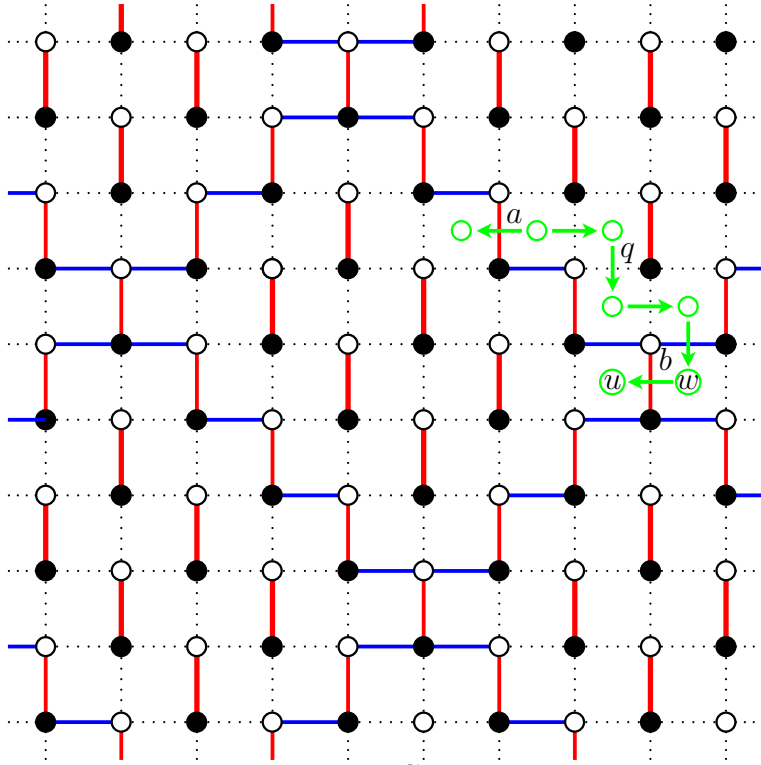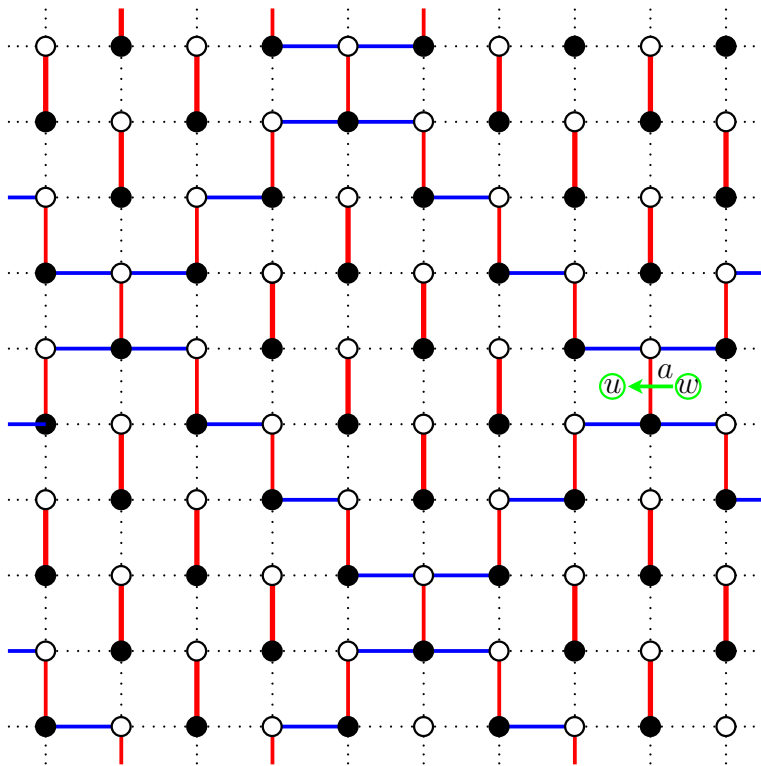
Figure 42: Case 2

Figure 43: Case 3

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- *Suppose q is a path from the source u of some large hexagon to $s(a)$ and q does not contain arrows in D. Let c be the arrow such that $s(c) = u$ and $t(c)$ is the sink of the adjacent large hexagon. Then, there is a path q' with $cq \equiv q'c$.*

