Cohen–Macaulay Representations

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The Krull-Remak-Schmidt Theorem

[Brief introduction here. A little history.]

§1 KRS in an additive category

Looking ahead to an application in Chapter 2, we will clutter things up slightly by working in an additive category, rather than a category of modules. An *additive* category is a category $\mathscr{A}$ with 0-object such that (i) $\mathrm{Hom}_{\mathscr{A}}(M_1, M_2)$ is an abelian group for each pair M_1, M_2 of objects, (ii) composition is bilinear, and (iii) every finite set of objects has a biproduct. A *biproduct* of $M_1, \ldots, M_m$ consists of an object M together with maps $u_i \colon M_i \longrightarrow M$ and $p_i \colon M \longrightarrow M_i$, $i = 1, \ldots, m$, such that $p_i u_j = \delta_{ij}$ and $u_1 p_1 + \cdots + u_m p_m = 1_M$. We denote the biproduct by $M_1 \oplus \cdots \oplus M_m$.

We will need an additional condition on our additive category, namely, that idempotents split (cf. [Bas68, Chapter I, §3, p. 19]). Given an object M and an idempotent $e \in \mathrm{End}_{\mathscr{A}}(M)$, we say that *e splits* provided there is a factorization $M \xrightarrow{p} K \xrightarrow{u} M$ such that $e = up$ and $pu = 1_K$.

The reader is probably familiar with the notion of an *abelian* category, that is, an additive category in which every map has a kernel and a cokernel, and in which every monomorphism (respectively epimorphism) is a kernel (respectively cokernel). Over any ring R the category R-Mod of all left R-modules is abelian; if R is left Noetherian, then the category R-mod of finitely generated left R-modules is abelian. It is easy to see that idempotents split in an abelian category. Indeed, suppose $e \colon M \longrightarrow M$ is an idempotent, and let $u: K \longrightarrow M$ be the kernel of $1_M - e$. Since $(1_M - e)e = 0$, the map e factors through u; that is, there is a map $p: M \longrightarrow K$ satisfying $up = e$. Then $upu = eu = eu + (1_M - e)u = u = u1_K$. Since u is a monomorphism (as kernels are always monomorphisms), it follows that $pu = 1_K$.

A non-zero object M in the additive category $\mathscr{A}$ is said to be *decomposable* if there exist non-zero objects M_1 and M_2 such that $M \cong M_1 \oplus M_2$. Otherwise, M is *indecomposable*. We leave the proof of the next result as an exercise:

1.1 Proposition. *Let M be a non-zero object in an additive category $\mathscr{A}$, and let $E = \mathrm{End}_{\mathscr{A}}(M)$.*

(i) If 0 and 1 are the only idempotents of E, then M is indecomposable.

(ii) Conversely, if $e = e^2 \in E$, if both e and $1 - e$ split, and if $e \neq 0, 1$, then M is decomposable.

We say that the Krull-Remak-Schmidt Theorem (KRS for short) holds in the additive category $\mathscr{A}$ provided

(i) every object in $\mathscr{A}$ is a biproduct of indecomposable objects, and

(ii) if $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n$, with each M_i and each N_j an indecomposable object in $\mathscr{A}$, then $m = n$ and, after renumbering, $M_i \cong N_i$ for each i.

It is easy to see that every Noetherian object is a biproduct of finitely many indecomposable objects (cf. Exercise 1.19), but there are easy examples to show that (ii) can fail. For perhaps the simplest example, let

$R = k[x,y]$, the polynomial ring in two variables over a field. Letting $\mathfrak{m} = Rx + Ry$ and $\mathfrak{n} = R(x-1) + Ry$, we get a short exact sequence

$$0 \longrightarrow \mathfrak{m} \cap \mathfrak{n} \longrightarrow \mathfrak{m} \oplus \mathfrak{n} \longrightarrow R \longrightarrow 0,$$

since $\mathfrak{m} + \mathfrak{n} = R$. This splits, so $\mathfrak{m} \oplus \mathfrak{n} \cong R \oplus (\mathfrak{m} \cap \mathfrak{n})$. Since neither $\mathfrak{m}$ nor $\mathfrak{n}$ is isomorphic to R as an R-module, KRS fails for finitely generated R-modules.

This example indicates that KRS is likely to fail for modules over rings that aren't local. It can fail even for finitely generated modules over local rings. An example due to R. G. Swan is in E. G. Evans's paper [Eva73]. In Chapter 12 we will see just how badly it can fail. G. Azumaya [Azu48] observed that the crucial property for guaranteeing KRS is that the endomorphism rings of the summands be local in the non-commutative sense. To avoid a conflict of jargon, we define a ring Λ (not necessarily commutative) to be *nc-local* provided $\Lambda/\mathscr{J}(\Lambda)$ is a division ring, where $\mathscr{J}(-)$ denotes the Jacobson radical. Equivalently (cf. Exercise 1.20) $\Lambda \neq \{0\}$ and $\mathscr{J}(\Lambda)$ is exactly the set of non-units of Λ. It is clear from Proposition 1.1 that any object with nc-local endomorphism ring must be indecomposable.

We'll model our proof of KRS after the proof of unique factorization in the integers, by showing that an object with nc-local endomorphism ring behaves like a prime element in an integral domain. We'll even use the same notation, writing "$M \mid N$", for objects M and N, to indicate that there is an object Z such that $N \cong M \oplus Z$. Our inductive proof depends on direct-sum cancellation ((ii) below), analogous to the fact that $mz = my \implies z = y$ for non-zero elements m, z, y in an integral domain. Later in the chapter (Corollary 1.15) we'll prove cancellation for arbitrary finitely generated modules over a local ring, but for now we'll prove only that objects with nc-local endomorphism rings can be cancelled.

1.2 Lemma. *Let $\mathscr{A}$ be an additive category in which idempotents split. Let M, X, Y, and Z be objects of $\mathscr{A}$, let $E = \operatorname{End}_{\mathscr{A}}(M)$, and assume that E is nc-local.*

(i) If $M \mid X \oplus Y$, then $M \mid X$ or $M \mid Y$ ("primelike").

(ii) If $M \oplus Z \cong M \oplus Y$, then $Z \cong Y$ ("cancellation").

Proof. We'll prove (i) and (ii) sort of simultaneously. In (i) we have an object Z such that $M \oplus Z \cong X \oplus Y$. In the proof of (ii) we set $X = M$ and again get an isomorphism $M \oplus Z \cong X \oplus Y$. Put $J = \mathscr{J}(E)$, the set of non-units of E.

Choose reciprocal isomorphisms $\varphi\colon M \oplus Z \longrightarrow X \oplus Y$ and $\psi\colon X \oplus Y \longrightarrow M \oplus Z$. Write

$$\varphi = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \qquad \text{and} \qquad \psi = \begin{bmatrix} \mu & \nu \\ \sigma & \tau \end{bmatrix},$$

where $\alpha\colon M \longrightarrow X$, $\beta\colon Z \longrightarrow X$, $\gamma\colon M \longrightarrow Y$, $\delta\colon Z \longrightarrow Y$, $\mu\colon X \longrightarrow M$, $\nu\colon Y \longrightarrow M$, $\sigma\colon X \longrightarrow Z$ and $\tau\colon Y \longrightarrow Z$. Since $\psi\varphi = 1_{M\oplus Z} = \left[\begin{smallmatrix} 1_M & 0 \\ 0 & 1_Z \end{smallmatrix}\right]$, we have $\mu\alpha + \nu\gamma = 1_M$. Therefore, as E is local, either $\mu\alpha$ or $\nu\gamma$ must be outside J and hence an automorphism of M. Assuming that $\mu\alpha$ is an automorphism, we will produce an object W and maps

$$M \xrightarrow{u} X \xrightarrow{p} M \qquad\qquad W \xrightarrow{v} X \xrightarrow{q} W$$

satisfying $pu = 1_M$, $pv = 0$, $qv = 1_W$, $qu = 0$, and $up + vq = 1_X$. This will show that $X = M \oplus W$. (Similarly, the assumption that $\nu\gamma$ is an isomorphism forces M to be a direct summand of Y.)

Letting $u = \alpha$, $p = (\mu\alpha)^{-1}\mu$ and $e = up \in \mathrm{End}_{\mathscr{A}}(X)$, we have $pu = 1_M$ and $e^2 = e$. By hypothesis, the idempotent $1-e$ splits, so we can write $1-e = vq$, where $X \xrightarrow{q} W \xrightarrow{v} X$ and $qv = 1_W$. From $e = up$ and $1-e = vq$, we get the equation $up + vq = 1_X$. Moreover, $qu = (qv)(qu)(pu) = q(vq)(up)u = q(1-e)eu = 0$; similarly, $pv = pupvqv = pe(1-e)v = 0$. We have verified all of the required equations, so $X = M \oplus W$. This proves (i).

To prove (ii) we assume that $X = M$. Suppose first that α is a unit of E. We use α to diagonalize φ:

$$\begin{bmatrix} 1 & 0 \\ -\gamma\alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} 1 & -\alpha^{-1}\beta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma\alpha^{-1}\beta + \delta \end{bmatrix}$$

Since all the matrices on the left are invertible, so must be the one on the right, and it follows that $-\gamma\alpha^{-1}\beta + \delta : Z \longrightarrow Y$ is an isomorphism.

Suppose, on the other hand, that $\alpha \in J$. Then $\nu\gamma \notin J$ (as $\mu\alpha + \nu\gamma = 1_M$), and it follows that $\alpha + \nu\gamma \notin J$. We define a new map

$$\psi' = \begin{bmatrix} 1_M & \nu \\ \sigma & \tau \end{bmatrix} : M \oplus Y \longrightarrow M \oplus Z,$$

which we claim is an isomorphism. Assuming the claim, we can diagonalize ψ' as we did φ, obtaining, in the lower-right corner, an isomorphism from Y onto Z, and finishing the proof. To prove the claim, we use the equation $\psi\varphi = 1_{M\oplus Z}$ to get

$$\psi'\varphi = \begin{bmatrix} \alpha + \nu\gamma & \beta + \nu\gamma \\ 0 & 1_Z \end{bmatrix}.$$

As $\alpha + \nu\gamma$ is an automorphism of M, $\psi'\varphi$ is clearly an automorphism of $M \oplus Z$. Therefore $\psi' = (\psi'\varphi)\varphi^{-1}$ is an isomorphism. □

1.3 Theorem (Krull-Remak-Schmidt-Azumaya)**.** *Let $\mathscr{A}$ be an additive category in which every idempotent splits. Let $M_1,\dots,M_m$ and $N_1,\dots,N_n$ be indecomposable objects of $\mathscr{A}$, with $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n$. Assume that $\operatorname{End}_{\mathscr{A}}(M_i)$ is nc-local for each i. Then $m = n$ and, after renumbering, $M_i \cong N_i$ for each i.*

Proof. We use induction on m, the case $m = 1$ being trivial. Assuming $m \geqslant 2$, we see that $M_m \mid N_1 \oplus \cdots \oplus N_n$. By (i) of Lemma 1.2, $M_m \mid N_j$ for some j; by renumbering, we may assume that $M_m \mid N_n$. Since N_n is indecomposable and $M_m \neq 0$, we must have $M_m \cong N_n$. Now (ii) of Lemma 1.2 implies that $M_1 \oplus \cdots \oplus M_{m-1} \cong N_1 \oplus \cdots \oplus N_{n-1}$, and the inductive hypothesis completes the proof. □

Azumaya [Azu48] actually proved the *infinite* version of Theorem 1.3: If $\bigoplus_{i\in I} M_i \cong \bigoplus_{j \in J} N_j$ and the endomorphism ring of each M_i is nc-local, and each N_j is indecomposable, then there is a bijection $\sigma\colon I \longrightarrow J$ such that $M_i \cong N_{\sigma(i)}$ for each i. (Cf. [Fac98, Chapter 2].)

We want to find some situations where indecomposables automatically have nc-local endomorphism rings. It is well known that idempotents lift modulo any nil ideal. A typical proof of this fact actually yields the following stronger result, which we will use in the next section.

1.4 Proposition. *Let I be a two-sided ideal of a (possibly non-commutative) ring Λ, and let e be an idempotent of Λ/I. Given any positive integer n, there is an element $x \in \Lambda$ such that $x + I = e$ and $x \equiv x^2 \pmod{I^n}$.*

Proof. Start with an arbitrary element $u \in \Lambda$ such that $u + I = e$, and let $v = 1 - u$. In the binomial expansion of $(u + v)^{2n-1}$, let x be the sum of the first n terms: $x = u^{2n-1} + \cdots + \binom{2n-1}{n-1} u^n v^{n-1}$. Putting $y = 1 - x$ (the other half of the expansion), we see that $x - x^2 = xy \in \Lambda(uv)^n \Lambda$. Since $uv = u(1-u) \in I$, we have $x - x^2 \in I^n$. □

Here is an easy consequence, which will be needed in Chapter 2:

1.5 Corollary. *Let M be an indecomposable object in an additive category $\mathscr{A}$. Assume that idempotents split in $\mathscr{A}$. If $E := \operatorname{End}_{\mathscr{A}}(M)$ is left or right Artinian, then E is nc-local.*

Proof. Since M is indecomposable, E has no non-trivial idempotents. Since $\mathscr{J}(E)$ is nilpotent, Proposition 1.4 implies that $E/\mathscr{J}(E)$ has no idempotents either. It follows easily from the Wedderburn–Artin Theorem [Lam91, (3.5)] that $E/\mathscr{J}(E)$ is a division ring, whence nc-local. □

§2 KRS over Henselian rings

We now proceed to prove KRS for finitely generated modules over complete and, more generally, Henselian local rings. Here we define a local ring $(R, \mathfrak{m}, k)$ to be *Henselian* provided, for every module-finite R-algebra Λ (not necessarily commutative), each idempotent of $\Lambda/\mathscr{J}(\Lambda)$ lifts to an idempotent of Λ. For the classical definition of "Henselian" in terms of factorization of polynomials, and for other equivalent conditions, see Theorem A.26.

1.6 Lemma. *Let R be a commutative ring and Λ a module-finite R-algebra (not necessarily commutative). Then $\Lambda\mathscr{J}(R) \subseteq \mathscr{J}(\Lambda)$.*

Proof. Let $f \in \Lambda\mathscr{J}(R)$. We want to show that $\Lambda(1 - \lambda f) = \Lambda$ for every $\lambda \in \Lambda$. Clearly $\Lambda(1 - \lambda f) + \Lambda\mathscr{J}(R) = \Lambda$, and now NAK completes the proof. □

1.7 Theorem. *Let* $(R, \mathfrak{m}, k)$ *be a Henselian local ring, and let* M *be an indecomposable finitely generated* R*-module. Then* $\operatorname{End}_R(M)$ *is nc-local. In particular, KRS holds for the category of finitely generated modules over a Henselian local ring.*

Proof. Let $E = \operatorname{End}_R(M)$ and $J = \mathscr{J}(E)$. Since E is a module-finite R-algebra (cf. Exercise 1.21), Lemma 1.6 implies that $\mathfrak{m}E \subseteq J$ and hence that E/J is a finite-dimensional k-algebra. It follows that E/J is semisimple Artinian. Moreover, since E has no non-trivial idempotents, neither does E/J. By the Wedderburn–Artin Theorem [Lam91, (3.5)], E/J is a division ring. □

1.8 Corollary (Hensel's Lemma)**.** *Every complete local ring is Henselian.*

Proof. Let $(R, \mathfrak{m}, k)$ be a complete local ring, let Λ be a module-finite R-algebra, and put $J = \mathscr{J}(\Lambda)$. Again, $\mathfrak{m}\Lambda \subseteq J$, and $J/\mathfrak{m}\Lambda$ is a nilpotent ideal of $\Lambda/\mathfrak{m}\Lambda$ (since $\Lambda/\mathfrak{m}\Lambda$ is Artinian). By Proposition 1.4 we can lift each idempotent of Λ/J to an idempotent of $\Lambda/\mathfrak{m}\Lambda$. Therefore it will suffice to show that every idempotent e of $\Lambda/\mathfrak{m}\Lambda$ lifts to an idempotent of Λ. Using Proposition 1.4, we can choose, for each positive integer n, an element $x_n \in \Lambda$ such that $x_n + \mathfrak{m}\Lambda = e$ and $x_n \equiv x_n^2 \pmod{\mathfrak{m}^n\Lambda}$. (Of course $\mathfrak{m}^n\Lambda = (\mathfrak{m}\Lambda)^n$.) We claim that (x_n) is a Cauchy sequence for the $\mathfrak{m}\Lambda$-adic topology on Λ. To see this, let n be an arbitrary positive integer. Given any $m \geqslant n$, put $z = x_m + x_n - 2x_m x_n$. Then $z \equiv z^2 \pmod{\mathfrak{m}^n\Lambda}$. Also, since $x_m \equiv x_n \pmod{\mathfrak{m}\Lambda}$, we see that $z \equiv 0 \pmod{\mathfrak{m}\Lambda}$, so $1 - z$ is a unit of Λ. Since $z(1-z) \in \mathfrak{m}^n\Lambda$, it follows that $z \in \mathfrak{m}^n\Lambda$. Thus we have

$$x_m + x_n \equiv 2x_m x_n, \qquad x_m \equiv x_m^2, \qquad x_n \equiv x_n^2 \pmod{\mathfrak{m}^n\Lambda}.$$

Multiplying the first congruence, in turn, by x_m and by x_n, we learn that $x_m \equiv x_m x_n \equiv x_n \pmod{\mathfrak{m}^n \Lambda}$. If, now, $\ell \geqslant n$ and $m \geqslant n$, we see that $x_\ell \equiv x_m \pmod{\mathfrak{m}^n \Lambda}$. This verifies the claim. Since Λ is $\mathfrak{m}\Lambda$-adically complete (cf. Exercise 1.24), we let x be the limit of the sequence (x_n) and check that x is an idempotent lifting e. □

1.9 Corollary. *KRS holds for finitely generated modules over complete local rings.*

Henselian rings are *almost* characterized by the Krull-Remak-Schmidt property. Indeed, a theorem due to E. G. Evans [Eva73] states that a local ring R is Henselian if and only if for every module-finite commutative local R-algebra A the finitely generated A-modules satisfy KRS.

§3 R-modules vs. $\widehat{R}$-modules

A major theme in this book is the study of direct-sum decompositions over local rings that are not necessarily complete. Here we record a few results that will allow us to use KRS over $\widehat{R}$ to understand R-modules.

We begin with a result due to Guralnick [Gur85] on lifting homomorphisms modulo high powers of the maximal ideal of a local ring. Given finitely generated modules M and N over a local ring $(R,\mathfrak{m})$, we define a *lifting number* for the pair (M,N) to be a non-negative integer e satisfying the following property: For each positive integer f and each R-homomorphism $\xi\colon M/\mathfrak{m}^{e+f}M \longrightarrow N/\mathfrak{m}^{e+f}N$, there exists $\sigma \in \operatorname{Hom}_R(M,N)$ such that σ and ξ induce the same homomorphism $M/\mathfrak{m}^f M \longrightarrow N/\mathfrak{m}^f N$. (Thus the outer and bottom squares in the diagram below both commute, though the top square may not.)

$$\begin{array}{ccc} M & \xrightarrow{\quad\sigma\quad} & N \\ \downarrow & & \downarrow \\ M/\mathfrak{m}^{e+f}M & \xrightarrow{\xi} & N/\mathfrak{m}^{e+f}N \\ \downarrow & & \downarrow \\ M/\mathfrak{m}^{f}M & \xrightarrow{\bar{\xi}=\bar{\sigma}} & N/\mathfrak{m}^{f}N \end{array}$$

For example, 0 is a lifting number for (M,N) if M is projective. Lemma 1.11 below shows that *every* pair of finitely generated modules has a lifting number.

1.10 Lemma. *If e is a lifting number for (M,N) and $e' \geqslant e$, then e' is also a lifting number for (M,N).*

Proof. Let f' be a positive integer, and let $\xi\colon M/\mathfrak{m}^{e'+f'}M \longrightarrow N/\mathfrak{m}^{e'+f'}N$ be an R-homomorphism. Put $f = f'+e'-e$. Since $e'+f' = e+f$ and e is a lifting number, there is a homomorphism $\sigma\colon M \longrightarrow N$ such that σ and ξ induce the same homomorphism $M/\mathfrak{m}^f M \longrightarrow N/\mathfrak{m}^f N$. Now $f \geqslant f'$, and it follows that σ and ξ induce the same homomorphism $M/\mathfrak{m}^{f'}M \longrightarrow N/\mathfrak{m}^{f'}N$. □

1.11 Lemma ([Gur85, Theorem A])**.** *Every pair (M,N) of modules over a local ring $(R,\mathfrak{m})$ has a lifting number.*

Proof. Choose exact sequences

$$F_1 \xrightarrow{\alpha} F_0 \longrightarrow M \longrightarrow 0\,,$$
$$G_1 \xrightarrow{\beta} G_0 \longrightarrow N \longrightarrow 0\,,$$

where F_i and G_i are finite-rank free R-modules. Define an R-homomorphism $\Phi\colon \operatorname{Hom}_R(F_0,G_0) \times \operatorname{Hom}_R(F_1,G_1) \longrightarrow \operatorname{Hom}_R(F_1,G_0)$ by $\Phi(\mu,\nu) = \mu\alpha - \beta\nu$.

Applying the Artin-Rees Lemma to the submodule $\operatorname{im}(\Phi)$ of $\operatorname{Hom}_R(F_1,G_0)$, we get a positive integer e such that

$$\operatorname{im}(\Phi)\cap \mathfrak{m}^{e+f}\operatorname{Hom}_R(F_1,G_0)\subseteq \mathfrak{m}^f\operatorname{im}(\Phi) \qquad \text{for each } f>0\,. \tag{1.11.1}$$

Suppose now that $f>0$ and $\xi\colon M/\mathfrak{m}^{e+f}M \longrightarrow N/\mathfrak{m}^{e+f}N$ is an R-homomorphism. We can lift ξ to homomorphisms $\overline{\mu_0}$ and $\overline{\nu_0}$ making the following diagram commute:

$$\begin{array}{ccccccc} F_1/\mathfrak{m}^{e+f}F_1 & \xrightarrow{\overline{\alpha}} & F_0/\mathfrak{m}^{e+f}F_0 & \longrightarrow & M/\mathfrak{m}^{e+f}M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \overline{\nu_0}} & & \downarrow{\scriptstyle \overline{\mu_0}} & & \downarrow{\scriptstyle \xi} & & \\ G_1/\mathfrak{m}^{e+f}G_1 & \xrightarrow[\overline{\beta}]{} & G_0/\mathfrak{m}^{e+f}G_0 & \longrightarrow & N/\mathfrak{m}^{e+f}N & \longrightarrow & 0 \end{array} \tag{1.11.2}$$

Now lift $\overline{\mu_0}$ and $\overline{\nu_0}$ to maps $\mu_0\in\operatorname{Hom}_R(F_0,G_0)$ and $\nu_0\in\operatorname{Hom}_R(F_1,G_1)$. The commutativity of (1.11.2) implies that the image of $\Phi(\mu_0,\nu_0)\colon F_1\longrightarrow G_0$ lies in $\mathfrak{m}^{e+f}G_0$. Choosing bases for F_1 and G_0, we see that the matrix representing $\Phi(\mu_0,\nu_0)$ has entries in $\mathfrak{m}^{e+f}$, so that $\Phi(\mu_0,\nu_0)\in\mathfrak{m}^{e+f}\operatorname{Hom}_R(F_1,G_0)$. By (1.11.1), $\Phi(\mu_0,\nu_0)\in\mathfrak{m}^f\operatorname{im}(\Phi)=\Phi(\mathfrak{m}^f(\operatorname{Hom}_R(F_0,G_0)\times\operatorname{Hom}_R(F_1,G_1)))$. Choose $(\mu_1,\nu_1)\in\mathfrak{m}^f(\operatorname{Hom}_R(F_0,G_0)\times\operatorname{Hom}_R(F_1,G_1))$ such that $\Phi(\mu_1,\nu_1)=\Phi(\mu_0,\nu_0)$, and set $(\mu,\nu)=(\mu_0,\nu_0)-(\mu_1,\nu_1)$. Then $\Phi(\mu,\nu)=0$, so μ induces an R-homomorphism $\sigma\colon M\longrightarrow N$. Since μ and μ_0 agree modulo $\mathfrak{m}^f$, it follows that σ and ξ induce the same map $M/\mathfrak{m}^fM\longrightarrow N/\mathfrak{m}^fN$. □

We denote by $e(M,N)$ the smallest lifting number for the pair (M,N).

1.12 Theorem ([Gur85, Corollary 2])**.** *Let $(R,\mathfrak{m})$ be a local ring, and let M and N be finitely generated R-modules. If $r\geqslant\max\{e(M,N),e(N,M)\}$ and $M/\mathfrak{m}^{r+1}M\mid N/\mathfrak{m}^{r+1}N$, then $M\mid N$.*

Proof. Choose R-module homomorphisms $\xi\colon M/\mathfrak{m}^{r+1}M \longrightarrow N/\mathfrak{m}^{r+1}N$ and $\eta\colon N/\mathfrak{m}^{r+1}N \longrightarrow M/\mathfrak{m}^{r+1}M$ such that $\eta\xi = 1_{M/\mathfrak{m}^{r+1}M}$. Since r is a lifting number (Lemma 1.10), there exist R-homomorphisms $\sigma\colon M \longrightarrow N$ and $\tau\colon N \longrightarrow M$ such that σ agrees with ξ modulo $\mathfrak{m}$ and τ agrees with η modulo $\mathfrak{m}$. By Nakayama's lemma, $\tau\sigma\colon M \longrightarrow M$ is surjective and therefore, by Exercise 1.27, an automorphism. It follows that $M \mid N$. □

1.13 Corollary. *Let $(R,\mathfrak{m})$ be a local ring and M, N finitely generated R-modules. If $M/\mathfrak{m}^n M \cong N/\mathfrak{m}^n N$ for all $n \gg 0$, then $M \cong N$.*

Proof. By Theorem 1.12, $M \mid N$ and $N \mid M$. In particular, we have surjections $N \xrightarrow{\alpha} M$ and $M \xrightarrow{\beta} N$. Then $\beta\alpha$ is a surjective endomorphism of N and therefore is an automorphism (cf. Exercise 1.27). It follows that α is one-to-one and therefore an isomorphism. □

1.14 Corollary. *Let $(R,\mathfrak{m})$ be a local ring and $(\widehat{R},\widehat{\mathfrak{m}})$ its $\mathfrak{m}$-adic completion. Let M and N be finitely generated R-modules.*

(i) If $\widehat{R}\otimes_R M \mid \widehat{R}\otimes_R N$, then $M \mid N$.

(ii) If $\widehat{R}\otimes_R M \cong \widehat{R}\otimes_R N$, then $M \cong N$. □

1.15 Corollary. *Let M, N and P be finitely generated modules over a local ring $(R,\mathfrak{m})$. If $M \oplus P \cong N \oplus P$, then $M \cong N$.*

Proof. We have $(\widehat{R}\otimes_R M)\oplus(\widehat{R}\otimes_R P) \cong (\widehat{R}\otimes_R N)\oplus(\widehat{R}\otimes_R P)$. Using KRS for complete rings (Corollary 1.8) we see easily that $\widehat{R}\otimes_R M \cong \widehat{R}\otimes_R N$. Now apply Corollary 1.14. □

Given an R-module M, let us say that two direct-sum decompositions $M \cong M_1 \oplus \cdots \oplus M_m$ and $M \cong N_1 \oplus \cdots \oplus N_n$ are *equivalent* provided $m = n$ and, after a permutation, $M_i \cong N_i$ for each i. (We do not require that the summands be indecomposable.)

1.16 Corollary. *A finitely generated module over a local ring has, up to isomorphism, only finitely many direct summands and, up to equivalence, only finitely many direct-sum decompositions.*

We leave the proof as an exercise.

§4 Exercises

1.17 Exercise. Prove Proposition 1.1: For a non-zero object M in an additive category $\mathscr{A}$, and $E = \operatorname{End}_{\mathscr{A}}(M)$, if 0 and 1 are the only idempotents of E, then M is indecomposable. Conversely, if $e = e^2 \in E$, if both e and $1-e$ split, and if $e \neq 0, 1$, then M is decomposable.

1.18 Exercise. Let M be an object in an additive category. Show that every direct-sum (i.e., coproduct) decomposition $M = M_1 \oplus M_2$ has a biproduct structure.

1.19 Exercise. Let M be an object in an additive category.

(i) Suppose that M has either the ascending chain condition or the descending chain condition on direct summands. Prove that M has an indecomposable direct summand.

(ii) Prove that M is a direct sum (biproduct) of finitely many indecomposable objects.

1.20 Exercise. Let Λ be a ring with $1 \neq 0$. Prove that the following conditions are equivalent:

(i) Λ is nc-local.

(ii) $\mathscr{J}(\Lambda)$ is the set of non-units of Λ.

(iii) The set of non-units of Λ is closed under addition.

(Warning: In a non-commutative ring one can have non-units x and y such that $xy = 1$.)

1.21 Exercise. Let M and N be finitely generated modules over a commutative Noetherian ring R. Prove that $\operatorname{Hom}_R(M,N)$ is finitely generated as an R-module.

1.22 Exercise. Let $(R,\mathfrak{m})$ be a local ring. Prove that the following two conditions are equivalent:

(i) Every module-finite commutative R-algebra is a direct product of local rings.

(ii) If Λ is a module-finite R-algebra (not necessarily commutative), then every idempotent of $\Lambda/\mathscr{J}(\Lambda)$ lifts to an idempotent of E.

Hint: To see that (i) implies (ii), let $x \in \Lambda$ be an arbitrary lifting of some idempotent $e \in \Lambda\mathscr{J}(\Lambda)$. Let $A = R[x]$, and show that $\mathscr{J}(A) = A \cap \mathscr{J}(\Lambda)$.

1.23 Exercise. Let $(R,\mathfrak{m})$ be a Henselian local ring and X, Y, M finitely generated R-modules. Let $\alpha\colon X \longrightarrow M$ and $\beta\colon Y \longrightarrow M$ be homomorphisms which are not split surjections. Prove that $[\alpha\ \beta]\colon X \oplus Y \longrightarrow M$ is not a split surjection.

1.24 Exercise. Let M be a finitely generated module over a complete local ring $(R,\mathfrak{m})$. Show that M is complete for the topology defined by the submodules $\mathfrak{m}^n M, n \geqslant 1$.

1.25 Exercise. Prove Fitting's Lemma: Let Λ be any ring and M a Λ-module of finite length n. If $f \in \operatorname{End}_\Lambda(M)$, then $M = \ker(f^n) \oplus f^n(M)$. Conclude that if M is indecomposable then every non-invertible element of $\operatorname{End}_\Lambda(M)$ is nilpotent.

1.26 Exercise. Use Exercise 1.20 and Fitting's Lemma (Exercise 1.25) to prove that the endomorphism ring of any indecomposable finite-length module is nc-local. Thus, over any ring R, KRS holds for the category of left R-modules of finite length. (Be careful: You're in a non-commutative setting, where the sum of two nilpotents might be a unit! If you get stuck, consult [Fac98, Lemma 2.21].)

1.27 Exercise. Let M be a Noetherian left Λ-module, and let $f \in \operatorname{End}_\Lambda(M)$.

(i) If f is surjective, prove that f is an automorphism of M. (Consider the ascending chain of submodules $\ker(f^n)$.)

(ii) If f is surjective and $f^2 = f$, prove that $f = 1_M$.

1.28 Exercise. Prove Corollary 1.16: A finitely generated module over a local ring has, up to isomorphism, only finitely many direct summands and, up to equivalence, only finitely many direct-sum decompositions.

2

Dimension zero

In this chapter we prove that the zero-dimensional commutative, Noetherian rings of finite representation type are exactly the Artinian principal ideal rings. We also introduce Artinian pairs, which will be used in the next chapter to classify the one-dimensional rings of finite Cohen-Macaulay type. The Drozd–Roĭter conditions (DR1) and (DR2) are shown to be necessary for finite representation type in Theorem 2.5, and in Theorem 2.21 we reduce the proof of their sufficiency to some special cases, where we can appeal to the matrix calculations of Green and Reiner.

§1 Artinian rings with finite Cohen-Macaulay type

A commutative Artinian ring R with finite Cohen-Macaulay type has only finitely many indecomposable finitely generated R-modules. To see that this condition forces R to be a principal ideal ring, and in several other constructions of indecomposable modules, we use the following result:

2.1 Lemma. *Let R be any commutative ring, n a positive integer and H the nilpotent $n \times n$ Jordan block with 1's on the superdiagonal and 0's elsewhere. If α is an $n \times n$ matrix over R and $\alpha H = H\alpha$, then $\alpha \in R[H]$.*

Proof. Let $\alpha = [a_{ij}]$. Left multiplication by H moves each row up one step and kills the bottom row, while right multiplication shifts each column to the right and kills the first column. The relation $\alpha H = H\alpha$ therefore yields the equations $a_{i,j-1} = a_{i+1,j}$ for $i, j = 1, \ldots, n$, with the convention that $a_{k\ell} = 0$ if $k = n+1$ or $\ell = 0$. These equations show (a) that each of the diagonals (of slope -1) is constant and (b) that $a_{21} = \cdots = a_{n1} = 0$. Combining (a) and (b), we see that α is upper triangular. Letting b_j be the constant on the diagonal $\begin{bmatrix} a_{1,j+1} & a_{2,j+2} & \ldots & a_{n-j,n} \end{bmatrix}$, for $0 \leqslant j \leqslant n-1$, we see that $\alpha = \sum_{j=0}^{n-1} b_j H^j$. □

When R is a field (the only case we will need), there is a fancy proof: H is "cyclic" or " non-derogatory", that is, its characteristic and minimal polynomials coincide. The centralizer of a non-derogatory matrix B is always just $R[B]$ (cf. [Jac75, Corollary, p. 107]).

2.2 Theorem. *Let R be a Noetherian ring. These are equivalent:*

(i) R is an Artinian principal ideal ring.

(ii) R has only finitely many indecomposable finitely generated modules, up to isomorphism.

(iii) R is Artinian, and there is a bound on the number of generators required for indecomposable finitely generated R-modules.

Under these conditions, the number of isomorphism classes of indecomposable finitely generated modules is exactly the length of R.

Proof. Assuming (i), we will prove (ii) and verify the last statement. Since R is a product of finitely many local rings, we may assume that R is local, with maximal ideal $\mathfrak{m}$. The length ℓ of R is the least integer t such that

$\mathfrak{m}^t = 0$. Since every finitely generated R-module is a direct sum of cyclic modules, the indecomposable modules are exactly the modules $R/\mathfrak{m}^t, 1 \leqslant t \leqslant \ell$.

To see that (ii) $\implies$ (iii), suppose R is not Artinian. Choose a maximal ideal $\mathfrak{m}$ of positive height. The ideals $\mathfrak{m}^t$, $t \geqslant 1$ then form a strictly descending chain of ideals (cf. Exercise 2.22). Therefore the R-modules $R/\mathfrak{m}^t$ are indecomposable and, since they have different annihilators, pairwise non-isomorphic, contradicting (ii).

To complete the proof, we show that (iii) $\implies$ (i). Again, we may assume that R is local with maximal ideal $\mathfrak{m}$. Supposing R is not a principal ideal ring, we will build, for every n, an indecomposable finitely generated R-module requiring exactly n generators. By passing to $R/\mathfrak{m}^2$, we may assume that $\mathfrak{m}^2 = 0$, so that now $\mathfrak{m}$ is a vector space over $k := R/\mathfrak{m}$. Choose two k-linearly independent elements $x, y \in \mathfrak{m}$.

Fix $n \geqslant 1$, let I be the $n \times n$ identity matrix, and let H be the $n \times n$ nilpotent Jordan block of Lemma 2.1. Put $\Psi = yI + xH$ and $M = \operatorname{cok}(\Psi)$. Since the entries of Ψ are in $\mathfrak{m}$, the R-module M needs exactly n generators.

To show that M is indecomposable, let $f = f^2 \in \operatorname{End}_R(M)$, and assume that $f \neq 1_M$. We will show that $f = 0$. There exist $n \times n$ matrices F and G over R making the following diagram commute:

$$\begin{array}{ccccccc}
R^{(n)} & \xrightarrow{\Psi} & R^{(n)} & \longrightarrow & M & \longrightarrow & 0 \\
{\scriptstyle G}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle f}\downarrow & & \\
R^{(n)} & \xrightarrow[\Psi]{} & R^{(n)} & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The equation $F\Psi = \Psi G$ yields $yF + xFH = yG + xHG$. Since x and y are linearly independent, we obtain, after reducing all entries of F, G and H

modulo $\mathfrak{m}$, that $\overline{F} = \overline{G}$ and $\overline{F}\,\overline{H} = \overline{H}\,\overline{G}$. Therefore $\overline{F}$ and $\overline{H}$ commute, and by Lemma 2.1 $\overline{F}$ is an upper-triangular matrix with constant diagonal.

Now f is not surjective, by Exercise 1.27, and therefore neither is F. By NAK, $\overline{F}$ is not surjective, so $\overline{F}$ must be strictly upper triangular. But then $\overline{F}^n = 0$, and it follows that $\operatorname{im}(f) = \operatorname{im}(f^n) \subseteq \mathfrak{m}M$. Now NAK implies that $1-f$ is surjective. Since $1-f$ is idempotent, Exercise 1.27 implies that $f = 0$. □

This construction is far from new. See, for example, the papers of Higman [Hig54], Heller and Reiner [HR61], and Warfield [War70]. Similar constructions can be found in the classification, up to simultaneous equivalence, of pairs of matrices. (Cf. Dieudonné's discussion [Die46] of the work of Kronecker [Kro74] and Weierstrass [Wei68].)

§2 Artinian pairs

Here we introduce the main computational tool for building indecomposable maximal Cohen-Macaulay modules over one-dimensional rings.

2.3 Definition. An *Artinian pair* is a module-finite extension $(A \hookrightarrow B)$ of commutative Artinian rings. Given an Artinian pair $\mathbf{A} = (A \hookrightarrow B)$, an $\mathbf{A}$-module is a pair $(V \hookrightarrow W)$, where W is a finitely generated projective B-module and V is an A-submodule of W with the property that $BV = W$. A morphism $(V_1 \hookrightarrow W_1) \longrightarrow (V_2 \hookrightarrow W_2)$ of $\mathbf{A}$-modules is a B-homomorphism from W_1 to W_2 that carries V_1 into V_2. We say that the $\mathbf{A}$-module $(V \hookrightarrow W)$ has *constant rank* n provided $W \cong B^{(n)}$.

With direct sums defined in the obvious way, we get an additive category **A**-mod. To see that Theorem 1.3 applies in this context, we note first that the endomorphism ring of every **A**-module is a module-finite A-algebra and therefore is left Artinian. Next, suppose ϵ is an idempotent endomorphism of an **A**-module $\mathfrak{X} = (V \hookrightarrow W)$. Then $\mathfrak{Y} = (\epsilon(V) \hookrightarrow \epsilon(W))$ is also an **A**-module. The projection $p : \mathfrak{X} \longrightarrow \mathfrak{Y}$ and inclusion $u : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ give a factorization $\epsilon = up$, with $pu = 1_{\mathfrak{Y}}$. Thus idempotents split in **A**-mod. Combining Theorem 1.3 and Corollary 1.5, we obtain the following:

2.4 Theorem. *Let* **A** *be an Artinian pair, and let* $\mathbf{M}_1,\dots,\mathbf{M}_s$ *and* $\mathbf{N}_1,\dots,\mathbf{N}_t$ *be indecomposable* **A***-modules such that* $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s \cong \mathbf{N}_1 \oplus \cdots \oplus \mathbf{N}_t$. *Then* $s = t$*, and, after renumbering,* $\mathbf{M}_i \cong \mathbf{N}_i$ *for each* i.

We say **A** has *finite representation type* provided there are, up to isomorphism, only finitely many indecomposable **A**-modules.

Our main result in this chapter is Theorem 2.5, which gives necessary conditions for an Artinian pair to have finite representation type. As we will see in the next chapter, these conditions are actually sufficient for finite representation type. The conditions were introduced by Drozd and Roĭter [DR67] in 1966, and we will refer to them as the Drozd–Roĭter conditions. (See the historical remarks in Section §2 of Chapter 3.)

2.5 Theorem. *Let* $\mathbf{A} = (A \hookrightarrow B)$ *be an Artinian pair in which* A *is local, with maximal ideal* $\mathfrak{m}$ *and residue field* k*. Assume that at least one of the following conditions fails:*

(dr1) $\dim_k(B/\mathfrak{m}B) \leqslant 3$

(dr2) $\dim_k\left(\dfrac{\mathfrak{m}B+A}{\mathfrak{m}^2B+A}\right) \leqslant 1.$

Let n be an arbitrary positive integer. Then there is an indecomposable **A**-*module of constant rank n. Moreover, if $|k|$ is infinite, there are at least $|k|$ pairwise non-isomorphic indecomposable* **A**-*modules of rank n.*

2.6 Remark. If k is infinite then the number of isomorphism classes of **A**-modules is *at most* $|k|$. To see this, note that there are, up to isomorphism, only countably many finitely generated projective B-modules W. Also, since any such W has finite length as an A-module, we see that $|W| \leqslant |k|$ and hence that W has at most $|k|$ A-submodules V. It follows that the number of possibilities for $(V \hookrightarrow W)$ is bounded by $\aleph_0|k| = |k|$.

The proof of Theorem 2.5 involves a basic construction and a dreary analysis of the many cases that must be considered in order to implement the construction.

2.7 Assumptions. Throughout the rest of this chapter, $\mathbf{A} = (A \hookrightarrow B)$ is an Artinian pair in which A is local, with maximal ideal $\mathfrak{m}$ and residue field k.

The next three results will allow us to pass to a more manageable Artinian pair $k \hookrightarrow D$, where D is a suitable finite-dimensional k-algebra. The proofs of the first two lemmas are exercises.

2.8 Lemma. *Let C be a subring of B containing A. The functor* $(V \hookrightarrow W) \rightsquigarrow (V \hookrightarrow B \otimes_C W)$ *from* $(A \hookrightarrow C)$*-mod to* $(A \hookrightarrow B)$*-mod is faithful and full. The functor is injective on isomorphism classes and preserves indecomposability.*

□

2.9 Lemma. *Let I be a nilpotent ideal of B, and put $\mathbf{E} = \left(\frac{A+I}{I} \hookrightarrow \frac{B}{I}\right)$. The functor $(V \hookrightarrow W) \rightsquigarrow \left(\frac{V+IW}{IW} \hookrightarrow \frac{W}{IW}\right)$, from $\mathbf{A}$-mod to $\mathbf{E}$-mod, is surjective on isomorphism classes and reflects indecomposable objects.* □

2.10 Proposition. *Let $A \hookrightarrow B$ be an Artinian pair for which either (dr1) or (dr2) fails. There is a ring C between A and B such that, with $D = C/\mathfrak{m}C$, we have either*

(i) $\dim_k(D) \geqslant 4$, *or*

(ii) D contains elements α and β such that $\{1, \alpha, \beta\}$ is linearly independent over k and $\alpha^2 = \alpha\beta = \beta^2 = 0$.

Proof. If (dr1) fails, we take $C = B$. Otherwise (dr2) fails, and we put $C = A + \mathfrak{m}B$. Since $\dim_k\left(\frac{\mathfrak{m}B+A}{\mathfrak{m}^2B+A}\right) \geqslant 2$, we can choose elements $x, y \in \mathfrak{m}B$ such that the images of x and y in $\frac{\mathfrak{m}B+A}{\mathfrak{m}^2B+A}$ are linearly independent. Since $D := C\mathfrak{m}C$ maps onto $\frac{\mathfrak{m}B+A}{\mathfrak{m}^2B+A}$, the images $\alpha, \beta \in D$ of x, y are linearly independent, and they obviously satisfy the required equations. □

Now let's begin the proof of Theorem 2.5. We have an Artinian pair $A \hookrightarrow B$, where $(A, \mathfrak{m}, k)$ is local and either (dr1) or (dr2) fails. We want to build indecomposable $\mathbf{A}$-modules $V \hookrightarrow W$, with $W = B^{(n)}$. By Lemmas 2.8 and 2.9, we can pass to the Artinian pair $k \hookrightarrow D$ provided by Proposition 2.10. We fix a positive integer n. Our goal is to build an indecomposable $(k \hookrightarrow D)$-module $(V \hookrightarrow D^{(n)})$ and, if k is infinite, a family $\{(V_t \hookrightarrow D^{(n)})\}_{t \in T}$ of pairwise non-isomorphic indecomposable $(k \hookrightarrow D)$-modules, with $|T| = |k|$.