12 Preservation of the tilting condition: the general case

Let Γ be the quiver with relations associated with a consistent dimer model and Γ' be obtained from Γ by adding inverse to the arrows from the sources of special large hexagons to the sinks of the neighboring large hexagons corresponding to "multiplication by z", as in §6. Let $\mathcal{M}$ be the corresponding moduli space with the stability condition chosen in §6. so that $\mathcal{M}$ contains $U = G\text{-Hilb}(\mathbb{C}^3)$ as an open subscheme and $Y = G\text{-Hilb}(\mathbb{C}^2)$ as a closed subscheme for some finite abelian small subgroup G of $GL_2(\mathbb{C})$. The McKay quiver of G as a subgroup of $SL_3(\mathbb{C})$ is denoted by Λ. $\mathcal{M}$ carries the tautological bundles $\mathcal{L}_v$ corresponding to vertices v of Γ. Let $\mathcal{M}^0$ be the complement $\mathcal{M} \setminus Y$ and $\mathcal{L}'_v$ be the restriction of $\mathcal{L}_v$ to $\mathcal{M}^0$. We prove the following in this section:

Proposition 12.1. $\bigoplus_v \mathcal{L}_v$ *is a tilting object if and only if so is* $\bigoplus_v \mathcal{L}'_v$.

Proof. In both directions, we use the long exact sequence

$$\cdots \to H^i_Y(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w) \to H^i(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w) \to H^i(\mathcal{M}^0, \mathcal{L}_v'^\vee \otimes \mathcal{L}'_w) \to \cdots. \quad (7)$$

Since Y is contained in U, one has $H^i_Y(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w) \cong H^i_Y(U, \mathcal{L}_v^\vee \otimes \mathcal{L}_w|_U)$ and the "only if" part follows immediately from Lemma 9.2 and Lemma 9.3.

To show the "if" part, assume that $\bigoplus_v \mathcal{L}'_v$ is a tilting object. In this case, (7) implies the vanishing of $H^i(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w)$ for $i \geq 2$, and it suffices to show the surjectivity of

$$H^0(\mathcal{M}^0, \mathcal{L}_v'^\vee \otimes \mathcal{L}'_w) \to H^1_Y(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w). \quad (8)$$

$H^1_Y(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w) \cong H^1_Y(U, \mathcal{L}_v^\vee \otimes \mathcal{L}_w|_U)$ has a basis consisting of Laurent monomials $x^a y^b z^c$ satisfying inequalities in section 10 with $c < 0$. Since $H^1(U, \mathcal{L}_v^\vee \otimes \mathcal{L}_w|_U)$ vanishes, it can be lifted to a section of $H^0(U \setminus Y, \mathcal{L}_v'^\vee \otimes \mathcal{L}'_w|_{U \setminus Y})$ and therefore is given by a path of Λ' by Proposition 10.1. Moreover, by virtue of Lemma 11.4 and the assumption $c < 0$, the path can be chosen so that it contains a reverse arrow in the intersection of the two zigzag paths (corresponding to "multiplication by z^{-1}") but not arrows in the corner perfect matching $\bar{D}$. Since "multiplication by z^{-1}" in Λ' can be lifted to an arrow of Γ' going from a sink to a source, Lemma 11.1 implies the path can be lifted to a path of Γ' from v to w and (8) is surjective.

Finally, we show that $\bigoplus_v \mathcal{L}_v$ is a generator. For an object $\alpha \in D(\mathcal{M})$, assume that $\mathbb{R}\mathrm{Hom}(\bigoplus_v \mathcal{L}_v, \alpha) = 0$. Let s be the source of the large hexagon corresponding to a special representation of G and t be the target of "multiplication by z" into the adjacent large hexagon. If ι denotes the inclusion $Y \to \mathcal{M}$, the dual to the exact sequence in Lemma 12.2 below shows that

$$\iota_* \iota^* \mathcal{L}_s^\vee \cong \{\mathcal{L}_t^\vee \to \mathcal{L}_s^\vee\},$$

so that one has

$$
\begin{aligned}
\mathbb{R}\mathrm{Hom}(\iota^*\mathcal{L}_s, \iota^*\alpha) &= \mathbb{R}\Gamma((\iota^*\mathcal{L}_s)^\vee \otimes \iota^*\alpha) \\
&= \mathbb{R}\Gamma(\iota^*\mathcal{L}_s^\vee \otimes \iota^*\alpha) \\
&= \mathbb{R}\Gamma(\iota_*(\iota^*\mathcal{L}_s^\vee \otimes \iota^*\alpha)) \\
&= \mathbb{R}\Gamma(\iota_*\iota^*\mathcal{L}_s^\vee \otimes \alpha)) \\
&= \mathbb{R}\Gamma(\{\mathcal{L}_t^\vee \to \mathcal{L}_s^\vee\} \otimes \alpha)) \\
&= 0.
\end{aligned}
$$