2.11 Construction. We describe a general construction, a modification of constructions found in [DR67, Wie89, ÇWW95]. Let n be a fixed positive integer, and suppose we have chosen $\alpha, \beta \in D$ with $\{1, \alpha, \beta\}$ linearly independent over k. Let I be the $n \times n$ identity matrix, and let H the $n \times n$ nilpotent Jordan block in Lemma 2.1. For $t \in k$, we consider the $n \times 2n$ matrix $\Psi_t = \left[I \quad | \quad \alpha I + \beta(tI + H) \right]$. Put $W = D^{(n)}$, viewed as columns, and let V_t be the k-subspace of W spanned by the columns of Ψ_t.

Suppose we have a morphism $(V_t \hookrightarrow W) \longrightarrow (V_u \hookrightarrow W)$, given by an $n \times n$ matrix φ over D. The requirement that $\varphi(V) \subseteq V$ says there is a $2n \times 2n$ matrix θ over k such that

$$\varphi \Psi_t = \Psi_u \theta. \tag{2.11.1}$$

Write $\theta = \left[\begin{smallmatrix} A & B \\ P & Q \end{smallmatrix} \right]$, where A, B, P, Q are $n \times n$ blocks. Then (2.11.1) gives the following two equations:

$$\begin{aligned} \varphi &= A + \alpha P + \beta(uI + H)P \\ \alpha\varphi + \beta\varphi(tI + H) &= B + \alpha Q + \beta(uI + H)Q \end{aligned} \tag{2.11.2}$$

Substituting the first equation into the second and combining terms, we get a mess:

$$\begin{aligned} -B + \alpha(A - Q) + \beta(tA - uQ + AH - HQ) + (\alpha + t\beta)(\alpha + u\beta)P \\ + \alpha\beta(HP + PH) + \beta^2(HPH + tHP + uPH) = 0. \end{aligned} \tag{2.11.3}$$

2.12 Case. *D satisfies (ii). (There exist $\alpha, \beta \in D$ such that $\{1, \alpha, \beta\}$ is linearly independent and $\alpha^2 = \alpha\beta = \beta^2 = 0$.)*

From (2.11.3) and the linear independence of $\{1, \alpha, \beta\}$, we get the equations

$$B = 0, \qquad A = Q, \qquad A((t-u)I + H) = HA. \tag{2.12.1}$$

If φ is an isomorphism, we see from (2.11.2) that A has to be invertible. If, in addition, $t \neq u$, the third equation in (2.12.1) gives a contradiction, since the left side is invertible and the right side is not. Thus $(V_t \hookrightarrow W) \ncong (V_u \hookrightarrow W)$ if $t \neq u$. To see that $(V_t \hookrightarrow W)$ is indecomposable, we take $u = t$ and suppose that φ, as above, is idempotent. Squaring the first equation in (2.11.2), and comparing "1" and "A" terms, we see that $A^2 = A$ and $P = AP + PA$. But equation (2.12.1) says that $AH = HA$, and it follows that A is in $k[H]$, which is a local ring. Therefore $A = 0$ or I, and either possibility forces $P = 0$. Thus $\varphi = 0$ or 1, as desired. Thus we may take $T = k$ in this case. □

2.13 Assumptions. Having dealt with the case (ii), we assume from now on that (i) holds, that is $\dim_k(D) \geqslant 4$.

2.14 Case. *D has an element α such that $\{1, \alpha, \alpha^2\}$ is linearly independent.*

Choose any element $\beta \in D$ such that $\{1, \alpha, \beta, \alpha^2\}$ is linearly independent. We let E be the set of elements $t \in k$ for which $\{1, \alpha, \beta, (\alpha + t\beta)^2\}$ is linearly independent. Then E is non-empty (since it contains 0). Also, E is open in the Zariski topology on k and therefore is cofinite in k. Moreover, if $t \in E$, the set $E_t = \{u \in E \mid \{1, \alpha, \beta, (\alpha + t\beta)(\alpha + u\beta)\}$ is linearly independent$\}$ is non-empty and cofinite in E. We will show that $(V_t \hookrightarrow W)$ is indecomposable for each $t \in E$, and that $(V_t \hookrightarrow W) \ncong (V_u \hookrightarrow W)$ if t and u are distinct elements of E with $u \in E_t$. Assuming this has been done we can complete the proof in this case as follows: Define an equivalence relation $\sim$ on E by declaring that $t \sim u$ if and only if $(V_u \hookrightarrow D) \cong (V_t \hookrightarrow D)$, and let T be a set of representatives. Then $T \neq \emptyset$, and $(V_t \hookrightarrow W)$ is indecomposable for each $t \in T$.

Moreover, each equivalence class is finite and E is cofinite in k. Therefore, if k is infinite, it follows that $|T| = |k|$.

Suppose $t \in E$ and $u \in E_t$ (possibly with $t = u$), and let $\varphi\colon (V_t \hookrightarrow W) \longrightarrow (V_u \hookrightarrow W)$ be a homomorphism. With the notation of (2.11.1)–(2.11.3), one can show, by descending induction on i and j, that $H^iPH^j = 0$ for all $i,j = 0,\dots,n$. (Cf. Exercise 2.28.) Therefore $P = 0$, and we again obtain equations (2.12.1). The rest of the proof proceeds exactly as in Case 2.12. $\square$

The following lemma, whose proof is left as an exercise, is useful in treating the remaining case, when every element of D satisfies a monic quadratic equation over k:

2.15 Lemma. *Let ℓ be a field, and let A be a finite-dimensional ℓ-algebra with $\dim_\ell(A) \geqslant 3$. Assume that $\{1,\alpha,\alpha^2\}$ is linearly dependent over ℓ for every $\alpha \in A$. Write $A = A_1 \times \cdots \times A_s$, where each A_i is local, with maximal ideal $\mathfrak{m}_i$. Let $\mathfrak{N} = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_s$, the nilradical of A.*

(i) If $x \in \mathfrak{N}$, then $x^2 = 0$.

(ii) There are at least $|\ell|$ distinct rings between ℓ and A.

(iii) If $s \geqslant 2$, then $A_i/\mathfrak{m}_i = \ell$ for each i.

(iv) If $s \geqslant 3$ then $|\ell| = 2$

$\square$

2.16 Assumptions. From now on, we assume that $\{1,\alpha,\alpha^2\}$ is linearly dependent over k for each $\alpha \in D$ (and that $\dim_k(D) \geqslant 4$). We write $D = D_1 \times \cdots \times D_s$, where each D_i is local, with maximal ideal $\mathfrak{m}_i$; we let $\mathfrak{N} = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_t$, the nilradical of D.

2.17 Case. $\dim_k(\mathfrak{N}) \geqslant 2$

Choose $\alpha, \beta \in \mathfrak{N}$ so that $\{1, \alpha, \beta\}$ is linearly independent. Then $\alpha^2 = \beta^2 = 0$ by Lemma 2.15. If $\{1, \alpha, \beta, \alpha\beta\}$ is linearly independent, we can use the mess (2.11.3) to complete the proof. Otherwise, we can write $\alpha\beta = a + b\alpha + c\beta$ with $a, b, c \in k$. Multiplying this equation first by α and then by β, we learn that $\alpha\beta = 0$, and we are in Case 2.12. □

2.18 Assumption. We assume from now on that $\dim_k(\mathfrak{N}) \leqslant 1$.

From Lemma 2.15 we see that s (the number of components) cannot be 2. Also, if $s = 3$, then, after renumbering if necessary, we have $\mathfrak{N} = \mathfrak{m}_1 \times 0 \times 0$ with $\mathfrak{m}_1 \neq 0$. Now put $\alpha = (x, 1, 0)$, where x is a non-zero element of $\mathfrak{m}_1$, and check that $\{1, \alpha, \alpha^2\}$ is linearly independent, contradicting Assumption 2.16. We have proved that either $s = 1$ or $s \geqslant 4$.

2.19 Case. $s = 1$. *(D is local.)*

By Assumptions 2.13 and 2.18, $K := D/\mathfrak{N}$ must have degree at least three over k. On the other hand, Assumption 2.16 implies that each element of K has degree at most 2 over k. Therefore K/k is not separable, $\operatorname{char}(k) = 2$, $\alpha^2 \in k$ for each $\alpha \in K$, and $[K : k] \geqslant 4$. Now choose two elements $\alpha, \beta \in K$ such that $[k(\alpha, \beta) : k] = 4$. By Lemma 2.9 we can safely pass to the Artinian pair (k, K) and build our modules there; for compatibility with the notation in Construction 2.11, we rename K and call it D. Now we have $\alpha, \beta \in D$ such that $\{1, \alpha, \beta, \alpha\beta\}$ is linearly independent and both α^2 and β^2 are in k. If, now, $\varphi\colon (V_t \hookrightarrow W) \longrightarrow (V_u \hookrightarrow W)$ is a morphism, the mess (2.11.3) provides the following equations:

$$B = (\alpha^2 + tu\beta^2)P + \beta^2(HPH + tHP + uPH), \qquad A = Q,$$
$$A((t-u)I + H) = HA, \quad (t+u)P + HP + PH = 0. \tag{2.19.1}$$

Suppose $t \neq u$. Then $t + u \neq 0$ (characteristic 2), and the fourth equation shows, via a descending induction argument as in Case 2.14, that $P = 0$. (Cf. Exercise 2.28.) Now the third equation shows, as in Case 2.12, that φ is not an isomorphism.

Now suppose $t = u$ and $\varphi^2 = \varphi$. Using the third and fourth equations of (2.19.1), the fact that $\mathrm{char}(k) = 2$, and Lemma 2.1, we see that both A and P are in $k[H]$. In particular, A, P and H commute, and, since we are in characteristic two, we can square both sides of (2.11.2) painlessly. Equating φ and φ^2, we see that $P = 0$ and $A = A^2$. Since $k[H]$ is local, $A = 0$ or I. □

One case remains:

2.20 Case. $s \geqslant 4$.

By Lemma 2.15, $|k| = 2$ and $D_i/\mathfrak{m}_i = k$ for each i. By Lemma 2.9 we can forget about the radical and assume that $D = k \times \cdots \times k$ (at least 4 components). Alas, this case does not yield to our general construction, but E. C. Dade's construction [Dad63] saves the day. (Dade works in greater generality, but the main idea is visible in the computation that follows. The key issue is that D has at least 4 components.)

Put $W = D^{(n)}$, and let V be the k-subspace of W consisting of all elements $(x, y, x+y, x+Hy, x, \ldots, x)$, where x and y range over $k^{(n)}$. (Again, H is the nilpotent Jordan block with 1's on the superdiagonal.) Clearly $DW = V$. To see that $(V \hookrightarrow W)$ is indecomposable, suppose φ is an endomorphism of $(V \hookrightarrow W)$, that is, a D-endomorphism of W carrying V into V. We write $\varphi = (\alpha, \beta, \gamma, \delta, \varepsilon_5, \dots, \varepsilon_s)$, where each component is an $n \times n$ matrix over k. Since $\varphi((x, 0, x, x, x, \dots, x))$ and $\varphi((0, y, y, Hy, 0, \dots, 0))$ are in V, there are matrices σ, τ, ξ, η satisfying the following two equations for all $x \in k^{(n)}$:

$$(\alpha x,\ 0,\ \gamma x,\ \delta x,\ \varepsilon_5 x,\ \dots,\ \varepsilon_s x) = (\sigma x,\ \tau x,\ (\sigma + \tau)x,\ (\sigma + H\tau)x,\ \sigma x,\ \dots,\ \sigma x)$$

$$(0,\ \beta y,\ \gamma y,\ \delta H y,\ 0,\ \dots,\ 0) = (\xi y,\ \eta y,\ (\xi + \eta)y,\ (\xi + H\eta)y,\ \xi y,\ \dots,\ \xi y)$$

The first equation shows that $\varphi = (\alpha,\ \alpha,\ \dots,\ \alpha)$, and the second then shows that $\alpha H = H\alpha$. By Lemma 2.1 $\alpha \in k[H] \cong k[x]/(x^n)$, which is a local ring. If, now, $\varphi^2 = \varphi$, then $\alpha^2 = \alpha$, and hence $\alpha = 0$ or I_n. This shows that $(V \hookrightarrow W)$ is indecomposable and completes the proof of Theorem 2.5. □

We close this chapter with the following partial converse to Theorem 2.5. This is due to Drozd–Roĭter [DR67] and Green-Reiner [GR78] in the special case where the residue field $A/\mathfrak{m}$ is finite. In this case they reduced to the situation where where $A/\mathfrak{m} \longrightarrow B/\mathfrak{n}$ is an isomorphism for each maximal ideal $\mathfrak{n}$ of B. In this situation they showed, via explicit matrix decompositions, that conditions (dr1) and (dr2) imply that $\mathbf{A}$ has finite representation type. These matrix decompositions depend only on the fact that the residue fields of B are all equal to k, and not on the fact that k is finite. The generalization stated here is due to R. Wiegand [Wie89] and depends crucially on the matrix decompositions in [GR78].

2.21 Theorem. *Let* $\mathbf{A} = (A \hookrightarrow B)$ *be an Artinian pair in which* A *is local, with maximal ideal* $\mathfrak{m}$ *and residue field* k. *Assume that* B *is a principal ideal ring and either* *(i) the field extension $k \hookrightarrow B/\mathfrak{n}$ is separable for every maximal ideal $\mathfrak{n}$ of B, or*

(ii) B is reduced (hence a direct product of fields).

If $\mathbf{A}$ satisfies (dr1) and (dr2), then $\mathbf{A}$ has finite representation type.

Proof. As in [GR78] we will reduce to the case where the residue fields of B are all equal to k. By (d1) B has at most three maximal ideals, and at most one of these has a residue field ℓ properly extending k. Moreover, $[\ell : k] \leqslant 3$. Assuming $\ell \neq k$, we choose a primitive element θ for ℓ/k, let $f \in A[T]$ be a monic polynomial reducing to the minimal polynomial for θ over k, and pass to the Artinian pair $\mathbf{A}' = (A' \hookrightarrow B')$, where $A' = A[T]/(f)$ and $B' = B \otimes_A A' = B[T]/(f)$. Each of the conditions (i), (ii) guarantees that B' is a principal ideal ring.

One checks that the Drozd–Roĭter conditions ascend to $\mathbf{A}'$, and finite representation type descends. (This is not difficult; the details are worked out in [Wie89].) If $k(\theta)/k$ is a separable, non-Galois extension of degree 3, then B' has a residue field that is separable of degree 2 over k, and we simply repeat the construction. Thus it suffices to prove the theorem in the case where each residue field of B is equal to k. For this case, we appeal to the matrix decompositions in [GR78], which work perfectly well over any field. □

§3 Exercises

2.22 Exercise. Let $\mathfrak{m}$ be a maximal ideal of a Noetherian ring R, and assume that $\mathfrak{m}$ is not a minimal prime ideal of R. Then $\{\mathfrak{m}^t \mid t \geqslant 1\}$ is an infinite strictly descending chain of ideals.

2.23 Exercise. Let $(R, \mathfrak{m}, k)$ be a commutative local Artinian ring, and assume k is infinite.

(i) If $\mathscr{G}$ is a set of pairwise non-isomorphic finitely generated R-modules, prove that $|\mathscr{G}| \leqslant |k|$.

(ii) Suppose R is not a principal ideal ring. Modify the proof of Theorem 2.2 to show that for each $n \geqslant 1$ there is a family $\mathscr{G}_n$ of pairwise non-isomorphic indecomposable modules, all requiring exactly n generators, with $|\mathscr{G}_n| = |k|$.

2.24 Exercise. Prove Lemmas 2.8 and 2.9.

2.25 Exercise. Prove Lemma 2.15. (For the second assertion, suppose there are fewer than $|\ell|$ intermediate rings. Mimic the proof of the primitive element theorem to show that $D = k[\alpha]$ for some α.)

2.26 Exercise. With E and E_t as in 2.14, prove that $|k - E| \leqslant 1$ and that $|E - E_t| \leqslant 1$.

2.27 Exercise. Let $\mathbf{A} = (A \hookrightarrow B)$ be an Artinian pair, and let C_1 and C_2 be distinct rings between A and B. Prove that the $\mathbf{A}$-modules $(C_1 \hookrightarrow B)$ and $(C_2 \hookrightarrow B)$ are not isomorphic.

2.28 Exercise. Work out the details of the descending induction arguments in Case 2.14 and Case 2.19. (In Case 2.14, assuming $H^{i+1}\gamma H^j = 0$ and $H^i\gamma H^{j+1} = 0$, multiply the mess (2.11.3) by H^i on the left and H^j on the right. In Case 2.19, use the fourth equation in (2.19.1) and do the same thing.

3

Dimension one

In this chapter we give necessary and sufficient conditions for a one-dimensional local ring to have finite Cohen-Macaulay type. In the main case of interest, where the completion $\widehat{R}$ is reduced, these conditions are simply the liftings of the Drozd–Roĭter conditions (dr1) and (dr2) of Chapter 2. Necessity of these conditions follows easily from Theorem 2.5. To prove that they are sufficient, we will reduce the problem to consideration of some special cases, where we can appeal to the matrix decompositions of Green and Reiner [GR78] and, in characteristic two, Çimen [Çim94, Çim98].

Throughout this chapter $(R, \mathfrak{m}, k)$ is a one-dimensional local ring (with maximal ideal $\mathfrak{m}$ and residue field k). Let K denote the total quotient ring $\{\text{non-zerodivisors}\}^{-1}R$ and $\overline{R}$ the integral closure of R in K. If R is reduced (hence CM), then $\overline{R} = \overline{R/\mathfrak{p}_1} \times \cdots \times \overline{R/\mathfrak{p}_s}$, where the $\mathfrak{p}_i$ are the minimal prime ideals of R, and each ring $\overline{R/\mathfrak{p}_i}$ is a semilocal principal ideal domain.

When R is CM, a finitely generated R-module M is MCM if and only if it is torsion-free, that is, the torsion submodule is zero.

We say that a finitely generated R-module M has *constant rank n* provided $K \otimes_R M \cong K^{(n)}$. If R is CM, then $K = R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_s}$; hence M has constant rank if and only if $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{(n)}$ for each minimal prime ideal $\mathfrak{p}$. (If R is *not* CM, then $K = R$, so free modules are the only modules with constant rank.)

The main result in this chapter is Theorem 3.10, which states that a one-dimensional local ring $(R, \mathfrak{m}, k)$, with reduced completion , has finite CM type if and only if R satisfies the following two conditions:

(DR1) $\mu_R(\overline{R}) \leqslant 3$, and

(DR2) $\frac{\mathfrak{m}\overline{R}+R}{R}$ is a cyclic R-module.

The first condition just says that the multiplicity of R is at most three (cf. Theorem A.23). When the multiplicity is three we have to consider the second condition. One can check, for example, that $k[\![t^3,t^5]\!]$ satisfies (DR2) but that $k[\![t^3,t^7]\!]$ does not.

The case where the completion is not reduced is dealt with separately, in Theorem 3.16. In particular, we find (Corollary 3.17) that a one-dimensional local ring R has finite CM type if and only if its completion does. The analogous statement fails badly in higher dimension; cf. Chapter 11. Furthermore, Proposition 3.15 shows that if a one-dimensional CM local ring has finite CM type, then its completion is reduced; in particular R is an isolated singularity, which property will appear again in Chapter 6. We also treat the case of multiplicity two directly, without any reducedness assumption.

As a look ahead to later chapters, in §3 we discuss the alternative classification of finite CM type in dimension one due to Greuel and Knörrer in terms of the ADE hypersurface singularities.

§1 Necessity of the Drozd–Roĭter conditions

Looking ahead to Chapter 15, we work in a somewhat more general context than is strictly required for Theorem 3.10. In particular, we will not assume that R is reduced, and $\overline{R}$ will be replaced by a more general extension ring S. By a *finite birational extension* of R we mean a ring S between R and its total quotient ring K such that S is finitely generated as an R-module.

3.1 Construction. Let $(R,\mathfrak{m},k)$ be a CM local ring of dimension one, and let S be a finite birational extension of R. Put $\mathfrak{c} = (R :_R S)$, the *conductor* of S into R. This is the largest common ideal of R and S. Set $A = R/\mathfrak{c}$ and $B = S/\mathfrak{c}$. Then the *conductor square of* $R \hookrightarrow S$

(3.1.1)
$$\begin{array}{ccc} R & \hookrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ A & \hookrightarrow & B \end{array}$$

is a pullback diagram, that is, $R = \pi^{-1}(A)$. Since S is a module-finite extension of R contained in the total quotient ring K, the conductor contains a non-zerodivisor (clear denominators), so that the bottom line $\mathbf{A} := (A \hookrightarrow B)$ is an Artinian pair in the sense of Chapter 2.

Suppose that M is a MCM R-module. Then M is torsion-free, so the natural map $M \longrightarrow K \otimes_R M$ is injective. Let SM be the S-submodule of $K \otimes_R M$ generated by the image of M; equivalently, $SM = (S \otimes_R M)/\text{torsion}$. If we furthermore assume that SM is a projective S-module, then the inclusion $M/\mathfrak{c}M \hookrightarrow SM/\mathfrak{c}M$ gives a module over the Artinian pair $A \hookrightarrow B$.

In the special case where S is the integral closure $\overline{R}$, the situation clarifies. Since $\overline{R}$ is a direct product of semilocal principal ideal domains, and $\overline{R}M$ is torsion-free for any MCM R-module M, it follows that $\overline{R}M$ is $\overline{R}$-projective. Thus $M/\mathfrak{c}M \hookrightarrow \overline{R}M/\mathfrak{c}M$ is automatically a module over the Artinian pair $R/\mathfrak{c} \hookrightarrow \overline{R}/\mathfrak{c}$. We dignify this special case with the notation $R_{\text{art}} = (R/\mathfrak{c} \hookrightarrow \overline{R}/\mathfrak{c})$ and $M_{\text{art}} = (M_{\mathfrak{c}}M \hookrightarrow \overline{R}M/\mathfrak{c}M)$.

Now return to the case of a general finite birational extension S, and let $V \hookrightarrow W$ be a module over the Artinian pair $\mathbf{A} = (A \hookrightarrow B) = (R/\mathfrak{c} \hookrightarrow S/\mathfrak{c})$. Assume that there exists a finitely generated projective S-module P such that $W \cong$

$P/\mathfrak{c}P$. (This is a real restriction; see the comments below.) We can then define an R-module M by a similar pullback diagram

(3.1.2)
$$\begin{array}{ccc} M & \hookrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ V & \hookrightarrow & W \end{array}$$

so that $M = \tau^{-1}(V)$. Using the fact that $BV = W$, one can check that $SM = P$, so that in particular M is a MCM R-module. Moreover, $M/\mathfrak{c}M = V$ and $SM/\mathfrak{c}M = W$, so that two non-isomorphic $\mathbf{A}$-modules that are both liftable have non-isomorphic liftings.

If in particular $V \hookrightarrow W$ is an $\mathbf{A}$-module of constant rank, so that $W \cong B^{(n)}$ for some n, then there is clearly a projective S-module P such that $P/\mathfrak{c}P \cong W$, namely $P = S^{(n)}$. Furthermore, in this case M has constant rank n over R. It follows that every $\mathbf{A}$-module of constant rank lifts to a MCM R-module of constant rank. Moreover, every $\mathbf{A}$-module is a direct summand of one of constant rank, so is a direct summand of a module extended from R.

By analogy with the terminology "weakly liftable" of [ADS93], we say that a module $V \hookrightarrow W$ over the Artinian pair $\mathbf{A} = R/\mathfrak{c} \hookrightarrow S/\mathfrak{c}$ is *weakly extended* (from R) if there exists a MCM R-module M such that $V \hookrightarrow W$ is a direct summand of the $\mathbf{A}$-module $M/\mathfrak{c}M \hookrightarrow SM/\mathfrak{c}M$. The discussion above shows that every $\mathbf{A}$-module is weakly extended from R.

Now we lift the Drozd–Roĭter conditions up to the finite birational extension $R \hookrightarrow S$.

3.2 Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension one, and let S be a finite birational extension of R. Assume that either* *(i)* $\mu_R(S) \geqslant 4$, *or*

(ii) $\mu_R\left(\frac{\mathfrak{m}S+R}{R}\right) \geqslant 2$.

Then R has infinite Cohen-Macaulay type. Moreover, given an arbitrary positive integer n, there is an indecomposable MCM R-module M of constant rank n; if k is infinite, there are at least $|k|$ pairwise non-isomorphic indecomposable MCM R-modules of constant rank n.

3.3 Remark. With R as in Theorem 3.2 and with k infinite, there are *at most* $|k|$ isomorphism classes of R-modules of constant rank. To see this, we note that there are at most $|k|$ isomorphism classes of finite-length modules and that every module of finite length has cardinality at most $|k|$. Given an arbitrary MCM R-module M of constant rank n, one can build an exact sequence

$$0 \longrightarrow T \longrightarrow M \longrightarrow R^{(n)} \longrightarrow U \longrightarrow 0,$$

in which both T and U have finite length. Let W be the kernel of $R^{(n)} \longrightarrow U$ (and the cokernel of $T \longrightarrow M$). Since $|U| \leqslant |k|$, we see that $|\operatorname{Hom}_R(R^{(n)},U)| \leqslant |k|$. Since there are at most $|k|$ possibilities for U, we see that there are at most $|k|^2 = |k|$ possibilities for W. Since there are at most $|k|$ possibilities for T, and since $\left|\operatorname{Ext}^1_R(W,T)\right| \leqslant |k|$, we see that there are at most $|k|$ possibilities for M.

Proof of Theorem 3.2. The assumptions imply immediately that either (dr1) or (dr2) of Theorem 2.5 fails for the Artinian pair $\mathbf{A} = (R/\mathfrak{c} \hookrightarrow S/\mathfrak{c})$. Therefore there exist indecomposable $\mathbf{A}$-modules of arbitrary constant rank n, in fact, $|k|$ of them if k is infinite. Each of these pulls back to R, so that there exist the same number of MCM R-modules of constant rank n for each $n \geqslant 1$.

Furthermore these MCM modules are pairwise non-isomorphic. Finally, we must show that if $V \hookrightarrow W$ is indecomposable and M is a lifting to R, then M is indecomposable as well. Suppose $M \cong X \oplus Y$. Then $SM = SX \oplus SY$, and it follows that $(V \hookrightarrow W)$ is the direct sum of the **A**-modules $(X/\mathfrak{c}X \hookrightarrow SX/\mathfrak{c}X)$ and $(Y/\mathfrak{c}Y \hookrightarrow SY/\mathfrak{c}Y)$. Therefore either $X/\mathfrak{c}X = 0$ or $Y/\mathfrak{c}Y = 0$. By NAK, either $X = 0$ or $Y = 0$. □

The requisite extension S of Theorem 3.2 always exists if R is CM of multiplicity at least 4, as we now show.

3.4 Proposition. *Let $(R, \mathfrak{m})$ be a one-dimensional CM local ring and set $e = \mathrm{e}(R)$, the multiplicity of R. (See Appendix A §2.) Then R has a finite birational extension S requiring e generators as an R-module.*

Proof. Let K again be the total quotient ring of R. Let $S_n = (\mathfrak{m}^n :_K \mathfrak{m}^n)$ for $n \geqslant 1$, and put $S = \bigcup_n S_n$. To see that this works, we may harmlessly assume that k is infinite. (This is relatively standard, but see Theorem 11.16 for the details on extending the residue field.) Let $Rf \subseteq \mathfrak{m}$ be a principal reduction of $\mathfrak{m}$. Choose n so large that

(a) $\mathfrak{m}^{i+1} = f\mathfrak{m}^i$ for $i \geqslant n$, and

(b) $\mu_R(\mathfrak{m}^i) = \mathrm{e}(R)$ for $i \geqslant n$.

Since f is a non-zerodivisor (as R is CM), it follows from (a) that $S = S_n$. We claim that $Sf^n = \mathfrak{m}^n$. We have $Sf^n = S_n f^n \subseteq \mathfrak{m}^n$. For the reverse inclusion, let $\alpha \in \mathfrak{m}^n$. Then $\frac{\alpha}{f^n}\mathfrak{m}^n \subseteq \frac{1}{f^n}\mathfrak{m}^{2n} = \frac{1}{f^n}f^n\mathfrak{m}^n = \mathfrak{m}^n$. This shows that $\frac{\alpha}{f^n} \in S_n$, and the claim follows. Therefore $S \cong \mathfrak{m}^n$ (as R-modules), and now (b) implies that $\mu(S) = \mathrm{e}(R)$. □

3.5 Remark. Observe that the proof of this Proposition shows more: for any one-dimensional CM local ring R and any ideal I of R containing a non-zerodivisor, there exists $n \geqslant 1$ such that I^n is projective as a module over its endomorphism ring $S = \mathrm{End}_R(I^n)$, which is a finite birational extension of R. (Ideals projective over their endomorphism ring are called *stable* in [Lip71] and [SV74].) Since S is semilocal, I^n is isomorphic to S as an S-module, whence as an R-module. Furthermore, n may be taken to be the least integer such that $\mu_R(I^n)$ achieves its stable value. Sally and Vasconcelos show in [SV74, Theorem 2.5] that this n is at most $\max\{1, \mathrm{e}(R)-1\}$, where $\mathrm{e}(R)$ is the multiplicity of R. This will be useful in Theorem 3.18 below.

§2 Sufficiency of the Drozd–Roĭter conditions

In this section we will prove, modulo the matrix calculations of Green and Reiner [GR78] and Çimen [Çim94, Çim98], that the Drozd–Roĭter conditions imply finite CM type. Recall that a local ring $(R, \mathfrak{m})$ is said to be *analytically unramified* provided its completion $\widehat{R}$ is reduced. The next result gives an equivalent condition—finiteness of the integral closure—for one-dimensional CM local rings.

3.6 Theorem ([Kru30]). *Let $(R, \mathfrak{m})$ be a local ring, and let $\overline{R}$ be the integral closure of R in its total quotient ring.*

(i) ([Nag58] If R is analytically unramified, then $\overline{R}$ is finitely generated as an R-module.

(ii) ([Kru30]) Suppose R is one-dimensional and CM. If $\overline{R}$ is finitely generated as an R-module then R is analytically unramified.

Proof. See [Mat86, p. 263] or [HS06, 4.6.2] for a proof of (i). With the assumptions in (ii), we'll show first that R is reduced. Suppose x is a non-zero nilpotent element of R and t a non-zerodivisor in $\mathfrak{m}$. Then

$$R\frac{x}{t} \subset R\frac{x}{t^2} \subset R\frac{x}{t^3} \subset \cdots$$

is an infinite strictly ascending chain of R-submodules of $\overline{R}$, contradicting finiteness of $\overline{R}$. Now assume that R is reduced and let $\mathfrak{p}_1 \dots, \mathfrak{p}_s$ be the minimal prime ideals of R. There are inclusions

$$R \hookrightarrow R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_s \hookrightarrow \overline{R/\mathfrak{p}_1} \times \cdots \times \overline{R/\mathfrak{p}_s} = \overline{R}\,.$$

Each of the rings $\overline{R/\mathfrak{p}_i}$ is a semilocal principal ideal domain. Since $\overline{R}$ is a finitely generated R-module, the $\mathfrak{m}$-adic completion of $\overline{R}$ is the product of the completions of the localizations of the $\overline{R/\mathfrak{p}_i}$ at their maximal ideals. In particular, the $\mathfrak{m}$-adic completion of $\overline{R}$ is a direct product of discrete valuation rings. The flatness of $\widehat{R}$ implies that $\widehat{R}$ is contained in the $\mathfrak{m}$-adic completion of $\overline{R}$, hence is reduced. □

In the proof of part (ii) of the following proposition we encounter the subtlety mentioned in Construction 3.1: not every projective module over B is of the form $P/\mathfrak{c}P$ for a projective S-module. This is because $\overline{R}$ might not be a direct product of local rings. For example, the integral closure $\overline{R}$ of the ring $R = \mathbb{C}[x,y]_{(x,y)}/(y^2 - x^3 - x^2)$ has two maximal ideals (cf. Exercise 3.22), and so $\overline{R}/\mathfrak{c}$ is a direct product $B_1 \times B_2$ of two local rings. Obviously $B_1 \times 0$ does not come from a projective $\overline{R}$-module and hence cannot be the second component of an R_{art}-module of the form M_{art}. The reader may recognize that exactly the same phenomenon gives rise to modules over the completion $\widehat{R}$ that don't come from R-modules, a situation that we will exploit shamelessly in Chapter 12.

Recall that we use the notation $M_1 \mid M_2$, introduced in Chapter 1, to indicate that M_1 is isomorphic to a direct summand of M_2.

3.7 Proposition. *Let $(R, \mathfrak{m}, k)$ be an analytically unramified local ring of dimension one, and assume $R \neq \overline{R}$. Let R_{art} be the Artinian pair $R/\mathfrak{c} \hookrightarrow \overline{R}/\mathfrak{c}$.*

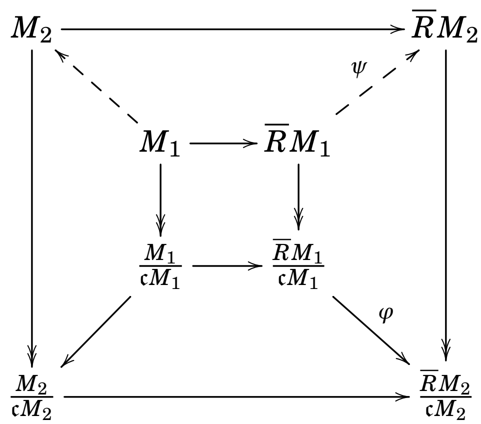
(i) The functor $M \rightsquigarrow M_{art} = (M/\mathfrak{c}M \hookrightarrow \overline{R}M/\mathfrak{c}M)$, for M a MCM R-module, is injective on isomorphism classes.

(ii) If M_1 and M_2 are MCM R-modules, then $M_1 \mid M_2$ if and only if $(M_1)_{art} \mid (M_2)_{art}$.

(iii) The ring R has finite CM type if and only if the Artinian pair R_{art} has finite representation type.

Proof. (i) First observe that $M \rightsquigarrow M_{\text{art}}$ is indeed well-defined: since $\overline{R}$ is a direct product of principal ideal rings, $\overline{R}M$ is a projective $\overline{R}$-module, so $\overline{R}M/\mathfrak{c}M$ is a projective $\overline{R}/\mathfrak{c}$-module. Thus M_{art} is a module over R_{art}. Let M_1 and M_2 be MCM R-modules, and suppose that $(M_1)_{\text{art}} \cong (M_2)_{\text{art}}$. Write $(M_i)_{\text{art}} = (V_i \hookrightarrow W_i)$, and choose an isomorphism $\varphi\colon W_1 \longrightarrow W_2$ such that $\varphi(V_1) = V_2$. Since $\overline{R}M_1$ is $\overline{R}$-projective, we can lift φ to an $\overline{R}$-homomorphism

$\psi\colon \overline{R}M_1 \longrightarrow \overline{R}M_2$ carrying M_1 into M_2.



Since $\mathfrak{c} \subseteq \mathfrak{m}$, the induced R-homomorphism $M_1 \longrightarrow M_2$ is surjective, by Nakayama's Lemma. (Here we need the assumption that $R \neq \overline{R}$.) Similarly, M_2 maps onto M_1, and it follows that $M_1 \cong M_2$ (cf. Exercise 3.25).

(ii) The "only if" direction is clear. For the converse, suppose there is an R_{art}-module $\mathfrak{X} = (V \hookrightarrow W)$ such that $(M_1)_{\text{art}} \oplus \mathfrak{X} \cong (M_2)_{\text{art}}$. Write $\overline{R} = D_1 \times \cdots \times D_s$, where each D_i is a semilocal principal ideal domain. Put $B_i = D_i/\mathfrak{c}D_i$, so that $\overline{R}/\mathfrak{c} = B_1 \times \cdots \times B_s$. Since $\overline{R}M_1$, and $\overline{R}M_2$ are projective $\overline{R}$-modules, there are non-negative integers e_i, f_i such that $\overline{R}M_1 \cong \prod_i D_i^{(e_i)}$ and $\overline{R}M_2 \cong \prod_i D_i^{(f_i)}$. Then $\overline{R}M_1/\mathfrak{c}M_1 \cong \prod_i B_i^{(e_i)}$, similarly $\overline{R}M_2/\mathfrak{c}M_2 \cong \prod_i B_i^{(f_i)}$, and $W = \prod_i B_i^{(f_i - e_i)}$. Letting $P = \prod_i D_i^{(f_i - e_i)}$, we see that $W \cong P/\mathfrak{c}P$. As discussed in Construction 3.1, it follows that there is a MCM R-module N such that $N_{\text{art}} \cong \mathfrak{X}$. We see from (i) that $M_1 \oplus N \cong M_2$.

(iii) Suppose R_{art} has finite representation type, and let $\mathbf{X}_1, \ldots, \mathbf{X}_t$ be a full set of representatives for the non-isomorphic indecomposable R_{art}-modules. Given a MCM R-module M, write $M_{\text{art}} \cong \mathbf{X}_1^{(n_1)} \oplus \cdots \oplus \mathbf{X}_t^{(n_t)}$, and put $j([M]) = (n_1, \ldots, n_t)$. By KRS (Theorem 2.4), j is a well-defined function from the set of isomorphism classes of MCM R-modules to $\mathbb{N}_0^{(t)}$, where $\mathbb{N}_0$

is the set of non-negative integers. Moreover, j is injective, by (i). Letting Σ be the image of j, we see, using (ii), that M is indecomposable if and only if $j[M]$ is a minimal non-zero element of Σ with respect to the product ordering. Dickson's Lemma (Exercise 3.26) says that every antichain in $\mathbb{N}_0^{(t)}$ is finite. In particular, Σ has only finitely many minimal elements, and R has finite CM type.

We leave the proof of the converse (which will not be needed here) as an exercise. □

3.8 Remark. It's worth observing that the proof of Proposition 3.7 uses KRS only over R_{art}, not over R (which is not assumed to be Henselian). In fact, if the completion $\widehat{R}$ is reduced, then $(\widehat{R})_{\mathrm{art}} = R_{\mathrm{art}}$. Indeed, the bottom row $R/\mathfrak{c} \hookrightarrow \overline{R}/\mathfrak{c}$ of the conductor square for R is unaffected by completion since $\overline{R}/\mathfrak{c}$ has finite length. Therefore the $\mathfrak{m}$-adic completion of the conductor square for R is

(3.8.1)
$$\begin{array}{ccc} \widehat{R} & \hookrightarrow & \widehat{R} \otimes_R \overline{R} \\ \downarrow & & \downarrow \widehat{\pi} \\ R/\mathfrak{c} & \hookrightarrow & \overline{R}/\mathfrak{c} \end{array}$$

which is still a pullback diagram by flatness of the completion. Note that $\widehat{R} \otimes_R \overline{R}$ is the integral closure of $\widehat{R}$. No non-zero ideal of $\overline{R}/\mathfrak{c}$ is contained in $R/\mathfrak{c}$, so $\ker \widehat{\pi}$ is the largest ideal of $\widehat{R} \otimes_R \overline{R}$ contained in $\widehat{R}$. Since also $\ker \widehat{\pi}$ contains a non-zerodivisor, $\ker \widehat{\pi}$ is the conductor for $\widehat{R}$ and (3.8.1) is the conductor square for $\widehat{R}$.

In particular, this shows that an analytically unramified R has finite CM type if and only if $\widehat{R}$ does. This is true as well in the case where $\widehat{R}$ is not reduced, cf. Corollary 3.17 below.

Returning now to sufficiency of the Drozd–Roĭter conditions, we will need the following observation from Bass's "ubiquity" paper [Bas63, (7.2)]:

3.9 Lemma (Bass)**.** *Let* $(R,\mathfrak{m})$ *be a one-dimensional Gorenstein local ring. Let* M *be a MCM* R*-module with no non-zero free direct summand. Let* $E = \operatorname{End}_R(\mathfrak{m})$*. Then* E *(viewed as multiplications) is a subring of* $\overline{R}$ *which contains* R *properly, and* M *has an* E*-module structure that extends the action of* R *on* M*.*

Proof. The inclusion $\operatorname{Hom}_R(M,\mathfrak{m}) \longrightarrow \operatorname{Hom}_R(M,R)$ is bijective, since a surjective homomorphism $M \longrightarrow R$ would produce a non-trivial free summand of M. Now $\operatorname{Hom}_R(M,\mathfrak{m})$ is an E-module via the action of E on $\mathfrak{m}$ by endomorphisms, and hence so is $M^* = \operatorname{Hom}_R(M,R)$. Therefore M^{**} is also an E-module, and since the canonical map; $M \longrightarrow M^{**}$ is bijective (as R is Gorenstein and M is MCM), M is an E-module. The other assertions regarding E are left to the reader. (Cf. Exercise 3.29. Note that the existence of the module M prevents R from being a discrete valuation ring.) □

Now we are ready for the main theorem of this chapter. We will not give a self-contained proof that the Drozd–Roĭter conditions imply finite CM type. Instead, we will reduce to a few special situations where the matrix decompositions of Green and Reiner [GR78] and Çimen [Çim94], [Çim98] apply.

3.10 Theorem. *Let* $(R,\mathfrak{m},k)$ *be an analytically unramified local ring of dimension one. These are equivalent:*

(i) *R has finite CM type.*

(ii) R satisfies both (DR1) and (DR2).

Let n be an arbitrary positive integer. If either (DR1) or (DR2) fails, there is an indecomposable MCM R-module of constant rank n; moreover, if $|k|$ is infinite, there are at least $|k|$ pairwise non-isomorphic indecomposable MCM R-modules of constant rank n.

Proof. By Theorem 3.6, $\overline{R}$ is a finite birational extension of R. The last statement of the theorem and the fact that (i) $\implies$ (ii) now follow immediately from Theorem 3.2 with $S = \overline{R}$.

Assume now that (DR1) and (DR2) hold. Let $A = R/\mathfrak{c}$ and $B = \overline{R}/\mathfrak{c}$, so that $R_{\text{art}} = (A \hookrightarrow B)$. Then R_{art} satisfies (dr1) and (dr2). By Proposition 3.7 it will suffice to prove that R_{art} has finite representation type. If every residue field of B is separable over k, then R_{art} has finite representation type by Theorem 2.21.

Now suppose that B has a residue field $\ell = B/\mathfrak{n}$ that is not separable over k. By (dr1), ℓ/k has degree 2 or 3, and ℓ is the only residue field of B that is not equal to k.

3.11 Case. *ℓ/k is purely inseparable of degree* 3.

If B is reduced (that is, R is *seminormal*), we can appeal to Theorem 2.21. Suppose now that B is *not* reduced. A careful computation of lengths (cf. Exercise 3.28) shows that R is Gorenstein, with exactly one ring S (the seminormalization of R) strictly between R and $\overline{R}$. By Lemma 3.9, $E := \operatorname{End}_R(\mathfrak{m}) \supseteq S$, and every non-free indecomposable MCM R-module is naturally an S-module. The Drozd–Roĭter conditions clearly pass to the seminormal ring S, which therefore has finite Cohen-Macaulay type. It follows that R itself has finite Cohen-Macaulay type.

3.12 Case. *ℓ/k is purely inseparable of degree 2.*

In this case, we appeal to Çimen's *tour de force* [Çim94],[Çim98], where he shows, by explicit matrix decompositions, that R_{art} has finite representation type. □

Let's insert here a few historical remarks. The conditions (DR1) and (DR2) were introduced by Drozd and Roĭter in a remarkable 1967 paper [DR67], where they classified the module-finite $\mathbb{Z}$-algebras having only finitely many indecomposable finitely generated torsion-free modules. Jacobinski [Jac67] obtained similar results at about the same time. The theorems of Drozd–Roĭter and Jacobinski imply the equivalence of (i) and (ii) in Theorem 3.10 for rings essentially module-finite over $\mathbb{Z}$. In the same paper they asserted the equivalence of (i) and (ii) in general. In 1978 Green and Reiner [GR78] verified the classification theorem of Drozd and Roĭter, giving more explicit details of the matrix decompositions needed to verify finite CM type. Their proof, like that of Drozd and Roĭter, depended crucially on arithmetic properties of algebraic number fields and thus did not provide immediate insight into the general problem. An important point here is that the matrix reductions of Green and Reiner work in arbitrary characteristics, as long as the integral closure $\overline{R}$ has no residue field properly extending that of R.