Since $\bigoplus \iota^*\mathcal{L}_s$ is a tilting object on Y by Theorem 2.8, we have $\iota^*\alpha = 0$. It follows that $\mathrm{Supp}\,\alpha \subset \mathcal{M}^0$ and we obtain $\alpha = 0$ by our assumption that $\bigoplus_v \mathcal{L}'_v$ is a tilting object. $\qquad\square$

Lemma 12.2. *Let s be the source of the large hexagon corresponding to a special representation of G and t be the target of "multiplication by z" into the adjacent large hexagon. Then we have an exact sequence*

$$
0 \to \mathcal{L}_s \to \mathcal{L}_t \to \mathcal{L}_t|_Y \to 0.
$$

Proof. Since the statement is true on $U = G\text{-Hilb}(\mathbb{C}^3)$ by Proposition 9.5, it suffices to show that the restriction of the map $\mathcal{L}_s \to \mathcal{L}_t$ to $\mathcal{M}^0$ is an isomorphism, which follows from the fact that $\mathcal{M}^0$ is the moduli of representations of Γ'. $\qquad\square$

13 Preservation of surjectivity: the general case

We use the same notation as in the previous section.

Proposition 13.1. *Assume that both $\bigoplus \mathcal{L}_v$ and $\bigoplus \mathcal{L}'_v$ are tilting objects. Then the map $\mathbb{C}\Gamma \to \mathrm{End}(\bigoplus \mathcal{L}_v)$ is surjective if and only if so is $\mathbb{C}\Gamma' \to \mathrm{End}(\bigoplus \mathcal{L}'_v)$.*

Proof. Take vertices v and w of Γ and consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & e_v\mathbb{C}\Gamma e_w & \xrightarrow{\ \gamma\ } & e_v\mathbb{C}\Gamma'e_w & \xrightarrow{\ \delta\ } & Q & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathcal{L}_v, \mathcal{L}_w) & \xrightarrow{\ \alpha\ } & \mathrm{Hom}(\mathcal{L}'_v, \mathcal{L}'_w) & \xrightarrow{\ \beta\ } & H^1_Y(\mathcal{L}_v^\vee \otimes \mathcal{L}_w) & \longrightarrow & 0
\end{array}
$$

where Q is defined as the cokernel of γ. The second row is exact by Proposition 12.1. Moreover, f and g are injective by consistency and hence the first row is also exact. The map k is defined so that the diagram is commutative, and it suffices to show that k is an isomorphism.

In the proof of the surjectivity of (8) ($= \beta$), we show that $\beta \circ g$ is surjective and hence k is surjective. To see that k is injective, consider the following commutative

diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow & e_v\mathbb{C}\Gamma e_w & \xrightarrow{\ \gamma\ } & e_v\mathbb{C}\Gamma' e_w & \xrightarrow{\ \delta\ } & Q & \longrightarrow 0 \\
& \downarrow & & i\downarrow & & j\downarrow & \\
0 \longrightarrow & e_{h(v)}\mathbb{C}\Lambda e_{h(w)} & \longrightarrow & e_{h(v)}\mathbb{C}\Lambda' e_{h(w)} & \xrightarrow{\ \delta'\ } & Q' & \longrightarrow 0 \\
& f'\downarrow & & g'\downarrow & & k'\downarrow & \\
0 \longrightarrow & \mathrm{Hom}(\mathcal{R}_{h(v)},\mathcal{R}_{h(w)}) & \longrightarrow & \mathrm{Hom}(\mathcal{R}'_{h(v)},\mathcal{R}'_{h(w)}) & \longrightarrow & H^1_Y(\mathcal{R}^\vee_{h(v)}\otimes\mathcal{R}_{h(w)}) & \longrightarrow 0
\end{array}
$$

where Λ is the McKay quiver whose vertices are large hexagons. Here k' is an isomorphism since f' and g' are isomorphisms.