In 1989 R. Wiegand [Wie89] proved necessity of the Drozd–Roĭter conditions (DR1) and (DR2) for a general one-dimensional local ring $(R,\mathfrak{m},k)$

and, via the separable descent argument in the proof of Theorem 2.21, sufficiency under the assumption that every residue field of the integral closure $\overline{R}$ is separable over k. By (DR1), this left only the case where k is imperfect of characteristic two or three. In [Wie94], he used the seminormality argument above to handle the case of characteristic three. Finally, in his 1994 Ph.D. dissertation [Çim94], N. Çimen solved the remaining case—characteristic two—by difficult matrix reductions. It is worth noting that Çimen's matrix decompositions work in all characteristics and therefore confirm the computations done by Green and Reiner in 1978. The existence of $|k|$ indecomposables of constant rank k, when $|k|$ is infinite and (DR) fails, was proved by Karr and Wiegand [KW09] in 2009.

§3 ADE singularities

Of course we have not really proved sufficiency of the Drozd–Roĭter conditions, since we have not presented all of the difficult matrix calculations of Green and Reiner [GR78] and Çimen [Çim94, Çim98]. If R contains the field of rational numbers, there is an alternate approach that uses the classification, which we present in Chapter 5, of the two-dimensional hypersurface singularities of finite Cohen-Macaulay type. First we recall the 1985 classification, by Greuel and Knörrer [GK85], of the complete, equicharacteristic-zero curve singularities of finite Cohen-Macaulay type. Suppose k is an algebraically closed field of characteristic different from $2, 3$ or 5. The complete ADE (or *simple*) plane curve singularities over k are the rings $k[[x,y]]/(f)$, where f is one of the following polynomials:

$$(A_n):\ x^2 + y^{n+1}\,, \qquad n \geqslant 1$$

$$(D_n):\ x^2y + y^{n-1}\,, \qquad \text{n} \geqslant 4$$

$$(E_6):\ x^3 + y^4$$

$$(E_7):\ x^3 + xy^3$$

$$(E_8):\ x^3 + y^5$$

We will encounter these singularities again in Chapter 5. Here we will discuss briefly their role in the classification of one-dimensional rings of finite CM type. Greuel and Knörrer [GK85] proved that the ADE singularities are exactly the complete plane curve singularities of finite CM type in equicharacteristic zero. In fact, they showed much more, obtaining, essentially, the conclusion of Theorem 3.2 in this context:

3.13 Theorem (Greuel and Knörrer)**.** *Let $(R, \mathfrak{m}, k)$ be a one-dimensional reduced complete local ring containing $\mathbb{Q}$. Assume that k is algebraically closed.*

(i) R satisfies the Drozd–Roĭter conditions if and only if R is a finite birational extension of an ADE singularity.

(ii) Suppose that R has infinite CM type.

(a) There are infinitely many rings between R and its integral closure.

(b) For every $n \geqslant 1$ there are infinitely many isomorphism classes of indecomposable MCM R-modules of constant rank n.

Greuel and Knörrer used Jacobinski's computations [Jac67] to prove that ADE singularities have finite CM type. The fact that finite CM type passes to finite birational extensions (in dimension one!) is recorded in Proposition 3.14 below. We note that (iia) can fail for infinite fields that are not algebraically closed. Suppose, for example, that ℓ/k is a separable field extension of degree $d > 3$. Put $R = k + x\ell[\![x]\!]$. Then $\overline{R} = \ell[\![x]\!]$ is minimally generated, as an R-module, by $\{1, x, \dots, x^{d-1}\}$. Theorem 3.10 implies that R has infinite CM type. There are, however, only finitely many rings between R and $\overline{R}$. Indeed, the conductor square (3.1.1) shows that the intermediate rings correspond bijectively to the intermediate fields between k and ℓ.

In Chapter 7 we will use the classification of two-dimensional hypersurface rings of finite CM type to show that the one-dimensional ADE singularities have finite CM type (even in characteristic p, as long as $p \geqslant 7$). Then, in Chapter 11, we will deduce that the Drozd–Roĭter conditions imply finite CM type for any one-dimensional local ring CM ring $(R, \mathfrak{m}, k)$ containing a field, provided k is perfect and of characteristic $\neq 2, 3, 5$. Together with Greuel and Knörrer's result and the next proposition, this will give a different, slightly roundabout, proof that the Drozd–Roĭter conditions are sufficient for finite CM type in dimension one.

3.14 Proposition. *Let R and S be one-dimensional local rings, and suppose S is a finite birational extension of R.*

(i) If M and N are MCM S-modules, then $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_S(M, N)$.

(ii) Every MCM S-module is a MCM R-module.

(iii) If M is a MCM S-module then M is indecomposable over S if and only if M is indecomposable over R.

(iv) If R has finite CM type, so has S.

Proof. We may assume that R is CM, else $R = S$, and everything is boring.

(i) We need only verify that $\operatorname{Hom}_R(M,N) \subseteq \operatorname{Hom}_S(M,N)$. Let $\varphi\colon M \longrightarrow N$ be an R-homomorphism. Given any $s \in S$, write $s = r/t$, where $r \in R$ and t is a non-zerodivisor of R. Then, for any $x \in M$, we have $t\varphi(sx) = \varphi(rx) = r\varphi(x) = ts\varphi(x)$. Since N is torsion-free, we have $\varphi(sx) = s\varphi(x)$. Thus φ is S-linear.

(ii) If M is a MCM S-module, then M is finitely generated and torsion-free, hence MCM, over R.

(iii) is clear from (i) and the fact that ${}_SM$ is indecomposable if and only if $\operatorname{Hom}_S(M,M)$ contains no idempotents. Finally, (iv) is clear from (iii), (ii) and the fact that by (i) non-isomorphic MCM S-modules are non-isomorphic over R. □

§4 The analytically ramified case

Let $(R,\mathfrak{m})$ be a local Noetherian ring of dimension one, let K be the total quotient ring $\{\text{non-zerodivisors}\}^{-1}R$, and let $\overline{R}$ be the integral closure of R in K. Suppose $\overline{R}$ is *not* finitely generated over R. Then, since algebra-finite integral extensions are module-finite, no finite subset of $\overline{R}$ generates $\overline{R}$ as an R-algebra, and we can build an infinite ascending chain of finitely generated R-subalgebras of $\overline{R}$. Each algebra in the chain is a maximal Cohen-Macaulay R-module, and it is easy to see (Exercise 3.31) that no two of the algebras are isomorphic as R-modules. Moreover, each of these algebras is isomorphic, as an R-module, to a faithful ideal of R. Therefore R has an infinite family of pairwise non-isomorphic faithful ideals. It follows (Exercise 3.32) that R has infinite CM type. Now Theorem 3.6 implies the following result:

3.15 Proposition. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional CM local ring with finite Cohen-Macaulay type. Then R is analytically unramified.* □

In particular, this proposition shows that R itself is reduced; equivalently, R is an isolated singularity: $R_\mathfrak{p}$ is a regular local ring (a field!) for every non-maximal prime ideal $\mathfrak{p}$. See Theorem 6.12.

What if R is *not* Cohen-Macaulay? The next theorem and Theorem 3.10 provide the full classification of one-dimensional local rings of finite Cohen-Macaulay type. We will leave the proof as an exercise.

3.16 Theorem ([Wie94, Theorem 1])**.** *Let $(R, \mathfrak{m})$ be a one-dimensional local ring, and let N be the nilradical of R. Then R has finite Cohen-Macaulay type if and only if*

(i) R/N has finite Cohen-Macaulay type, and

(ii) $\mathfrak{m}^i \cap N = (0)$ for $i \gg 0$. □

For example, $k[[x, y]]/(x^2, xy)$ has finite Cohen-Macaulay type, since (x) is the nilradical and $(x, y)^2 \cap (x) = (0)$. However $k[[x, y]]/(x^3, x^2y)$ has infinite CM type: For each $i \geqslant 1$, xy^{i-1} is a non-zero element of $(x, y)^i \cap (x)$.

3.17 Corollary ([Wie94, Corollary 2]). *Let* $(R,\mathfrak{m})$ *be a one-dimensional local ring. Then* R *has finite CM type if and only if the* $\mathfrak{m}$*-adic completion* $\widehat{R}$ *has finite CM type.* □

Proof. Suppose first that R is analytically unramified. Since the bottom lines of the conductor squares for R and for $\widehat{R}$ are identical (Remark 3.8), it follows from (iii) of Proposition 3.7 that R has finite CM type if and only if $\widehat{R}$ has finite CM type.

For the general case, let N be the nilradical of R. Suppose R has finite CM type. The CM ring R/N then has finite CM type by Theorem 3.16 and hence is analytically unramified by Proposition 3.15. It follows that $\widehat{N}$ is the nilradical of $\widehat{R}$. By the first paragraph, $\widehat{R}/\widehat{N}$ has finite CM type; moreover, $\widehat{\mathfrak{m}}^i \cap \widehat{N} = (0)$ for $i \gg 0$. Therefore $\widehat{R}$ has finite CM type. For the converse, assume that $\widehat{R}$ has finite CM type. Since every MCM $\widehat{R}/\widehat{N}$-module is also a MCM $\widehat{R}$-module, we see that $\widehat{R/N} = \widehat{R}/\widehat{N}$ has finite CM type. Since R/N is CM, so is $\widehat{R/N}$, and now Theorem 3.15 implies that $\widehat{R/N}$ is reduced. By the first paragraph, R/N has finite CM type. Now $\widehat{N}$ is contained in the nilradical of $\widehat{R}$, so Theorem 3.16 implies that $\widehat{\mathfrak{m}}^i \cap \widehat{N} = (0)$ for $i \gg 0$. It follows that $\mathfrak{m}^i \cap N = (0)$ for $i \gg 0$, and hence that R has finite CM type. □

It is interesting to note that the proof of the corollary does not depend on the characterization (Theorem 3.10) of one-dimensional analytically unramified local rings of finite CM type. We remark that in higher dimensions finite CM type does not always ascend to the completion (cf. Example 11.14).

§5 Multiplicity two

Suppose $(R,\mathfrak{m})$ is an analytically unramified one-dimensional local ring and that $\dim_k(\overline{R}/\mathfrak{m}\overline{R}) = 2$. One can show (cf. Exercise 3.30) that R automatically satisfies (DR2) and therefore has finite CM type. Here we will give a direct proof of finite CM type in multiplicity two, using some results in Bass's "ubiquity" paper [Bas63]. We don't assume that $\overline{R}$ is a finitely generated R-module.

We refer the reader to Appendix A, §2 for basic stuff on multiplicities, particularly for one-dimensional rings.

3.18 Theorem. *Let* $(R,\mathfrak{m},k)$ *be a one-dimensional Cohen–Macaulay local ring with* $\mathrm{e}(R) = 2$.

(i) Every ideal of R *is generated by at most two elements.*

(ii) Every ring S *with* $R \subseteq S \subsetneq \overline{R}$ *and finitely generated over* R *is local and Gorenstein. In particular* R *itself is Gorenstein.*

(iii) Every MCM R*-module is isomorphic to a direct sum of ideals of* R*. In particular, every indecomposable MCM* R*-module has multiplicity at most* 2 *and is generated by at most* 2 *elements.*

(iv) The ring R *has finite CM type if and only if* R *is analytically unramified.*

Proof. Item (i) follows from [Sal78, Chap. 3, Theorem 1.1] or [Gre82].

(ii) Let S be a module-finite R-algebra properly contained in $\overline{R}$. Every ideal I of S is isomorphic to an ideal of R (clear denominators) and hence is two-generated as an R-module; therefore I is generated by two elements as an ideal of S. Since the maximal ideal of S is two-generated, Exercise 3.34 guarantees that S is Gorenstein. Moreover, the multiplicity of S as a module over itself is two. If S is not local, then its multiplicity is the sum of the multiplicities of its localizations at maximal ideals, so S is regular, contradicting $S \neq \overline{R}$.

(iii) Let M first be a faithful MCM R-module. As M is torsion-free, the map $j\colon M \longrightarrow K \otimes_R M$ is injective. Let $H = \{t \in K \mid tj(M) \subseteq j(M)\}$; then M is naturally an H-module. Since M is faithful, $H \hookrightarrow \operatorname{Hom}_R(M, M)$, and thus H is a module-finite extension of R contained in $\overline{R}$. Suppose first that $H = \overline{R}$. Then R is reduced by Lemma 3.6, and hence $\overline{R}$ is a principal ideal ring. It follows from the structure theory for modules over a principal ideal ring that M has a copy of H as a direct summand, and of course H is isomorphic to an ideal of R. If H is properly contained in $\overline{R}$, then, since H/R has finite length, we can apply Lemma 3.9 repeatedly, eventually getting a copy of some subring of H as a direct summand of M. In either case, we see that M has a faithful ideal of R as a direct summand.

Suppose, now, that M is an arbitrary MCM R-module, and let $I = (0 :_R M)$. Then R/I embeds in a direct product of copies of M (one copy for each generator); therefore R/I has depth 1 and hence is a one-dimensional CM ring. Of course $\mathrm{e}(R) \leqslant 2$, and, since M is a faithful MCM R/U-module, M has a non-zero ideal of R/I as a direct summand. To complete the proof, it will suffice to show that R/I is isomorphic to an ideal of R. By basic duality theory over the Gorenstein ring R, the type of R/I is equal to the number of generators $\mu_R((R/I)^*)$ of its dual $(R/I)^*$. Since R/I is Gorenstein, this implies that $(R/I)^*$ is cyclic. Choosing a surjection $R^* \twoheadrightarrow (R/I)^*$ and dualizing again, we have (since R/I is MCM and R is Gorenstein) $R/I \hookrightarrow R$ as desired.

(iv) The "only if" implication is Proposition 3.15. For the converse, we assume that R is analytically unramified, so that $\overline{R}$ is a finitely generated R-module by Theorem 3.6. It will suffice, by item (iii), to show that R has only finitely many ideals up to isomorphism. We first observe that every submodule of $\overline{R}/R$ is cyclic. Indeed, if H is an R-submodule of $\overline{R}$ and $H \supseteq R$, then H is isomorphic to an ideal of R, whence is generated by two elements, one of which can be chosen to be 1_R. Since $\overline{R}/R$ in particular is cyclic, it follows that $\overline{R}/R \cong \overline{R}/(R :_R \overline{R}) = R/\mathfrak{c}$. Thus every submodule of $R/\mathfrak{c}$ is cyclic; but then $R/\mathfrak{c}$ is an Artinian principal ideal ring and hence $R/\mathfrak{c}$ has only finitely many ideals. Since $\overline{R}/R \cong R/\mathfrak{c}$, we see that there are only finitely many R-modules between R and $\overline{R}$.

Given a faithful ideal I of R, put $E = (I :_R I)$, the endomorphism ring of I. Then I is a projective E-module by Remark 3.5. Since E is semilocal, I is isomorphic to E as an E-module and therefore as an R-module. In particular, R has only finitely many faithful ideals up to R-isomorphism.

Suppose now that J is a non-zero unfaithful ideal; then R is not a domain. Notice that if R had more than two minimal primes $\mathfrak{p}_i$, the direct product of the $R/\mathfrak{p}_i$ would be an R-submodule of $\overline{R}$ requiring more than two generators. Therefore R has exactly two minimal prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. Exercise 3.35 implies that J is a faithful ideal of either $R/\mathfrak{p}$ or $R/\mathfrak{q}$. Now $R/\mathfrak{p}$ and $R/\mathfrak{q}$ are discrete valuation rings: if, say, $R/\mathfrak{p}$ were properly contained in $\overline{R/\mathfrak{p}}$, then $\overline{R/\mathfrak{p}} \times R/\mathfrak{q}$ would need at least three generators as an R-module.

Therefore there are, up to isomorphism, only two possibilities for J. □

§6 Ranks of indecomposable MCM modules

Suppose $(R,\mathfrak{m},k)$ is a reduced local ring of dimension one, and let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ be the minimal prime ideals of R. Let us define the *rank* of a finitely generated R-module M to be the s-tuple $\operatorname{rank}_R(M) = (r_1,\dots,r_s)$, where r_i is the dimension of $M_{\mathfrak{p}_i}$ as a vector space over the field $R_{\mathfrak{p}_i}$. If R has finite CM type, it follows from (DR1) and Theorem A.23 that $s \leqslant \mathrm{e}(R) \leqslant 3$. There are universal bounds on the ranks of the indecomposable MCM R-modules, as R varies over one-dimensional reduced local rings with finite CM type. The precise ranks that occur have recently been worked out by N. Baeth and M. Luckas.

3.19 Theorem ([BL10]). *Let $(R,\mathfrak{m})$ be a one-dimensional, analytically unramified local ring with finite CM type. Let $s \leqslant 3$ be the number of minimal prime ideals of R.*

(i) If R is a domain, then every indecomposable finitely generated torsion-free R-module has rank 1, 2, *or* 3.

(ii) If $s = 2$, then the rank of every indecomposable finitely generated torsion-free R-module is $(0,1)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,1)$, *or* $(2,2)$.

(iii) If $s = 3$, then one can choose a fixed ordering of the minimal prime ideals so that the rank of every indecomposable finitely generated torsion-free R-module is $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$, *or* $(2,1,1)$.

Moreover, there are examples showing that each of the possibilities listed actually occurs. □

The lack of symmetry in the last possibility is significant: One cannot have, for example, both an indecomposable of rank $(2,1,1)$ and one of rank $(1,2,1)$. An interesting consequence of the theorem is a universal bound on modules of constant rank, even in the non-local case. First we note the following local-global theorem:

3.20 Theorem ([WW94]). *Let R be a one-dimensional reduced ring with finitely generated integral closure, let M be a finitely generated torsion-free R-module, and let r be a positive integer. If, for each maximal ideal $\mathfrak{m}$ of R, the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ has a direct summand of constant rank r, then M has a direct summand of constant rank r.*

3.21 Corollary ([BL10]). *Let R be a one-dimensional reduced ring with finitely generated integral closure. Assume that $R_{\mathfrak{m}}$ has finite Cohen-Macaulay type for each maximal ideal $\mathfrak{m}$ of R. Then every indecomposable finitely generated torsion-free R-module of* constant *rank has rank* 1, 2, 3, 4, 5 *or* 6.

Theorem 3.19 and Corollary 3.21 correct an error in a 1994 paper of R. and S. Wiegand [WW94] where it was claimed that the sharp universal bounds were 4 in the local case and 12 in general.

If one allows non-constant ranks, there is no universal bound, even if one assumes that all localizations have multiplicity two (cf. [Wie88]). An interesting phenomenon is that in order to achieve rank $(r_1,\dots,r_s)$ with *all* of the r_i large, one must have the ranks sufficiently spread out. For example [BL10, Theorem 5.5], if R has finite CM type locally and $n \geqslant 8$, every finitely generated torsion-free R module whose local ranks are between n and $2n-8$ has a direct summand of constant rank 6.

§7 Exercises

3.22 Exercise. Let $R = \mathbb{C}[x,y]_{(x,y)}/(y^2-x^3-x^2)$. Prove that the integral closure $\overline{R}$ is $R\left[\frac{y}{x}\right]$ and that $\overline{R}$ has two maximal ideals. Prove that the completion $\widehat{R} = \mathbb{C}[\![x,y]\!]/(y^2-x^2-x^3)$ has two minimal prime ideals. Show that the conductor square for R is

$$\begin{array}{ccc} R & \hookrightarrow & k[t]_U \\ \downarrow & & \downarrow \pi \\ k & \underset{\Delta}{\hookrightarrow} & k \times k \end{array}$$

where Δ is the diagonal embedding, U is a certain multiplicatively closed set, and the right-hand vertical map sends t to $(1,-1)$.

3.23 Exercise. Let R be a one-dimensional CM local ring with integral closure $\overline{R}$, and let M be a torsion-free R-module. Show that $\overline{R} \otimes_R M$ is torsion-free over $\overline{R}$ if and only if M is free.

3.24 Exercise. Let $c_1,\dots,c_n$ be distinct real numbers, and let S be the subring of $\mathbb{R}[t]$ of real polynomial functions f satisfying

$$f^{(k)}(c_i) = f^{(k)}(c_j)$$

for all $i,j = 1,\dots,n$ and all $k = 0,\dots,3$, where $f^{(k)}$ denotes the k^{th} derivative. Let S' be the semilocalization of S at the union of prime ideals $(t-c_1)\cup\cdots\cup$

$(t - c_n)$. Let $\mathfrak{m} = \{f \in S \mid f(c_1) = 0\}$, and set $R = S_{\mathfrak{m}}$. Show that $\mathfrak{m}$ is a maximal ideal of S and that

$$\begin{array}{ccc} R & \hookrightarrow & S' \\ \downarrow & & \downarrow \pi \\ k & \hookrightarrow & k[t_1]/(t_1^4) \times \cdots \times k[t_n]/(t_n^4) \end{array}$$

is the conductor square for R.

3.25 Exercise. Let Λ be a ring (not necessarily commutative), and let M_1 and M_2 be Noetherian left Λ-modules. Suppose there exist surjective Λ-homomorphisms $M_1 \twoheadrightarrow M_2$ and $M_2 \twoheadrightarrow M_1$. Prove that $M_1 \cong M_2$.

3.26 Exercise. A subset C of a poset X is called a *clutter* (or *antichain*) provided no two elements of C are comparable. Consider the following property of a poset X: (†) X has the descending chain condition, and every clutter in X is finite. Prove that if X and Y both satisfy (†), then $X \times Y$ (with the product partial ordering: $(x_1, y_1) \leqslant (x_2, y_2) \iff x_1 \leqslant x_2$ and $y_1 \leqslant y_2$) satisfies (†). Deduce Dickson's Lemma [Dic13]: Every clutter in $\mathbb{N}_0^{(t)}$ is finite.

3.27 Exercise. Prove the "only if" direction of (iii) in Proposition 3.7. (Hint: Use the fact that any indecomposable R_{art}-module is weakly extended from R, and use KRS (Theorem 2.4). See Proposition 11.7 if you get stuck.)

3.28 Exercise ([Wie94, Lemma 4])**.** Let $(R, \mathfrak{m}, k)$ be a one-dimensional reduced local ring satisfying (DR1) and (DR2). Assume that $\overline{R}$ has a maximal ideal $\mathfrak{n}$ such that $\ell = \overline{R}/\mathfrak{n}$ has degree 3 over k. Further, assume that R is not seminormal (equivalently, $\overline{R}/\mathfrak{c}$ is not reduced). Prove the following:

(i) $\overline{R}$ is local and $\mathfrak{m}\overline{R} = \mathfrak{n}$.

(ii) There is exactly one ring strictly between R and $\overline{R}$, namely $S = R + \mathfrak{n}$.

(iii) R is Gorenstein.

(iv) S is seminormal.

3.29 Exercise. Let $(R, \mathfrak{m})$ be a one-dimensional CM local ring which is not a discrete valuation ring. Let $\overline{R}$ be the integral closure of R in its total quotient ring K. Identify $E = \{c \in K \mid c\mathfrak{m} \subseteq \mathfrak{m}\}$ with $\operatorname{End}_R(\mathfrak{m})$ via the isomorphism taking c to multiplication by c. Prove that $E \subseteq \overline{R}$ and that E contains R properly.

3.30 Exercise. Let $(R, \mathfrak{m}, k)$ be a one-dimensional reduced local ring for which $\overline{R}$ is generated by two elements as an R-module. Prove that R satisfies the second Drozd–Roĭter condition (DR2). (Hint: Pass to $R/\mathfrak{c}$ and count lengths carefully.)

3.31 Exercise. Let R be a commutative ring with total quotient ring $K = \{\text{non-zerodivisors}\}^{-1}R$.

(i) Let M be an R-submodule of K. Assume that M contains a non-zerodivisor of R. Prove that $\operatorname{Hom}_R(M, M)$ is naturally identified with $\{\alpha \in K | \alpha M \subseteq M\}$, so that every endomorphism of M is given by multiplication by an element of K.

(ii) ([Wie94, Lemma 1]) Suppose A and B are subrings of K with $R \subseteq A \cap B$. Prove that if A and B are isomorphic as R-modules then $A = B$.

3.32 Exercise. Let R be a reduced one-dimensional local ring. Suppose R has an infinite family of ideals that are pairwise non-isomorphic as R-modules. Prove that R has infinite CM type. (Hint: the *Goldie dimension* of R is the least integer s such that every ideal of R is a direct sum of at most s indecomposable ideals. Prove that $s < \infty$.)

3.33 Exercise. Prove Theorem 3.16.

3.34 Exercise ([Bas63, Theorem 6.4])**.** Let $(R, \mathfrak{m})$ be a one-dimensional CM ring, and suppose $\mathfrak{m}$ can be generated by two elements. Prove that R is Gorenstein.

3.35 Exercise. Let $(R, \mathfrak{m})$ be a reduced one-dimensional local ring, and let M be a MCM R-module. Prove that $(0 :_R M)$ is the intersection of the minimal prime ideals $\mathfrak{p}$ for which $M_{\mathfrak{p}} \neq 0$.

4
Invariant Theory

In this chapter we describe an abundant source of maximal Cohen–Macaulay modules coming from invariant theory. We consider subrings of elements of a power series ring $S = k[[x_1, \dots, x_n]]$ left fixed by the action of a finite group of linear changes of variable $G \subseteq \mathrm{GL}(n, k)$. We assume that $|G|$ is invertible in k. Then the invariant subring $R = S^G$ is a Cohen–Macaulay complete local normal domain of dimension n, and comes equipped with a natural MCM module, namely the ring S considered as an R-module. The main goal of this chapter is a collection of one-one correspondences between:

- the indecomposable R-direct summands of S;
- the indecomposable projective modules over the endomorphism ring $\mathrm{End}_R(S)$;
- the indecomposable projective modules over the twisted group ring $S\#G$; and
- the irreducible k-representations of G.

We also introduce two directed graphs (quivers) associated with the data above, the McKay quiver and the Gabriel quiver, and show that they are isomorphic.

In the next chapter we will specialize to the case $n = 2$, and show that in fact every indecomposable MCM R-module is a direct summand of S, so that the correspondences above classify all the MCM R-modules. We will also see that the McKay–Gabriel quiver is isomorphic to the Auslander–Reiten quiver of R.

We begin with a little general invariant theory, then define a central object, the skew group ring.

§1 The skew group ring

We intend to investigate *invariant subrings* of elements of a power series ring $S = k[\![x_1,\dots,x_n]\!]$ left immobile by the action of a finite group of ring automorphisms $G \subseteq \operatorname{Aut}(S)$. Denote this fixed ring by $R = S^G$. First we observe that we may assume the action of G is linear on the variables x_i, via an argument going back to Cartan [Car57].

4.1 Lemma. *Let k be a field, $S = k[\![x_1,\dots,x_n]\!]$ a power series ring over k, and $G \subseteq \operatorname{Aut}(S)$ a finite group of ring automorphisms of S with $|G|$ invertible in k. Then there exists a finite group $G' \subseteq \operatorname{GL}(n,k)$, acting on S via linear changes of variable such that $S^{G'} \cong S^G$.*

Proof. Let $V = (x_1,\dots,x_n)/(x_1,\dots,x_n)^2$ be the vector space of linear forms of S. Then G acts on V, giving a group homomorphism $\rho\colon G \longrightarrow \operatorname{GL}(V)$. Set $G' = \rho(G)$, and extend the action of G' linearly to all of S. For $\sigma \in G$, denote by $\widehat{\sigma}$ its image in G'.

Define a ring homomorphism $\theta\colon S \longrightarrow S$ by the rule

$$\theta(f) = \frac{1}{|G|}\sum_{\sigma\in G} \widehat{\sigma}^{-1}\sigma(f).$$

Since θ restricts to the identity on V, it is an automorphism of S. For an invariant $f \in S^G$, $\theta(f)$ is the average of the G'-orbit of f, so is invariant under the action of G'. Restricting θ to S^G thus delivers the isomorphism $S^G \cong S^{G'}$. □

4.2 Notation. Here is our primary setup for the entire chapter. Let k be a field, and fix the power series ring $S = k[[x_1, \dots, x_n]]$ of dimension n over k. Let $\mathfrak{n} = (x_1, \dots, x_n)$ be the maximal ideal of S, and $V = \mathfrak{n}/\mathfrak{n}^2$ the vector space of linear forms. Let G be a finite subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}(n, k)$, and assume that $|G|$ is non-zero in k. Let G act on V by left-multiplication. Then G acts naturally on the left on elements of S by extending the action on V multiplicatively. Set $R = S^G$, the invariant ring.

4.3 Remarks. Here is a laundry list of properties of $R = S^G$ in the notation of 4.2. Most of the unproved assertions can be found in D. Benson's admirable book [Ben93]. (Some adjustments are necessary for passage from the case of polynomials to that of power series.) First observe that when $n = 1$, R is again a regular local ring. We consider this situation dull, and rule it out from now on.

The assumption that $|G|$ is invertible in k is essential for what we do below; virtually everything breaks terribly in the "modular" situation. Since we do insist upon it, we have an R-linear *Reynolds operator* $\rho\colon S \longrightarrow R$, defined by sending $f \in S$ to the average of its orbit:

$$\rho(f) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(f).$$

This splits the inclusion $R \subseteq S$, thereby making R an R-direct summand of S. It follows (cf. Exercise 4.24) that $IS \cap R = I$ for every ideal I of R, whence R is Noetherian, local, and even complete, with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n) \cap R$. (The Reynolds operator is not strictly necessary for the Noetherian property, but it simplifies matters significantly and will definitely be needed below.)

Let K be the quotient field of S and F the quotient field of R. Then G acts naturally on K with fixed field equal to F. Thus K/F is a Galois extension of F with Galois group G. Any element in F which is integral over R is also integral over S; since S is integrally closed in K, this means that R is integrally closed in F, i.e. a normal domain. Now every element $f \in S$ is a root of the monic polynomial $\prod_{\sigma\in G}(X - \sigma(f))$, whose coefficients are elementary symmetric polynomials in the conjugates $\{\sigma(f)\}$. This shows that S is an integral extension of R. In particular $\dim R = \dim S = n$.

If S were replaced by a ring of polynomials, rather than power series, then clearly S would be a finitely generated R-algebra, whence a module-finite R-algebra since it is integral. Our current situation is just the completion of the polynomial case, so we see that here too S is a finitely generated R-module. (Alternatively, one can use the "complete Nakayama's Lemma" for this assertion.) The equality $IS \cap R = I$ for ideals I of R then shows that S is a maximal Cohen–Macaulay R-module (cf. Exercise 4.24). The rank of S as an R-module is equal to $\dim_F(K) = |G|$.

By Cohen's structure theorems, R has a power series subring T over which it is a finitely generated module; since S is finite over R it is also finite over T. The power series generating T form s system of parameters in S, hence a regular sequence there. Thus in particular S is a free T-module. The Reynolds operator $\rho\colon S \longrightarrow R$ is also T-linear, so that R is a direct summand of S over T, and is T-free as well. In particular R is Cohen–Macaulay. (This argument is essentially due to Hochster and

Eagon [HE71]. By the way, it is true that even if G is not invertible in k then R has depth at least two; see [Ben93, Prop. 4.3.7].)

4.4 Remark. Many of the results in this chapter remain true under weaker assumptions than those in 4.2. In particular the fundamental result, Theorem 4.13 of Auslander, does not require S to be complete, to contain a field, or even to be regular. This additional generality is used in Appendix B. Accordingly, we will state some results more generally than is strictly necessary for this chapter.

Generally speaking, what is essential is that $(S,\mathfrak{n})$ be a local ring, $G \subseteq \mathrm{GL}(\mathfrak{n}/\mathfrak{n}^2)$ a finite group with $|G|$ invertible in S, and that the action of G be compatible with the relations among the basis elements of $\mathfrak{n}/\mathfrak{n}^2$, so that G acts via linear automorphisms of S. We will refer briefly to this scenario with "G acts via linear automorphisms on S."

To understand the subring R, we move to an extension of S. Perhaps surprisingly, we choose a non-commutative one.

4.5 Definition. Let S be a local ring and G a finite group with order invertible in S, acting via linear automorphisms on S. Let $S\#G$ denote the *skew group ring* of S and G. As an S-module, $S\#G = \bigoplus_{\sigma \in G} S \cdot \sigma$ is free on the elements of G; the product of two elements $s \cdot \sigma$ and $t \cdot \tau$ is

$$(s \cdot \sigma)(t \cdot \tau) = s\sigma(t) \cdot \sigma\tau \,.$$

Thus moving σ past t "twists" the ring element. See the next section for an explanation of this rule.

4.6 Remarks. In the notation of Definition 4.5, a left $S\#G$-module M is nothing but an S-module with a compatible action of G, in the sense that $\sigma(sm) = \sigma(s)\sigma(m)$ for all $\sigma \in G$, $s \in S$, $m \in M$. Since the action of G on S is defined on the variables and extended multiplicatively, we have $\sigma(st) = \sigma(s)\sigma(t)$ for all s and t in S, and so S itself is a left $S\#G$-module. Of course $S\#G$ is also a left module over itself.

Similarly, an $S\#G$-linear map between left $S\#G$-modules is an S-module homomorphism $f\colon M \longrightarrow N$ respecting the action of G, so that $f(\sigma(m)) = \sigma(f(m))$. This allows us to define a left $S\#G$-module structure on $\operatorname{Hom}_S(M,N)$, when M and N are $S\#G$-modules, by $\sigma(f)(m) = \sigma(f(\sigma^{-1}(m)))$. It follows that an S-linear map $f\colon M \longrightarrow N$ between $S\#G$-modules is $S\#G$-linear if and only if it is invariant under the G-action. Indeed, if $\sigma(f) = f$ for all $\sigma \in G$, then $f(m) = \sigma(f(\sigma^{-1}(m)))$, so that $\sigma^{-1}(f(m)) = f(\sigma^{-1}(m))$ for all $\sigma \in G$. Concisely,

$$\operatorname{Hom}_{S\#G}(M,N) = \operatorname{Hom}_S(M,N)^G\,. \tag{4.6.1}$$

Since the order of G is invertible in k, taking G-invariants of an $S\#G$-modules is an exact functor. (To see this, first note that $-^G$ is clearly left-exact. Then for an $S\#G$-linear surjection $f\colon M \longrightarrow N$, and $n \in N^G$, note that $f(\sigma(m)) = \sigma(f(m) = \sigma(n) = n$ for every preimage $m \in M$ of n. Taking the average of the orbit of such preimages, then, gives an element of M^G mapping to n.) In particular, $-^G$ commutes with taking cohomology, so (4.6.1) extends to higher Exts:

$$\operatorname{Ext}^i_{S\#G}(M,N) = \operatorname{Ext}^i_S(M,N)^G \tag{4.6.2}$$

for all $S\#G$-modules M and N and all $i \geqslant 0$. This has the following wonderful consequence, the easy proof of which we leave as an exercise.

4.7 Proposition. *An $S\#G$-module M is projective if and only if it is projective (that is, free) as an S-module. If in particular $S = k[\![x_1,\dots,x_n]\!]$ is regular, then the twisted group ring $S\#G$ has finite global dimension, equal to n.*

The next example may come in handy when proving the last assertion of the proposition.

4.8 Example. Set $S = k[\![x_1,\dots,x_n]\!]$. The Koszul complex $K_\bullet$ on the sequence of variables $\mathbf{x} = x_1,\dots,x_n$ is a minimal $S\#G$-linear resolution of the residue field k of S (with trivial action of G). In detail, let $V = \mathfrak{n}/\mathfrak{n}^2$ again be the k-vector space with basis $x_1,\dots,x_n$, and

$$K_p = K_p(\mathbf{x}, S) = S \otimes_k \bigwedge^p V$$

for $p \geqslant 0$. The differential $\partial_p : K_p \longrightarrow K_{p-1}$ is given by

$$\partial_p(x_{i_1} \wedge \cdots \wedge x_{i_p}) = \sum_{j=1}^{p} (-1)^{j+1} x_{i_j} (x_{i_1} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_p}),$$

where $\{x_{i_1} \wedge \cdots \wedge x_{i_p}\}$, $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$, are the natural basis vectors for $\bigwedge^p V$. Since the x_i form an S-regular sequence, K_p is acyclic, minimally resolving k.

The exterior powers $\bigwedge^p V$ carry a natural action of G, by $\sigma(x_{i_1} \wedge \cdots \wedge x_{i_p}) = \sigma(x_{i_1}) \wedge \cdots \wedge \sigma(x_{i_p})$, and it's easy to see that the differentials ∂_p are $S\#G$-linear for this action. Since the modules appearing in $K_\bullet$ are free S-modules, they are projective over $S\#G$, so we see that $K_\bullet$ resolves the trivial module k over $S\#G$. Since every projective over $S\#G$ is free over S, the Depth Lemma then shows that $\operatorname{pd}_{S\#G} k$ cannot be any smaller than n.

4.9 Remark. Let S be local and G a finite group with order invertible in S, acting via linear automorphisms on S. The ring S sits inside $S\#G$ naturally via $S = S\cdot 1$. However, it also sits in a more symmetric fashion via a modified version of the Reynolds operator. Define $\widehat{\rho}\colon S \longrightarrow S\#G$ by

$$\widehat{\rho}(s) = \frac{1}{|G|}\sum_{\sigma\in G}\sigma(s)\cdot\sigma\,.$$

One checks easily that $\widehat{\rho}$ is an injective ring homomorphism, and that the image of $\widehat{\rho}$ is precisely equal to $(S\#G)^G$, the fixed points of $S\#G$ under the left action of G. In particular, $\widehat{\rho}(1)$ is an idempotent of $S\#G$.

§2 The endomorphism algebra

The "twisted" multiplication on the skew group ring $S\#G$ is cooked up precisely so that the homomorphism

$$\gamma\colon S\#G \longrightarrow \operatorname{End}_R(S)\,, \qquad \gamma(s\cdot\sigma)(t) = s\sigma(t)\,,$$

is a ring homomorphism extending the group homomorphism $G \longrightarrow \operatorname{End}_R(S)$ defining the action of G on S. Colloquially, γ simply considers an element of $S\#G$ as an endomorphism of S.

In general, γ is neither injective nor surjective, even when S is a power series ring. Under an additional assumption on G, however, it is both, by a theorem of Auslander [Aus62]. We turn now to this additional assumption, explaining which will necessitate a brief detour through classical invariant theory and ramification theory. We banish all the details to Appendix B.

4.10 Definition. An element $\sigma \in \mathrm{GL}(V)$ of finite order is called a *pseudo-reflection* provided the fixed subspace $V^\sigma = \{v \in V \mid \sigma(v) = v\}$ has codimension one in V. Equivalently, $\sigma - 1_V$ has rank 1. A pseudo-reflection σ is a *reflection* if it has order 2. We say a subgroup $G \subseteq \mathrm{GL}(V)$ is *small* if it contains no pseudo-reflections.

If a non-identity pseudo-reflection σ is diagonalizable, then σ is similar to a diagonal matrix with diagonal entries $1,\dots,1,\lambda$ with $\lambda \neq 1$ a root of unity.

The importance of pseudo-reflections in invariant theory generally begins with the foundational theorem of Shephard–Todd (Theorem B.28), which says that, the case $S = k[\![x_1,\dots,x_n]\!]$, the invariant ring $R = S^G$ is a regular local ring if and only if G is generated by pseudo-reflections.

More apposite for our immediate application, pseudo-reflections control the "large ramification" of the extension $R \hookrightarrow S$. To explain this, recall (Definition B.1) that a local homomorphism of local rings $(A,\mathfrak{m},k) \longrightarrow (B,\mathfrak{n},\ell)$ which is essentially of finite type is called *unramified* provided $\mathfrak{m}B = \mathfrak{n}$ and the induced homomorphism $A/\mathfrak{m} \longrightarrow B/\mathfrak{m}B$ is a finite separable field extension. Equivalently (Proposition B.9), the exact sequence

$$0 \longrightarrow \mathscr{J} \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0\,, \tag{4.10.1}$$

where $\mu\colon B \otimes_A B \longrightarrow B$ is the *diagonal map* defined by $\mu(b \otimes b') = bb'$ and $\mathscr{J}$ is generated by all elements of the form $b\otimes 1 - 1\otimes b$, splits as $B\otimes_A B$-modules. We say that a ring homomorphism $A \longrightarrow B$ which is essentially of finite type is *unramified in codimension one* if the induced local homomorphism $A_{\mathfrak{q}\cap A} \longrightarrow B_\mathfrak{q}$ is unramified for every prime ideal $\mathfrak{q}$ of height one in B. If

$A \longrightarrow B$ is module-finite, then it is equivalent to quantify over height-one primes in A.

Here is the connection with invariant subrings. See Theorem B.30.

4.11 Proposition. *In the notation of 4.2, the group G is small if and only if the extension $R \longrightarrow S$ is unramified in codimension one.*

In fact, by a theorem of Prill, we could always assume that G is small. Specifically, we may replace S, V, and G by another power series ring S', vector space V', and finite group G', respectively, so that $G \subseteq \mathrm{GL}(V')$ is small and $S'^{G'} \cong S^G$. See Appendix B for this, which will we will not use in this chapter.

In order to leverage codimension-one information into a global conclusion, we will use a general lemma about normal domains due to Auslander and Buchsbaum [AB59], which will reappear repeatedly in other contexts.

4.12 Lemma. *Let A be a normal domain and let $f : M \longrightarrow N$ be a homomorphism of finitely generated A-modules such that M satisfies the condition (S_2) and N satisfies (S_1). If $f_\mathfrak{p}$ is an isomorphism for every prime ideal $\mathfrak{p}$ of codimension 1 in A, then f is an isomorphism.*

Proof. Set $K = \ker f$ and $C = \operatorname{cok} f$, so that we have the exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{f} N \longrightarrow C \longrightarrow 0\,. \tag{4.12.1}$$

Since $f_{(0)}$ is an isomorphism, $K_{(0)} = 0$, which means that K is annihilated by a non-zero element of A. But M is torsion-free, so $K = 0$. As for C, suppose that $C \neq 0$ and choose $\mathfrak{p} \in \operatorname{Ass} C$. Then $\mathfrak{p}$ has height at least 2.

Localize (4.12.1) at $\mathfrak{p}$:

$$0 \longrightarrow M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0\,.$$

As M is reflexive, it satisfies (S_2), so $M_{\mathfrak{p}}$ has depth at least 2. On the other end, however, $C_{\mathfrak{p}}$ has depth 0, which contradicts the Depth Lemma. □

4.13 Theorem (Auslander [Aus62, Prop. 3.4]). *Let* $(S,\mathfrak{n})$ *be a local normal domain and let* G *be a finite subgroup of* $\mathrm{GL}(\mathfrak{n}/\mathfrak{n}^2)$ *with order invertible in* S. *Assume that* G *acts via linear automorphisms on* S, *and set* $R = S^G$. *If* $R \longrightarrow S$ *is unramified in codimension one, then the ring homomorphism* $\gamma\colon S\#G \longrightarrow \mathrm{End}_R(S)$ *defined by* $\gamma(s\cdot\sigma)(t) = s\sigma(t)$ *is an isomorphism. If in particular* $S = k[[x_1,\dots,x_n]]$ *as in 4.2 and* G *is small, then* γ *is an isomorphism.*

Proof. Since $S\#G$ is isomorphic to a direct sum of copies of S as an S-module, it in particular satisfies (S_2) over R. The endomorphism ring $\mathrm{End}_R(S)$ has depth at least 2 over R by Exercise 4.25, so satisfies (S_1). Thus by Lemma 4.12 it suffices to prove that γ is an isomorphism in height one. At height one primes, the extension is unramified, so we may assume for the proof that $R \longrightarrow S$ is unramified.

The strategy of the proof is to define a right splitting $\mathrm{End}_R(S) \longrightarrow S\#G$ for $\gamma\colon S\#G \longrightarrow \mathrm{End}_R(S)$ based on the diagram below.

(4.13.1)
$$\begin{array}{ccc}
S\#G & \xrightarrow{\quad\gamma\quad} & \mathrm{End}_R(S) \\
\Big\uparrow{\scriptstyle\widetilde{\mu}} & & \Big\downarrow{\scriptstyle f \mapsto f\otimes\widehat{\rho}} \\
S\otimes_R (S\#G) & \xleftarrow[\mathrm{ev}_\epsilon]{} & \mathrm{Hom}_S(S\otimes_R S, S\otimes_R (S\#G))
\end{array}$$

We now define each of the arrows in (4.13.1) in turn. Recall from Remark 4.9 that the homomorphism

$$\widehat{\rho}\colon S \longrightarrow S\#G, \qquad \widehat{\rho}(s) = \frac{1}{|G|}\sum_{\sigma\in G}\sigma(s)\cdot\sigma$$

embeds S as the fixed points $(S\#G)^G$ of $S\#G$. Thus $-\otimes\widehat{\rho}$ defines the right-hand vertical arrow in (4.13.1).

Since we assume $R \longrightarrow S$ is unramified, the short exact sequence

$$0 \longrightarrow \mathscr{J} \longrightarrow S\otimes_R S \xrightarrow{\mu} S \longrightarrow 0\,, \tag{4.13.2}$$

splits as $S\otimes_R S$-modules, where as before $\mu\colon S\otimes_R S \longrightarrow S$ is the diagonal map and $\mathscr{J}$ is generated by all elements of the form $s\otimes 1 - 1\otimes s$ for $s \in S$. Tensoring (4.13.2) on the right with $S\#G$ thus gives another split exact sequence

$$0 \longrightarrow \mathscr{J}\otimes_S (S\#G) \longrightarrow S\otimes_R (S\#G) \xrightarrow{\widetilde{\mu}} S\#G \longrightarrow 0 \tag{4.13.3}$$

with $\widetilde{\mu}(t\otimes s\cdot\sigma) = ts\cdot\sigma \in S\#G$ defining the left-hand vertical arrow in (4.13.1).

Let $j\colon S \longrightarrow S\otimes_R S$ be a splitting for (4.13.2), and set $\epsilon = j(1)$. Then $\mu(\epsilon) = 1$ and

$$(1\otimes s - s\otimes 1)\epsilon = 0 \tag{4.13.4}$$

for all $s \in S$. Evaluation at $\epsilon \in S\otimes_R S$ defines

$$\mathrm{ev}_\epsilon\colon \mathrm{Hom}_S(S\otimes_R S, S\otimes_R (S\#G)) \longrightarrow S\otimes_R (S\#G)\,,$$

the bottom row of the diagram. Now we show that for an arbitrary $f \in \mathrm{End}_R(S)$, we have

$$\gamma\left(\widetilde{\mu}\left(\mathrm{ev}_\epsilon\left(f\otimes\widehat{\rho}\right)\right)\right) = \frac{1}{|G|}f\,.$$

Write $\epsilon = \sum_i x_i \otimes y_i$ for some elements $x_i, y_i \in S$. We claim first that

$$\sum_i x_i \sigma(y_i) = \begin{cases} 1 & \text{if } \sigma = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

To see this, recall from (4.13.4) that $\mathscr{J}\epsilon = 0$, so that

$$(s \otimes 1)\left(\sum_i x_i \otimes y_i\right) = (1 \otimes s)\left(\sum_i x_i \otimes y_i\right)$$

for every $s \in S$. Apply the endomorphism $1 \otimes \sigma$ to both sides, obtaining

$$\sum_i s x_i \otimes \sigma(y_i) = \sum_i x_i \otimes \sigma(s)\sigma(y_i).$$

Collapse the tensor products with $\mu \colon S \otimes_R S \longrightarrow S$, and factor each side, getting

$$s\left(\sum_i x_i \sigma(y_i)\right) = \sigma(s)\left(\sum_i x_i \sigma(y_i)\right).$$

This holds for every $s \in S$, so that either $\sigma = 1$ or $\sum_i x_i \sigma(y_i) = 0$, proving the claim.

Now fix $f \in \operatorname{End}_R(S)$ and $s \in S$. Then unravelling all the definitions, we find

$$\begin{aligned} \gamma\left[\widetilde{\mu}\left[(f \otimes \widehat{\rho})(\epsilon)\right]\right](s) &= \gamma\left[\widetilde{\mu}\left[(f \otimes \widehat{\rho})\left(\textstyle\sum_i x_i \otimes y_i\right)\right]\right](s) \\ &= \gamma\left[\widetilde{\mu}\left(\textstyle\sum_i f(x_i) \otimes \widehat{\rho}(y_i)\right)\right](s) \\ &= \gamma\left[\left(\textstyle\sum_i f(x_i)\widehat{\rho}(y_i)\right)\right](s) \\ &= \gamma\left[\left(\textstyle\sum_i f(x_i)\left(\frac{1}{|G|}\sum_\sigma \sigma(y_i)\cdot\sigma\right)\right)\right](s) \\ &= \frac{1}{|G|}\textstyle\sum_i f(x_i)\left(\sum_\sigma \sigma(y_i)\sigma(s)\right). \end{aligned}$$

Now, since the sum over σ is fixed by G, it lives in R, so

$$= \frac{1}{|G|} f\left(\sum_i x_i \left(\sum_\sigma \sigma(y_i)\sigma(s)\right)\right)$$
$$= \frac{1}{|G|} f\left(\sum_\sigma \left(\sum_i x_i \sigma(y_i)\right)\sigma(s)\right)$$
$$= \frac{1}{|G|} f\left(\sum_i x_i y_i s\right)$$

by the claim. By the definition of $\epsilon = \sum x_i \otimes y_i$, this last expression is equal to $\frac{1}{|G|} f(s)$, as desired. Therefore $\gamma\colon S\#G \longrightarrow \operatorname{End}_R(S)$ is a split surjection. Since both source and target of γ are R-modules of rank equal to $|G|^2$, this forces γ to be an isomorphism. □

4.14 Corollary. *With notation as in 4.2, assume that G acts without pseudo-reflections. Then we have ring isomorphisms*

$$S\#G \xrightarrow{\iota} (S\#G)^{op} \xrightarrow{\nu} \operatorname{End}_{S\#G}(S\#G) \xrightarrow{\text{res}} \operatorname{End}_R(S)$$

where $\iota(s\cdot\sigma) = \sigma^{-1}(s)\cdot\sigma^{-1}$, $\nu(s\cdot\sigma)(t\cdot\tau) = (t\cdot\tau)(s\cdot\sigma)$, *and* res *is restriction to the subring* $\widehat{\rho}(S)$. *The composition of these maps is the isomorphism* γ. *These isomorphisms induce one-one correspondences among*

- *the indecomposable direct summands of S as an R-module;*
- *the indecomposable direct summands of* $\operatorname{End}_R(S)$ *as an* $\operatorname{End}_R(S)$*-module; and*
- *the indecomposable direct summands of $S\#G$ as an $S\#G$-module.*

Explicitly, if $P_0,\dots,P_d$ are the indecomposable direct summands of $S\#G$, then P_j^G, for $j = 0,\dots,d$ are the direct summands of S as an R-module. They are in particular MCM R-modules.

Proof. It's easy to check that ι and ν are isomorphisms, and that the composition $\operatorname{res}\circ\nu\circ\iota$ is equal to γ. The primitive idempotents of $\operatorname{End}_R(S)$ correspond both to the indecomposable R-direct summands of S and to the indecomposable $\operatorname{End}_R(S)$-projectives, while those of $\operatorname{End}_{S\#G}(S\#G)$ correspond to the indecomposable $S\#G$-projective modules. The fact that $(S\#G)^G = S$ implies the penultimate statement, and the fact that S is MCM over R was observed already. □

We have not yet shown that the indecomposable direct summands of $S\#G$ as an $S\#G$-module are all the indecomposable projective $S\#G$-modules. This will follow from the first result of the next section, where we prove that $S\#G$ (and hence $\operatorname{End}_R(S)$) satisfies the Krull-Remak-Schmidt Theorem.

§3 Group representations and the McKay–Gabriel quiver

The module theory of the skew group ring faithfully reflects the representation theory of G, in a precise sense. Let's keep all the notation established in 4.2. The extra generality mentioned in Remark 4.4 will not be useful in this section.

4.15 Definition. Let M be an $S\#G$-module and W a k-representation of G, that is, a module over the group algebra kG. Define an $S\#G$-module structure on $M\otimes_k W$ by the diagonal action

$$s\sigma(m\otimes w) = s\sigma(m)\otimes\sigma(w)\,.$$

Define a functor $\mathscr{F}$ from the category of finite-dimensional k-representations W of G to that of finitely generated $S\#G$-modules by

$$\mathscr{F}(W) = S \otimes_k W$$

and similarly for homomorphisms. For any W, $\mathscr{F}(W)$ is obviously a free S-module and thus a projective $S\#G$-module.

In the opposite direction, let P be a finitely generated projective $S\#G$-module. Then $P/\mathfrak{n}P$ is a finite-dimensional k-vector space with an action of G, that is, a k-representation of G. Define a functor $\mathscr{G}$ from projective $S\#G$-modules to k-representations of G by

$$\mathscr{G}(P) = P/\mathfrak{n}P$$

and correspondingly on homomorphisms.

4.16 Proposition. *The functors $\mathscr{F}$ and $\mathscr{G}$ form an adjoint pair, that is,*

$$\operatorname{Hom}_{kG}(\mathscr{G}(P), W) = \operatorname{Hom}_{S\#G}(P, \mathscr{F}(W)),$$

and are inverses of each other on objects. Concretely, for a projective $S\#G$-module P and a k-representation W of G, we have

$$S \otimes_k P/\mathfrak{n}P \cong P$$

and

$$(S \otimes_k W)/\mathfrak{n}(S \otimes_k W) \cong W\,.$$

In particular, there is a one-one correspondence between the isomorphism classes of indecomposable projective $S\#G$-modules and the irreducible k-representations of G.

Proof. It is clear that $\mathscr{G}(\mathscr{F}(W)) \cong W$, since

$$(S \otimes_k W)/\mathfrak{n}(S \otimes_k W) \cong S/\mathfrak{n} \otimes_k W \cong W.$$

To show that the other composition is also the identity, let P be a projective $S\#G$-module. Then $\mathscr{F}(\mathscr{G}(P)) = S \otimes_k P/\mathfrak{n}P$ is a projective $S\#G$-module, with a natural projection onto $P/\mathfrak{n}P$. Of course, the original projective P also maps onto $P/\mathfrak{n}P$. This latter is in fact a projective cover of $P/\mathfrak{n}P$ (since idempotents in kG lift to $S\#G$ via the retraction $kG \longrightarrow S\#G \longrightarrow kG$). There is thus a lifting $S \otimes_k P/\mathfrak{n}P \longrightarrow P$, which is surjective modulo $\mathfrak{n}P$. Nakayama's Lemma then implies that the lifting is surjective, so split, as P is projective. Comparing ranks over S, we must have $S \otimes_k P/\mathfrak{n}P \cong P$. □

4.17 Corollary. *Let $V_0, \dots, V_d$ be a complete set of non-isomorphic simple kG-modules. Then*

$$S \otimes_k V_0, \dots, S \otimes_k V_d$$

is a complete set of non-isomorphic indecomposable finitely generated projective $S\#G$-modules. Furthermore, the category of finitely generated projective $S\#G$-modules satisfies the Krull–Remak–Schmidt property, i.e. each finitely generated projective P is isomorphic to a unique direct sum $\bigoplus_{i=0}^{d}(S \otimes_k V_i)^{n_i}$.

Putting together the one-one correspondences obtained so far, we have

4.18 Corollary. *Let k be a field, $S = k[[x_1, \dots, x_n]]$, and $G \subseteq \mathrm{GL}(n,k)$ a finite group acting linearly on S without pseudo-reflections and such that $|G|$ is invertible in k. Then there are one-one correspondences between*

- *the indecomposable direct summands of S as an R-module;*

- *the indecomposable finitely generated projective* $\operatorname{End}_R(S)$*-modules;*
- *the indecomposable finitely generated projective* $S\#G$*-modules; and*
- *the irreducible* kG*-modules.*

The correspondence between the first and last items is induced by the equivalence of categories between k*-representations of* G *and* $\operatorname{add}_R(S)$ *defined by* $W \mapsto (S \otimes_k W)^G$.

Explicitly, if $V_0, \dots, V_d$ *are the non-isomorphic irreducible representations of* G *over* k*, then*

$$M_j = (S \otimes_k V_j)^G, \qquad j = 0, \dots, d$$

are the indecomposable R*-direct summands of* S*. They are in particular MCM* R*-modules. Furthermore, we have* $\operatorname{rank}_R M_j = \dim_k V_j$.

The one-one correspondence between projectives, representations, and certain MCM modules obtained so far extends to an isomorphism of two graphs naturally associated to these data, as we now explain. We will meet a third incarnation of these graphs in Chapter 10.

We keep all the notation from 4.2, and additionally let $V_0, \dots, V_d$ be a complete set of the non-isomorphic irreducible k-representations of G, with V_0 the trivial representation k. The given linear action of G on S is induced from an n-dimensional representation of G on the space $V = \mathfrak{n}/\mathfrak{n}^2$ of linear forms.

4.19 Definition. The *McKay quiver* of $G \subseteq \mathrm{GL}(V)$ has

- vertices $V_0, \dots, V_d$, and

- m_{ij} arrows $V_i \longrightarrow V_j$ if the multiplicity of V_i in an irreducible decomposition of $V \otimes_k V_j$ is equal to m_{ij}.

In case k is algebraically closed, the multiplicities m_{ij} in the McKay quiver can also be computed from the characters $\chi, \chi_0, \dots, \chi_d$ for $V, V_0, \dots, V_d$ [FH91, 2.10]:

$$m_{ij} = \langle \chi_i, \chi\chi_j \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_i(\sigma)\chi(\sigma^{-1})\chi_j(\sigma^{-1}).$$

For each $i = 0, \dots, d$, we set $P_i = S \otimes_k V_i$, the corresponding indecomposable projective $S\#G$-module. Then in particular $P_0 = S \otimes_k V_0 = S$, and $\{P_0, \dots, P_d\}$ is a complete set of non-isomorphic indecomposable projective $S\#G$-modules by Prop. 4.16. The V_j are simple $S\#G$-modules via the surjection $S\#G \longrightarrow kG$, with minimal projective cover P_j. Since $\operatorname{pd}_{S\#G} V_j \leqslant n$ by Proposition 4.7, the minimal projective resolution of V_j over $S\#G$ thus has the form

$$0 \longrightarrow Q_n^{(j)} \longrightarrow Q_{n-1}^{(j)} \longrightarrow \cdots \longrightarrow Q_1^{(j)} \longrightarrow P_j \longrightarrow V_j \longrightarrow 0$$

with projective $S\#G$-modules $Q_i^{(j)}$ for $i = 1, \dots, n$ and $j = 0, \dots, d$.

4.20 Definition. The *Gabriel quiver* of $G \subseteq \mathrm{GL}(V)$ has

- vertices $P_0, \dots, P_d$, and
- m_{ij} arrows $P_i \longrightarrow P_j$ if the multiplicity of P_i in $Q_1^{(j)}$ is equal to m_{ij}.

4.21 Theorem ([Aus86b])**.** *The McKay quiver and the Gabriel quiver of R are isomorphic directed graphs.*

Proof. First consider the trivial module $V_0 = k$. The minimal $S\#G$-resolution of k was computed in Example 4.8; it is the Koszul complex

$$K_\bullet : \qquad 0 \longrightarrow S \otimes_k \bigwedge^n V \longrightarrow \cdots \longrightarrow S \otimes_k V \longrightarrow S \longrightarrow 0\,.$$

To obtain the minimal $S\#G$-resolution of V_j, we simply tensor the Koszul complex with V_j over k, obtaining

$$0 \longrightarrow S \otimes_k \left(\bigwedge^n V \otimes_k V_j\right) \longrightarrow \cdots \longrightarrow S \otimes_k \left(V \otimes_k V_j\right) \longrightarrow S \otimes_k V_j \longrightarrow 0\,.$$

This displays $Q_1^{(j)} = S \otimes_k \left(V \otimes_k V_j\right)$, so that the multiplicity of P_i in $Q_1^{(j)}$ is equal to that of V_i in $V \otimes_k V_j$. □

4.22 Example. Take $n = 3$, and write $S = k[[x,y,z]]$. Let $G = \mathbb{Z}/2\mathbb{Z}$, with the generator acting on $V = kx \oplus ky \oplus kz$ by negating each variable. Then $R = S^G = k[[x^2, xy, xz, y^2, yz, z^2]]$. There are only two irreducible representations of G, namely the trivial representation k and its negative, which is isomorphic to the inverse determinant representation $V_1 = \det(V)^{-1} = \bigwedge^3 V^*$. The Koszul complex

$$0 \longrightarrow S \otimes \bigwedge^3 V \longrightarrow S \otimes_k \bigwedge^2 V \longrightarrow S \otimes_k V \longrightarrow S \longrightarrow 0$$

resolves k, while the tensor product

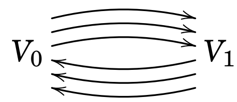
$$0 \longrightarrow S \otimes \left(\wedge^3 V \otimes_k \wedge^3 V^*\right) \longrightarrow S \otimes_k \left(\wedge^2 V \otimes_k \wedge^3 V^*\right) \longrightarrow$$

$$S \otimes_k \left(V \otimes_k \wedge^3 V^*\right) \longrightarrow S \otimes_k \wedge^3 V^* \longrightarrow 0$$

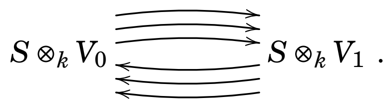
is canonically isomorphic to

$$0 \longrightarrow S \longrightarrow S \otimes_k V^* \longrightarrow S \otimes_k \bigwedge^2 V^* \longrightarrow S \otimes_k \bigwedge^3 V^* \longrightarrow 0\,.$$

Since the given representation $V = \left(\bigwedge^3 V^*\right)^{(3)}$ is just 3 copies of V_1, we obtain the McKay quiver



or the Gabriel quiver



Taking fixed points as specified in Corollary 4.18, we find MCM modules

$$M_0 \cong R \qquad \text{and} \qquad M_1 = (S \otimes_k V_1)^G .$$

Since V_1 is the negative of the trivial representation, the fixed points of $S \otimes_k V_1$, with the diagonal action, are generated over R by those elements $f \otimes \alpha$ such that $\sigma(f) = -f$. These are generated by the linear forms of S, so that M_1 is the submodule of S generated by (x, y, z). This is isomorphic to the ideal (x^2, xy, xz) of R. In particular we recover the obvious R-direct sum decomposition $S = R \oplus R(x, y, z)$ of S.

From now on, we draw the McKay quiver for a group G, and refer to it as the McKay–Gabriel quiver.

4.23 Example. Let $n = 2$ now, and write $S = k[\![u, v]\!]$. Let $r \geqslant 2$ be an integer not divisible by $\operatorname{char}(k)$, and choose $0 < q < r$ with $(q, r) = 1$. Take $G = \langle g \rangle \cong \mathbb{Z}/r\mathbb{Z}$ to be the cyclic group of order r generated by

$$g = \begin{pmatrix} \zeta_r & \\ & \zeta_r^q \end{pmatrix} \qquad \in \mathrm{GL}(2, k),$$

where ζ_r is a primitive r^{th} root of unity. Let $R = k[\![u, v]\!]^G$ be the corresponding ring of invariants, so that R is generated by the monomials $u^a v^b$ satisfying $a + bq \equiv 0 \bmod r$.

As G is Abelian, it has exactly r irreducible representations, each of which is one-dimensional. We label them $V_0,\dots,V_{r-1}$, where the generator g is sent to ζ_r^i in V_i. The given representation V of G is isomorphic to $V_1 \oplus V_q$, so that for any j we have

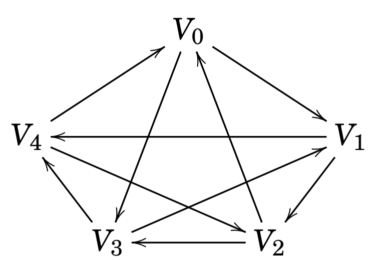
$$V \otimes_k V_j \cong V_{j+1} \oplus V_{j+q}\,,$$

where the indices are of course to be taken modulo r. The corresponding MCM R-modules are $M_j = (S \otimes_k V_j)^G$, each of which is an R-submodule of S:

$$M_j = R\left(u^a v^b \;\middle|\; a + qb \equiv -j \mod r\right).$$

The general picture is a bit chaotic, so here are a few particular examples.

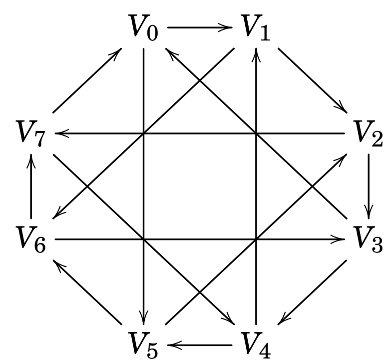
Take $r = 5$ and $q = 3$. Then $R = k[[u^5, u^2v, uv^3, v^5]]$. The McKay–Gabriel quiver takes the following shape.



The associated indecomposable MCM R-modules appearing as R-direct summands of S are the ideals

$$\begin{aligned}
M_0 &= R\\
M_1 &= R(u^4, uv, v^3) \cong (u^5, u^2v, uv^3)\\
M_2 &= R(u^3, v) \cong (u^5, u^2v)\\
M_3 &= R(u^2, uv^2, v^4) \cong (u^5, u^4v^2, u^3v^4)\\
M_4 &= R(u, v^2) \cong (u^5, u^4v^2).
\end{aligned}$$

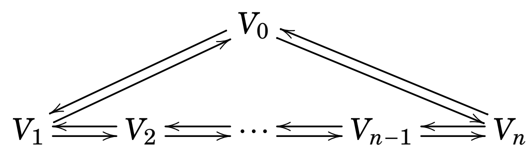
For another example, take $r = 8, q = 5$, so that $R = k[[u^8, u^3v, uv^3, v^8]]$. The McKay–Gabriel quiver looks like



and the indecomposable MCM R-modules arising as direct summands of S are

$$
\begin{aligned}
M_0 &= R \\
M_1 &= R(u^7, u^2v, v^3) \cong (u^8, u^3v, uv^3) \\
M_2 &= R(u^6, uv, v^6) \cong (u^8, u^3v, u^2v^6) \\
M_3 &= R(u^5, v) \cong (u^8, u^3v) \\
M_4 &= R(u^4, u^2v^2, v^4) \cong (u^8, u^6v^2, u^4v^4) \\
M_5 &= R(u^3, uv^2, v^7) \cong (u^8, u^6v^2, u^5v^7) \\
M_6 &= R(u^2, u^5v, v^2) \cong (u^2v^6, u^5v^7, v^8) \\
M_7 &= R(u, v^5) \cong (uv^3, v^8).
\end{aligned}
$$

Finally, take $r = n+1$ arbitrary, and $q = n$. Then $R = k[[u^{n+1}, uv, v^{n+1}]] \cong k[[x, y, z]]/(xz - y^{n+1})$ is isomorphic to an (A_n) hypersurface singularity (see the next chapter). There are $n+1$ irreducible representations $V_0, \dots, V_n$, and the McKay–Gabriel quiver looks like the one below.



The non-free indecomposable MCM R-modules take the form

$$M_j = R\left(u^a v^b \;\middle|\; b - a \equiv j \mod n+1\right)$$

for $j = 1, \ldots, n$. They have presentation matrices over $k[[u^{n+1} uv, v^{n+1}]]$

$$\varphi_j = \begin{pmatrix} (uv)^{n+1-j} & -u^{n+1} \\ -v^{n+1} & (uv)^j \end{pmatrix}$$

or over $k[[x, y, z]]/(xz - y^{n+1})$

$$\varphi_j = \begin{pmatrix} y^{n+1-j} & -x \\ -z & y^j \end{pmatrix}.$$

§4 Exercises

4.24 Exercise. In the notation of 4.2, prove the following statements.

(i) $IS \cap R = I$ for every ideal I of R.

(ii) R is Noetherian and local with maximal ideal $\mathfrak{m} = \mathfrak{n} \cap R$.

(iii) R is complete (use Cauchy sequences: if $\{r_i\}$ is Cauchy in R, then it converges in S to, say, s; apply Krull Intersection to $\sigma(s) - s$).

(iv) S is a finitely generated R-module (use the complete version of Nakayama's Lemma).

(v) S is a MCM R-module and R is a CM ring. (Hint: $x_1^N,\dots,x_n^N$ is an S-sequence contained in R for some $N \gg 0$; the colon ideal $x_{i+1}^N :_R (x_1^N,\dots,x_i^N)$ can be computed in R or S.)

By the way, the hypothesis that $|G|$ be invertible in k is not essential for item (ii) (cf. [Ben93, Theorem 1.3.1]), but is definitely needed for (v). Fogarty [Fog81] has given an example of a finite group G acting on a CM ring S such that S^G is Noetherian but not CM.

4.25 Exercise. Let A be a local ring and M, N two finitely generated A-modules. Then $\operatorname{depth}\operatorname{Hom}_A(M,N) \geqslant \min\{2, \operatorname{depth} N\}$.

5

Kleinian Singularities and Finite Representation Type

In the previous chapter we saw that when $S = k[\![x_1,\dots,x_n]\!]$ is a power series ring endowed with a linear action of a finite group G whose order is invertible in k, and $R = S^G$ is the invariant subring, then the R-direct summands of S are MCM R-modules, and are closely linked to the representation theory of G. In dimension two, we shall see in this chapter that *every* indecomposable MCM R-modules is a direct summand of S. This is due to Herzog [Her78b]. Thus in particular two-dimensional rings of invariants under finite non-modular group actions have finite CM type. In the next chapter we shall prove that in fact every two-dimensional complete normal domain containing $\mathbb{C}$ and having finite CM type arises in this way.

In the present chapter, we first recall some basic facts on reflexive modules over normal domains, then prove the theorem of Herzog mentioned above. Next we discuss the two-dimensional invariant rings $k[\![u,v]\!]^G$ that are Gorenstein; by a result of Watanabe [Wat74] these are the ones for which $G \subseteq \mathrm{SL}(2,k)$. The finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ are well-known, their classification going back to Klein, so here we call the resulting invariant rings *Kleinian singularities,* and we derive their defining equations following [Kle93]. It turns out that the resulting equations are precisely the three-variable versions of the ADE hypersurface rings from Chapter 3 §3. This section owes many debts to previous expositions, particularly [Slo83].

In the last two sections, we describe two incarnations of the so-called *McKay correspondence*: first, the identification of the McKay–Gabriel quiver of $G \subseteq \mathrm{SL}(2,\mathbb{C})$ with the corresponding ADE Coxeter-Dynkin diagram, and then the original observation of McKay that both these are the same as the desingularization graph of $\operatorname{Spec} k[[u,v]]^G$.

§1 Invariant rings in dimension two

In the last chapter we considered invariant rings $R = k[[x_1,\dots,x_n]]^G$, where G is a finite group with order invertible in k acting linearly on the power series ring $S = k[[x_1,\dots,x_n]]$. In general, the direct summands of S as an R-module are MCM modules. Here we prove that in dimension two, every indecomposable MCM module is among the R-direct summands of S.

First we recall some background on reflexive modules over normal domains. See Chapter 13 for some extensions to the non-normal case.

5.1 Remarks. Recall (from, for example, Appendix A) that for a normal domain R, if a finitely generated R-module M is MCM then it is *reflexive,* that is the natural map

$$\sigma_M \colon M \longrightarrow M^{**} = \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R),$$

defined by $\sigma_M(m)(f) = f(m)$, is an isomorphism. If moreover $\dim R = 2$, then the converse holds as well, so that M is MCM if and only if it is reflexive.

The first assertion of the next proposition is due to Herzog [Her78b], and will imply that two-dimensional rings of invariants have finite CM type.

5.2 Proposition. *Let $R \longrightarrow S$ be a module-finite extension of two-dimensional normal domains, and assume that R is a direct summand of S as an R-module. Then every finitely generated reflexive R-module is a direct summand of a finitely generated reflexive S-module. If in particular R is complete and S has finite CM type, then R has finite CM type as well.*

Proof. Let M be a reflexive R-module and set $M^* = \mathrm{Hom}_R(M,R)$. Then the split monomorphism $R \longrightarrow S$ induces a split monomorphism $M = \mathrm{Hom}_R(M^*,R) \longrightarrow \mathrm{Hom}_R(M^*,S)$. Now $\mathrm{Hom}_R(M^*,S)$ is an S-module via the action on the codomain, and Exercise 4.25 shows that it satisfies (S_2) as an R-module, hence as an S-module, so is reflexive over S.

Let $N_1,\dots,N_n$ be representatives for the isomorphism classes of indecomposable MCM S-modules. Then each N_i is a MCM R-module as well, so we write $N_i = M_{i,1} \oplus \cdots \oplus M_{i,m_i}$ for indecomposable MCM R-modules $M_{i,j}$. By the first statement of the Proposition, every indecomposable MCM R-module is a direct summand of a direct sum of copies of the N_i, so is among the $M_{i,j}$ by KRS. □

5.3 Theorem. *Let $S = k[[u,v]]$ be a power series ring in two variables over a field, G a finite subgroup of $\mathrm{GL}(2,k)$ acting linearly on S, and $R = S^G$. Assume that R is a direct summand of S as an R-module. Then every indecomposable finitely generated reflexive R-module is a direct summand of S as an R-module. In particular, R has finite CM type.*[1]

[1]Note: add a counterexample in dim 3 to the section where we do Herzog's theorem that Gorenstein rings of finite CM type are hypersurfaces: $\mathbb{C}[x,y,z]$, $g\colon x \mapsto -x, y \mapsto iy, z \mapsto iz$. Then R is Gorenstein but not a hypersurface.

Proof. Let M be an indecomposable reflexive R-module. By Proposition 5.2 M is an R-summand of a reflexive S-module N. But S is regular, so in fact N is free over S. Since R is complete, the KRS Theorem 1.8 implies that M is a direct summand of S. □

The one-one correspondences of Corollary 4.18 can thus be extended in dimension two.

5.4 Corollary. *Let k be a field, $S = k[[x, y]]$, and $G \subseteq \mathrm{GL}(2, k)$ a finite group, with $|G|$ invertible in k, acting linearly on S without pseudo-reflections. Put $R = S^G$. Then there are one-one correspondences between*

- *the indecomposable reflexive (MCM) R-modules;*
- *the indecomposable direct summands of S as an R-module;*
- *the indecomposable projective $\mathrm{End}_R(S)$-modules;*
- *the indecomposable projective $S\#G$-modules; and*
- *the irreducible kG-modules.*

Observe that while we need the assumption that $|G|$ be invertible in k for Corollary 5.4, Proposition 5.2 requires only the weaker assumption that R be a direct summand of S as an R-module. We will make use of this in Remark 5.21 below.

§2 Kleinian singularities

Having seen the privileged position that dimension two holds in the story so far, we are ready to define and study the two-dimensional hypersurface rings of finite CM type. These turn out to coincide with a class of rings ubiquitous throughout algebra and geometry, variously called *Kleinian singularities, Du Val singularities,* two-dimensional *rational double points,* and other names. Even more, they are the two-dimensional analogues of the ADE hypersurfaces seen in the previous chapter.

For historical reasons, we introduce the Kleinian singularities in a slightly opaque fashion. The rest of the section will clarify matters. For the first part of this chapter, we work over $\mathbb{C}$ for ease of exposition. We will in the end define the complete Kleinian singularities over any algebraically closed field of characteristic not 2, 3, or 5 (see Definition 5.20).

5.5 Definition. A *complete complex Kleinian singularity* (also *rational double point*, *Du Val singularity*) over k is a ring of the form $\mathbb{C}[[u,v]]^G$, where G is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$.

The reason behind the restriction to $\mathrm{SL}(2,\mathbb{C})$ rather than $\mathrm{GL}(2,\mathbb{C})$ as in the previous chapter is the fact, due to Watanabe [Wat74], that $R = S^G$ is Gorenstein when $G \subseteq \mathrm{SL}(n,k)$, and the converse holds if G is small. Thus the complete Kleinian singularities are the two-dimensional complete Gorenstein rings of invariants of finite group actions.

In order to make sense of this definition, we recall the fact that the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ are the "binary polyhedral" groups, which are double covers of the rotational symmetry groups of the Platonic solids, together with two degenerate cases.

The classification of the Platonic solids goes back to Theaetetus around 400 BCE, and is at the center of Plato's *Timaeus*; the final book of Euclid's *Elements* is devoted to their properties. According to Bourbaki [Bou02], the determination of the finite groups of rotations in $\mathbb{R}^3$ goes back to Hessel, Bravais, and Möbius in the early 19th century, though they did not yet have the language of group theory. Jordan [Jor77] was the first to explicitly classify the finite groups of rotations of $\mathbb{R}^3$.

5.6 Theorem. *The finite subgroups of the group $SO(3)$, of rotations of $\mathbb{R}^3$, are up to conjugacy the following rotational symmetry groups.*

C_{n+1}: *The cyclic group of order $n+1$ for $n \geqslant 0$, the symmetry group of a pyramid (or of a regular plane polygon).*

D_{n-2}: *The dihedral group of order $2(n-2)$ for $n \geqslant 4$, the symmetry group of a beach ball ("hosohedron").*

T: *The symmetry group of a tetrahedron, which is isomorphic to the alternating group A_4 of order 12.*

O: *the symmetry group of the octahedron, which is isomorphic to the symmetric group S_4 of order 24.*

I: *The symmetry group of the icosahedron, which is isomorphic to the alternating group A_5 of order 60.*

In order to leverage this classification into a description of the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$, we recall some basics of classical group theory. Recall first that the *unitary group* $\mathrm{U}(n)$ is the subgroup of $\mathrm{GL}(n,\mathbb{C})$ consisting of unitary transformations, i.e. those preserving the standard Hermitian dot product on $\mathbb{C}^n$. The *special unitary group* $\mathrm{SU}(n)$ is $\mathrm{SL}(n,\mathbb{C}) \cap \mathrm{U}(n)$. We first observe that to classify the finite subgroups of $\mathrm{SL}(n,\mathbb{C})$, it suffices to classify those of $\mathrm{SU}(n)$.

5.7 Lemma. *Every finite subgroup of* $\mathrm{GL}(n,\mathbb{C})$ *(resp.,* $\mathrm{SL}(n,\mathbb{C})$*) is conjugate to a subgroup of* $\mathrm{U}(n)$ *(resp.,* $\mathrm{SU}(n)$*).*

Proof. Let G be a finite subgroup of $\mathrm{GL}(n,\mathbb{C})$. Denote the usual Hermitian inner product on $\mathbb{C}^n$ by $\langle\ ,\ \rangle$. It suffices to define a new inner product $\{\ ,\ \}$ on $\mathbb{C}^n$ such that $\{\sigma u,\ \sigma v\} = \{u,\ v\}$ for every $\sigma \in G$ and $u,\ v \in \mathbb{C}^n$. Indeed, if we find such an inner product, let $\mathscr{B}$ be an orthonormal basis for $\{\ ,\ \}$, and let $\rho\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the change-of-basis operator taking $\mathscr{B}$ to the standard basis. Then $\rho G \rho^{-1} \subseteq \mathrm{U}(n)$, as

$$\begin{aligned}\left\langle \rho\sigma\rho^{-1}u,\ \rho\sigma\rho^{-1}v\right\rangle &= \left\{\sigma\rho^{-1}u,\ \sigma\rho^{-1}v\right\}\\ &= \left\{\rho^{-1}u,\ \rho^{-1}v\right\}\\ &= \langle u,\ v\rangle\end{aligned}$$

for every $\sigma \in G$ and $u, v \in \mathbb{C}^n$. Define the desired new product by

$$\{u,v\} = \frac{1}{|G|}\sum_{\sigma\in G}\langle\sigma(u),\ \sigma(v)\rangle\ .$$

Then it is easy to check that $\{\ ,\ \}$ is again an inner product on $\mathbb{C}^n$, and that $\{\sigma u,\ \sigma v\} = \{u,\ v\}$ for every $\sigma,\ u,\ v$. □

The special unitary group $\mathrm{SU}(2)$ acts on the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ by fractional linear transformations (Möbius transformations):

$$\begin{pmatrix}\alpha & -\beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix}[z : w] = \left[\alpha z - \beta w : \overline{\beta} z + \overline{\alpha} w\right]\ .$$

Since the matrices $\pm I$ act trivially, the action factors through $\mathrm{PSU}(2) = \mathrm{SU}(2)/\{\pm I\}$. We claim now that $\mathrm{PSU}(2) \cong \mathrm{SO}(3)$, the group of symmetries of the 2-sphere S^2. Position S^2 with its south pole at the origin, and consider the stereographic projection onto the equatorial plane, which we identify with $\mathbb{C}$. Extend this to an isomorphism $S^2 \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ by sending the north pole to the point at infinity. This isomorphism identifies the conformal transformations of $\mathbb{P}^1_{\mathbb{C}}$ with the rotations of the sphere, and gives a double cover of SO(3).

5.8 Proposition. *There is a surjective group homomorphism* $\pi\colon \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ *with kernel* $\{\pm I\}$.

5.9 Lemma. *The only element of order* 2 *in* SU(2) *is* $-I$.

Proof. This is a direct calculation using the general form $\left(\begin{smallmatrix}\alpha & -\beta \\ \beta & \overline{\alpha}\end{smallmatrix}\right)$ of an arbitrary element of SU(2). □

5.10 Lemma. *Let* Γ *be a finite subgroup of* SU(2). *Then either* Γ *is cyclic of odd order, or* $|\Gamma|$ *is even and* $\Gamma = \pi^{-1}(\pi(\Gamma))$ *is the preimage of a finite subgroup* G *of* SO(3).

Proof. If Γ has odd order, then $-I \notin \Gamma$, so $\Gamma \cap \ker \pi = \{I\}$, and the restriction of π to Γ is an isomorphism of Γ onto its image. By the classification of finite subgroups of SO(3), we see that the only ones of odd order are the cyclic groups C_{n+1} with $n+1$ odd. If $|\Gamma|$ is even, then by Cauchy's Theorem there is an element of order 2 in Γ, which must be $-I$. Thus $\ker \pi \subseteq \Gamma$ and $\Gamma = \pi^{-1}(\pi(\Gamma))$. □

5.11 Theorem. *The finite non-trivial subgroups of* $\mathrm{SL}(2,\mathbb{C})$*, up to conjugacy, are the following groups, called* binary polyhedral groups. *Let* ζ_r *denote a primitive* r^{th} *root of unity in* $\mathbb{C}$.

$\mathscr{C}_m$: *The cyclic group of order m for $m \geqslant 2$, generated by*

$$\begin{pmatrix} \zeta_m & \\ & \zeta_m^{-1} \end{pmatrix}.$$

$\mathscr{D}_m$: *The binary dihedral group of order $4m$ for $m \geqslant 1$, generated by $\mathscr{C}_{2m}$ and*

$$\begin{pmatrix} & i \\ i & \end{pmatrix}.$$

$\mathscr{T}$: *The binary tetrahedral group of order 24, generated by $\mathscr{D}_2$ and*

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{pmatrix}.$$

$\mathscr{O}$: *The binary octahedral group of order 48, generated by $\mathscr{T}$ and*

$$\begin{pmatrix} \zeta_8^3 & \\ & \zeta_8^5 \end{pmatrix}.$$

$\mathscr{I}$: *The binary icosahedral group of order 120, generated by*

$$\frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^4-\zeta_5 & \zeta_5^2-\zeta_5^3 \\ \zeta_5^2-\zeta_5^3 & \zeta_5-\zeta_5^4 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^2-\zeta_5^4 & \zeta_5^4-1 \\ 1-\zeta_5 & \zeta_5^3-\zeta_5 \end{pmatrix}.$$

5.12 Theorem. *The complete complex Kleinian singularities are the rings of invariants of the groups above acting linearly on the power series ring $S = \mathbb{C}[[u,v]]$. We name them as follows:*

Singularity Name	*Group Name*
A_n	$\mathscr{C}_{n+1}$, *cyclic* ($n \geqslant 1$)
D_n	$\mathscr{D}_{n-2}$, *binary dihedral* ($n \geqslant 4$)
E_6	$\mathscr{T}$, *binary tetrahedral*
E_7	$\mathscr{O}$, *binary octahedral*
E_8	$\mathscr{I}$, *binary icosahedral*

At this point the naming system is utterly mysterious, but we continue anyway.

It is a classical fact from invariant theory that the Kleinian singularities "embed in codimension one," that is, are isomorphic to hypersurface rings.[2] We can make this explicit by writing down a set of generating invariants for each of the binary polyhedral groups. These calculations go back to Klein [Kle93], and are also found in Du Val's book [DV64]; for a more modern treatment see [Lam86]. We like the concreteness of having actual invariants in hand, so we present them here. The details of the derivations are quite involved, so we only sketch them.

5.13 (A_n)**.** In this case, the only monomials fixed by the generator $(u,v) \longmapsto (\zeta_{n+1}u, \zeta_{n+1}^{-1}v)$ are uv, u^{n+1}, and v^{n+1}. Thus we set

$$X_{\mathscr{C}}(u,v) = u^{n+1} + v^{n+1} \qquad Y_{\mathscr{C}}(u,v) = uv, \qquad \text{and} \qquad Z_{\mathscr{C}}(u,v) = u^{n+1} - v^{n+1}.$$

[2]Abstractly, we can see this from the connection with Platonic solids as follows [McK01, Dic59]: drawing a sphere around the platonic solid, we project from the north pole to the equatorial plane, which we interpret as $\mathbb{C}$. Thus the projection of each vertex v gives a complex number z_v, and we form the homogeneous polynomial $V(x,y) = \prod_v (x - z_v y)$. Similarly, the center of each edge e gives a complex number z_e, and the center of each face f a corresponding z_f, which we compile into the polynomials $E(x,y) = \prod_e (x - z_e y)$ and $F(x,y) = \prod_f (x - z_f y)$. These are three functions in two variables, and so there must be a relation $f(V,E,F) = 0$.

These generate all the invariants, and satisfy the relation

$$Z_{\mathscr{C}}^2 = X_{\mathscr{C}}^2 - 4Y_{\mathscr{C}}^{n+1}.$$

5.14 (D_n)**.** The subgroup $\mathscr{C}_{2(n-2)}$ of $\mathscr{D}_{n-2}$ has invariants $a = u^{2(n-2)} + v^{2(n-2)}$, $b = uv$, and $c = u^{2(n-2)} - v^{2(n-2)}$ as in the case above. The additional generator $(u,v) \mapsto (iv, iu)$ changes the sign of b, multiplies a by $(-1)^n$, and sends c to $-(-1)^n c$. Now we have two cases to consider depending on the parity of n. If n is even, then c, a^2, ab, and b^2 are all fixed, but we can throw out b^2 since $b^2 = c^2 - 4(a^2)^{n-2}$. In the other case, when n is odd, similar considerations imply that the invariants are generated by b, a^2, and ac. Thus in this case we set

$$X_{\mathscr{D}}(u,v) = u^{2(n-2)} + (-1)^n v^{2(n-2)}, \qquad Y_{\mathscr{D}}(u,v) = u^2v^2$$
$$Z_{\mathscr{D}}(u,v) = uv\left(u^{2(n-2)} - (-1)^n v^{2(n-2)}\right).$$

For these generating invariants we have the relation

$$Z_{\mathscr{D}}^2 = Y_{\mathscr{D}} X_{\mathscr{D}}^2 + 4(-Y_{\mathscr{D}})^{n-1}.$$

5.15 (E_6)**.** The invariants (D_4) of the subgroup $\mathscr{D}_2$ are

$$u^4 + v^4, \qquad u^2v^2, \quad \text{and} \quad uv\left(u^4 - v^4\right).$$

The third of these is invariant under the whole group $\mathscr{T}$, so we set

$$Y_{\mathscr{T}}(u,v) = uv\left(u^4 - v^4\right).$$

Searching for an invariant (or coinvariant) of the form $P(u,v) = X_{\mathscr{D}} + tY_{\mathscr{D}} = u^4 + tu^2v^2 + v^4$, we find that if $t = \sqrt{-12}$, and we set

$$P(u,v) = u^4 + \sqrt{-12}\, u^2v^2 + v^4 \qquad \text{and} \qquad \overline{P}(u,v) = u^4 - \sqrt{-12}\, u^2v^2 + v^4,$$

then

$$X_{\mathscr{T}}(u,v) = P(u,v)\,\overline{P}(u,v) = u^8 + 14u^4v^4 + v^8$$

is invariant.