As in the proof of the surjectivity of (8), any path in $\mathbb{C}\Gamma'\backslash\mathbb{C}\Gamma$ is equivalent to a path that contains a reverse arrow in the intersection of the two zigzag paths (corresponding to "multiplication by z^{-1}") but not arrows in the corner perfect matching D. This implies that Q (resp. Q') is isomorphic to the subspace of $e_v\mathbb{C}\Gamma' e_w$ (resp. $e_{h(v)'}\mathbb{C}\Lambda' e_{h(w)'}$) spanned by paths that contain reverse arrows but not arrows contained in D. Therefore, the injectivity of i implies the injectivity of j. Now $H^1_Y(\mathcal{R}^\vee_{h(v)}\otimes\mathcal{R}_{h(w)})$ coincides with $H^1_Y(\mathcal{L}^\vee_v\otimes\mathcal{L}_w)$ and $k=k'\circ j$ is injective. $\qquad\square$

14   Proof of derived equivalences

In this section, we give a proof of Theorem 1.1 using the "only if" parts of Propositions 12.1 and 13.1. We also show that consistency implies derived equivalence using the "if" parts of Propositions 12.1 and 13.1.

Proof of Theorem 1.1: First we embed a given lattice polygon Δ into a lattice triangle Δ_0. Bridgeland, King and Reid [4] establishes a derived equivalence between the McKay quiver Γ_0 corresponding to Δ_0 and the toric Calabi-Yau 3-fold. Then we repeat the operation in §7 on the dimer model B_0 corresponding to Γ_0 until we arrive at a dimer model B associated with Δ:

$$
B_0 \to B_1 \to \cdots \to B_k = B
$$

At each step, consistency is preserved by Proposition 8.1. Assume that we have established a derived equivalence

$$
D^b\,\mathrm{mod}\,\mathbb{C}\Gamma_i \cong D^b\,\mathrm{coh}\,\mathcal{M}_{\theta_i}
$$

via the tautological bundle between the quiver Γ_i associated with the dimer model B_i and the moduli space of representations with a stability parameter θ_i. Then $\mathbb{C}\Gamma_i$ is of finite global dimension and the 3-shift is a Serre functor on the subcategory of finite dimensional representations, so that the argument in [4] proves the derived equivalence for any generic θ. Now choose θ as in §6 and put $\mathcal{M}=\mathcal{M}_\theta$, $Y=G\text{-Hilb}(\mathbb{C}^2)$ be as in §6. Then $\mathcal{M}^0=\mathcal{M}\setminus Y$ is the moduli of representations of the quiver associated with the dimer model B_{i+1} for an appropriate stability parameter θ_{i+1}. A derived equivalence for B_{i+1} is ensured by Proposition 12.1, Proposition 13.1 and Lemma 5.16. Repeating the argument, we obtain the result for the dimer model B. $\qquad\square$

Theorem 14.1. *Let θ be a generic stability parameter for a consistent dimer model. Then the tautological bundle is a tilting object on $\mathcal{M}_\theta$ and gives an equivalence from the derived category of coherent sheaves on $\mathcal{M}_\theta$ to the derived category of finitely generated modules over $\mathbb{C}\Gamma$.*

Proof. Start with the given dimer model and repeat the operation in §7. Proposition 8.1, Proposition 12.1, Proposition 13.1 and Lemma 5.16 ensure that it suffices to consider the assertion in the case where the corresponding lattice polygon is a lattice triangle with the minimum area. In this case, there are three zigzag paths and the large hexagon constructed from two of them is a unique vertex of the quiver. If we remove all the nodes of valence two, then the corresponding quiver is that of McKay quiver for the trivial group. Therefore, the path algebra is isomorphic to the polynomial algebra in three variables and we are done. □

15　Crepant resolutions as moduli spaces

In this section, we extend the main result of [8] to general dimer models:

Theorem 15.1. *Let X be the affine toric variety corresponding to the lattice polygon coming from a consistent dimer model. Then for any projective crepant resolution $\widetilde{X} \to X$, there is a generic parameter θ such that $\widetilde{X} \cong \mathcal{M}_\theta$.*

Proof. Almost the same proof as in [8] works in our new situation, with the following minor modifications:

- In [8, Proposition 4.4], assertions on non-compact exceptional divisors are reduced to the two-dimensional case. In the present situation, this can be done by considering the division of the torus by the zigzag paths perpendicular to a fixed edge of the polygon, which determines the McKay quiver of type A in the two-dimensional case.

- In [8, Lemma 3.10], the existence of certain Quot-scheme is used, so that the argument there works even for non-abelian groups G. This is not necessary for the moduli of representations with dimension vector $(1, \ldots, 1)$; the subscheme is defined just by setting the values of some arrows to zero.

- In [8, Lemma 5.7], the symmetry of the McKay quiver (by tensoring representations) is used only for the sake of simplicity, and we do not need to use such a symmetry.

- The use of the tessellation of the torus by "diamonds" in the argument of [8, §10] can be replaced by hexagons, and as such can be generalized to dimer models.

□

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