Furthermore, $\left[\frac{1}{4}(t-2)\right]^3 = 1$, so that every linear combination of P^3 and $\overline{P}^3$ is invariant, such as

$$\begin{aligned} Z_{\mathscr{T}}(u,v) &= \frac{1}{2}\left[P^3 + \overline{P}^3\right] \\ &= u^{12} - 33u^8v^4 - 33u^4v^8 + v^{12}. \end{aligned}$$

These three invariants generate all others, and satisfy the relation

$$Z_{\mathscr{T}}^2 = X_{\mathscr{T}}^3 + 108Y_{\mathscr{T}}^4.$$

5.16 (E_7)**.** Begin with the above invariants for $\mathscr{T}$. The additional generator for $\mathscr{O}$ leaves $X_{\mathscr{T}}$ fixed but changes the signs of $Y_{\mathscr{T}}$ and $Z_{\mathscr{T}}$. We therefore obtain generating invariants

$$\begin{aligned} X_{\mathscr{O}}(u,v) &= Y_{\mathscr{T}}(u,v)^2 = \left(u^5v - uv^5\right)^2 \\ Y_{\mathscr{O}}(u,v) &= X_{\mathscr{T}}(u,v) = u^8 + 14v^4v^4 + v^8 \\ Z_{\mathscr{O}}(u,v) &= Y_{\mathscr{T}}(u,v)Z_{\mathscr{T}}(u,v) = uv\left(u^4 - v^4)(u^{12} - 33u^8v^4 - 33u^4v^8 + v^{12}\right) \end{aligned}$$

(of degrees 8, 12, and 18, respectively). These satisfy

$$Z_{\mathscr{O}}^2 = -X_{\mathscr{O}}\left(108X_{\mathscr{O}}^2 - Y_{\mathscr{O}}^3\right).$$

5.17 (E_8)**.** From the geometry of the 12 vertices of the icosahedron, Klein derives an invariant of degree 12:

$$\begin{aligned} Y_{\mathscr{I}}(u,v) &= uv(u^5 + \varphi^5v^5)(u^5 - \varphi^{-5}v^5) \\ &= uv(u^{10} + 11u^5v^5 + v^{10}), \end{aligned}$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The Hessian of this form is also invariant, and takes the form $-121X_{\mathscr{I}}(u,v)$, where

$$X_{\mathscr{I}}(u,v) = \left| \begin{pmatrix} \partial^2/\partial u^2 & \partial^2/\partial v\partial u \\ \partial^2/\partial u\partial v & \partial^2/\partial v^2 \end{pmatrix} \right|$$
$$= \left(u^{20}+v^{20}\right) - 228\left(u^{15}v^5 - u^5v^{15}\right) + 494u^{10}v^{10}.$$

The Jacobian of these two forms (i.e. the determinant of the 2×2 matrix of partial derivatives) is invariant as well:

$$Z_{\mathscr{I}}(u,v) = \left(u^{30}+v^{30}\right) + 522\left(u^{25}v^5 - u^5v^{25}\right) - 10005\left(u^{20}v^{10} + u^{10}v^{20}\right).$$

Now one checks that[3]

$$Z_{\mathscr{I}}^2 = X_{\mathscr{I}}^3 + 1728Y_{\mathscr{I}}^5.$$

It's interesting to note that in each case above, we have $\deg X \cdot \deg Y = 2\,|G|$, namely $2(n+1), 8(n-2), 48, 96, 240$.

Adjusting the polynomials by certain n^{th} roots (of integers at most 5), one obtains the following normal forms for the Kleinian singularities

5.18 Theorem. *The complete complex Kleinian singularities are the hypersurface rings defined by the following polynomials in* $\mathbb{C}[[x,y,z]]$.

(A_n): $x^2 + y^{n+1} + z^2$, $n \geqslant 1$

(D_n): $x^2y + y^{n-1} + z^2$, $n \geqslant 4$

(E_6): $x^3 + y^4 + z^2$

(E_7): $x^3 + xy^3 + z^2$

[3]tempting one to call E_8 the *great gross singularity* ($1728 = 12 \times 144$, a dozen gross, aka a great gross).

(E_8): $x^3+y^5+z^2$

We summarize the information we have on the Kleinian singularities so far in Table 5.1.

Table 5.1: Complete Kleinian Singularities

Name	$f(x,y,z)$	G	$\|G\|$	(p,q,r)
(A_n), $n \geqslant 1$	$x^2+y^{n+1}+z^2$	$\mathscr{C}_{n+1}$, cyclic	$n+1$	$(1,1,n)$
(D_n), $n \geqslant 4$	$x^2y+y^{n-1}+z^2$	$\mathscr{D}_{n-2}$, b. dihedral	$4(n-2)$	$(2,2,n-2)$
(E_6)	$x^3+y^4+z^2$	$\mathscr{T}$, b. tetrahedral	24	$(2,3,3)$
(E_7)	$x^3+xy^3+z^2$	$\mathscr{O}$, b. octahedral	48	$(2,3,4)$
(E_8)	$x^3+y^5+z^2$	$\mathscr{I}$, b. icosahedral	120	$(2,3,5)$

5.19 Remark. Now we relax our requirement that we work over $\mathbb{C}$. Assume from now on only that k is an algebraically closed field of characteristic different from 2, 3, and 5.

With this restriction on the characteristic, the groups defined by generators in Theorem 5.11 exist equally well in $\mathrm{SL}(2,k)$, with two exceptions: $\mathscr{C}_n$ and $\mathscr{D}_n$ are not defined if $\operatorname{char} k$ divides n. We therefore use the generating invariants X, Y, and Z listed in 5.13 and 5.14 to determine the (A_{n-1}) and (D_{n+2}) singularities in positive characteristic. The derivation of the normal forms listed in Theorem 5.18 involves only taking roots of or inverting integers a for $a \leqslant 5$, so are equally valid for $\operatorname{char} k \neq 2,\ 3,\ 5$.

5.20 Definition. Let k be an algebraically closed field of characteristic not equal to 2, 3, or 5. The *complete Kleinian singularities over k* are the hypersurface rings $k[\![x,y,z]\!]/(f)$, where f is one of the polynomials listed in Theorem 5.18.

5.21 Remark. There is one further technicality to address. In the cases $\mathscr{C}_n$ and $\mathscr{D}_n$ where n is divisible by the characteristic of k, we lose the ability to define the Reynolds operator. However, in each case we can verify that the Kleinian singularity is a direct summand of the regular ring $k[\![u,v]\!]$ by using the generating invariants X, Y, and Z.

The case (A_{n-1}) was mentioned in passing already in Example 4.23. Set $R = k[\![u^n, uv, v^n]\!]$. Then $k[\![u,v]\!]$ is isomorphic as an R-module to $\bigoplus_{j=0}^{n-1} M_j$, where M_j is the R-span of the monomials $u^a v^b$ such that $b - a \equiv j \bmod n$. In particular, R is a direct summand of $k[\![u,v]\!]$ in any characteristic.

For the case (D_{n+2}), we have $R = k[\![u^{2n}+v^{2n}, u^2v^2, uv\left(u^{2n}-v^{2n}\right)]\!]$. Then R is a direct summand of $A = k[\![u^{2n}, uv, v^{2n}]\!]$: observe that $A = R \oplus R\left(uv, u^{2n}-v^{2n}\right)$ and that the second summand is generated by elements negated by $\tau\colon (u,v) \mapsto (v,-u)$. As A is an (A_{2n-1}) singularity, it is a direct summand of $k[\![u,v]\!]$ by the previous case.

Combined with Herzog's theorem 5.3, these observations prove the following theorem.

5.22 Theorem. *Let k be an algebraically closed field of characteristic not equal to 2, 3, or 5, and let R be a complete Kleinian singularity over k. Then R has finite CM type.*

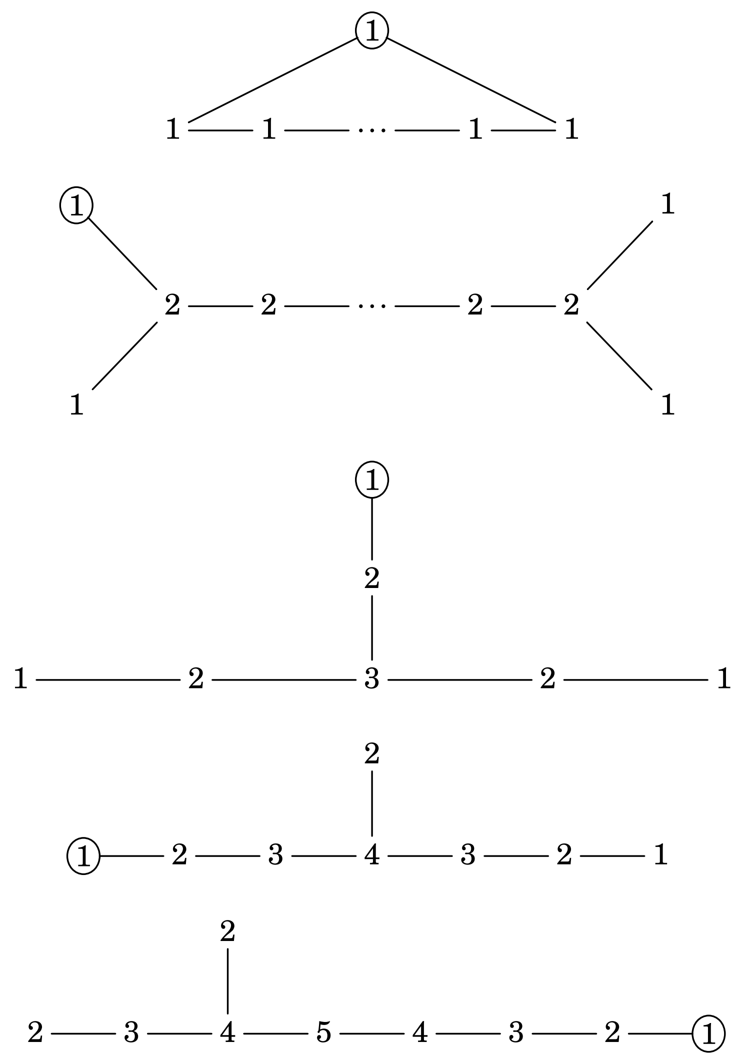
§3 McKay–Gabriel quivers of the Kleinian singularities

In this section we compute the McKay–Gabriel quivers (defined in Chapter 4) for the complete complex Kleinian singularities. We will recover McKay's observation that the underlying graphs of the quivers are exactly the *extended* (also *affine*, or *Euclidean*) *Coxeter–Dynkin diagrams* $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, corresponding to the name of the singularity from Table 5.1.

For background on the Coxeter–Dynkin diagrams A_n, D_n, E_6, E_7, E_8, and their extended counterparts $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, we recommend I. Reiten's survey article in the Notices [Rei97]. They have their vertices in far too many pies for us to enumerate. Beyond the connections we will make explicitly in this and the next section, we will content ourselves with the following brief description. The extended ADE diagrams are the finite connected graphs with no loops (a loop is a single edge with both ends at the same vertex) bearing an additive function, i.e. a function f from the vertices $\{1,\dots,n\}$ to $\mathbb{N}$ satisfying $2f(i) = \sum_j f(j)$ for every i, where the sum is taken over all neighbors j of i. Similarly, the (non-extended) ADE diagrams are the graphs bearing a sub-additive but not additive function, that is, one satisfying $2f(i) \geqslant \sum_j f(j)$ for each i, with strict inequality for at least one i. The non-extended diagrams are obtained by removing a single distinguished vertex and its incident edges from the extended ADE diagrams.

They're all listed in Table 5.2, with their (sub-)additive functions labeling the vertices. The distinguished vertex to be removed in obtaining the ordinary diagrams from the extended ones is circled. We shall see that, furthermore, the ranks of the irreducible representations (that is, indecomposable MCM modules) attached to each vertex of the quiver gives the (sub)additive function on the diagram.

Table 5.2: ADE and Extended ADE Diagrams



(A_n) $(\widetilde{A}_n)$

(D_n) $(\widetilde{D}_n)$

(E_6) $(\widetilde{E}_6)$

(E_7) $(\widetilde{E}_7)$

(E_8) $(\widetilde{E}_8)$

Recall from Definition 4.19 that the vertices of the McKay–Gabriel quiver of a two-dimensional representation $G \hookrightarrow \mathrm{GL}(V)$ are the irreducible representations $V_0, \dots, V_d$ of the group G, with an arrow $V_i \longrightarrow V_j$ for each copy of V_i in the direct-sum decomposition of $V \otimes_k V_j$. The number of arrows $V_i \longrightarrow V_j$ will (temporarily) be denoted m_{ij}. Recall that when k is algebraically closed

$$m_{ij} = \langle \chi_i, \chi\chi_j \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_i(\sigma)\chi(\sigma^{-1})\chi_j(\sigma^{-1}),$$

where $\chi, \chi_0, \dots, \chi_d$ are the characters of $V, V_0, \dots, V_d$.

5.23 Lemma. *Let G be a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *other than the two-element cyclic group. Then $m_{ij} \in \{0,1\}$ and $m_{ij} = m_{ji}$ for all $i, j = 1, \dots, d$. In other words, the arrows in the McKay–Gabriel quiver appear in opposed pairs.*

Proof. Let G be one of the subgroups of $\mathrm{SL}(2,\mathbb{C})$ listed in Theorem 5.11; in particular, the given two-dimensional representation V is defined by the matrices listed there. By Schur's Lemma and the Hom-tensor adjointness, we have

$$\begin{aligned} m_{ij} &= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V \otimes_{\mathbb{C}G} V_j, V_i) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V_j, \mathrm{Hom}_{\mathbb{C}G}(V, V_j)). \end{aligned}$$

The inner Hom has dimension equal to the number of copies of V_i appearing in the irreducible decomposition of V. These irreducible decompositions are easily read off from the listed matrices; the only one consisting of two copies of a single irreducible is (A_1), which corresponds to the two-element cyclic subgroup $\mathscr{C}_2$. Thus $\mathrm{Hom}_{\mathbb{C}G}(V_i, V)$ has dimension at most 1 for all i, and so $m_{ij} \leqslant 1$ for all i, j.

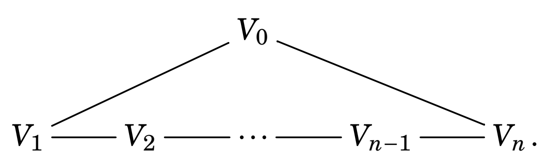
Since the trace of a matrix in SL(2, $\mathbb{C}$) is the same as that of its inverse, the given representation V satisfies $\chi(\sigma^{-1}) = \chi(\sigma)$ for every σ. Thus

$$m_{ij} = \langle \chi_i, \chi\chi_j \rangle = \langle \chi_i\chi, \chi_j \rangle = m_{ji}$$

for every i and j. □

In displaying the McKay–Gabriel quivers for the Kleinian singularities, we replace each opposed pair of arrows by a simple edge. This has the effect, thanks to Lemma 5.23, of reducing the quiver to a simple graph with no multiple edges.

5.24 (A_n)**.** We have already calculated the McKay–Gabriel quiver for the (A_n) singularities $xz - y^{n+1}$, for $n \geqslant 1$, in Example 4.23. Replacing the pairs of arrows there by single edges, we obtain



5.25 (D_n)**.** The binary dihedral group $\mathscr{D}_{n-2}$ is generated by two elements

$$\alpha = \begin{pmatrix} \zeta_{2(n-2)} & \\ & \zeta_{2(n-2)}^{-1} \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} & i \\ i & \end{pmatrix}$$

satisfying the relations

$$\alpha^{n-2} = \beta^2 = (\alpha\beta)^2, \qquad \text{and} \qquad \beta^4 = 1\,.$$

There are four natural one-dimensional representations as follows:

$$
\begin{array}{lclcl}
V_0 & : & \alpha \mapsto 1, & & \beta \mapsto 1\,; \\
V_1 & : & \alpha \mapsto 1, & & \beta \mapsto -1\,; \\
V_{n-1} & : & \alpha \mapsto -1, & & \beta \mapsto i\,; \\
V_n & : & \alpha \mapsto -1, & & \beta \mapsto -i\,.
\end{array}
$$

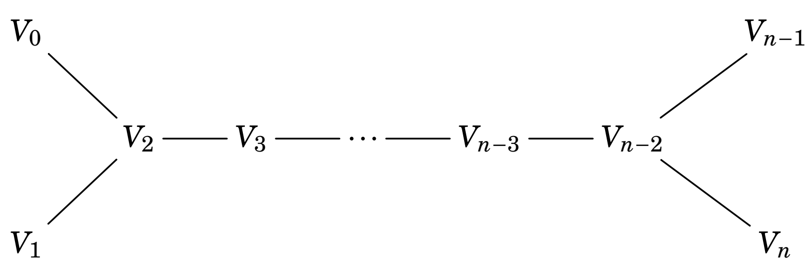
Furthermore, there is for each $j = 2, \dots, n-2$ an irreducible two-dimensional representation V_j given by

$$
a \mapsto \begin{pmatrix} \zeta_{2(n-2)}^{j-1} & \\ & \zeta_{2(n-2)}^{-j+1} \end{pmatrix} \qquad \text{and} \qquad b \mapsto \begin{pmatrix} & i^{j-1} \\ i^{j-1} & \end{pmatrix}.
$$

In particular, the given representation V is isomorphic to V_2. It's easy to compute now that

$$
V \otimes_k V_j \cong V_{j+1} \oplus V_{j-1}
$$

for $2 \leqslant j \leqslant n-2$, leading to the McKay–Gabriel quiver for the (D_n) singularity.



For the remaining examples, we will take the character table of G as given (see, for example, [Hum94], [IN99], or [GAP08]). From these data, we will be able to calculate the McKay–Gabriel quiver, since the character of a tensor product is the product of the characters and the irreducible representations are uniquely determined up to equivalence by their characters.

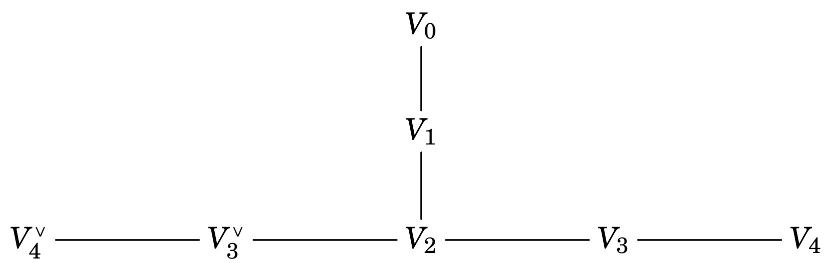
5.26 (E_6)**.** The given presentation of $\mathscr{T}$ is defined by the generators

$$\alpha = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad \beta = \begin{pmatrix} & i \\ i & \end{pmatrix}, \quad \text{and} \quad \gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{pmatrix}.$$

The character table has the following form.

representative	I	$-I$	β	γ	γ^2	γ^4	γ^5
|class|	1	1	6	4	4	4	4
order	1	2	4	6	3	3	6
V_0	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	-1	1
V_2	3	3	-1	0	0	0	0
V_3	2	-2	0	ζ_3	$-\zeta_3$	$-\zeta_3^2$	ζ_3^2
$V_3^\vee$	2	-2	0	ζ_3^2	$-\zeta_3^2$	$-\zeta_3$	ζ_3
V_4	1	1	1	ζ_3	ζ_3	ζ_3^2	ζ_3^2
$V_4^\vee$	1	1	1	ζ_3^2	ζ_3^2	ζ_3	ζ_3

Here $V = V_1$ is the given two-dimensional representation. Now one verifies for example that the character of $V_1 \otimes_k V_4$, that is the element-wise product of the second and sixth rows of the table, is equal to the character of V_3. Hence $V_1 \otimes_k V_4 \cong V_3$ and the McKay–Gabriel quiver contains an edge connecting V_3 and V_4. Similarly, $V_1 \otimes_k V_2 \cong V_1 \oplus V_3 \oplus V_3^\vee$, so V_2 is a vertex of degree three. Continuing in this way gives the following McKay–Gabriel quiver.



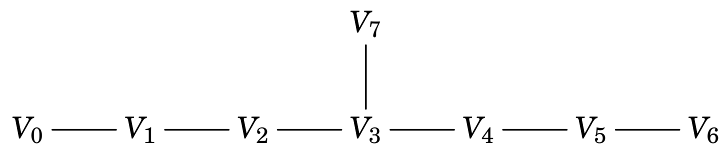
5.27. (E_7) The binary octahedral group $\mathscr{O}$ is generated by α, β, and γ from the previous case together with

$$\delta = \begin{pmatrix} \zeta_8^3 & \\ & \zeta_8^5 \end{pmatrix}.$$

This time the character table is as follows.

representative	I	$-I$	β	γ	γ^2	δ	$\beta\delta$	δ^3
|class|	1	1	6	8	8	6	12	6
order	1	2	4	6	3	8	4	8
V_0	1	1	1	1	1	1	1	1
V_1	2	2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$
V_2	3	3	-1	0	0	1	-1	1
V_3	4	-4	0	-1	1	0	0	0
V_4	3	3	-1	0	0	-1	1	-1
V_5	2	-2	0	1	-1	$\sqrt{2}$	0	$-\sqrt{2}$
V_6	1	1	1	1	1	-1	-1	-1
V_7	2	2	2	-1	-1	0	0	0

Again $V = V_1$ is the given two-dimensional representation. Now we compute the McKay–Gabriel quiver to be the following.



5.28. (E_8) Finally, we consider the binary icosahedral group, generated by

$$\sigma = \frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix} \qquad \text{and} \qquad \tau = \frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^2 - \zeta_5^4 & \zeta_5^4 - 1 \\ 1 - \zeta_5 & \zeta_5^3 - \zeta_5 \end{pmatrix}.$$

Set $\varphi^+ = (1+\sqrt{5})/2$, the golden ratio, and $\varphi^- = (1-\sqrt{5})/2$. The character table for $\mathscr{I}$ is below.

representative	I	$-I$	σ	τ	τ^2	$\sigma\tau$	$(\sigma\tau)^2$	$(\sigma\tau)^3$	$(\sigma\tau)^4$
|class|	1	1	30	20	20	12	12	12	12
order	1	2	4	6	3	10	5	10	5
V_0	1	1	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	φ^+	$-\varphi^-$	φ^-	$-\varphi^+$
V_2	3	3	-1	0	0	φ^+	φ^-	φ^-	φ^+
V_3	4	-4	0	-1	1	1	-1	1	-1
V_4	5	5	1	-1	-1	0	0	0	0
V_5	6	-6	0	0	0	-1	0	-1	0
V_6	4	4	0	1	1	-1	-1	-1	-1
V_7	2	-2	0	1	-1	φ^-	$-\varphi^+$	φ^+	$-\varphi^-$
V_8	3	3	-1	0	0	φ^-	φ^+	φ^+	φ^-

We find that the McKay–Gabriel quiver is the extended Coxeter–Dynkin diagram $\widetilde{E}_8$.

$$\begin{array}{ccccccccccccccc} & & & & V_8 & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ V_7 & — & V_6 & — & V_5 & — & V_4 & — & V_3 & — & V_2 & — & V_1 & — & V_0 \end{array}$$

We have verified the first sentence of the following result, and the rest is straightforward to check from the definitions.

5.29 Proposition. *The McKay–Gabriel quivers of the finite subgroups of* $\mathrm{SL}(2,\mathbb{C})$ *are the extended Coxeter-Dynkin diagrams. The dimensions of the irreducible representations appearing in the McKay–Gabriel quiver define an additive function on the quiver: Twice the dimension at a given vertex is equal to the sum of the dimensions at the neighboring vertices. In accordance with Corollary 4.18, these dimensions coincide with the ranks of the indecomposable MCM modules over the Kleinian singularity.*

§4 Geometric McKay correspondence

The one-one correspondences derived in Chapter 4 in general, and in this chapter in dimension two, connect the representation theories of a finite subgroup of $\mathrm{SL}(2,k)$ and of its ring of invariants to the (extended) ADE Coxeter–Dynkin diagrams. These diagrams were known to be related to the geometry of the Kleinian singularities much earlier. P. Du Val's three-part 1934 paper [DV34] showed that the desingularization graphs of surfaces "not affecting the conditions of adjunction" are of ADE type; these are exactly the Kleinian singularities [Art66].

The first direct link between the representation theory of a Kleinian singularity and geometric information is due to G. Gonzalez-Sprinberg and J.-L. Verdier [GSV81]. They constructed, on a case-by-case basis, a one-one correspondence between the irreducible representations of a binary polyhedral group and the irreducible components of the exceptional fiber in a minimal resolution of singularities of the invariant ring. (See below for definitions.) At the end of this section we describe M. Artin and Verdier's direct argument linking MCM modules and exceptional components.

This section is significantly more geometric than other parts of the book; in particular, we omit many of the proofs which would take us too far afield to justify. Most unexplained terminology can be found in [Har77].

Throughout the section, $(R,\mathfrak{m},k)$ will be a two-dimensional normal local domain with algebraically closed residue field k. We do not assume $\operatorname{char} k = 0$. Let $X = \operatorname{Spec} R$, a two-dimensional affine scheme, that is, a surface. In particular, since R is normal, X is regular in codimension one, so $\mathfrak{m}$ is the unique singular point of X.

A *resolution of singularities* of X is a non-singular surface Y with a proper birational map $\pi\colon Y \longrightarrow X$ such that the restriction of π to $Y \setminus \pi^{-1}(\mathfrak{m})$ is an isomorphism. Since $\dim X = 2$, resolutions of X exist as long as R is excellent [Lip78]. The *geometric genus* $g(X)$ of X is the k-dimension of the first cohomology group $\mathrm{H}^1(Y,\mathscr{O}_Y)$. This number is finite, and is independent of the choice of a resolution Y. Again since $\dim X = 2$, there is among all resolutions of X a *minimal resolution* $\pi\colon \widetilde{X} \longrightarrow X$ such that any other resolution factors through π.

5.30 Definition. We say that X and R have (or are) *rational singularities* if $g(X) = 0$, that is, $\mathrm{H}^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = 0$.

We can rephrase this definition in a number of ways. Since $X = \operatorname{Spec} R$ is affine, the cohomology $\mathrm{H}^i(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ is isomorphic to the higher direct image $R^i\pi_*(\mathscr{O}_{\widetilde{X}})$, so R has a rational singularity if and only if $R^1\pi_*(\mathscr{O}_{\widetilde{X}}) = 0$. This is equivalent to the condition that $R^i\pi_*(\mathscr{O}_{\widetilde{X}}) = 0$ for all $i \geqslant 1$, since the fibers of a resolution π are at most one-dimensional [Har77, III.11.2]. The direct image $\pi_*\mathscr{O}_{\widetilde{X}}$ itself is easy to compute: it is a coherent sheaf of R-algebras, so $S = \Gamma(X, \pi_*\mathscr{O}_{\widetilde{X}})$ is a module-finite R-algebra. But since π is birational, S has the same quotient field as R. Thus S is an integral extension, whence equal to R by normality, and so $\pi_*\mathscr{O}_{\widetilde{X}} = \mathscr{O}_X$.

Alternatively, recall that the *arithmetic genus* of a scheme Y is defined by $p_a(Y) = \chi(\mathscr{O}_Y) - 1$, where χ is the Euler characteristic, defined by the alternating sum of the k-dimensions of the $\mathrm{H}^i(Y, \mathscr{O}_Y)$. It follows from the Leray spectral sequence, for example, that if $\pi \colon \widetilde{X} \longrightarrow X$ is a resolution of singularities, then

$$p_a(X) - p_a(\widetilde{X}) = \dim_k \mathrm{H}^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}),$$

so that X is a rational singularity if and only if the arithmetic genus of X is not changed by resolving the singularity.

For a more algebraic criterion, assume momentarily that R is a non-negatively graded ring over a field $R_0 = k$ of characteristic zero. Flenner [Fle81] and Watanabe [Wat83] independently proved that R has a rational singularity if and only if the *a-invariant* $a(R)$ is negative. In general, $a(R)$ is the largest n such that the n^{th} graded piece of the local cohomology module $\mathrm{H}_{\mathfrak{m}}^{\dim R}(R)$ is non-zero. For a two-dimensional weightedhomogeneous hypersurface singularity such as the Kleinian singularities in Theorem 5.18, the definition is particularly easy to apply:

$$a(k[x,y,z]/(f)) = \deg f - \deg x - \deg y - \deg z\,.$$

In particular, we check from Table 5.1 that the Kleinian singularities have rational singularities in characteristic zero.

More generally, any two-dimensional quotient singularity $k[u,v]^G$ or $k[\![u,v]\!]^G$, where G is a finite group with $|G|$ invertible in k, has rational singularities [Bur74, Vie77]. In fact, the restriction on $|G|$ is unnecessary for the Kleinian singularities: if S has rational singularities and R is a subring of S which is a direct summand as R-module, then R has rational singularities [Bou87]. Thus the Kleinian singularities have rational singularities in any characteristic in which they are defined.

As a final bit of motivation for the study of rational surface singularities, we point out that a normal surface $X = \operatorname{Spec} R$ is a rational singularity if and only if the divisor class group $\operatorname{Cl}(R)$ is finite, if and only if R has only finitely many rank-one MCM modules up to isomorphism [Mum61, Lip69].

Return now to our two-dimensional normal domain R, its spectrum X, and $\pi\colon \widetilde{X} \longrightarrow X$ the minimal resolution of singularities. With $0 \in X$ the unique singular point of X, set $E = \pi^{-1}(0)$, the *exceptional fiber* of π. Then E is connected by Zariski's Main Theorem [Har77, III.5.2], and is one-dimensional since π is birational. In other words, E is a union of irreducible curves on $\widetilde{X}$, so we write $E = \bigcup_{i=1}^n E_i$.

5.31 Lemma ([Bri68, Lemma 1.3]). *Let $\pi\colon \widetilde{X} \longrightarrow X$ be the minimal resolution of a rational singularity X, and let $E = \bigcup_{i=1}^n E_i$ be the exceptional* *fiber.*

(i) Each E_i is non-singular, in particular reduced, and furthermore is a rational curve, i.e. $E_i \cong \mathbb{P}^1$.

(ii) $E_i \cap E_j \cap E_k = \emptyset$ for pairwise distinct i, j, k.

(iii) $E_i \cap E_j$ is either empty or a single reduced point for $i \neq j$, that is, the E_i meet transversely if at all.

(iv) E is cycle-free.

To describe the intersection properties of the exceptional curves more precisely, recall a bit of the intersection theory of curves on non-singular surfaces. Let C and D be curves on $\widetilde{X}$ with no common component. The *intersection multiplicity of C and D at a closed point $x \in \widetilde{X}$* is the length of the quotient $\mathscr{O}_{\widetilde{X},x}/(f, g)$, where $f = 0$ and $g = 0$ are local equations of C and D at x. The *intersection number $C \cdot D$* of C and D is the sum of intersection multiplicities at all common points x. The *self-intersection C^2*, a special case, is defined to be the degree of the normal bundle to C in $\widetilde{X}$. Somewhat counter-intuitively, this can be negative; see [Har77, V.1.9.2] for an example.

The first part of the next theorem is due to Du Val [DV34] and Mumford [Mum61, Hir95a]; it immediately implies the second and third parts [Art66, Prop. 2 and Thm. 4].

5.32 Theorem. *Let $\pi \colon \widetilde{X} \longrightarrow X$ be the minimal resolution of a surface singularity (not necessarily rational) with exceptional fiber $E = \bigcup_{i=1}^{n} E_i$. Define the* intersection matrix *of X to be the symmetric matrix $E(X)_{ij} = (E_i \cdot E_j)$.*

(i) The matrix $E(X)$ is negative definite with off-diagonal entries either 0 or 1.

(ii) There exist positive divisors supported on E (that is, divisors of the form $Z = \sum_{i=1}^{n} m_i E_i$ with $m_i \geqslant 1$ for all i) such that $Z \cdot E_i \leqslant 0$ for all i.

(iii) Among all such Z as in (ii), there is a unique smallest one, which is called the fundamental divisor *of X and denoted Z_f.*

To find the fundamental divisor there is a straightforward combinatorial algorithm: begin with $m_i = 1$ for all i, so that $Z_1 = \sum_i E_i$. If $Z_1 \cdot E_i \leqslant 0$ for each i, we set $Z_f = Z_1$ and stop; otherwise $Z_1 \cdot E_j > 0$ for some j. In that case, we put $Z_2 = Z_1 + E_j$ and continue. The process terminates by the negative definiteness of the matrix $E(X)$. See below for two examples.

For a rational singularity, we can identify Z_f more precisely, and this will allow us to identify the Gorenstein rational singularities.

5.33 Proposition ([Art66, Thm 4]). *The fundamental divisor Z_f of a surface X with a rational singularity satisfies*

$$\left(\mathscr{O}_{\tilde{X}} \otimes_{\mathscr{O}_X} \mathfrak{m}\right)/\textit{torsion} = \mathscr{O}_{\tilde{X}}(-Z_f)\,.$$

In particular, we have formulas for the multiplicity and the embedding dimension $\mu_R(\mathfrak{m})$ of R:

$$\begin{aligned} \mathrm{e}(R) &= -Z_f^2 \\ \mathrm{embdim}(R) &= -Z_f^2 + 1 \end{aligned}$$

5.34 Corollary. *A two-dimensional normal local domain R with a rational singularity has* minimal multiplicity *[Abh67]:*

$$\mathrm{e}(R) = \mu_R(\mathfrak{m}) - \dim R + 1\,.$$

5.35 Corollary. *Let $(R,\mathfrak{m})$ be a two-dimensional normal local domain, and assume that R is Gorenstein. If R is a rational singularity, then R is a hypersurface ring of multiplicity two (a "double point").*

Proof. By the Proposition, we have $\mathrm{e}(R) = -Z_f^2$ and $\mu_R(\mathfrak{m}) = -Z_f^2 + 1$. Cut down by a regular sequence of length two in $\mathfrak{m} \setminus \mathfrak{m}^2$ to arrive at an Artinian local ring $\overline{R}$ with $\mathrm{e}(R) = \ell(\overline{R})$ and $\mu_{\overline{R}}(\overline{\mathfrak{m}}) = \mu_R(\mathfrak{m}) - 2$. These together imply that $\mu_{\overline{R}}(\overline{\mathfrak{m}}) = \ell(\overline{R}) - 1$, so the Hilbert function of $\overline{R}$ is $(1, -Z_f^2 - 1, 0, \dots)$. However, $\overline{R}$ is Gorenstein, so has socle dimension equal to 1. This forces $Z_f^2 = -2$, which gives $\mathrm{e}(R) = 2$ and $\mu_R(\mathfrak{m}) = 3$. In particular R is a hypersurface ring. □

5.36 Corollary. *Let R be a Gorenstein rational surface singularity. The self-intersection number E_i^2 of each exceptional component is -2. Equivalently the normal bundle $\mathcal{N}_{E_i/\widetilde{X}}$ is $\mathcal{O}_{E_i}(-2)$.*

Proof. This is a straightforward calculation using the adjunction formula and Riemann–Roch Theorem, see [Dur79, A3], together with $Z_f^2 = -2$. □

5.37 Remark. At this point, we can describe the connection between Gorenstein rational surface singularities and the ADE Coxeter–Dynkin diagrams. To do this, we define the *desingularization graph* of a surface X to be the dual graph of the exceptional fiber in a minimal resolution of singularities. Precisely, let $\pi\colon \widetilde{X} \longrightarrow X$ be the minimal resolution of singularities, and let

$E_1,\dots,E_n$ be the irreducible components of the exceptional fiber. Then the desingularization graph has vertices $E_1,\dots,E_n$, with an edge joining E_i to E_j for $i \neq j$ if and only if $E_i \cap E_j \neq \emptyset$.

Let $Z_f = \sum_i m_i E_i$ be the fundamental divisor of X, and define a function f from the vertices $\{E_1,\dots,E_n\}$ to $\mathbb{N}$ by $f(E_i) = m_i$. Then for $i = 1,\dots,n$ we have

$$0 \geqslant Z \cdot E_i = -2m_i + \sum_j m_j (E_i \cdot E_j) = -2m_i + \sum_j m_j ,$$

where the sum is over all $j \neq i$ such that $E_i \cap E_j \neq \emptyset$. This gives $2f(E_i) \geqslant \sum_j f(E_j)$, and the negative definiteness of the intersection matrix (Theorem 5.32) implies that f is a sub-additive, non-additive function on the graph. Thus the graph is ADE.

We illustrate the general facts described so far with two examples of resolutions of rational double points: the (A_1) and (D_4) hypersurfaces. We will also draw the desingularization graphs for these two examples.

5.38 Example. Let X be the hypersurface in $\mathbb{A}^3$ defined by the (A_1) polynomial $x^2 + y^2 + z^2$. To resolve the singularity of X at the origin, we blow up the origin in $\mathbb{A}^3$. Precisely, we set

$$\widetilde{\mathbb{A}}^3 = \left\{ ((x,y,z),(a:b:c)) \in \mathbb{A}^3 \times \mathbb{P}^2 \;\middle|\; xb = ya,\ xc = za,\ yc = zb \right\}.$$

(See [Har77] for basics on blowups.) The projection $\varphi \colon \widetilde{\mathbb{A}}^3 \longrightarrow \mathbb{A}^3$ is an isomorphism away from the origin in $\mathbb{A}^3$, while $\varphi^{-1}(0,0,0)$ is the projective plane $\mathbb{P}^2 \subseteq \widetilde{\mathbb{A}}^3$.

Let $\widetilde{X}$ be the blowup of X at the origin. That is, $\widetilde{X}$ is the Zariski closure of $\varphi^{-1}(X \setminus (0,0,0))$ in $\widetilde{\mathbb{A}}^3$. Then $\widetilde{X}$ is defined in $\widetilde{\mathbb{A}}^3$ by the vanishing of $a^2 +$

b^2+c^2. The restriction of φ gives $\pi\colon \widetilde{X} \longrightarrow X$, and the exceptional fiber E is the preimage of $(0,0,0)$ in $\widetilde{X}$. We claim that $\widetilde{X}$ is smooth, and that E is a single projective line $\mathbb{P}^1$.

The blowup $\widetilde{X}$ is covered by three affine charts U_a, U_b, U_c, defined by $a \neq 0$, $b \neq 0$, $c \neq 0$ respectively, or equivalently by $a = 1$, $b = 1$, $c = 1$. In the chart U_a, we have $y = xb$ and $z = xc$, so that the defining equation of X becomes

$$x^2 + x^2b^2 + x^2z^2 = x^2\left(1+b^2+c^2\right)$$

Above $X \setminus (0,0,0)$, we have $x \neq 0$, so the preimage of $X \setminus (0,0,0)$ is defined by $x \neq 0$ and $1+b^2+c^2 = 0$. The Zariski closure of $\varphi^{-1}(X \setminus (0,0,0))$ is thus in this chart the cylinder $1+b^2+c^2 = 0$ in $U_a \cong \mathbb{A}^3$. Since all three charts are symmetric, we conclude that $\widetilde{X}$ is smooth.

Remaining in the chart U_a, we see that the exceptional fiber E is defined in $\widetilde{X}$ by $x = 0$, so is defined in U_a by $1+b^2+c^2 = x = 0$. Again we use the symmetry of the three charts to conclude that E is smooth, and even rational, so $E \cong \mathbb{P}^1$.

Drawing the desingularization graph of X is thus quite trivial: it has a single node and no edges.

$$E$$

Observe that this is the (A_1) Coxeter–Dynkin diagram. Since $E^2 = -2$ by Corollary 5.36, we find that $Z_f = E$ is the fundamental divisor.

5.39 Example. For a slightly more sophisticated example, consider the (D_4) hypersurface $X \subseteq \mathbb{A}^3$ defined by the vanishing of $x^2y + y^3 + z^2$. Again blowing up the origin in $\mathbb{A}^3$, we obtain as before

$$\widetilde{\mathbb{A}}^3 = \left\{((x,y,z),(a:b:c)) \in \mathbb{A}^3 \times \mathbb{P}^2 \;\middle|\; xb = ya,\ xc = za,\ yc = zb\right\},$$

with projection $\varphi\colon \widetilde{\mathbb{A}}^3 \longrightarrow \mathbb{A}^3$. This time let X_1 be the Zariski closure of $\varphi^{-1}(X \setminus (0,0,0))$. In the affine chart U_a where $a = 1$, we again have $y = xb$ and $z = xc$, so the defining polynomial becomes

$$x^3b + x^3b^3 + x^2c^2 = x^2\left(x\left(b + b^3\right) + c^2\right).$$

Thus X_1 is defined by $x(b+b^3)+c^2$ in U_a, so is a *singular* surface. In fact, an easy change of variables reveals that in this chart X_1 is isomorphic to an (A_1) hypersurface singularity (in the variables $\frac{1}{2}(x+(b+b^3))$, $\frac{i}{2}(x-(b+b^3)$, and c). In particular, X_1 has three singular points, with coordinates $x = c = 0$ and $b + b^3 = 0$. In the coordinates of $\widetilde{\mathbb{A}}^3$, they are at $((0,0,0),(1:b:0))$, where $b^3 = -b$. The exceptional fiber, which we denote E_1, corresponds in this chart to $x = 0$, whence $c = 0$, so is just the b-axis.

In the other charts, we find no further singularities. On U_b, the defining polynomial is

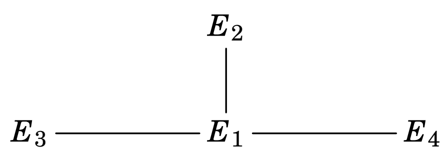
$$y^3a + y^3 + y^2c^2 = y^2\left(ya + y + c^2\right)$$

so that X_1 is defined in U_b by $ya+y+c^2 = 0$. This is also an (A_1) singularity, this time with a single singular point at $y = c = 0$. However, we've already seen this point, as its $\widetilde{\mathbb{A}}^3$ coordinates are $((0,0,0),(-1:1:0))$, which is in U_a. The exceptional fiber here is the a-axis. Finally, in the chart U_c, we find

$$z^3a^2b + z^3b^3 + z^2 = z^2\left(za^2b + zb^3 + 1\right)$$

so that X_1 is smooth in this chart and E_1 is not visible. In particular we find that $E_1 \cong \mathbb{P}^1$.

Since the first blowup X_1 is not smooth, we continue, resolving the singularities of the surface $x(b+b^3)+c^2=0$ by blowing up its three singular points. Since each singular point is locally isomorphic to an (A_1) hypersurface, we appeal to the previous example to see that the resulting surface $\widetilde{X}$ is smooth, and that each of the three new exceptional fibers E_2, E_3, E_4 intersects the original one E_1 transversely. The desingularization graph thus has the shape of the (D_4) Coxeter–Dynkin diagram:



To compute the fundamental divisor Z_f, we begin with $Z_1 = E_1+E_2+E_3+E_4$. Since $E_i^2=-2$ and $E_j \cdot E_1 = 1$ for each $j=2,3,4$, we find

$$Z_1 \cdot E_1 = -2+1+1+1 = 1 > 0\,.$$

Thus we replace Z_1 by $Z_2 = 2E_1+E_2+E_3+E_4$. Now

$$Z_2 \cdot E_1 = -4+1+1+1 \leqslant 0\,,$$

and for $j=2,3,4$ we have $Z_2 \cdot E_j = 2-2+0+0 \leqslant 0$, so that $Z_f = Z_2 = 2E_1+E_2+E_3+E_4$ is the fundamental divisor.

The calculations in the examples can be carried out for each of the Kleinian singularities in Table 5.1, and one verifies the next result, which was McKay's original observation.

5.40 Theorem (McKay). *Let G be a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *and let $R=\mathbb{C}[[u,v]]^G$ be the corresponding ring of invariants. Then the desingularization graph of $X=\operatorname{Spec} R$ is an ADE Coxeter–Dynkin diagram. In* *particular, it is equal to the McKay–Gabriel quiver of G with the vertex corresponding to the trivial representation removed. Furthermore, the coefficients of the fundamental divisor Z_f coincide with the dimensions of the corresponding irreducible representations of G, and with the ranks of the corresponding indecomposable MCM R-modules.*

We can now state the theorem of Artin and Verdier on the geometric McKay correspondence. Here is the notation in effect through the end of the section:

5.41 Notation. Let $(R,\mathfrak{m},k)$ be a complete local normal domain of dimension two, which is a rational singularity. Let $\pi\colon \widetilde{X} \longrightarrow X = \operatorname{Spec} R$ be its minimal resolution of singularities, and $E = \pi^{-1}(\mathfrak{m})$ the exceptional fiber, with irreducible components $E_1,\dots,E_n$. Let $Z_f = \sum_i m_i E_i$ be the fundamental divisor of X. We identify a reflexive R-module M with the associated coherent sheaf of $\mathscr{O}_X$-modules, and define the *strict transform* of M by

$$\widetilde{M} = (M \otimes_{\mathscr{O}_X} \mathscr{O}_{\widetilde{X}})/\text{torsion},$$

a sheaf on $\widetilde{X}$.

5.42 Theorem (Artin-Verdier)**.** *With notation as above, assume moreover that R is Gorenstein. Then there is a one-one correspondence, induced by the first Chern class $c_1(-)$, between indecomposable non-free MCM R-modules and irreducible components E_i of the exceptional fiber. Precisely: Let M be an indecomposable non-free MCM R-module, and let* $[C] = c_1(\widetilde{M}) \in \operatorname{Pic}(\widetilde{X})$. *Then there is a unique index i such that $C \cdot E_i = 1$ and $C \cdot E_j = 0$ for $i \neq j$. Furthermore, we have* $\operatorname{rank}_R(M) = C \cdot Z_f = m_i$.

The first Chern class mentioned in the Theorem is a mechanism for turning a locally free sheaf $\mathscr{E}$ into a divisor $c_1(\mathscr{E})$ in the Picard group $\operatorname{Pic}(\widetilde{X})$. In particular, $c_1(-)$ is additive on short exact sequences over $\widetilde{X}$.

The main ingredients of the proof of Theorem 5.42 are compiled in the next propositions.

5.43 Proposition. *With notation as in 5.41, $\widetilde{M}$ enjoys the following properties.*

(i) $\widetilde{M}$ is a locally free $\mathscr{O}_{\widetilde{X}}$-module, generated by its global sections.

(ii) $\Gamma(\widetilde{X},\widetilde{M}) = M$ and $\mathrm{H}^1(\widetilde{X},\widetilde{M}^*) = 0$.

(iii) There is a short exact sequence of sheaves on $\widetilde{X}$

$$0 \longrightarrow \mathscr{O}_{\widetilde{X}}^{(r)} \longrightarrow \widetilde{M} \longrightarrow \mathscr{O}_C \longrightarrow 0\,, \tag{5.43.1}$$

where $r = \operatorname{rank}_R(M)$, and C is a closed one-dimensional subscheme of $\widetilde{X}$ which meets the exceptional fiber E transversely. Furthermore, the global sections of (5.43.1) give an exact sequence of R-modules

$$0 \longrightarrow R^{(r)} \longrightarrow M \longrightarrow \Gamma(\widetilde{X},\mathscr{O}_C) \longrightarrow 0 \tag{5.43.2}$$

Observe that the class $[C]$ of the curve C in the Picard group $\operatorname{Pic}(\widetilde{X})$ is equal to the first Chern class $c_1(\widetilde{M})$ of $\widetilde{M}$, since $c_1(-)$ is additive on short exact sequences and $c_1(\mathscr{L}) = [\mathscr{L}] \in \operatorname{Pic}(\widetilde{X})$ for any line bundle $\mathscr{L}$.

5.44 Proposition. *Keep all the notation of 5.41, and assume moreover that R is Gorenstein. Fix a reflexive R-module M, and let C be the curve guaranteed by Prop. 5.43. Then*

(i) *$C \cdot Z_f \leqslant r$, with equality if and only if M has no non-trivial free direct summands.*

(ii) *If $C = C_1 \cup \cdots \cup C_s$ is the decomposition of C into irreducible components, then M can be decomposed accordingly: $M \cong M_1 \oplus \cdot \oplus M_s$, with each M_i indecomposable and $c_1(\widetilde{M_i}) = [C_i]$ for each i.*

The proofs of Propositions 5.43 and 5.44 are relatively straightforward algebraic geometry. The key observation giving the existence of the short exact sequence (5.43.1) is a general-position argument: r generically chosen sections of $\widetilde{M}$ generate a free subsheaf $\mathscr{O}_{\widetilde{X}}^{(r)}$, and one checks that the choice can be made so that the restriction of the kernel to each E_i is isomorphic to a direct sum of residue fields at points distinct from each other and from the intersections $E_i \cap E_j$. The statements in Proposition 5.44 follow from the fact that $Z_f \cdot C$ is equal to the minimal number of generators of the R-module $\Gamma(\widetilde{X}, \mathscr{O}_C)$ by Proposition 5.33, together with duality for the proper map π.

§5 Exercises

5.45 Exercise. Let R be a reduced Noetherian ring and let M be a finitely generated R-module. Set $-^* = \operatorname{Hom}_R(-, R)$.

(i) Prove that there is an exact sequence

$$0 \longrightarrow \operatorname{tor}(M) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow T \longrightarrow 0$$

where $\sigma_M : M \longrightarrow M^{**}$ is the natural biduality homomorphism $\sigma_M(m)(f) = f(m)$, and T is a torsion module.

(ii) If in addition R is CM with canonical module ω, prove that there is an exact sequence

$$0 \longrightarrow \operatorname{tor}(M) \longrightarrow M \xrightarrow{\tau_M} M^{\vee\vee} \longrightarrow T' \longrightarrow 0$$

where $-^{\vee} = \operatorname{Hom}_R(-,\omega)$ and $\tau_M \colon M \longrightarrow M^{\vee\vee}$ is another biduality map defined analogously to σ_M. Again, T' is torsion.

5.46 Exercise. Let R be a reduced Noetherian ring satisfying (S_2) and let M, N be finitely generated R-modules with N reflexive. Prove that

$$\operatorname{Hom}_R(M,N) = \operatorname{Hom}_R(M^{**},N).$$

If in addition R is CM with canonical module ω, then

$$\operatorname{Hom}_R(M,N) = \operatorname{Hom}_R(M^{\vee\vee},N).$$

(Hint: first reduce to the torsion-free case, then show that when M is torsion-free, $T_{\mathfrak{p}} = 0$ for every prime $\mathfrak{p}$ of height one, and conclude $\operatorname{Ext}^1_R(T,N) = 0$.)

5.47 Exercise. Let R be a reduced CM local ring with canonical module ω, and assume that R is Gorenstein on the punctured spectrum. Prove that $M^{**} \cong M^{\vee\vee}$ for every finitely generated R-module M.

5.48 Exercise. Let R be a reduced Noetherian ring and M, N, P finitely generated reflexive R-modules. Define the *reflexive product* of M and N by

$$M \cdot N = (M \otimes_R N)^{**}$$

Prove the following isomorphisms.

(i) $M \cdot N \cong N \cdot M$.

(ii) $\operatorname{Hom}_R(M \cdot N, P) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, P))$.

(iii) $M \cdot (N \cdot P) \cong (M \cdot N) \cdot P$.

5.49 Exercise. Let $R \longrightarrow S$ be a homomorphism of reduced Noetherian rings satisfying (S_2), and such that S is a finitely generated reflexive R-module. Let M be a finitely generated reflexive R-module. Define $S \cdot M$ to be the S-module $(S \otimes_R M)^{\dagger\dagger}$, where $-^{\dagger} = \operatorname{Hom}_S(-, S)$.

(i) Prove that if N is a reflexive S-module, then $\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_S(S \cdot M, N)$.

(ii) Conclude that we may compute $S \cdot M$ as $(S \otimes_R M)^{**}$, with the double dual occurring over R.

5.50 Exercise. In the setup of 4.2, prove that if M is a reflexive R-module such that $\operatorname{Ext}^i_R(M^*, S) = 0$ for $i = 1, \ldots, n-2$, then $M \in \operatorname{add}_R(S)$.

6

Isolated Singularities and Classification in Dimension Two

In this chapter we present a pair of celebrated theorems due originally to Auslander. The first, Theorem 6.12, states that a CM local ring of finite CM type has at most an isolated singularity. We give the simplified proof due to Huneke and Leuschke, which requires some easy general preliminaries on elements of Ext^1. The second, Theorem 6.19, gives a strong converse to Herzog's Theorem 5.3, namely that in dimension two over a field of characteristic zero, every CM complete local algebra having finite CM type is a ring of invariants.

§1 Miyata's theorem

The classical Yoneda correspondence (see [ML95]) allows us to identify elements of an Ext-module $\mathrm{Ext}^i_R(M,N)$ as equivalence classes of i-fold extensions of N by M. In the case $i = 1$, this is particularly simple: an element $\alpha \in \mathrm{Ext}^1_R(M,N)$ is an equivalence class of short exact sequences $0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$, where we declare two such sequences, with middle terms X, X', to be equivalent if they fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & X' & \longrightarrow & M & \longrightarrow & 0\,. \end{array}$$

It follows from the Snake Lemma that in this situation $X \cong X'$, so the middle term X_α is determined by the element α. The converse is false (cf. Exercise 6.22), but Miyata's Theorem [Miy67] gives a partial converse: if a short exact sequence "looks" split—the middle term is isomorphic to the direct sum of the other two—then it is split.

6.1 Theorem (Miyata). *Let R be a commutative Noetherian ring and let*

$$\alpha: \qquad N \xrightarrow{p} X_\alpha \xrightarrow{q} M \longrightarrow 0$$

be an exact sequence of finitely generated R-modules. If $X_\alpha \cong M \oplus N$, then α is a split short exact sequence.

Proof. It suffices to show that $p\colon N \longrightarrow X_\alpha$ is a *pure* homomorphism, that is, $Z \otimes_R p\colon Z \otimes_R N \longrightarrow Z \otimes_R X_\alpha$ is injective for every finitely generated R-module Z. Indeed, taking $Z = R$ will show that p is injective, and by Exercise 6.23 (or Exercise 10.50), pure submodules with finitely-presented quotients are direct summands.

Fix a finitely generated R-module Z. To show that $Z \otimes_R p$ is injective, we may localize at a maximal ideal and assume that $(R, \mathfrak{m})$ is local. Suppose $c \in Z \otimes N$ is a non-zero element of the kernel of $Z \otimes_R p$. Take n so large that $c \notin \mathfrak{m}^n(Z \otimes_R N) = \mathfrak{m}^n Z \otimes_R N$. Tensoring further with $R/\mathfrak{m}^n$ gives the right-exact sequence

$$(Z/\mathfrak{m}^n Z) \otimes_R N \xrightarrow{\overline{p}} (Z/\mathfrak{m}^n Z) \otimes_R X_\alpha \longrightarrow (Z/\mathfrak{m}^n Z) \otimes_R M \longrightarrow 0$$

of finite length R-modules. Counting lengths shows that $\overline{p}$ is injective, contradicting the presence of the nonzero element $\overline{c}$ in the kernel. $\square$

Let

$$\alpha: \qquad 0 \longrightarrow N \longrightarrow X_\alpha \longrightarrow M \longrightarrow 0$$

$$\beta: \qquad 0 \longrightarrow N \longrightarrow X_\beta \longrightarrow M \longrightarrow 0$$

be two extensions of N by M, with $X_\alpha \cong X_\beta$. As mentioned above, α and β need not represent the same element of $\operatorname{Ext}^1_R(M,N)$. In the rest of this section we describe a result of Striuli [Str05] giving a partial result in that direction.

6.2 Remark. We recall briefly a few more details of the Yoneda correspondence for Ext^1. First, recall that if $\alpha \in \operatorname{Ext}^1_R(M,N)$ is represented by the short exact sequence

$$\alpha: \qquad 0 \longrightarrow N \longrightarrow X_\alpha \longrightarrow M \longrightarrow 0,$$

then for $r \in R$, the product $r\alpha$ can be computed via either a pullback or a pushout. Precisely, $r\alpha$ is represented either by the top row of the diagram

$$\begin{array}{rccccccccc} r\alpha: & 0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle r} & & \\ \alpha: & 0 & \longrightarrow & N & \underset{p}{\longrightarrow} & X & \underset{q}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

or the bottom row of the diagram

$$\begin{array}{rccccccccc} \alpha: & 0 & \longrightarrow & N & \overset{p}{\longrightarrow} & X & \overset{q}{\longrightarrow} & M & \longrightarrow & 0 \\ & & & {\scriptstyle r}\downarrow & & \downarrow & & \| & & \\ r\alpha: & 0 & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where

$$P = \{(x,m) \in X \oplus M \mid q(x) = rm\}$$

and

$$Q = X \oplus N / \langle (p(n), -rn) \mid n \in N \rangle \, .$$

More generally, the same sorts of diagrams define actions of $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$ on $\mathrm{Ext}^1_R(M,N)$, on the right and left respectively, replacing r by an endomorphism of the appropriate module.

Pullbacks and pushouts also define the connecting homomorphisms δ in the long exact sequences of Ext. If $\alpha \in \mathrm{Ext}^1_R(M,N)$ is as above, then for an R-module Z the long exact sequence looks like

$$\cdots \longrightarrow \mathrm{Hom}_R(Z,X) \xrightarrow{q_*} \mathrm{Hom}_R(Z,M) \xrightarrow{\delta} \mathrm{Ext}^1_R(Z,N) \longrightarrow \cdots .$$

The image of a homomorphism $g\colon Z \longrightarrow M$ in $\mathrm{Ext}^1_R(M,N)$ is the top row of the pullback diagam below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & N & \xrightarrow[p]{} & X & \xrightarrow[q]{} & M & \longrightarrow & 0 \end{array}$$

In particular, when $Z = M$ we find that $\delta(1_M) = \alpha$. Similar considerations apply for the long exact sequence attached to $\mathrm{Hom}_R(-,Z)$.

Here is the result that will occupy the rest of the section. In fact Striuli's result holds for arbitrary Noetherian rings; we leave the straightforward extension to the interested reader.

6.3 Theorem (Striuli)**.** *Let R be a local ring. Let*

$$\alpha\colon \qquad 0 \longrightarrow N \longrightarrow X_\alpha \longrightarrow M \longrightarrow 0$$

$$\beta\colon \qquad 0 \longrightarrow N \longrightarrow X_\beta \longrightarrow M \longrightarrow 0$$

be two short exact sequences of finitely generated R-modules. Suppose that $X_\alpha \cong X_\beta$ and that $\beta \in I\operatorname{Ext}^1_R(M,N)$ for some ideal I of R. Then the complex $\alpha \otimes_R R/I$ is a split exact sequence.

We need one preliminary result.

6.4 Proposition. *Let $(R,\mathfrak{m})$ be a local ring and I an ideal of R. Let*

$$\alpha: \qquad 0 \longrightarrow N \xrightarrow{p} X_\alpha \xrightarrow{q} M \longrightarrow 0$$

be a short exact sequence of finitely generated R-modules, and denote by $\overline{\alpha} = \alpha \otimes_R R/I$ the complex

$$\overline{\alpha}: \qquad 0 \longrightarrow N/IN \xrightarrow{\overline{p}} X_\alpha/IX_\alpha \xrightarrow{\overline{q}} M/IM \longrightarrow 0\,.$$

If $\alpha \in I\operatorname{Ext}^1_R(M,N)$, then $\overline{\alpha}$ is a split exact sequence.

Proof. By Miyata's Theorem 6.1 it is enough to show that $X_\alpha/IX_\alpha \cong M/IM \oplus N/IN$. Let

$$\xi: \qquad 0 \longrightarrow Z \xrightarrow{i} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

be the beginning of a minimal resolution of M over R, so that $Z = \operatorname{syz}^R_1(M)$ is the first syzygy of M. Then applying $\operatorname{Hom}_R(-,N)$ gives a surjection $\operatorname{Hom}_R(Z,N) \longrightarrow \operatorname{Ext}^1_R(M,N)$. In particular $I\operatorname{Hom}_R(Z,N)$ maps onto $I\operatorname{Ext}^1_R(M,N)$, so there exists $\varphi \in I\operatorname{Hom}_R(Z,N)$ such that α is obtained from the pushout diagram below.

$$\begin{array}{rccccccccc}
\xi: & 0 & \longrightarrow & Z & \xrightarrow{i} & F_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \| & & \\
\alpha: & 0 & \longrightarrow & N & \xrightarrow[p]{} & X_\alpha & \xrightarrow[q]{} & M & \longrightarrow & 0
\end{array}$$

In particular, we have $\varphi(Z) \subseteq IN$. The pushout diagram also induces an exact sequence

$$\nu: \qquad 0 \longrightarrow Z \xrightarrow{\left[\begin{smallmatrix} i \\ -\varphi \end{smallmatrix}\right]} F_0 \oplus N \xrightarrow{[\psi\ p]} X_\alpha \longrightarrow 0\,.$$

Let L be an arbitrary R/I-module of finite length, and tensor both ξ and ν with L:

$$Z \otimes_R L \xrightarrow{i\otimes 1_L} F_0 \otimes_R L \xrightarrow{d_0 \otimes 1_L} M \otimes_R L \longrightarrow 0$$

$$Z \otimes_R L \xrightarrow{\left[\begin{smallmatrix} i\otimes 1_L \\ -\varphi\otimes 1_L \end{smallmatrix}\right]} (F_0 \otimes_R L) \oplus (N \otimes_R L) \xrightarrow{\left[\begin{smallmatrix} \psi\otimes 1_L \\ p\otimes 1_L \end{smallmatrix}\right]^T} X_\alpha \otimes_R L \longrightarrow 0\,.$$

Since $\varphi(Z) \subset IN$ and $IL = 0$, the image of $-\varphi \otimes 1_L$ is zero in $N \otimes_R L$. Denoting the image of $i \otimes 1$ by K, we get exact sequences

$$0 \longrightarrow K \longrightarrow F_0 \otimes_R L \longrightarrow M \otimes_R L \longrightarrow 0$$

$$0 \longrightarrow K \longrightarrow (F_0 \otimes_R L) \oplus (N \otimes_R L) \longrightarrow X_\alpha \otimes_R L \longrightarrow 0\,.$$

Counting lengths (over either R or R/I, equally) now gives

$$\ell(X_\alpha \otimes_R L) = \ell(M \otimes_R L) + \ell(N \otimes_R L)\,.$$

In particular, since L is an R/I-module, we have

$$\ell(X_\alpha/IX_\alpha \otimes_{R/I} L) = \ell(M/IM \otimes_{R/I} L) + \ell(N/IN \otimes_{R/I} L)\,.$$

Proposition **??** now applies, as L was arbitrary, to give $X_\alpha/IX_\alpha \cong M/IM \oplus N/IN$. □

Proof of Theorem 6.3. Since $\beta \in I\operatorname{Ext}^1_R(M,N)$, Miyata's Theorem 6.1 implies that $X_\beta/IX_\beta \cong M/IM \oplus N/IN$ and hence $X_\alpha/IX_\alpha \cong M/IM \oplus N/IN$. Applying Miyata's Theorem 6.1, we have that $\alpha \otimes_R R/I$ is split exact. □

Here is an amusing consequence.

6.5 Corollary. *Let* $(R,\mathfrak{m})$ *be a local ring and* M *a non-free finitely generated module. Let* α *be the short exact sequence*

$$\alpha: \qquad 0 \longrightarrow M_1 \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F *is a finitely generated free module and* $M_1 \subseteq \mathfrak{m}F$*. Then* α *is a minimal generator of* $\operatorname{Ext}^1_R(M, M_1)$*.*

Proof. If $\alpha \in \mathfrak{m}\operatorname{Ext}^1_R(M, M_1)$, then $\overline{\alpha} = \alpha \otimes R/\mathfrak{m}$ is split exact. But since $M_1 \subseteq \mathfrak{m}F$, the image of $M_1 \otimes R/\mathfrak{m}$ is zero, a contradiction. □

6.6 Example. The converse of Lemma 6.4 fails. Consider the one-dimensional (A_2) singularity $R = k[\![t^2, t^3]\!]$. Since R is Gorenstein, $\operatorname{Ext}^1_R(k,R) \cong k$, and so every nonzero element of $\operatorname{Ext}^1_R(k,R)$ is a minimal generator. Define α to be the bottom row of the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m} & \longrightarrow & R & \longrightarrow & k & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & R & \longrightarrow & X & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

where φ is defined by $\varphi(t^2) = t^3$ and $\varphi(t^3) = t^4$. Then α is non-split, since there is no map $R \longrightarrow R$ extending φ, whence $\alpha \notin \mathfrak{m}\operatorname{Ext}^1_R(k,R)$. On the other hand, $\mu(X) = 2$ and hence $X/\mathfrak{m}X \cong k \oplus k$.

These results raise the following question, which will be particularly relevant in Chapter 14.

6.7 Question. *Let* $(R,\mathfrak{m})$ *be a CM local ring and let* M *and* N *be MCM* R*-modules. Take a maximal regular sequence* $\mathbf{x}$ *on* R*,* M*, and* N*, and take*

$\alpha \in \operatorname{Ext}^1_R(M,N)$. *Is it true that* $\alpha \in \mathbf{x}\operatorname{Ext}^1_R(M,N)$ *if and only if* $\alpha \otimes R/(\mathbf{x})$ *is split exact?*

§2 Isolated singularities

Now we come to the first major theorem in the general theory of CM local rings of finite CM type: that they have at most isolated singularities. The result is due originally to Auslander [Aus86a] for complete local rings, though as Yoshino observed, the original proof relies only on the KRS property, hence works equally well for Henselian rings by Theorem 1.7. Auslander's argument is a tour de force of functorial imagination, and an early vindication of the use of almost split sequences in commutative algebra (cf. Chapter 10). Here we give a simple argument due to Huneke and Leuschke [HL02], valid for all CM local rings, using the results of the previous section.

6.8 Definition. Let $(R,\mathfrak{m})$ be a local ring. We say that R is, or has, an *isolated singularity* provided $R_\mathfrak{p}$ is a regular local ring for all non-maximal prime ideals $\mathfrak{p}$.

Note that we include the case where R is regular under the definition above. We also say R has "at most" an isolated singularity to explicitly allow this possibility.

The next lemma is standard, and we leave its proof as an exercise (Exercise 6.25).

6.9 Lemma. *Let $(R,\mathfrak{m})$ be a CM local ring. Then the following conditions are equivalent.*

(i) *The ring R has at most an isolated singularity.*

(ii) *Every MCM R-module is locally free on the punctured spectrum.*

(iii) *For all MCM R-modules M and N, $\operatorname{Ext}^1_R(M,N)$ has finite length.* $\square$

6.10 Lemma. *Let $(R,\mathfrak{m})$ be a local ring, $r \in \mathfrak{m}$, and*

$$\begin{array}{lccccccccc} \alpha: & 0 & \longrightarrow & N & \xrightarrow{i} & X_\alpha & \longrightarrow & M & \longrightarrow & 0 \\ & & & {\scriptstyle r}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\ r\alpha: & 0 & \longrightarrow & N & \xrightarrow[j]{} & X_{r\alpha} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

a commutative diagram of short exact sequences of finitely generated R-modules. Assume that $X_\alpha \cong X_{r\alpha}$ (not necessarily via the map f). Then $\alpha \in r\operatorname{Ext}^1_R(M,N)$.

Note that the case $r = 0$ is Miyata's Theorem 6.1.

Proof. The pushout diagram gives an exact sequence

$$0 \longrightarrow N \xrightarrow{\left[\begin{smallmatrix} r \\ -i \end{smallmatrix}\right]} N \oplus X_\alpha \xrightarrow{[j\ f]} X_{r\alpha} \longrightarrow 0\,.$$

Since $N \oplus X_\alpha \cong N \oplus X_{r\alpha}$, Miyata's Theorem 6.1 implies that the sequence splits. In particular, the induced map on Ext,

$$\left[\begin{smallmatrix} r \\ -i_* \end{smallmatrix}\right] : \operatorname{Ext}^1_R(M,N) \longrightarrow \operatorname{Ext}^1_R(M,N) \oplus \operatorname{Ext}^1_R(M,X_\alpha)\,,$$

is a split injection. Let h be a right inverse for $\left[\begin{smallmatrix} r \\ -i_* \end{smallmatrix}\right]$.

Now apply $\operatorname{Hom}_R(M,-)$ to α, getting an exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_R(M,M) \xrightarrow{\delta} \operatorname{Ext}^1_R(M,N) \xrightarrow{i_*} \operatorname{Ext}^1_R(M,X_\alpha) \longrightarrow \cdots .$$

The connecting homomorphism δ takes 1_M to α, so $i_*(\alpha) = 0$. Thus

$$\alpha = h(r\alpha, 0) = rh(\alpha, 0) \quad \in r\operatorname{Ext}^1_R(M,N). \qquad \square$$

6.11 Theorem. *Let $(R,\mathfrak{m})$ be local and M, N finitely generated R-modules. Suppose there are only finitely many isomorphism classes of modules X for which there exists a short exact sequence*

$$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0.$$

Then $\operatorname{Ext}^1_R(M,N)$ has finite length.

Proof. Let $\alpha \in \operatorname{Ext}^1_R(M,N)$, and let $r \in \mathfrak{m}$. By Exercise 6.24, it will suffice to prove that $r^n\alpha = 0$ for $n \gg 0$. For any integer $n \geqslant 0$, we consider a representative for $r^n\alpha$, namely

$$r^n\alpha: \qquad 0 \longrightarrow N \longrightarrow X_n \longrightarrow M \longrightarrow 0.$$

Since there are only finitely many such isomorphism classes of X_n, there exists an infinite sequence $n_1 < n_2 < \cdots$ such that $X_{n_i} \cong X_{n_j}$ for every i,j. Set $\beta = r^{n_1}\alpha$, and let $i > 1$. Note that $r^{n_i}\alpha = r^{n_i - n_1}\beta$. Hence we get the commutative diagram

$$\begin{array}{llcccccccc}
\beta: & 0 & \longrightarrow & N & \longrightarrow & X_{n_1} & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \big\downarrow{\scriptstyle r^{n_i-n_1}} & & \big\downarrow & & \big\| & & \\
r^{n_i-n_1}\beta: & 0 & \longrightarrow & N & \longrightarrow & X_{n_i} & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

for each i. By Lemma 6.10, $X_{n_1} \cong X_{n_i}$ implies $\beta \in r^{n_i - n_1} \operatorname{Ext}^1_R(M,N)$ for every i. This implies $\beta \in \mathfrak{m}^t \operatorname{Ext}^1_R(M,N)$ for every $t \geqslant 1$, whence, by the Krull Intersection Theorem, $\beta = 0$. □

If R has finite CM type, then for all MCM modules M and N, there exist only finitely many MCM modules X generated by at most $\mu_R(M) + \mu_R(N)$ elements, thus finitely many potential middle terms for short exact sequences. Thus we obtain Auslander's theorem:

6.12 Theorem (Auslander)**.** *Let* $(R, \mathfrak{m})$ *be a CM ring with finite CM type. Then* R *has at most an isolated singularity.*

6.13 Remark. A non-commutative version of Theorem 6.12 is easy to state, and the same proof applies. This was Auslander's original context [Aus86a]. Specifically, Auslander considers the following situation: Let T be a complete regular local ring and let Λ be a (possibly non-commutative) T-algebra which is a finitely generated free T-module. Say that Λ is *nonsingular* if $\operatorname{gldim} \Lambda = \dim T$, and that Λ has *finite representation type* if there are only finitely many isomorphism classes of indecomposable finitely generated (left) Λ-modules that are free as T-modules. If Λ has finite representation type, then $\Lambda_{\mathfrak{p}}$ is nonsingular for all non-maximal primes $\mathfrak{p}$ of T.

We mention here a few further applications of Theorem 6.11, all based on the same observation. Suppose that R is a CM local ring and M is a MCM R-module such that there are only finitely many non-isomorphic MCM modules of multiplicity less than or equal to $\mu_R(M) \cdot \mathrm{e}(R)$; then M is locally free on the punctured spectrum. This follows immediately from Theorem 6.12 upon taking N to be the first syzygy of M in a minimal free resolution. If in addition R is a domain, then the criterion simplifies to the existence of only finitely many MCM modules of rank at most $\mu_R(M)$.

Obvious candidates for M are the canonical module ω, the conormal I/I^2 of a regular presentation $R = A/I$, and the module of Kähler differentials $\Omega^1_{R/k}$ if R is essentially of finite type over a field k. Since the freeness of these modules implies that R is Gorenstein, resp. complete intersection [Mat86, 19.9], resp. regular [Kun86, Theorem 7.2], we obtain the following corollaries.

6.14 Corollary. *Let $(R,\mathfrak{m})$ be a CM local ring with canonical module ω. If R has only finitely many non-isomorphic MCM modules of multiplicity up to $r(R)\,\mathrm{e}(R)$, where $r(R) = \dim_k \operatorname{Ext}^{\dim R}_R(k,R)$ denotes the Cohen–Macaulay type of R, then R is Gorenstein on the punctured spectrum.*

6.15 Corollary. *Let $(A,\mathfrak{n})$ be a regular local ring, and suppose $I \subseteq \mathfrak{n}^2$ is an ideal such that $R = A/I$ is CM. Assume that I/I^2 is a MCM R-module. If R has only finitely many non-isomorphic MCM modules of multiplicity at most $\mu_A(I)\cdot\mathrm{e}(R)$, then R is complete intersection on the punctured spectrum.*

6.16 Corollary. *Let k be a field of characteristic zero, and let R be a k-algebra essentially of finite type. Let $\Omega^1_{R/k}$ be the module of Kähler differentials of R over k. Assume that Ω is a MCM R-module. If R has only finitely many non-isomorphic MCM modules of multiplicity up to* $\operatorname{embdim}(R)\cdot\mathrm{e}(R)$*, then R has at most an isolated singularity.*

The second corollary naturally raises the question of when I/I^2 is a MCM A/I-module for an ideal I in a regular local ring A. Herzog [Her78a] showed that this is the case if A/I is Gorenstein and I has height three; see [HU89] and [Buc81, 6.2.10] for some further results in this direction.

§3 Classification of two-dimensional CM rings of finite CM type

Our aim in this section is to prove a converse to Herzog's Theorem 5.3, which states that rings of invariants in dimension two have finite CM type. The result, due to Auslander and Esnault, is that if a complete local ring R of dimension two, with a coefficient field k of characteristic zero, has finite CM type, then $R \cong k[[u,v]]^G$ for some finite group $G \subseteq \mathrm{GL}(n,k)$.

Auslander's proof relies on a deep topological result of Mumford [Mum61], [Hir95b]. We give Mumford's theorem below, followed by the interpretation and more general statement in commutative algebra due to Flenner [Fle75], see also [CS93].

6.17 Theorem (Mumford)**.** *Let V be a normal complex space of dimension 2 and $x \in V$ a point. Then the following properties hold.*

(i) The local fundamental group $\pi(V,x)$ is finitely generated.

(ii) If the local homology group $H_1(V,x)$ vanishes, then $\pi(V,x)$ is isomorphic to the fundamental group of a valued tree with negative definite intersection matrix.

(iii) If $\pi(V,x) = \{1\}$ is trivial, then x is a regular point.

To translate Mumford's result into commutative algebra, we recall the definition of the *étale fundamental group*, also called the algebraic fundamental group. See [Mil08] for more details. (We will not attempt maximal generality in this brief sketch; in particular, we will ignore the need to choose a base point.) For a connected normal scheme X, the étale fundamental group $\pi_1^{et}(X)$ classifies the finite étale coverings of X in a manner analogous to the usual fundamental group classifying the covering spaces of a topological space.

The construction of π_1^{et} is clearest when $X = \operatorname{Spec} A$ for a normal domain A. Let K be the quotient field of A, and fix an algebraic closure Ω of K. Then $\pi_1^{et}(X) \cong \operatorname{Gal}(L/K)$, where L is the union of all the finite separable field extensions K' of K contained in Ω, and such that the integral closure of A in K' is étale over A. There is a Galois correspondence between subgroups $H \subseteq \pi_1^{et}(X)$ of finite index and finite étale covers $A \longrightarrow B$ of A. In particular, $\pi_1^{et}(X) = 0$ if and only if A has no non-trivial finite étale covers.

With some extra work, the étale fundamental group can be defined for arbitrary schemes X. In particular, one may take X to be the punctured spectrum $\operatorname{Spec}^\circ A = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ of a local ring $(A, \mathfrak{m})$. We say that the local ring $(A, \mathfrak{m})$ is *pure* if the induced morphism of étale fundamental groups $\pi_1^{et}(\operatorname{Spec}^\circ A) \longrightarrow \pi_1^{et}(\operatorname{Spec} A)$ is an isomorphism. (Unfortunately this usage of the word "pure" has nothing to do with the usage of the same word earlier in this chapter.) The point is the surjectivity: A is pure if and only if every étale cover of the punctured spectrum extends to an étale cover of the whole spectrum.

6.18 Theorem (Flenner)**.** *Let $(A, \mathfrak{m}, k)$ be an excellent Henselian local nor-*

mal domain of dimension two. Assume that $\operatorname{char} k = 0$. *Consider the following conditions.*

(i) $\pi_1^{et}(\operatorname{Spec}^\circ A) = 0$;

(ii) *A is pure;*

(iii) *A is a regular local ring.*

Then $(a) \implies (b) \iff (c)$, *and the three conditions are equivalent if k is algebraically closed.*

The implication "A regular $\implies$ A pure" is a restatement of the theorem on the purity of the branch locus (Theorem B.12). The content of the theorem of Mumford and Flenner is in the other implications, in particular, a converse to purity of the branch locus.

Now we come to Auslander's characterization of the two-dimensional complete local rings of finite CM type in characteristic zero.

6.19 Theorem. *Let R be a complete CM local ring of dimension two with a coefficient field k. Assume that k has characteristic zero. If R has finite CM type, then there exists a power series ring* $S = k[\![u,v]\!]$ *and a finite group G acting on S by linear changes of variables such that* $R \cong S^G$.

Proof. First, notice that by Theorem 6.12 R is regular in codimension one, whence a normal domain.

Let K be the quotient field of the normal domain R, and fix an algebraic closure Ω. Consider the family of all finite field extensions K' of K, contained in Ω, and such that the integral closure of R in K' is unramified in codimension one over R. Let L be the field generated by all these K', and let S be the integral closure of R in L.

We will show that L is a finite Galois extension of K, so that in particular S is a module-finite R-algebra. Observe that if we show this, then by construction S has no module-finite ring extensions which are unramified in codimension one; indeed, any such ring extension would also be module-finite and unramified in codimension one over R. (See Appendix B.) In other words, we will have $\pi_1^{et}(\operatorname{Spec} S \setminus \{\mathfrak{m}_S\}) = 0$ and it will follow that S is a regular local ring, hence $S \cong k[\![u,v]\!]$.

To show that L/K is a finite Galois extension, assume that there is an infinite ascending chain

$$K \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L$$

of finite Galois extensions of K inside L. Let S_i be the integral closure of R in L_i. Then we have a corresponding infinite ascending chain

$$R \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S$$

of module-finite ring extensions. Each S_i is a normal domain, so in particular a reflexive R-module. By Exercise 3.31, the S_i are pairwise non-isomorphic as R-modules, contradicting the assumption that R has finite CM type. Thus L/K is finite, and it's easy to see it is a Galois extension. Let G be the Galois group of L over K. Then G acts on S with fixed ring R, and the argument of Lemma 4.1 allows us to assume the action is linear. □

Theorem 6.19 is false in positive characteristic. Artin [Art77] has given counterexamples to Mumford's characterization of smoothness in characteristic $p > 0$; the simplest is the (A_{p-1}) singularity $x^2 + y^p + z^2 = 0$, which has trivial étale fundamental group, and which has finite CM type by Theorem 5.22. Thus in particular $k[[x,y,z]]/(x^2+y^p+z^2)$ is not a ring of invariants when k has characteristic p.

Among other things, Theorem 6.19 implies that the two-dimensional CM local rings of finite CM type with residue field $\mathbb{C}$ have *rational singularities* (see Definition 5.30). This suggests the following conjecture.

6.20 Conjecture. *Let $(R,\mathfrak{m})$ be a CM local ring of dimension at least two. Assume that R has finite CM type. Then R has a rational singularity.*

The assumption $\dim R \geqslant 2$ is necessary to allow for the existence of non-normal, that is, non-regular, one-dimensional rings of finite CM type.

To add some evidence for this conjecture, we recall that by Mumford [Mum61] (in characteristic zero) and Lipman [Lip69] (in characteristic $p>0$), a normal surface singularity $X = \operatorname{Spec} R$ has a rational singularity if and only if there are only finitely many rank one MCM R-modules up to isomorphism.

Here is a weaker version of Conjecture 6.20.

6.21 Conjecture. *Let $(R,\mathfrak{m})$ be a CM local ring of dimension at least two. If R has finite CM type, then R has* minimal multiplicity, *that is,*

$$\mathrm{e}(R) = \mu_R(\mathfrak{m}) - \dim R + 1\,.$$

Recall that rational singularity implies minimal multiplicity, Corollary 5.34. We will prove Conjecture 6.21 for hypersurfaces in **??**, and in fact Conjecture 6.20 for the hypersurface case will follow from the classification in Chapter 8.

§4 Exercises

6.22 Exercise. Prove that the $p-1$ non-zero elements of $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ all have isomorphic middle terms. Find an example of two elements of $\operatorname{Ext}^1_{\mathbb{Z}}(A, B)$ with isomorphic middle terms but different annihilators. (See [Str05] for one example, due to G. Caviglia.)

6.23 Exercise. Let $N \subset M$ be modules over a commutative ring R. Prove that N is a pure submodule of M if and only if the following condition is satisfied: Whenever $x_1, \dots, x_t$ is a sequence of elements in N, and $x_i = \sum_{j=1}^{s} r_{ij} m_j$ for some $r_{ij} \in R$ and $m_j \in M$, there exist $y_1, \dots, y_s \in N$ such that $x_i = \sum_{j=1}^{s} r_{ij} y_j$ for $i = 1, \dots, t$. Conclude that if M/N is finitely presented and $N \subset M$ is pure, then the inclusion of N into M splits. (See also Exercise 10.50.)

6.24 Exercise. Let $(R, \mathfrak{m})$ be local, and let M be a finitely generated R-module. Show that M has finite length if and only if for all $r \in \mathfrak{m}$ and for all $x \in M$, there exists an integer n such that $r^n x = 0$.

6.25 Exercise. Prove a slightly more general version of Lemma 6.9: if R is a local ring, then for the conditions below we have (i) $\implies$ (ii) $\implies$ (iii), and (iii) $\implies$ (i) if R is CM.

(i) The ring R has at most an isolated singularity.

(ii) Every MCM R-module is locally free on the punctured spectrum.

(iii) For all MCM R-modules M and N, $\operatorname{Ext}^1_R(M, N)$ has finite length.

7

The Double Branched Cover

In this chapter we introduce two key tools in the representation theory of hypersurface rings: matrix factorizations and the double branched cover. We fix the following notation for the entire chapter.

7.1 Conventions. Let $(S, \mathfrak{n}, k)$ be a regular local ring and let f be a non-zero element of $\mathfrak{n}^2$. Put $R = S/(f)$ and $\mathfrak{m} = \mathfrak{n}/(f)$. We let $d = \dim(R) = \dim(S) - 1$.

§1 Matrix factorizations

With the notation of 7.1, suppose M is a MCM R-module. Then M has depth d when viewed as an R-module or as an S-module. By the Auslander-Buchsbaum formula, M has projective dimension 1 over S. Therefore the minimal free resolution of M as an S-module is of the form

(7.1.1)
$$0 \longrightarrow G \xrightarrow{\varphi} F \longrightarrow M \longrightarrow 0\,,$$

where G and F are finitely generated free S-modules. Since $f \cdot M = 0$, M is a torsion S-module, so $\operatorname{rank}_S G = \operatorname{rank}_S F$.

For any $x \in F$, the image of fx in M vanishes, so there is a unique element $y \in G$ such that $\varphi(y) = fx$. Since the element y is linearly determined by x, we get a homomorphism $\psi\colon F \longrightarrow G$ satisfying $\varphi\psi = f\,1_F$. It follows from the injectivity of the map φ that $\psi\varphi = f\,1_G$ too. This construction motivates the following definition [Eis80].

7.2 Definition. Let $(S, \mathfrak{n}, k)$ be a regular local ring, and let f be a non-zero element of $\mathfrak{n}^2$. A *matrix factorization* of f is a pair (φ, ψ) of homomorphisms between free S-modules of the same rank, $\varphi \colon G \longrightarrow F$ and $\psi \colon F \longrightarrow G$, such that

$$\psi\varphi = 1_G \qquad \text{and} \qquad \varphi\psi = 1_F\,.$$

Equivalently (after choosing bases), φ and ψ are square matrices of the same size over S, say $n \times n$, such that

$$\psi\varphi = \varphi\psi = I_n\,.$$

Let (φ, ψ) be a matrix factorization of f as in Definition 7.2. Since f is a non-zerodivisor, it follows that φ and ψ are injective, and we have short exact sequences

$$\begin{array}{c} 0 \longrightarrow G \xrightarrow{\ \varphi\ } F \longrightarrow \operatorname{cok}\varphi \longrightarrow 0 \\ 0 \longrightarrow F \xrightarrow{\ \psi\ } G \longrightarrow \operatorname{cok}\psi \longrightarrow 0 \end{array} \tag{7.2.1}$$

of S-modules. As $fF = \varphi\psi(F)$ is contained in the image of φ, the cokernel of φ is annihilated by f. Similarly, $f \cdot \operatorname{cok}\psi = 0$. Thus $\operatorname{cok}\varphi$ and $\operatorname{cok}\psi$ are naturally finitely generated modules over $R = S/(f)$.

7.3 Proposition. *Let* $(S, \mathfrak{n})$ *be a regular local ring and let* f *be a non-zero element of* $\mathfrak{n}^2$.

(i) *For every MCM* R*-module, there is a matrix factorization* (φ, ψ) *of* f *with* $\operatorname{cok}\varphi \cong M$.

(ii) *If* (φ, ψ) *is a matrix factorization of* f*, then* $\operatorname{cok}\varphi$ *and* $\operatorname{cok}\psi$ *are MCM* R*-modules.*

Proof. Only the second statement needs verification. The exact sequences (7.2.1) and the fact that $f \cdot \operatorname{cok}\varphi = 0 = f \cdot \operatorname{cok}\psi$ imply that the cokernels have projective dimension one over S. By the Auslander-Buchsbaum formula, they have depth equal to $\dim S - 1 = \dim R$ and therefore are MCM R-modules. □

7.4 Notation. When we wish to emphasize the provenance of a presentation matrix φ as half of a matrix factorization (φ, ψ), we write $\operatorname{cok}(\varphi, \psi)$ in place of $\operatorname{cok}\varphi$. We also write $(\varphi\colon G \longrightarrow F,\ \psi\colon F \longrightarrow G)$ to include the free S-module G and F in the notation.

There are two distinguished *trivial* matrix factorizations of any element f, namely $(f, 1)$ and $(1, f)$. Note that $\operatorname{cok}(1, f) = 0$, while $\operatorname{cok}(f, 1) \cong R$.

7.5 Definition. Let $(\varphi\colon G \longrightarrow F,\ \psi\colon F \longrightarrow G)$ and $(\varphi'\colon G' \longrightarrow F',\ \psi'\colon F' \longrightarrow G')$ be matrix factorizations of $f \in S$. A *homomorphism of matrix factorizations* between (φ, ψ) and (φ', ψ') is a pair of S-module homomorphisms $\alpha\colon F \longrightarrow F'$ and $\beta\colon G \longrightarrow G'$ rendering the diagram

(7.5.1)
$$\begin{array}{ccccc} F & \xrightarrow{\psi} & G & \xrightarrow{\varphi} & F \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ F' & \xrightarrow[\psi']{} & G' & \xrightarrow[\varphi']{} & F' \end{array}$$

commutative. (In fact, commutativity of just one of the squares is sufficient; see Exercise 7.31.)

A homomorphism of matrix factorizations $(\alpha, \beta)\colon (\varphi, \psi) \longrightarrow (\varphi', \psi')$ induces a homomorphism of R-modules $\operatorname{cok}(\varphi, \psi) \longrightarrow \operatorname{cok}(\varphi', \psi')$, which we denote $\operatorname{cok}(\alpha, \beta)$. Conversely, every S-module homomorphism $\operatorname{cok}(\varphi, \psi) \longrightarrow$

$\operatorname{cok}(\varphi', \psi')$ lifts to give a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\varphi} & F & \longrightarrow & \operatorname{cok}(\varphi,\psi) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow & & \\ 0 & \longrightarrow & G' & \xrightarrow[\varphi']{} & F' & \longrightarrow & \operatorname{cok}(\varphi',\psi) & \longrightarrow & 0 \end{array}$$

with exact rows, and thus a homomorphism of matrix factorizations.

Two matrix factorizations (φ, ψ) and (φ', ψ') are *equivalent* if there is a homomorphism of matrix factorizations $(\alpha, \beta)\colon (\varphi, \psi) \longrightarrow (\varphi', \psi')$ in which both α and β are isomorphisms.

Direct sums of matrix factorizations are defined in the natural way:

$$(\varphi,\psi) \oplus (\varphi',\psi') = \left(\begin{pmatrix} \varphi & \\ & \varphi' \end{pmatrix}, \begin{pmatrix} \psi & \\ & \psi' \end{pmatrix} \right).$$

We say that a matrix factorization is *reduced* provided it is not equivalent to a matrix factorizations having a trivial direct summand $(f, 1)$ or $(1, f)$. It's straightforward to see that (φ, ψ) is reduced if and only if all the entries of φ and ψ are in the maximal ideal of S. See Exercise 7.32. In particular, φ has no unit entries if and only if $\operatorname{cok}(\varphi, \psi)$ has no non-zero R-free direct summands.

Letting bars denote reduction modulo f, a matrix factorization $(\varphi\colon G \longrightarrow F,\ \psi\colon F \longrightarrow G)$ induces a complex

$$\cdots \longrightarrow \overline{G} \xrightarrow{\overline{\varphi}} \overline{F} \xrightarrow{\overline{\psi}} \overline{G} \xrightarrow{\overline{\varphi}} \overline{F} \longrightarrow \operatorname{cok}(\varphi,\psi) \longrightarrow 0 \tag{7.5.2}$$

in which $\overline{G}$ and $\overline{F}$ are finitely generated free modules over $R = S/(f)$. In fact (Exercise 7.33), this complex is exact, hence is a free resolution of $\operatorname{cok}(\varphi, \psi)$. If (φ, ψ) is a reduced matrix factorization, then (7.5.2) is a minimal R-free resolution of $\operatorname{cok}(\varphi, \psi)$.

The reversed pair (ψ,φ) is also a matrix factorization of f, and the resolution (7.5.2) exhibits $\operatorname{cok}(\psi,\varphi)$ as a first syzygy of $\operatorname{cok}(\varphi,\psi)$ and vice versa:

$$\begin{array}{c} 0 \longrightarrow \operatorname{cok}(\psi,\varphi) \longrightarrow \overline{F} \longrightarrow \operatorname{cok}(\varphi,\psi) \longrightarrow 0 \\ 0 \longrightarrow \operatorname{cok}(\varphi,\psi) \longrightarrow \overline{G} \longrightarrow \operatorname{cok}(\psi,\varphi) \longrightarrow 0 \end{array} \tag{7.5.3}$$

are exact sequences of R-modules. This gives the first assertion of the next result; we leave the rest, and the proof of the theorem following, as exercises. Recall that an R-module M is *stable* provided it does not have a direct summand isomorphic to R. We remark that a direct sum of two stable modules is again stable, by KRS (or directly, cf. Exercise 7.34).

7.6 Proposition. *Keep the notation of 7.1.*

(i) Let M be a MCM R-module. Then M has a free resolution which is periodic of period at most two.

(ii) Let M be a stable MCM R-module. Then the minimal free resolution of M is periodic of period at most two.

(iii) Let M be a MCM R-module. Then $\operatorname{syz}_1^R M$ is a stable MCM R-module. If M is indecomposable, so is $\operatorname{syz}_1^R M$.

(iv) Let M be a finitely generated R-module. Then the minimal free resolution of M is eventually periodic of period at most two. In particular the minimal free resolution of M is bounded.

(v) Let M and N be R-modules with M finitely generated. For each $i \geqslant \dim R$, we have $\operatorname{Ext}_R^i(M,N) \cong \operatorname{Ext}_R^{i+2}(M,N)$ and $\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_{i+2}^R(M,N)$.

In the next chapter we will see a converse to (iii): If every minimal free resolution over a local ring R is bounded, then (the completion of) R is a hypersurface ring.

7.7 Theorem ([Eis80, Theorem 6.3]). *Keep the notation of 7.1. The association*

$$(\varphi,\psi) \longleftrightarrow \operatorname{cok}(\varphi,\psi)$$

induces an equivalence of categories between reduced matrix factorizations of f up to equivalence and of stable MCM R-modules up to isomorphism. In particular, it gives a bijection between equivalence classes of reduced matrix factorizations and isomorphism classes of stable MCM modules. □

7.8 Remark. If in addition f is a prime/irreducible element of S, so that R is an integral domain, then from $\varphi\psi = f \cdot I_n$ it follows that both $\det\varphi$ and $\det\psi$ are, up to units, powers of f. Specifically, we must have $\det\varphi = uf^k$ and $\det\psi = u^{-1}f^{n-k}$ for some unit $u \in S$ and $k \leqslant n$. In this case the R-module $\operatorname{cok}(\varphi,\psi)$ has rank k, while $\operatorname{cok}(\psi,\varphi)$ has rank $n-k$. To see this, localize at the prime ideal (f). Then over the discrete valuation ring $S_{(f)}$, φ is equivalent to $f \cdot 1_k \oplus 1_{n-k}$ and so $\operatorname{cok}\varphi$ has rank k over the field $R_{(f)}$.

Similar remarks hold when f is merely reduced, provided we consider $\operatorname{rank} M$ as the tuple $(\operatorname{rank}_{R_\mathfrak{p}} M_\mathfrak{p})$ as $\mathfrak{p}$ runs over the minimal primes in R.

7.9 Remark. Let $(\varphi\colon G \longrightarrow F,\ \psi\colon F \longrightarrow G)$ and $(\varphi'\colon G' \longrightarrow F',\ \psi'\colon F' \longrightarrow G')$ be two matrix factorizations of f. Put $M = \operatorname{cok}(\varphi,\psi)$, $N = \operatorname{cok}(\psi,\varphi)$, $M' = \operatorname{cok}(\varphi',\psi')$, and $N' = \operatorname{cok}(\psi',\varphi')$. Then any homomorphism of matrix factorizations $(\alpha,\beta)\colon (\psi,\varphi) \longrightarrow (\varphi',\psi')$ (note the order!) defines a pushout diagram

$$
(7.9.1)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & \overline{F} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M' & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

of R-modules, the bottom row of which is the image of $\operatorname{cok}(\alpha,\beta)$ under the surjective connecting homomorphism

$$\operatorname{Hom}_R(N,M') \longrightarrow \operatorname{Ext}^1_R(M,M').$$

In particular, every extension of M' by M arises in this way.

The middle module Q is of course MCM as well. Splicing (7.9.1) together with the R-free resolutions of N and M', we obtain a morphism of exact sequences

$$
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\overline{\varphi}} & \overline{F} & \xrightarrow{\overline{\psi}} & \overline{G} & \xrightarrow{\overline{\varphi}} & \overline{F} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \beta\downarrow & & \downarrow\alpha & & \downarrow & & \| & & \\
\cdots & \xrightarrow[\overline{\psi'}]{} & \overline{G'} & \xrightarrow[\overline{\varphi'}]{} & \overline{F'} & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

defined, after the first step, by α and β. The *mapping cone* of this morphism is thus the exact complex

$$\cdots \longrightarrow \overline{F'} \oplus \overline{F} \xrightarrow{\begin{bmatrix} \overline{\psi'} & \beta \\ & -\overline{\varphi} \end{bmatrix}} \overline{G'} \oplus \overline{G} \xrightarrow{\begin{bmatrix} \overline{\varphi'} & \alpha \\ & -\overline{\psi} \end{bmatrix}} \overline{F'} \oplus \overline{F} \longrightarrow Q \oplus M \longrightarrow M \longrightarrow 0\,.$$

We may cancel the two occurrences of M (since the map between them is the identity) and find that

$$Q \cong \operatorname{cok}\left(\begin{pmatrix} \overline{\varphi'} & \alpha \\ & -\overline{\psi} \end{pmatrix}, \begin{pmatrix} \overline{\psi'} & \beta \\ & -\overline{\varphi} \end{pmatrix}\right).$$

§2 The double branched cover

We continue with the notation and conventions established in 7.1 and assume, in addition, that S is complete. Thus $(S, \mathfrak{n}, k)$ is a complete regular local ring of dimension $d+1$, $0 \neq f \in \mathfrak{n}^2$, and $R = S/(f)$. We will refer to a ring R of this form as a *complete hypersurface singularity*.

7.10 Definition. The *double branched cover* of R is

$$R^{\sharp} = S[[z]]/(f + z^2),$$

a complete hypersurface singularity of dimension $d+1$.

7.11 Warning. It is important to have a particular defining equation in mind, since different equations defining the same hypersurface R can lead to non-isomorphic rings $R^{\sharp}$. For example, we have $\mathbb{R}[[x]]/(x^2) = \mathbb{R}[[x]]/(-x^2)$, yet $\mathbb{R}[[x,z]]/(x^2+z^2) \ncong \mathbb{R}[[x,z]]/(-x^2+z^2)$. (One is a domain; the other is not.) Exercise 7.36 shows that such oddities cannot occur if k is algebraically closed and of characteristic different from two.

We want to compare the MCM modules over $R^{\sharp}$ with those over R. Observe that we have a surjection $R^{\sharp} \longrightarrow R$, killing the class of z. There is no homomorphism the other way in general. However, $R^{\sharp}$ is a finitely generated free S-module, generated by the images of 1 and z; cf. Exercise 7.38.

7.12 Definition. Let N be a MCM $R^{\sharp}$-module. Set

$$N^{\flat} = N/zN,$$

a MCM R-module. Contrariwise, let M be a MCM R-module. View M as an $R^\sharp$-module via the surjection $R^\sharp \longrightarrow R$, and set

$$M^\sharp = \operatorname{syz}_1^{R^\sharp} M\,.$$

Notice that there is no conflict of notation if we view R as an R-module and sharp it: Since z is a non-zerodivisor of $R^\sharp$ (cf. Exercise 7.37), we have a short exact sequence

$$0 \longrightarrow R^\sharp \xrightarrow{\ z\ } R^\sharp \longrightarrow R \longrightarrow 0\,.$$

Thus $R^\sharp$ is indeed the first syzygy of R as an $R^\sharp$-module.

7.13 Notation. Let $\varphi\colon G \longrightarrow F$ be a homomorphism of finitely generated free S-modules, or equivalently a matrix with entries in S. We use the same symbol φ for the induced homomorphism $S[\![z]\!] \otimes_S G \longrightarrow S[\![z]\!] \otimes_S F$; as a matrix, they are identical. In particular we abuse the notation 1_F, using it also for the identity map $S[\![z]\!] \otimes_S 1_F$.

Furthermore let $\widetilde{\varphi}\colon \widetilde{G} \longrightarrow \widetilde{F}$ denote the corresponding homomorphism over $R^\sharp$, obtained via the composition of natural homomorphisms $S \longrightarrow S[\![z]\!] \longrightarrow S[\![z]\!]/(f+z^2) = R^\sharp$. Finally, as in §1, we let $\overline{\varphi}\colon \overline{G} \longrightarrow \overline{F}$ denote the matrix over $R = S/(f)$ obtained via the natural map $S \longrightarrow R$. Thus $\overline{F} = \widetilde{F}/z\widetilde{F}$.

7.14 Lemma. *Let $(\varphi\colon G \longrightarrow F,\ \psi\colon F \longrightarrow G)$ be a matrix factorization of f, let $M = \operatorname{cok}(\varphi,\psi)$, and let $\pi\colon \widetilde{F} \twoheadrightarrow M$ be the composition $\widetilde{F} \twoheadrightarrow \overline{F} \twoheadrightarrow M$.*

(i) There is an exact sequence

$$\widetilde{F} \oplus \widetilde{G} \xrightarrow{\begin{bmatrix} \widetilde{\psi} & -z1_{\widetilde{G}} \\ z1_{\widetilde{F}} & \widetilde{\varphi} \end{bmatrix}} \widetilde{G} \oplus \widetilde{F} \xrightarrow{\begin{bmatrix} \widetilde{\varphi} & z1_{\widetilde{F}} \end{bmatrix}} \widetilde{F} \xrightarrow{\ \pi\ } M \longrightarrow 0$$

of $R^\sharp$-modules.

(ii) The matrices over $S[\![z]\!]$

$$\begin{pmatrix} \psi & -z\,1_G \\ z\,1_F & \varphi \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} \varphi & z\,1_F \\ -z\,1_G & \psi \end{pmatrix}$$

form a matrix factorization of $f + z^2$ over $S[\![z]\!]$.

(iii) We have

$$M^\sharp \cong \operatorname{cok}\left(\begin{pmatrix} \psi & -z\,1_G \\ z\,1_F & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & z\,1_F \\ -z\,1_G & \psi \end{pmatrix} \right)$$

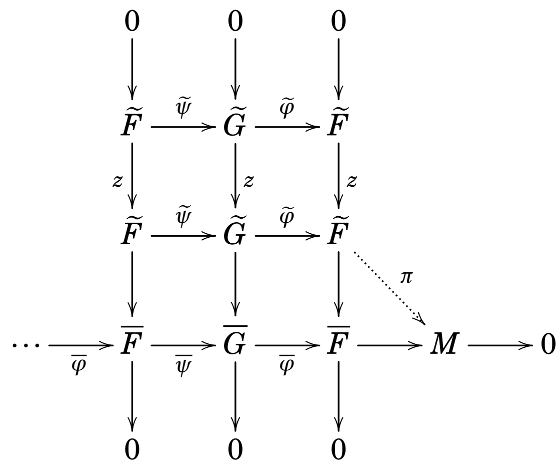
and

$$\operatorname{syz}_1^{R^\sharp}(M^\sharp) \cong M^\sharp\,.$$

Proof. The proof of (ii) amounts to matrix multiplication, and (iii) is an immediate consequence of (i), (ii), and the matrix calculation

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \varphi & z\,1_F \\ -z\,1_G & \psi \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} \psi & -z\,1_G \\ z\,1_F & \varphi \end{pmatrix}$$

over $S[\![z]\!]$. It thus suffices to prove (i). First we note that z is a nonzerodivisor of $R^\sharp$ (Exercise 7.37). Therefore the columns of the following commutative diagram are exact.



The bottom row is also exact by (7.5.2), but the first two rows aren't even complexes. In fact,

$$\widetilde{\varphi}\widetilde{\psi} = -z^2\, 1_F \,. \tag{7.14.1}$$

An easy diagram chase shows that $\ker \pi = \operatorname{im} \widetilde{\varphi} + z\widetilde{F} = \operatorname{im}\left[\begin{smallmatrix}\widetilde{\varphi} & z\,1_{\widetilde{F}}\end{smallmatrix}\right]$. Also,

$$\ker \begin{bmatrix}\widetilde{\varphi} & z\,1_{\widetilde{F}}\end{bmatrix} \supseteq \operatorname{im} \begin{bmatrix}\widetilde{\psi} & -z\,1_{\widetilde{G}} \\ z\,1_{\widetilde{F}} & \widetilde{\varphi}\end{bmatrix}$$

by (7.14.1). For the opposite inclusion, let $\left[\begin{smallmatrix}x \\ y\end{smallmatrix}\right] \in \ker\left[\begin{smallmatrix}\widetilde{\varphi} & z\,1_{\widetilde{F}}\end{smallmatrix}\right]$, so that $\widetilde{\varphi}(x) = -zy$. A diagram chase yields elements $a \in \widetilde{F}$ and $b \in \widetilde{G}$ such that $\left[\begin{smallmatrix}\widetilde{\psi} & -z\,1_{\widetilde{G}}\end{smallmatrix}\right]\left[\begin{smallmatrix}a \\ b\end{smallmatrix}\right] = x$. We need to show that $\left[\begin{smallmatrix}z\,1_{\widetilde{F}} & \widetilde{\varphi}\end{smallmatrix}\right]\left[\begin{smallmatrix}a \\ b\end{smallmatrix}\right] = y$. Using (7.14.1), we obtain the equations

$$z(za + \widetilde{\varphi}(b)) = -\widetilde{\varphi}\widetilde{\psi}(a) + z\widetilde{\varphi}(b) = -\widetilde{\varphi}(\widetilde{\psi}(a) - zb) = -\widetilde{\varphi}(x) = zy.$$

Cancelling the non-zerodivisor z, we get the desired result. □

This allows us already to prove one "natural" relation between sharping and flatting.

7.15 Proposition. *Let M be a MCM R-module. Then*

$$M^{\sharp\flat} \cong M \oplus \operatorname{syz}_1^R M\,.$$

Moreover, M is a stable R-module if and only if $M^\sharp$ is a stable $R^\sharp$-module.

Proof. The R-module $M^{\sharp\flat}$ is presented by the matrix factorization $(\Phi \otimes_{R^\sharp} R,\ \Psi \otimes_{R^\sharp} R)$, where (Φ, Ψ) is the matrix factorization for $M^\sharp$ given in Lemma 7.14. Killing z in that matrix factorization gives

$$M^{\sharp\flat} \cong \operatorname{cok}\left(\begin{pmatrix} \varphi & \\ & \psi \end{pmatrix}, \begin{pmatrix} \psi & \\ & \varphi \end{pmatrix}\right),$$

as desired. The "Moreover" statement follows form Exercise 7.32, since the entries of the matrix factorization for $M^\sharp$ are those in the matrix factorization for M, together with z. □

Now we turn to the other "natural" relation. Recall that $R^\sharp$ is a free S-module of rank 2; in particular any MCM $R^\sharp$-module is a finitely generated free S-module.

7.16 Lemma. *Let N be a MCM $R^\sharp$-module. Let $\varphi\colon N \longrightarrow N$ be an S-linear homomorphism representing multiplication by z on N.*

(i) The pair $(\varphi, -\varphi)$ is a matrix factorization of f with $\operatorname{cok}(\varphi, -\varphi) \cong N^\flat$.

(ii) If N is stable, then

$$N^\flat \cong \operatorname{syz}_1^R(N^\flat)\,.$$

(iii) Consider $z\,1_N \pm \varphi$ *as an endomorphism of* $S[\![z]\!] \otimes_S N$*, a finitely generated free* $S[\![z]\!]$*-module. Then*

$$(z\,1_N - \varphi,\ z\,1_N + \varphi)$$

is a matrix factorization of $f + z^2$ *with* $\operatorname{cok}(z\,1_N - \varphi, z\,1_N + \varphi) \cong N$*. If* N *is stable, then it is a reduced matrix factorization.*

Proof. On the S-module N, $-\varphi^2$ corresponds to multiplication by $-z^2$. But since N is an $R^\sharp$-module, the action of $-z^2$ on N agrees with that of f. In other words, $-\varphi^2 = f\,1_N$. Now φ and $-\varphi$ obviously have isomorphic cokernels, each isomorphic to $N/zN = N^\flat$, so (i) and (ii) are proved. We leave the first assertion of (iii) as Exercise 7.39. For the final sentence, note that if $z\,1_N - \varphi$ contains a unit of $S[\![z]\!]$, then φ contains a unit of S as an entry. But then $z\,1_N + \varphi$ has a unit entry, so that the trivial matrix factorization $(f + z^2, 1)$ is a direct summand of $(z\,1_N - \varphi, z\,1_N + \varphi)$ up to equivalence. This exhibits $R^\sharp$ as a direct summand of N, contradicting the stability of N. □

7.17 Proposition. *Let* N *be a stable MCM* $R^\sharp$*-module. Assume that* $\operatorname{char} k \neq 2$*. Then*

$$N^{\flat\sharp} \cong N \oplus \operatorname{syz}_1^{R^\sharp} N\,.$$

Proof. Let $\varphi : N \longrightarrow N$ be the homomorphism of free S-modules representing multiplication by z as in Lemma 7.16. Then $(\varphi, -\varphi)$ is a matrix factorization of f with $\operatorname{cok}(\varphi, -\varphi) \cong N^\flat$ by the Lemma, so that

$$\begin{aligned} N^{\flat\sharp} &= \operatorname{syz}_1^{R^\sharp}(N^\flat) \\ &\cong \operatorname{cok}\left(\begin{pmatrix} -\varphi & -z\,1_N \\ z\,1_N & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & z\,1_N \\ -z\,1_N & -\varphi \end{pmatrix} \right) \end{aligned}$$

by (iii) of Lemma 7.14. Passing to the equivalent matrix

$$\begin{bmatrix} z\,1_N - \varphi & 0 \\ 0 & z\,1_N + \varphi \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\varphi & -z\,1_N \\ z\,1_N & \varphi \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

(this is legal since $1/2 \in R$ and hence the matrix $\left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$ is invertible over R), we see that

$$\begin{aligned} N^{\flat\sharp} &\cong \operatorname{cok}(z\,1_N - \varphi,\ z\,1_N + \varphi) \oplus \operatorname{cok}(z\,1_N + \varphi,\ z\,1_N - \varphi) \\ &\cong N \oplus \operatorname{syz}_1^{R^\sharp} N \end{aligned}$$

by (iii) of Lemma 7.16. □

7.18 Theorem (Knörrer). *Let $(S, \mathfrak{n}, k)$ be a complete regular local ring, f a non-zero element of $\mathfrak{n}^2$, and $R = S/(f)$.*

(i) If $R^\sharp$ has finite CM type, then so has R.

(ii) If R has finite CM type and $\operatorname{char} k \neq 2$, then $R^\sharp$ has finite CM type.

Proof. We will prove (ii), leaving the almost identical proof of (i) to the reader. Let $M_1, \dots, M_t$ be a representative list of the indecomposable non-free MCM R-modules. Write ${M_i}^\sharp = N_{i1} \oplus \cdots \oplus N_{ir_i}$, where each N_{ij} is an indecomposable $R^\sharp$-module. We will show that every indecomposable non-free MCM $R^\sharp$-module is isomorphic to some N_{ij}.

Let N be an indecomposable non-free MCM $R^\sharp$-module. Then $N \oplus \operatorname{syz}_1^{R^\sharp}(N)$ is stable, by (v) of Proposition 7.6. It follows from Proposition 7.17 that $N^\flat$ is a stable R-module. For, if $N^\flat \cong X \oplus R$, then $N \oplus \operatorname{syz}_1^{R^\sharp}(N) \cong N^{\flat\sharp} \cong X^\sharp \oplus R^\sharp$, a contradiction.

Write $N^\flat \cong M_1^{(e_1)} \oplus \cdots \oplus M_t^{(e_t)}$, where the e_i are non-negative integers. Then $N \oplus \operatorname{syz}_1^{R^\sharp}(N) \cong N^{\flat\sharp} \cong ({M_1}^\sharp)^{(e_1)} \oplus \cdots \oplus ({M_t}^\sharp)^{(e_t)}$. By KRS, N is isomorphic to a direct summand of some $M_i{}^\sharp$, and therefore isomorphic to some N_{ij}. □

7.19 Corollary (ADE Redux)**.** *Let* $(R,\mathfrak{m},k)$ *be an ADE (or simple) plane curve singularity (cf. Chapter 3, §3) over a field k of characteristic different from* 2, 3 *or* 5. *Then R has finite CM type.*

Proof. The hypersurface $R^\sharp$ is a complete Kleinian singularity and therefore has finite CM type by Theorem 5.22. By Theorem 7.18, R has finite CM type. □

7.20 Example. Assume k is a field with $\operatorname{char} k \neq 2$, and let n and d be integers with $n \geqslant 1$ and $d \geqslant 0$. Put $R_{n,d} = k[\![x,z_1,\dots,z_d]\!]/(x^{n+1}+z_1^2+\cdots+z_d^2)$. The ring $R_{n,0} = k[\![x]\!]/(x^{n+1})$ obviously has finite CM type (see Theorem 2.2). By applying Theorem 7.18 repeatedly, we see that the d-dimensional (A_n)-singularity $R_{n,d}$ has finite CM type for every d. Consequently, the ring $R = k[\![x_1,\dots,x_t,y_1,\dots,y_t]\!]/(x_1y_1+\cdots+x_ty_t)$ also has finite CM type: The change of variables $x_i = u_i+\sqrt{-1}v_i$, $y_i = u_i-\sqrt{-1}v_i$ shows that $R \cong R_{1,2d+2}$.

§3 Knörrer's periodicity

The results of the previous section on the double branched cover imply that if M and N are indecomposable MCM modules over R and $R^\sharp$, respectively, then $M^{\sharp\flat}$ and $N^{\flat\sharp}$ both decompose into precisely two indecomposable MCM modules. However, we do not yet know whether this splitting occurs on the way up or the way down. In this section we clarify this point, and use the result to prove Knörrer's theorem that the MCM modules over R are in bijection with those over the *double* double branched cover $R^{\sharp\sharp}$.

7.21 Notation. We keep all the notations of the last section, so that $(S,\mathfrak{n},k)$ is a complete regular local ring, $f \in \mathfrak{n}^2$ is a non-zero element, and $R = S/(f)$ is the corresponding complete hypersurface singularity. In addition, we assume throughout this section that k is an *algebraically closed* field of characteristic different from 2.

We first prove a sort of converse to Lemma 7.16.

7.22 Lemma. *Let M be a MCM R-module such that $M \cong \operatorname{syz}_1^R M$. Then $M \cong \operatorname{cok}(\varphi_0,\varphi_0)$ for an $n\times n$ matrix φ_0 satisfying $\varphi_0^2 = f\,I_n$.*

Proof. We may assume that M is indecomposable, and write $M = \operatorname{cok}(\varphi\colon G \longrightarrow F,\ \psi\colon F \longrightarrow G)$ by Theorem 7.7. By assumption there is an equivalence of matrix factorizations $(\alpha,\beta)\colon (\varphi,\psi) \longrightarrow (\psi,\varphi)$, i.e. a commutative diagram of free S-modules

$$\begin{array}{ccccc} F & \xrightarrow{\psi} & G & \xrightarrow{\varphi} & F \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ G & \xrightarrow[\varphi]{} & F & \xrightarrow[\psi]{} & G \end{array}$$

with α and β isomorphisms. Thus $\operatorname{cok}(\beta\alpha,\alpha\beta)$ is an automorphism of M. Since M is indecomposable and R is complete, $\operatorname{End}_R(M)$ is a nc-local ring. Furthermore, $\operatorname{End}_R(M)/\operatorname{rad}\operatorname{End}_R(M) \cong k$ since k is algebraically closed. Hence we may write

$$(\beta\alpha,\alpha\beta) = (1_F,1_G) + (\rho_1,\rho_2)$$

with $\operatorname{cok}(\rho_1,\rho_2) \in \operatorname{rad}\operatorname{End}_R(M)$. In particular $\alpha\rho_1 = \rho_2\alpha$ and $\beta\rho_2 = \rho_1\beta$.

Choose a (convergent) power series representing $(1+x)^{-1/2}$ and set

$$\alpha' = \alpha(1_F + \rho_1)^{-1/2} = (1_G + \rho_2)^{-1/2}\alpha$$

$$\beta' = \beta(1_G + \rho_2)^{-1/2} = (1_G + \rho_1)^{-1/2}\beta\,.$$

Then the homomorphism of matrix factorizations $(\alpha', \beta')\colon (\varphi, \psi) \longrightarrow (\psi, \varphi)$ satisfies $\beta'\alpha' = 1_F$ and $\alpha'\beta' = 1_G$. Finally choose an automorphism γ of the free module $F \cong S^{(n)} \cong G$ such that $\gamma^2 = \alpha'$. Then

$$\varphi_0 := \gamma\psi\gamma = \gamma^{-1}\varphi\gamma^{-1}$$

satisfies $\varphi_0^2 = f\,I_n$ and $\operatorname{cok}(\varphi_0, \varphi_0) \cong M$. □

Let $R^\sharp = S[\![z]\!]/(f + z^2)$ be the double branched cover of the previous section. Then $R^\sharp$ carries an involution σ, which fixes S and sends z to $-z$. Denote by $R^\sharp[\sigma]$ the *twisted group ring* of the two-element group generated by σ (cf. Chapter 4), i.e. $R^\sharp[\sigma] = R^\sharp \oplus (R^\sharp \cdot \sigma)$ as $R^\sharp$-modules, with multiplication

$$(r + s\sigma)(r' + s'\sigma) = (rr' + s\sigma(s')) + (rs' + s\sigma(r'))\sigma\,.$$

The modules over $R^\sharp[\sigma]$ are precisely the $R^\sharp$-modules carrying a compatible action of the involution σ. We will call a $R^\sharp[\sigma]$-module N *maximal Cohen–Macaulay* (MCM, as usual) if it is MCM as an $R^\sharp$-module.

Let N be a finitely generated $R^\sharp[\sigma]$-module, and set

$$N^+ = \{x \in M \mid \sigma(x) = x\}$$

$$N^- = \{x \in M \mid \sigma(x) = -x\}\,.$$

Then $N = N^+ \oplus N^-$ as $R^\sharp$-modules. If N is a MCM $R^\sharp[\sigma]$-module, then it follows that N^+ and N^- are MCM modules over $(R^\sharp)^+ = S$, i.e. free S-modules of finite rank.

7.23 Definition. Let R, $R^\sharp$, and $R^\sharp[\sigma]$ be as above.

(i) Let N be a MCM $R^\sharp[\sigma]$-module. Define a MCM R-module $\mathscr{A}(N)$ as follows: Multiplication by z, resp. $-z$, defines an S-linear map between finitely generated free S-modules

$$\varphi\colon N^+ \longrightarrow N^-, \quad \text{resp.} \quad \psi\colon N^- \longrightarrow N^+$$

which together constitute a matrix factorization of f. Set

$$\mathscr{A}(N) = \operatorname{cok}(\varphi, \psi).$$

(ii) Let M be a MCM R-module, and define a MCM $R^\sharp$-module $\mathscr{B}(M)$ with compatible σ-action as follows: Write $M = \operatorname{cok}(\varphi\colon G \longrightarrow F, \psi\colon F \longrightarrow G)$ with F and G finitely generated free S-modules. Set

$$\mathscr{B}(M) = G \oplus F,$$

with multiplication by z defined via

$$z(x, y) = (-\psi(y), \varphi(x))$$

and σ-action

$$\sigma(x, y) = (x, -y).$$

7.24 Proposition. *The mappings $\mathscr{A}(-)$ and $\mathscr{B}(-)$ induce mutually inverse bijections between the isomorphism classes of MCM R-modules and the isomorphism classes of MCM $R^\sharp[\sigma]$-modules having no direct summand isomorphic to $R^\sharp$.*

Proof. It is easy to verify that $\mathscr{A}(R^\sharp) = \operatorname{cok}(1, f) = 0$ (here $R^\sharp$ has the natural σ-action), and that $\mathscr{A}\mathscr{B}$ and $\mathscr{B}\mathscr{A}$ are naturally the identities otherwise. $\square$

In fact $\mathscr{A}$ and $\mathscr{B}$ can be used to define equivalences of categories between the MCM $R^\sharp[\sigma]$-modules and the matrix factorizations of f, though we will not need this fact.

7.25 Lemma. *Let M be a MCM R-module. Then*

$$M^\sharp \cong \mathscr{B}(M)$$

as $R^\sharp$-modules, ignoring the action of σ on the right-hand side. Thus $M^\sharp$ admits the structure of a $R^\sharp[\sigma]$-module for every MCM R-module M.

Proof. Write $M = \operatorname{cok}(\varphi\colon G \longrightarrow F,\ \psi\colon F \longrightarrow G)$, so that $\mathscr{B}(M) = G \oplus F$ as S-modules, with $z(x,y) = (-\psi(y), \varphi(x))$. On the other hand, by Lemma 7.14,

$$M^\sharp \cong \operatorname{cok}\left(\begin{pmatrix} \psi & -z\,1_G \\ z\,1_F & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & z\,1_F \\ -z\,1_G & \psi \end{pmatrix}\right).$$

Choose bases for the free modules to write

$$M^\sharp \cong (R^\sharp)^{(n)} \oplus (R^\sharp)^{(n)} \Big/ \operatorname{span}\big((\psi(u), -zu), (zu, \varphi(u))\big)$$

where u runs over $(R^\sharp)^{(n)}$. Now $R^\sharp \cong S \oplus S\cdot z$ as S-modules, so writing $u = v + wz$ gives

$$M^\sharp \cong \frac{S^{(n)} \oplus S^{(n)} \oplus S^{(n)} \oplus S^{(n)}}{\operatorname{span}\big((\psi(v),0,0,-v), (0,\psi(w), fw, 0), (0, v, \varphi(v), 0, 0), (-fw, 0, 0, \varphi(w))\big)}$$

as v and w run over $S^{(n)}$. Multiplication by z on this representation of $M^\sharp$ is defined by

$$z(s_1, s_2, s_3, s_4) = (-fs_2, s_1, -fs_4, s_3)$$

for $s_1, s_2, s_3, s_4 \in S^{(n)}$. We therefore define a homomorphism of $R^\sharp$-modules $M^\sharp \longrightarrow \mathscr{B}(M)$ by

$$(s_1, s_2, s_3, s_4) \mapsto (-fs_4 - \psi(s_1), -fs_2 - \varphi(s_3)).$$

This is easily checked to be well-defined and surjective, hence an isomorphism of $R^\sharp$-modules. $\square$

7.26 Proposition. *Let N be a stable MCM $R^\sharp$-module. Then N is in the image of $(-)^\sharp$, that is, $N \cong M^\sharp$ for some MCM R-module M, if and only if $N \cong \operatorname{syz}_1^{R^\sharp} N$.*

Proof. If $N \cong M^\sharp$ and N is stable, then $N \cong \operatorname{syz}_1^{R^\sharp} N$ by Lemma 7.14(iii). For the converse, it suffices to show that if N is an indecomposable MCM $R^\sharp$-module such that $N \cong \operatorname{syz}_1^{R^\sharp} N$, then N has the structure of an $R^\sharp[\sigma]$-module. Indeed, in that case $N \cong \mathscr{B}(\mathscr{A}(N)) \cong \mathscr{A}(N)^\sharp$ by Proposition 7.24 and Lemma 7.25, so that N is in the image of $(-)^\sharp$.

By assumption, there is an isomorphism of $R^\sharp$-modules $\alpha\colon N \longrightarrow \operatorname{syz}_1^{R^\sharp} N$, which induces an isomorphism $\beta = \operatorname{syz}_1^{R^\sharp}(\alpha)\colon \operatorname{syz}_1^{R^\sharp} N \longrightarrow N$. As N is indecomposable, $R^\sharp$ is complete, and k is algebraically closed we may, as in Lemma 7.22, assume that

$$\beta\alpha = 1_N + \rho\,,$$

where $\rho \in \operatorname{rad}\operatorname{End}_{R^\sharp}(N)$. Choose again a convergent power series for $(1+x)^{-1/2}$, and set

$$\widetilde{\alpha} = \alpha(1_N + \rho)^{-1/2}\,.$$

Then $\widetilde{\alpha}$ itself induces an isomorphism $\widetilde{\beta} = \operatorname{syz}_1^{R^\sharp}(\widetilde{\alpha})\colon \operatorname{syz}_1^{R^\sharp} N \longrightarrow N$, which is easily seen to be

$$\widetilde{\beta} = (1_N + \rho)^{-1/2}\beta$$

so that $\widetilde{\beta}\widetilde{\alpha} = 1_N$. Therefore $\widetilde{\alpha}$ defines an action of σ on N, whence N has a structure of $R^\sharp[\sigma]$-module. $\square$

Now we can say exactly which modules decompose upon sharping or flatting.

7.27 Proposition. *Keep all the notation of 7.21. In particular, assume that k is an algebraically closed field of characteristic not equal to 2.*

(i) Let M be an indecomposable non-free MCM R-module. Then $M^\sharp$ is indecomposable if, and only if, $M \cong \operatorname{syz}_1^R M$. In this case $M^\sharp \cong N \oplus \operatorname{syz}_1^{R^\sharp} N$ for an indecomposable $R^\sharp$-module N such that $N \not\cong \operatorname{syz}_1^{R^\sharp} N$.

(ii) Let N be a non-free indecomposable MCM $R^\sharp$-module. Then $N^\flat$ is indecomposable if, and only if, $N \cong \operatorname{syz}_1^{R^\sharp} N$. In this case $N^\flat \cong M \oplus \operatorname{syz}_1^R M$ for an indecomposable R-module M such that $M \not\cong \operatorname{syz}_1^R M$.

Proof. First let ${}_R M$ be indecomposable, MCM, and non-free. If $M \cong \operatorname{syz}_1^R M$, then $M \cong \operatorname{cok}(\varphi, \varphi)$ for some φ by Lemma 7.22, so that by Lemma 7.14

$$\begin{aligned} M^\sharp &\cong \operatorname{cok}\left(\begin{pmatrix} \varphi & -z\,1_F \\ z\,1_F & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & z\,1_F \\ -z\,1_F & \varphi \end{pmatrix} \right) \\ &\cong \operatorname{cok}\left(\varphi + iz\,1_F,\ \varphi - iz\,1_F\right) \oplus \operatorname{cok}\left(\varphi - iz\,1_F,\ \varphi + iz\,1_F\right) \end{aligned}$$

is decomposable, where i is a square root of -1 in k. Conversely, suppose $M^\sharp \cong N_1 \oplus N_2$ for non-zero MCM $R^\sharp$-modules N_1 and N_2. Then

$$N_1{}^\flat \oplus N_2{}^\flat \quad \cong \quad M^{\sharp\flat} \quad \cong \quad M \oplus \operatorname{syz}_1^R M$$

by Proposition 7.15. Since M is indecomposable and R is complete, by KRS we may interchange N_1 and N_2 if necessary to assume that $N_1{}^\flat \cong M$ and $N_2{}^\flat \cong \operatorname{syz}_1^R M$. Note that N_1 is stable since M is not free. Then $\operatorname{syz}_1^R(N_1{}^\flat) \cong N_1{}^\flat$ by Lemma 7.16(ii), so

$$M \quad \cong \quad N_1{}^\flat \quad \cong \quad \operatorname{syz}_1^R(N_1{}^\flat) \quad \cong \quad \operatorname{syz}_1^R M,$$

as desired.

Next let N be a non-free indecomposable MCM $R^\sharp$-module. By Proposition 7.26, if $N \cong \operatorname{syz}_1^{R^\sharp} N$ then $N \cong M^\sharp$ for some ${}_R M$, whence

$$N^\flat \quad \cong \quad M^{\sharp\flat} \quad \cong \quad M \oplus \operatorname{syz}_1^R M$$

is decomposable by Proposition 7.15. The converse is shown as above.

To complete the proof of (i), suppose $M \cong \operatorname{syz}_1^R M$, so that $M^\sharp \cong N \oplus \operatorname{syz}_1^{R^\sharp} N$ for some ${}_{R^\sharp}N$. Then $M^{\sharp\flat} \cong M \oplus \operatorname{syz}_1^R M$ has exactly two indecomposable direct summands, so $N^\flat$ must be indecomposable. Hence $N \not\cong \operatorname{syz}_1^{R^\sharp} N$ by the part of (ii) we have already proved. The last sentence of (ii) follows similarly. □

7.28 Definition. In the notation of 7.21, set

$$R^{\sharp\sharp} = S[[u,v]]/(f + uv)\,.$$

(Since we assume k is algebraically closed of characteristic not 2, this is isomorphic to $(R^\sharp)^\sharp$.) For a MCM R-module $M = \operatorname{cok}(\varphi\colon G \longrightarrow F,\ \psi\colon F \longrightarrow G)$, we define a MCM $R^{\sharp\sharp}$-module $M^{\maltese}$ by

$$M^{\maltese} = \operatorname{cok}\left(\begin{pmatrix} \varphi & -v\,1_F \\ u\,1_G & \psi \end{pmatrix}, \begin{pmatrix} \psi & v\,1_G \\ -u\,1_F & \varphi \end{pmatrix}\right).$$

Here we continue our convention (cf. 7.13) of using 1_F and 1_G for the identity maps on the free $S[[u,v]]$-modules induced from F and G.

We leave verification of the next lemma as an exercise.

7.29 Lemma. *Keep the notation of the Definition.*

(i) $(M^{\sharp})^{\sharp} \cong M^{\maltese} \oplus \operatorname{syz}_1^{R^{\sharp\sharp}}(M^{\maltese})$.

(ii) $(M^{\maltese})^{\flat\flat} \cong M \oplus \operatorname{syz}_1^R M$.

(iii) $(\operatorname{syz}_1^R M)^{\maltese} \cong \operatorname{syz}_1^{R^{\sharp\sharp}}(M^{\maltese})$.

Now we can prove a more precise version of Theorem 7.18.

7.30 Theorem (Knörrer)**.** *The association $M \mapsto M^{\maltese}$ defines a bijection between the isomorphism classes of indecomposable non-free MCM modules over R and over $R^{\sharp\sharp}$.*

Proof. Let M be a non-free indecomposable MCM R-module. Then $M^{\sharp\sharp}$ splits into precisely two indecomposable direct summands by Proposition 7.27(i), so that $M^{\maltese}$ is indecomposable by Lemma 7.29(i).

If M' is another indecomposable MCM R-module with $(M')^{\maltese} \cong M^{\maltese}$, then by Lemma 7.29(ii) we have either $M' \cong M$ or $M' \cong \operatorname{syz}_1^R M$. Assume $M \not\cong M' \cong \operatorname{syz}_1^R M$. Then by Proposition 7.27 $M^{\sharp}$ is indecomposable. Therefore the two indecomposable direct summands of $M^{\sharp\sharp}$ are non-isomorphic by Proposition 7.27 again. It follows from Lemma 7.29(i) and (iii) that

$$M^{\maltese} \quad \not\cong \quad \operatorname{syz}_1^{R^{\sharp\sharp}}(M^{\maltese}) \quad \cong \quad (\operatorname{syz}_1^R)^{\maltese} \quad \cong \quad (M')^{\maltese},$$

a contradiction.

Finally let N be an indecomposable non-free MCM $R^{\sharp\sharp}$-module. We must show that N is a direct summand of $M^{\maltese}$ for some ${}_R M$. From Lemma 7.29(i) we find

$$\begin{aligned}(N^{\flat\flat})^{\sharp\sharp} &\cong (N^{\flat\flat})^{\maltese} \oplus \operatorname{syz}_1^{R^{\sharp\sharp}}((N^{\flat\flat})^{\maltese}) \\ &\cong (N^{\flat\flat} \oplus \operatorname{syz}_1^{R^{\sharp\sharp}}(N^{\flat\flat}))^{\maltese}.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(N^{\flat\flat})^{\sharp\sharp} &\cong \left((N^{\flat})^{\flat\sharp}\right)^{\sharp} \\
&\cong \left(N^{\flat} \oplus \operatorname{syz}_1^{R^\sharp}(N^{\flat})\right)^{\sharp} \\
&\cong N^{\flat\sharp} \oplus \operatorname{syz}_1^{R^\sharp}(N^{\flat\sharp}) \\
&\cong N^{(2)} \oplus (\operatorname{syz}_1^{R^\sharp} N)^{(2)}.
\end{aligned}
$$

Hence N is in the image of $(-)^{\maltese}$. □

We will not prove Knörrer's stronger result than in fact $M \longleftrightarrow M^{\maltese}$ induces an equivalence between the stable categories of MCM modules; see [Knö87] for details.

§4 Exercises

7.31 Exercise. Prove that commutativity of one of the squares in the diagram (7.5.1) implies commutativity of the other.

7.32 Exercise. Prove that a matrix factorization (φ, ψ) is reduced if and only if all entries of φ and ψ are in the maximal ideal $\mathfrak{n}$ of S.

7.33 Exercise. Verify exactness of the sequence (7.5.2).

7.34 Exercise. Let Λ be a ring, not necessarily commutative, with exactly one maximal left ideal, and let M and N be left Λ-modules. If $M \oplus N$ has a direct summand isomorphic to ${}_\Lambda\Lambda$, then either M or N has a direct summand isomorphic to ${}_\Lambda\Lambda$. Is this still true if, instead, Λ has exactly one maximal two-sided ideal?

7.35 Exercise. Fill in the details of the proofs of Proposition 7.6 and Theorem 7.7.

7.36 Exercise. Let $(S, \mathfrak{n}, k)$ be a complete local ring, let $f \in \mathfrak{n}^2 \setminus \{0\}$, and put $g = uf$, where u is a unit of R. If k is closed under square roots and has characteristic different from 2, show that $S[[z]]/(f + z^2) \cong S[[z]]/(g + z^2)$.

7.37 Exercise. Prove that z is a non-zerodivisor of $R^\sharp = S[[z]]/(f + z^2)$.

7.38 Exercise. Prove that the natural map $S[z]/(f + z^2) \longrightarrow S[[z]]/(f + z^2)$ is an isomorphism. In particular, $R^\sharp$ is a free S-module with basis $\{1, z\}$. Show by example that if S is not assumed to be complete then $S[[z]]/(f + z^2)$ need not be finitely generated as an S-module.

7.39 Exercise. With notation as in the proof of (iii) of Lemma 7.16, show that the sequence

$$S[[z]]^{(n)} \xrightarrow{zI_n - \varphi} S[[z]]^{(n)} \longrightarrow N \longrightarrow 0$$

is exact. (Hint: Use Exercise 7.38 and choose bases.)

7.40 Exercise. Prove Lemma 7.29.

8

Hypersurfaces with finite CM type

In this chapter we will show that the complete, equicharacteristic hypersurface singularities with finite CM type are exactly the higher-dimensional ADE singularities. In any characteristic but two, Theorem 8.6 shows that such a hypersurface of dimension $d \geqslant 2$ is the double branched cover (Chapter 7) of one with dimension $d-1$. In Theorem 8.7, proved in 1987 by Buchweitz, Greuel, Knörrer and Schreyer [Knö87, BGS87], we restrict to rings having an algebraically closed coefficient field of characteristic different from 2, 3, and 5, and show that finite CM type is equivalent to *simplicity* (Definition 8.1), and to being an ADE singularity. We'll also prove Herzog's theorem [Her78b]: Gorenstein rings of finite CM type are abstract hypersurfaces. In §3 we derive matrix factorizations for the Kleinian singularities (two-dimensional ADE hypersurface singularities). At the end of the chapter we will discuss the situation in characteristics $2,3$ and 5. Later, in Chapter 11 we will see how to eliminate the assumption that R be complete, and also we'll weaken "algebraically closed" to "perfect".

§1 Hypersurfaces in characteristics $\neq 2,3,5$

For this section k is an algebraically closed field and d is a positive integer. Put $S = k[[x_0,\dots,x_d]]$ and $\mathfrak{n} = (x_0,\dots,x_d)$. We will consider d-dimensional hypersurface singularities: rings of the form $S/(f)$ where $0 \neq f \in \mathfrak{n}^2$.

8.1 Definition. A non-zero power series $f \in \mathfrak{n}^2$ is *simple* provided S has only finitely many ideals I for which $f \in I^2$. A complete local ring R is a (d-dimensional) *simple hypersurface singularity* provided R is isomorphic to a ring of the form $S/(f)$ for some simple power series f.

8.2 Theorem (Buchweitz, Greuel and Schreyer [BGS87])**.** *Let f be a non-zero non-unit of $S = k[[x_0,\dots,x_d]]$. If $S/(f)$ has finite CM type, then f is a simple power series.*

Proof. Given a reduced matrix factorization (φ,ψ) of f, let $L(\varphi,\psi)$ be the ideal of S generated by the entries of $[\varphi\,|\,\psi]$. By Theorem 7.7, f has, up to equivalence, only finitely many indecomposable reduced matrix factorizations $(\varphi_1,\psi_1),\dots,(\varphi_t,\psi_t)$. Let $\mathscr{S}$ be the set of ideals that are ideal sums of subsets of $\{L(\varphi_1,\psi_1),\dots,L(\varphi_t,\psi_t)\}$. Then $\mathscr{S}$ is finite, and we claim that every proper ideal L for which $f \in L^2$ belongs to $\mathscr{S}$. To see this, let $a_0,\dots,a_r$ generate L, and write $f = a_0b_0 + \cdots + a_rb_r$, with $b_i \in L$. For $0 \leqslant s \leqslant r$, let $f_s = a_0b_0 + \cdots + a_sb_s$. Put $\sigma_0 = a_0$, $\tau_0 = b_0$, and for $1 \leqslant s \leqslant r$ define, inductively, a $2^s \times 2^s$ matrix factorization of f_s by

(8.2.1) $$\sigma_s = \begin{bmatrix} a_sI_{2^{s-1}} & \sigma_{s-1} \\ \tau_{s-1} & -b_sI_{2^{s-1}} \end{bmatrix} \quad \text{and} \quad \tau_s = \begin{bmatrix} b_sI_{2^{s-1}} & \sigma_{s-1} \\ \tau_{s-1} & -a_sI_{2^{s-1}} \end{bmatrix}.$$

Letting $\sigma = \sigma_r$ and $\tau = \tau_r$, we see that (σ,τ) is a reduced matrix factorization of f with $L(\sigma,\tau) = L$. Write (σ,τ) as a direct sum of the indecomposable matrix factorizations, $(\sigma,\tau) = (\varphi_1,\psi_1)^{(n_1)} \oplus \cdots \oplus (\varphi_t,\psi_t)^{(n_t)}$, and note that $L = L(\sigma,\tau) = \sum \{L(\varphi_j,\psi_j)\,|\,n_j > 0\} \in \mathscr{S}$. □

The following lemma (cf. [Yos90, Lemma 8.2]), together with the Weierstrass Preparation Theorem, will show that every simple singularity of dimension $d \geqslant 2$ is a double branched cover of a $(d-1)$-dimensional simple singularity:

8.3 Lemma. *Let R be a simple singularity of dimension $d \geqslant 1$.*

(i) *R is reduced.*

(ii) $\mathrm{e}(R) \leqslant 3$.

(iii) *If $d \geqslant 2$, then* $\mathrm{e}(R) = 2$.

Proof. Write $R = S/(f)$, where $S = k[\![x_0,\dots,x_d]\!]$ and f is a simple power series in S.

(i) Suppose R is not reduced. Then f has a repeated prime factor, and we can write $f = gh^2$, where $g \in S$ and $h \in \mathfrak{n}$. Now $\dim(S/(h)) = d \geqslant 1$, so $S/(h)$ has infinitely many ideals. Therefore S has infinitely many ideals that contain h, and f is in the square of each, a contradiction.

(ii) Suppose $e(R) \geqslant 4$. Then $f \in \mathfrak{n}^4$ (cf. Example 8.28). If L is any ideal such that $\mathfrak{n}^2 \subsetneq L \subsetneq \mathfrak{n}$, then $f \in J^2$. These ideals correspond to non-zero proper subspaces of the k-vector space $\mathfrak{n}/\mathfrak{n}^2$, so there are infinitely many of them, a contradiction.

(iii) We know that $\mathrm{e}(R)$ is either 2 or 3, so we suppose $\mathrm{e}(R) = 3$. Write $f = f_3 + f_4 + \cdots$, where $f_i \in T := k[x_0,x_1,\dots,x_d]$ is homogeneous of degree i and $f_3 \neq 0$. Set $V = \{p \in \mathbb{P}^d_k \mid f_3(p) = 0\}$. Then $\dim(V) = d-1 \geqslant 1$; in particular, V is infinite. Given $\lambda \in V$, let $I(\lambda)$ be the ideal of T generated by forms vanishing at λ. Then $I(\lambda) = (\ell_1,\dots,\ell_d)T$, where the ℓ_i form a basis for the d-dimensional vector space of linear forms in $I(\lambda)$. Now put $L_\lambda = (\ell_1,\dots,\ell_d)S + \mathfrak{n}^2$. Since $f_3 \in (\ell_1,\dots,\ell_d)\mathfrak{n}^2$ and $f - f_3 \in \mathfrak{n}^4$, we see that

$f \in L_\lambda^2$. Finally, if λ and μ are two distinct points of V, we can choose a linear form ℓ vanishing at λ but not at μ. Then $\ell \in L_\lambda \setminus L_\mu$. Thus we have infinitely many distinct ideals L_λ with $f \in L_\lambda^2$ for each λ, and once again simplicity is contradicted. □

We refer to [Lan02, Chapter IV, Theorem 9.2] for the following version of the Weierstrass Preparation Theorem:

8.4 Theorem (WPT). *Let $(D, \mathfrak{m})$ be a complete local ring, and let $g \in D[\![x]\!]$. Suppose $g = a_0 + a_1 x + \cdots + a_e x^e +$ higher degree terms, with $a_0, a_1, \ldots, a_{e-1} \in \mathfrak{m}$ and $a_e \in D \setminus \mathfrak{m}$. Then there exist $b_1, \ldots, b_e \in \mathfrak{m}$ and a unit $u \in D[\![x]\!]$ such that $g = (x^e + b_1 x^{e-1} + \cdots + b_e)u$.*

8.5 Corollary. *Let k be an infinite field, and let g be a non-zero power series in $k[\![x_0, \ldots, x_n]\!]$, $n \geqslant 1$. Assume that the order e of g is at least 2 and is not a multiple of* char(k). *Then $R = \ell[\![x_0, \ldots, x_n]\!]/(g)$ is isomorphic as a k-algebra to a ring of the form $k[\![x_0, \ldots, x_n]\!]/(h)$, where $h = x_n^e + b_2 x_n^{e-2} + b_3 x_n^{e-3} + \cdots + b_{e-1} x_n + b_e$ and where $b_2, \ldots, b_e$ are non-units of $D = k[\![x_0, \ldots, x_{n-1}]\!]$.*

Proof. We will make a linear change of variables, following Zariski and Samuel [ZS75, p. 147], so that Theorem 8.4 applies, with respect to the new variables. Write $g = g_e + g_{e+1} + \cdots$, where each g_j is a homogeneous polynomial of degree j and $g_e \neq 0$. Then $x_n g_e \neq 0$, and, since k is infinite, there is a point $(c_0, c_1, \ldots, c_n) \in k^{n+1}$ such that $(x_n g_e)(c_0, \ldots, c_n) \neq 0$. Then $c_n \neq 0$, and since $x_n g_e$ is homogeneous we can scale and assume that $c_n = 1$.

We change variables as follows:

$$\varphi\colon x_i \mapsto \begin{cases} x_i + c_i x_n & \text{if } i < n \\ x_n & \text{if } i = n\,. \end{cases}$$

Now, $\varphi(g) = \varphi(g_e) + \text{higher-order terms}$, and $\varphi(g_e)$ contains the term $g_e(c_0, c_1, \dots, c_{n-1}, 1)x_n^e = cx_n^e$, where $c \in k^\times$. It follows that $\varphi(g)$ has the form required in Theorem 8.4, with $D = k[\![x_0,\dots,x_{d-1}]\!]$ and $x = x_n$. Replacing g by $\varphi(g)$, we now have $g = (x_n^e + b_1 x_n^{e-1} + \cdots + b_e)u$, where the b_i are non-units of D and u is a unit of $k[\![x_0,\dots,x_n]\!]$. Finally, we put $h = gu^{-1}$, and make the substitution $x_n \mapsto x_n - \frac{b_1}{e}x_n^{e-1}$ to eliminate the term of degree $e-1$ in h. □

Here is the main theorem of this chapter, proved in 1987 by Buchweitz, Greuel, Knörrer and Schreyer [Knö87, BGS87].

8.6 Theorem. *Let k be an algebraically closed field of characteristic different from 2, and put $S = k[\![x_0,\dots,x_d]\!]$, where $d \geqslant 2$. Let $R = S/(f)$, where $0 \neq f \in (x_0,\dots,x_d)^2$. Then R has finite CM type if and only if there is a non-zero element $g \in (x_0,x_1)^2 k[\![x_0,x_1]\!]$ such that $k[\![x_0,x_1]\!]/(g)$ has finite CM type and $R \cong k[\![x_0,\dots,x_d]\!]/(g + x_2^2 + \cdots + x_d^2)$.*

Proof. The "if" direction follows from Theorem 7.18 and induction on d. For the converse, we assume that R has finite CM type. Then R is a simple singularity (Theorem 8.2), and (iii) of Lemma 8.3 implies that $\mathrm{e}(R) = 2$. Since $\operatorname{char}(k) \neq 2$, we may assume, by Corollary 8.5, that $f = x_d^2 + b$, with $b \in (x_0,\dots,x_{d-1})^2 k[\![x_0,x_1,\dots,x_{d-1}]\!]$. Also, $b \neq 0$, by (i) of Lemma 8.3. Then $R = A^\#$, where $A = k[\![x_0,x_1,\dots,x_{d-1}]\!]/(b)$. Now Theorem 7.18 implies that B has finite CM type. If $d = 2$ we set $g = b$, and we're done. Otherwise, we simply repeat the procedure, eventually getting f into the desired form. □

If the characteristic of k is different from 2, 3 and 5, we get a more explicit version of the theorem.

8.7 Theorem. *Let k be an algebraically closed field with* $\operatorname{char}(k) \neq 2,3,5$*, let $d \geqslant 1$, and let $R = k[\![x,y,x_2,\dots,x_d]\!]/(f)$, where $0 \neq f \in (x,y,x_2,\dots,x_d)^2$. These are equivalent:*

(i) R has finite CM type.

(ii) f is a simple power series.

(iii) R is a simple singularity.

(iv) $R \cong k[\![x,y,x_2,\dots,x_d]\!]/(g+x_2^2+\cdots+x_d^2)$, where $g \in k[x,y]$ defines a one-dimensional ADE singularity (cf. Chapter 3, §3).

The proof of this theorem will occupy the rest of the section.

Proof. (i) $\implies$ (ii) by Theorem 8.2, and (ii) $\implies$ (iii) trivially.

(iii) $\implies$ (iv): Suppose first that $d \geqslant 2$; then $\mathrm{e}(R) = 2$ by (iii) of Lemma 8.3. By Corollary 8.5, we may assume that $f = x_d^2 + b$, where b is a non-zero non-unit of $k[\![x_0,x_1,\dots,x_{d-1}]\!]$. Then $R = A^{\#}$, where $A = k[\![x_0,x_1,\dots,x_{d-1}]\!]/(b)$. Simplicity passes from R to A: If there were an infinite number of ideals L_i of $k[\![x_0,x_1,\dots,x_{d-1}]\!]$ with $b \in L_i^2$ for each i, we would have $x_d^2 + b \in (L_iS + x_dS)^2$ for each i, where $S = k[\![x_0,\dots,x_d]\!]$. Since $(L_iS+x_dS) \cap k[\![x_0,x_1,\dots,x_{d-1}]\!] = L_i$, the extended ideals would be distinct, contradicting simplicity of R. Thus we can continue the process, dropping dimensions till we reach dimension one. It suffices, therefore, to prove that (iii) $\implies$ (iv) when $d = 1$.

Changing notation, we set $S = k[[y,x]]$ and $\mathfrak{n} = (y,x)S$. (The silly ordering of the variables stems from the choice of the normal forms for the ADE singularities in Chapter 3, §3.) We have a simple power series $f \in \mathfrak{n}^2 \setminus \{0\}$, and we want to show that $R = S/(f)$ is an ADE singularity. We will follow Yoshino's proof of [Yos90, Proposition 8.5] closely, adding a few details and making a few necessary modifications (some of them to accommodate non-zero characteristic $p > 5$).

Suppose first that $\mathrm{e}(R) = 2$. By Corollary 8.5, we may assume that $f = x^2 + g$, where $g \in yk[[y]]$. Then $g \neq 0$ by (i) of Lemma 8.3, and we write $g = x^t u$, where $u \in k[[x]]^\times$. Then $t \geqslant 2$, else R would be a discrete valuation ring. Replacing f by $u^{-1}f$, we now have $f = u^{-1}x^2 + y^t$. Now we let $v \in k[[y]]^\times$ be a square root of u^{-1} (using Corollary A.27) and make the change of variables $x \mapsto vx$. Then $f = x^2 + y^t$, so R is an (A_{t-1})-singularity.

Before taking on the more challenging case $\mathrm{e}(R) = 3$, we pause for a primer on tangent directions of an analytic curve. Given any non-zero, non-unit power series $g \in K[[x,y]]$, where K is any algebraically closed field, let g_e be the initial form of g. Thus g_e is a non-zero homogeneous polynomial of degree $e \geqslant 1$ and $g = g_e +$ higher-degree forms. We can factor g_e as a product of powers of distinct linear forms:

$$g_e = \ell_1^{m_1} \cdot \cdots \cdot \ell_h^{m_h},$$

where each $m_i > 0$ and the linear forms ℓ_i are not associates in $K[x,y]$. (To do this, dehomogenize, then factor, then homogenize.) The *tangent lines* to the curve $g = 0$ are the lines $\ell_i = 0$, $1 \leqslant i \leqslant h$. We will need the "Tangent Lemma" (cf. [Abh90, p. 141]):

8.8 Lemma. *Let g be a non-zero non-unit in $K[[x,y]]$, where K is an algebraically closed field. If g is irreducible, then g has a unique tangent line.*

The lemma is exemplified by the nodal cubic $g = y^2 - x^2 - x^3 = y^2 - x^2(1+x)$, which, though irreducible in $K[x,y]$, factors in $K[[x,y]]$, as long as $\operatorname{char}(K) \neq 2$. It has two distinct tangent lines, $x+y=0$ and $x-y=0$; and indeed it factors: If h is a square root of $1+x$ (obtained from the binomial expansion of $(1+x)^{\frac{1}{2}}$, or via Hensel's Lemma—Corollaries 1.8 and A.27), then $g = (y+xh)(y-xh)$.

We will use the following lemma (cf. [Yos90, Lemma 8.4]) to control the order of the higher-degree terms in the normal forms for f:

8.9 Lemma. *Let $f \in k[[x,y]]$ be a simple power series. Let $\alpha, \beta \in (x,y)k[[x,y]]$. Then $f \notin (\alpha, \beta^2)^3$.*

Proof. For each $\lambda \in k$, put $I_\lambda = (\alpha + \lambda\beta^2, \beta^3)$, and check easily that $(\alpha,\beta^2)^3 \subseteq I_\lambda^2$. Moreover, as shown in the proof of [Yos90, Lemma 8.4], $I_\lambda \neq I_\mu$ if λ and μ are distinct elements of k. By simplicity, $f \notin (\alpha,\beta^2)^3$. □

Now assume $\mathrm{e}(R) = 3$, and write $f = x^3 + xa + b$, where $a, b \in yk[[y]]$. Since f has order 3, we have $a \in y^2k[[y]]$ and $b \in y^3k[[y]]$.

8.10 Case. *f is irreducible.*

Then $b \neq 0$. The initial form f_3 of f is a power of a single linear form by Lemma 8.8, and it follows that $f_3 = x^3$. Therefore the order of a is at least 3, and b has order $n \geqslant 4$. If $a = 0$ we have $f = x^3 + uy^n$, where $u \in k[[y]]^\times$. By extracting a cube root of u^{-1} (using Corollary A.27), we may assume that $f = x^3 + y^n$. Now Lemma 8.9 implies that n must be 4 or 5, and R is an (E_6)

or (E_8) singularity. If $a \neq 0$ we can assume that $f = x^3 + uxy^m + y^n$, where $m \geqslant 3$ and $u \in k[[y]]^\times$. Suppose for a moment that $m = 3$ and $n \geqslant 5$. Then one can find a root $\xi \in k[[y]]^\times$ of $T^3 + uT^2 + y^{2n-9} = 0$ by lifting the simple root $-\overline{u}$ of $T^3 - \overline{u}T^2 \in k[T]$. One checks that then $x = \xi^{-1}y^{m-3}$ is a root of f, contradicting irreducibility. Thus either $m \geqslant 4$ or $n = 4$.

Suppose $n = 4$, so $f = x^4 + uxy^m + y^4$. After the transformation $y \mapsto y - \frac{1}{4}uxy^{m-3}$, f takes the form

$$f = \begin{cases} x^3 + bx^2y^2 + y^4 & (b \in k[[x,y]]) & \text{if } m > 3 \\ vx^3 + cx^2y^2 + y^4 & (c \in k[[x,y]],\ v \in k[[x,y]]^\times) & \text{if } m = 3\,. \end{cases}$$

If $m = 3$, we use the transformation $x \mapsto v^{\frac{1}{3}}x$ to eliminate the unit v (modifying c along the way). Thus in either case we have $f = x^3 + bx^2y^2 + y^4$, and now the transformation $x \mapsto x - \frac{1}{3}by^2$ puts f into the form $f = x^3 + wy^4$, where $w \in k[[x,y]]^\times$. Replacing y by $w^{\frac{1}{4}}y$, we obtain the (E_6)-singularity.

Now assume that $n \neq 4$ (and, consequently, $m \geqslant 4$). Lemma 8.9 implies that $n = 5$. The transformation $y \mapsto y - \frac{1}{5}uxy^{m-4}$ (with a unit adjustment to x if $m = 4$) puts f in the form $x^3 + bx^2y^3 + y^5$. The change of variable $x \mapsto x - \frac{1}{3}by^3$ now transforms f to $x^3 + wy^5$, where $w \in k[[x,y]]^\times$. On replacing y with $w^{\frac{1}{5}}y$, we obtain the (E_8) singularity, finishing this case.

8.11 Case. *f is reducible but has only one tangent line.*

Changing notation, we may assume that $f = x(x^2 + ax + b)$, where a and b are non-units of $k[[y]]$. As before, x^3 must be the initial form of f, so $f = x(x^2 + cxy^2 + dy^3)$, where $c, d \in k[[y]]$. By Lemma 8.9 d must be a unit. After replacing y by $d^{\frac{1}{3}}y$, we can write $f = x(x^2 + exy^2 + y^3)$, where $e \in k[[y]]$. Next do the change of variable $y \mapsto y - \frac{1}{3}ex$ to eliminate the y^2 term. Now

$f = x(ux^2 + y^3)$, where $u \in k[\![x,y]\!]^\times$. Replacing x by $u^{-\frac{1}{2}}x$, we have, up to a unit multiple, an (E_7) singularity.

8.12 Case. *f is reducible and has more than one tangent line.*

Write $f = \ell q$, where ℓ is linear in x and q is quadratic. If the tangent line of ℓ happens to be a tangent line of q, then, by Lemma 8.8, q factors as a product of two linear polynomials with distinct tangent lines. In any case, we can write $f = (x-r)(x^2+sx+t)$, where $r,s,t \in yk[\![y]\!]$, and where the tangent line to $x-r$ is *not* a tangent line of x^2+sx+t. After the usual changes of variables and multiplication by a unit, we may assume that $f = (x-r)(x^2+y^n)$, where $n \geqslant 2$. If $n = 2$, then f is a product of three distinct lines, and we get (D_4). Assume now that $n \geqslant 3$. Then $x = 0$ is the tangent line to x^2+y^n and therefore cannot be the tangent line to $x-r$. Hence $r = uy$ for some unit $u \in k[\![y]\!]^\times$. We make the coordinate change $y \mapsto x-uy$. Now $f = y(ax^2+bxy^{n-1}+cy^n)$, where a and c are units of $k[\![x,y]\!]$. Better, up to the unit multiple c, we have $f = y(ac^{-1}x^2+bc^{-1}xy^{n-1}+y^n)$. Replace x by $(ac^{-1})^{\frac{1}{2}}$; now $f = y(x^2+dxy^{n-1}+y^n)$. After the change of coordinates $x \mapsto x - \frac{1}{2}dy^{n-1}$, we have $f = y(x^2 - \frac{1}{4}d^2y^{2n-2}+y^n)$. Since $2n-2 > n$, we can rewrite this as $f = y(x^2+ey^n)$, where $e \in k[\![x,y]\!]^\times$. Finally, we factor out e and replace x by $e^{-\frac{1}{2}}x$, bringing f into the form $y(x^2+y^n)$, the equation for the (D_{n+2}) singularity.

To finish the cycle and complete the proof of Theorem 8.7, we now show that (iv) $\implies$ (i). If $d = 1$ we invoke Corollary 7.19. Assuming inductively that $k[\![x_0,\dots,x_r]\!]/(g+z_2^2+\cdots+z_r^2)$ has finite CM type for some $r \geqslant 1$, we see, by (ii) of Theorem 7.18, that $k[\![x_0,\dots,x_{r+1}]\!]/(g+z_2^2+\cdots+z_{r+1}^2)$ has finite CM type as well. □

§2 Gorenstein singularities of finite CM type

In this section we will prove Herzog's theorem [Her78b] stating that the rings of the title are hypersurfaces. Before giving the proof, we establish the following result (also from [Her78b]) of independent interest. Recall that a MCM module M is *stable* provided it has no non-zero free summands.

8.13 Lemma. *Let $(R,\mathfrak{m})$ be a CM local ring, let M be a stable MCM R-module, and let $N = \operatorname{syz}_1^R(M)$.*

(i) N is stable.

(ii) Assume M is indecomposable, that $\operatorname{Ext}_R^1(M,R) = 0$, and that $R_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p}$ of R with $\operatorname{height}\mathfrak{p} \leqslant 1$*. Then N is indecomposable.*

Proof. We have a short exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0\,, \tag{8.13.1}$$

where F is free and $N \subseteq \mathfrak{m}F$. Let $(\underline{x}) = (x_1,\dots,x_d)$ be a maximal R-regular sequence in $\mathfrak{m}$. Since M is MCM, $(\underline{x})$ is M-regular, and it follows that $\underline{x}N = \underline{x}F \cap N$. The map $N/\underline{x}N \longrightarrow F/\underline{x}F$ is therefore injective, and it gives an injection $N/\underline{x}N \hookrightarrow \mathfrak{m}F/\underline{x}F$. Since $(\underline{x})$ is a maximal N-regular sequence, $\mathfrak{m} \in \operatorname{Ass} N/\underline{x}N$, so $\mathfrak{m} \in \operatorname{Ass}(\mathfrak{m}F/\underline{x}F) = \operatorname{Ass}(\mathfrak{m}/(\underline{x}))$. It follows that $\mathfrak{m}/\underline{x}$ is an unfaithful $R/(\underline{x})$-module and hence that $N/\underline{x}N$ is unfaithful too. But then $N/\underline{x}N$ cannot have have $R/\underline{x}$ as a direct summand, and item (i) follows.

For the second statement, we note at the outset that both M and N are reflexive R-modules, by Proposition A.15. We dualize (8.13.1), using the vanishing of $\operatorname{Ext}^1_R(M,R)$, to get an exact sequence

(8.13.2) $$0 \longrightarrow M^* \longrightarrow F^* \longrightarrow N^* \longrightarrow 0\,.$$

Suppose $N = N_1 \oplus N_2$, with both summands non-zero. By (i), neither summand is free. Since N is reflexive, neither N_1^* nor N_2^* is free, and it follows from (8.13.2) that M^* decomposes non-trivially. As M is reflexive, this contradicts indecomposability of M. □

8.14 Theorem (Herzog)**.** *Let $(R,\mathfrak{m},k)$ be a Gorenstein local ring with a bound on the number of generators required for indecomposable MCM modules. Then $\widehat{R}$ is a hypersurface ring.*

Proof. Let $M = \operatorname{syz}^R_d(k)$, and write $M = M_1 \oplus \cdots \oplus M_t$, where each M_i is indecomposable and the summands are indexed so that $M_i \cong R$ if and only if $i > s$. By Lemma 8.13, $\operatorname{syz}^R_j(M)$ is a direct sum of at most s indecomposable modules for $j > d$. (The requisite vanishing of Ext comes from Corollary A.19.) It follows that the Betti numbers of k are bounded. The fact that they have polynomial growth implies, by [Gul80], that $\widehat{R}$ is a complete intersection, and now [Tat57] implies that $\widehat{R}$ is a hypersurface. □

8.15 Theorem. *Let $(R,\mathfrak{m},k)$ be a Gorenstein complete local ring of finite CM type. Assume that k is algebraically closed of characteristic different from 2, 3, and 5, and that R contains k as a coefficient field. Then R is a complete ADE hypersurface singularity.*

8.16 Corollary. *Let R be as in Theorem 8.15. Then R has rational singularities.*

§3 Matrix factorizations for the Kleinian singularities

Theorem 5.22 shows that the complete Kleinian singularities $k[\![x,y,z]\!]/(f)$ have finite CM type, where f is one of the polynomials listed in Table 5.1 and k is an algebraically closed field of characteristic not 2, 3, or 5. This was the key step in the classification of Gorenstein rings of finite CM type in the previous section. Given their central importance, it is worthwhile to have a complete listing of the matrix factorizations for the indecomposable MCM modules over these rings.

To describe the matrix factorizations, we return to the setup of Definition 5.5: Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, that is, one of the binary polyhedral groups of Theorem 5.11. Let G act linearly on the power series ring $S = \mathbb{C}[\![u,v]\!]$, and set $R = S^G$. Then R is generated over $\mathbb{C}$ by three invariants $x(u,v)$, $y(u,v)$, and $z(u,v)$, which satisfy a relation $z^2 + g(x,y) = 0$ for some polynomial g depending on G, so that $R \cong \mathbb{C}[\![x,y,z]\!]/(z^2 + g(x,y))$.

Set $A = \mathbb{C}[\![x(u,v), y(u,v)]\!] \subset R$. Then A is a power series ring, in particular a regular local ring. Since $z^2 \in A$, we see that as in Chapter 7, R is a free A-module of rank 2. Moreover, any MCM R-module is A-free as well. It is known [ST54, Coh76] that A is also a ring of invariants of a finite group $G' \subset \mathrm{U}(2,\mathbb{C})$, generated by complex reflections of order 2 and containing G as a subgroup of index 2.

Let $V_0,\dots,V_d$ be a full set of the non-isomorphic irreducible representations of G; then we know from Corollary 4.18 and Theorem 5.3 that $M_j = (S \otimes_{\mathbb{C}} V_j)^G$, for $j = 0,\dots,d$, are precisely the direct summands of S

as R-module and are also precisely the indecomposable MCM R-modules. To get a handle on the M_j, we can express them as $(S \otimes_{\mathbb{C}} \operatorname{Ind}_G^{G'} V_j)^{G'}$. Being free over A, each M_j will have a basis of G'-invariants. These, and the identities of the representations $\operatorname{Ind}_G^{G'} V_j$, are computed in [GSV81].

Now we show how to obtain the matrix factorization corresponding to each M_j, following [GSV81]. The proof of the next proposition is a straightforward verification, mimicking the proof (see B.6(i)) that the kernel of the multiplication map $\mu\colon B \otimes_A B \longrightarrow B$ is generated by all elements of the form $b \otimes 1 - 1 \otimes b$. The essential observation is that $z^2 = -g(x,y) \in A$.

8.17 Proposition. *Let $\sigma: S \longrightarrow S$ be the R-module endomorphism sending z to $-z$, and let ${}^{\sigma}S$ be the R-module with underlying abelian group S, but with R-module structure given by $r \cdot s = \sigma(r)s$. Then we have two exact sequences of R-modules:*

$$0 \longrightarrow {}^{\sigma}S \xrightarrow{i^-} R \otimes_A S \xrightarrow{p^+} S \longrightarrow 0$$

and

$$0 \longrightarrow S \xrightarrow{i^+} R \otimes_A S \xrightarrow{p^-} {}^{\sigma}S \longrightarrow 0,$$

where $i^-(s) = i^+(s) = z \otimes s - 1 \otimes zs$, $j^+(r \otimes s) = rs$, and $j^-(r \otimes s) = \sigma(r)s$.

From this proposition one deduces the following theorem. We omit the details.

8.18 Theorem. *Let $S = \mathbb{C}[[u,v]]$, G a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ acting linearly on S, and $R = S^G$. Let x, y, and z be generating invariants for R satisfying the relation $z^2 + g(x,y) = 0$, and let $A = \mathbb{C}[[x,y]]$. Then the R-free* *resolution of S has the form*

$$\cdots \xrightarrow{T^-} R \otimes_A S \xrightarrow{T^+} R \otimes_A S \xrightarrow{T^-} R \otimes_A S \xrightarrow{p^+} S \longrightarrow 0\,,$$

where

$$T^{\pm}(r \otimes s) = zr \otimes s \pm r \otimes zs\,.$$

Moreover, the R-free resolution of each indecomposable R-direct summand M_j of S is the direct summand of the above resolution of the form

$$\cdots \xrightarrow{T_j^-} R \otimes_A M_j \xrightarrow{T_j^+} R \otimes_A M_j \xrightarrow{T_j^-} R \otimes_A M_j \xrightarrow{p_j^+} M_j \longrightarrow 0\,.$$

□

In terms of matrices, the resolution and corresponding matrix factorization of the MCM R-module M_j can be deduced from the theorem as follows. Let $\Phi\colon S \longrightarrow S$ denote the R-linear homomorphism given by multiplication by z, and let $\Phi_j\colon M_j \longrightarrow M_j$ be the restriction to M_j. Then each Φ_j is an A-linear map of free A-modules. Choose a basis and represent Φ_j by an $n \times n$ matrix φ_j with entries in x and y. Then φ_j^2 is equal to multiplication by $z^2 = -g(x,y) \in A$ on M_j, so that

$$(zI_n - \varphi_j,\ zI_n + \varphi_j)$$

is a matrix factorization of $z^2 + g(x,y)$ with cokernel M_j.

Our task is thus reduced to computing the matrix representing multiplication by z on each M_j. As in Chapter 5, we treat each case separately.

8.19 (A_n)**.** We have already computed the presentation matrices of the MCM modules over $\mathbb{C}[[x,y,z]]/(xz - y^{n+1})$ in Example 4.23, but we illustrate

Theorem 8.18 in this easy case before proceeding to the more involved ones below. The cyclic group $\mathscr{C}_{n+1}$, generated by

$$\epsilon_{n+1} = \begin{pmatrix} \zeta_{n+1} & \\ & \zeta_{n+1}^{-1} \end{pmatrix},$$

has invariants $x = u^{n+1} + v^{n+1}$, $y = uv$, and $z = u^{n+1} - v^{n+1}$, satisfying

$$z^2 - (x^2 - 4y^{n+1}) = 0\,.$$

Set $A = \mathbb{C}[\![x,y]\!] \subset R = k[\![x,y,z]\!]$. Then $A = \mathbb{C}[u^{n+1}+v^{n+1}, uv]\!]$ is an invariant ring of the group G' generated by ϵ_{n+1} and the additional reflection $s = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$.

Let V_j, for $j = 0, \dots, n$, be the irreducible representation of $\mathscr{C}_{n+1}$ with character $\chi_j(g) = \zeta_{n+1}^j$. Then the MCM R-modules $M_j = (S \otimes_{\mathbb{C}} V_j)^G$ are generated over R by the monomials $u^a v^b$ such that $b - a \equiv j \mod n+1$. Over A, each M_j is freely generated by u^j and v^{n+1-j}. Since

$$zu^j = (u^{n+1} - v^{n+1})u^j = (u^{n+1} + v^{n+1})u^j - 2(uv)^j v^{n+1-j}$$

and

$$zv^{n+1-j} = (u^{n+1} - v^{n+1})v^{n+1-j} = 2(uv)^{n+1-j}u^j - (u^{n+1} + v^{n+1})v^{n+1-j}\,,$$

the matrix φ_j representing the action of z on M_j is

$$\varphi_j = \begin{pmatrix} x & 2y^{n+1-j} \\ -2y^j & -x \end{pmatrix}.$$

One checks that $\varphi_j^2 = (x^2 - 4y^{n+1})I_2$, so $(zI_2 - \varphi_j, zI_2 + \varphi_j)$ is the matrix factorization corresponding to M_j.

Making a linear change of variables, we find that the indecomposable matrix factorizations of the (A_n) singularity defined by $x^2+y^{n+1}+z^2=0$ are $(zI_2-\varphi_j, zI_2+\varphi_j)$, where

$$\varphi_j = \begin{pmatrix} ix & y^{n+1-j} \\ -y^j & -ix \end{pmatrix},$$

for $j=0,\dots,n$, and where i denotes a square root of -1.

8.20 (D_n)**.** The dihedral group $\mathscr{D}_{n-2}$ is generated by

$$\alpha = \begin{pmatrix} \zeta_{2(n-2)} & \\ & \zeta_{2(n-2)}^{-1} \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} & i \\ i & \end{pmatrix},$$

where again i denotes a square root of -1. The invariants of α and β are $x = u^{2(n-2)} + (-1)^n v^{2(n-2)}$, $y = u^2v^2$, and $z = uv(u^{2(n-2)} - (-1)^n v^{2(n-2)}$, which satisfy

$$z^2 - y(x^2 - 4(-1)^n y^{n-2}) = 0\,.$$

Again we set $A = \mathbb{C}[\![x,y]\!] = \mathbb{C}[\![u^{2(n-2)} + (-1)^n v^{2(n-2)}, u^2v^2]\!]$ and again A is the ring of invariants of the group G' generated by $\epsilon_{2(n-2)}$, τ, and $s = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$.

In the matrices below, we will implicitly make the linear changes of variable necessary to put the defining equation of R into the form

$$z^2 - \left(-y\left(x^2 + y^{n-2}\right)\right) = 0\,.$$

Consider first the one-dimensional representation V_1 given by $\alpha \mapsto 1$ and $\beta \mapsto -1$. The MCM R-module $M_1 = (S \otimes_{\mathbb{C}} V_1)^G$ has A-basis $(uv, u^{2(n-2)} - (-1)^n v^{2(n-2)})$, and after the change of variable the matrix φ_1 for multiplication by z is

$$\varphi_1 \begin{pmatrix} & -x^2 - y^{n-1} \\ y & \end{pmatrix}.$$

Next consider the two-dimensional irreducible representations V_j, for $j = 2,\dots,n-2$, given by

$$\alpha \mapsto \begin{pmatrix} \zeta_{2(n-2)}^{j-1} & \\ & \zeta_{2(n-2)}^{-j+1} \end{pmatrix} \qquad \text{and} \qquad \beta \mapsto \begin{pmatrix} & i^{j-1} \\ i^{j-1} & \end{pmatrix}.$$

For each j, the corresponding MCM R-module M_j has A-basis $(u^{j-1}, uv^{2n-j-2}, u^jv, v^{2n-j-3})$. The matrix φ_j depends on the parity of j; for j even, it is

$$\varphi_j = \begin{pmatrix} 0 & 0 & -xy & -y^{n-1-j/2} \\ 0 & 0 & -y^{j/2} & x \\ x & y^{n-1-j/2} & 0 & 0 \\ y^{j/2} & -xy & 0 & 0 \end{pmatrix}$$

while if j is odd we have

$$\varphi_j = \begin{pmatrix} 0 & 0 & -xy & -y^{n-1-(j-1)/2} \\ 0 & 0 & -y^{(j+1)/2} & xy \\ x & y^{n-2-(j-1)/2} & 0 & 0 \\ y^{(j-1)/2} & -x & 0 & 0 \end{pmatrix}.$$

Finally consider V_{n-1} and V_n, which are the irreducible components of the two-dimensional reducible representation

$$\alpha \mapsto \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \qquad \beta \mapsto \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

The MCM R-modules M_{n-1} and M_n have bases $(uv(u^{n-2}+(-1)^{n+1}v^{n-2}), u^{n-2}+(-1)^n v^{n-2})$ and $(uv(u^{n-2}+(-1)^n v^{n-2}), u^{n-2}+(-1)^{n+1}v^{n-2})$, respectively. Again the corresponding matrices φ_{n-1} and φ_n depend on parity: for n odd we have

$$\varphi_{n-1} = \begin{pmatrix} iy^{(n-1)/2} & -x \\ xy & -iy^{(n-1)/2} \end{pmatrix} \qquad \text{and} \qquad \varphi_n = \begin{pmatrix} iy^{(n-1)/2} & -xy \\ x & -iy^{(n-1)/2} \end{pmatrix},$$

and for n even

$$\varphi_{n-1} = \begin{pmatrix} 0 & -x - iy^{(n-2)/2} \\ xy - iy^{n/2} & 0 \end{pmatrix} \qquad \text{and} \qquad \varphi_n = \begin{pmatrix} 0 & -x + iy^{(n-2)/2} \\ xy + iy^{n/2} & 0 \end{pmatrix}.$$

For the E-series examples, we suppress the details of the complex reflection group G' and the A-bases for the M_j. See [ST54] and [GSV81].

8.21 (E_6)**.** The defining equation of the (E_6) singularity is $z^2-(-x^3-y^4)=0$. For each of the six non-trivial irreducible representations $V_1, V_2, V_3, V_3^\vee, V_4$, and $V_{4^\vee}$, one can choose A-bases for M_j so that multiplication by z is given by the following matrices. The matrix factorizations for the corresponding MCM R-modules are given by $(zI_n - \varphi, zI_n + \varphi)$.

$$\varphi_1 = \begin{pmatrix} 0 & 0 & -x^2 & -y^3 \\ 0 & 0 & -y & x \\ x & y^3 & 0 & 0 \\ y & -x^2 & 0 & 0 \end{pmatrix} \qquad \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & -x^2 & -y^3 & xy^2 \\ 0 & 0 & 0 & xy & -x^2 & -y^3 \\ 0 & 0 & 0 & -y^2 & xy & -x^2 \\ x & 0 & y^2 & 0 & 0 & 0 \\ y & x & 0 & 0 & 0 & 0 \\ 0 & y & x & 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_3 = \begin{pmatrix} iy^2 & 0 & -x^2 & 0 \\ 0 & iy^2 & -xy & -x^2 \\ x & 0 & -iy^2 & 0 \\ -y & x & 0 & -iy^2 \end{pmatrix} \qquad \varphi_3^\vee = \begin{pmatrix} -iy^2 & 0 & -x^2 & 0 \\ 0 & -iy^2 & -xy & -x^2 \\ x & 0 & iy^2 & 0 \\ -y & x & 0 & iy^2 \end{pmatrix}$$

$$\varphi_4 = \begin{pmatrix} iy^2 & -x^2 \\ x & -iy^2 \end{pmatrix} \qquad \varphi_4^\vee = \begin{pmatrix} -iy^2 & -x^2 \\ x & iy^2 \end{pmatrix}$$

8.22 (E_7)**.** The (E_7) singularity is defined by $z^2-(-x^3-xy^3)=0$. There are 7 non-trivial irreducible representations $V_1, \dots, V_7$, and the matrices φ_j corresponding to multiplication by z are given below. The matrix factorizations for the corresponding MCM R-modules are given by $(zI_n-\varphi, zI_n+\varphi)$.

$$\varphi_1=\begin{pmatrix}0&0&-x^2&-xy^2\\0&0&-y&x\\x&xy^2&0&0\\y&-x^2&0&0\end{pmatrix}\qquad \varphi_2=\begin{pmatrix}0&0&0&-x^2&-xy^2&x^2y\\0&0&0&xy&-x^2&-xy^2\\0&0&0&-y^2&xy&-x^2\\x&0&xy&0&0&0\\y&x&0&0&0&0\\0&y&x&0&0&0\end{pmatrix}$$

$$\varphi_3=\begin{pmatrix}0&0&0&0&0&0&-x^2&-xy^2\\0&0&0&0&0&0&-xy&x^2\\0&0&0&0&-x&-y^2&0&xy\\0&0&0&0&-y&x&-x&0\\0&-xy&x^2&xy^2&0&0&0&0\\x&0&xy&-x^2&0&0&0&0\\x&y^2&0&0&0&0&0&0\\y&-x&0&0&0&0&0&0\end{pmatrix}$$

$$\varphi_4=\begin{pmatrix}0&0&0&xy&-x^2&-xy^2\\0&0&0&-y^2&xy&-x^2\\0&0&0&-x&-y^2&xy\\0&xy&x^2&0&0&0\\x&0&xy&0&0&0\\y&x&0&0&0&0\end{pmatrix}\qquad \varphi_5=\begin{pmatrix}0&0&-xy&-x^2\\0&0&-x&y^2\\y^2&x^2&0&0\\x&-xy&0&0\end{pmatrix}$$

$$\varphi_6 = \begin{pmatrix} 0 & y^3+x^2 \\ -x & 0 \end{pmatrix} \qquad \varphi_7 = \begin{pmatrix} 0 & 0 & -x^2 & -xy^2 \\ 0 & 0 & -xy & x^2 \\ x & y^2 & 0 & 0 \\ y & -x & 0 & 0 \end{pmatrix}$$

8.23 (E_8)**.** The defining equation of the (E_6) singularity is $z^2-(-x^3-y^5)=0$. Here are the matrices φ_j representing multiplication by z on the 8 non-trivial indecomposable MCM R-modules. The matrix factorizations are given by $(zI_n-\varphi, zI_n+\varphi)$.

$$\varphi_1 = \begin{pmatrix} 0 & 0 & -x^2 & -y^4 \\ 0 & 0 & -y & x \\ x & y^4 & 0 & 0 \\ y & -x^2 & 0 & 0 \end{pmatrix} \qquad \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & -x^2 & -y^4 & xy^3 \\ 0 & 0 & 0 & xy & -x^2 & -y^4 \\ 0 & 0 & 0 & -y^2 & xy & -x^2 \\ x & 0 & y^3 & 0 & 0 & 0 \\ y & x & 0 & 0 & 0 & 0 \\ 0 & y & x & 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & xy & -y^2 & -x^2 & 0 \\ 0 & 0 & 0 & 0 & -y^3 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & x^2 & 0 & 0 & -y^2 \\ 0 & 0 & 0 & 0 & 0 & x & -y^3 & -y \\ 0 & y^2 & -x & 0 & 0 & 0 & 0 & 0 \\ y^3 & xy & 0 & -x^2 & 0 & 0 & 0 & 0 \\ x & 0 & -y & y^2 & 0 & 0 & 0 & 0 \\ 0 & x^2 & y^3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -y^3 & x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y^3 & -x^2 & xy^2 & -y^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -xy & -y^3 & -x^2 & xy^2 \\ 0 & 0 & 0 & 0 & 0 & y^2 & 0 & xy & -y^3 & -x^2 \\ 0 & 0 & 0 & 0 & 0 & -x & -y^2 & 0 & 0 & 0 \\ y^2 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & y^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & x & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & x & y^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x^2 & xy^2 & -y^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y^3 & -x^2 & xy^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy & -y^3 & -x^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x & -y^2 & 0 & 0 & 0 & y^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x & -y^2 & y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -y & 0 & -x & 0 & y^2 & 0 \\ 0 & 0 & y^3 & x^2 & -xy^2 & y^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ y^2 & 0 & 0 & y^3 & x^2 & -xy^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y^2 & 0 & -xy & y^3 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -y^3 & -x^2 & 0 \\ 0 & 0 & 0 & 0 & -y^2 & 0 & xy & -x^2 \\ 0 & 0 & 0 & 0 & -x & -y^2 & 0 & y^3 \\ 0 & 0 & 0 & 0 & 0 & -x & y^2 & 0 \\ 0 & y^3 & x^2 & -xy^2 & 0 & 0 & 0 & 0 \\ y^2 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & -y^3 & 0 & 0 & 0 & 0 \\ y & x & -y^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \varphi_7 = \begin{pmatrix} 0 & 0 & -y^3 & -x^2 \\ 0 & 0 & x & -y^2 \\ y^2 & -x^2 & 0 & 0 \\ x & y^3 & 0 & 0 \end{pmatrix}$$

$$\varphi_8 = \begin{pmatrix} 0 & 0 & 0 & -x^2 & xy^2 & -y^4 \\ 0 & 0 & 0 & -y^3 & -x^2 & xy^2 \\ 0 & 0 & 0 & xy & -y^3 & -x^2 \\ x & y^2 & 0 & 0 & 0 & 0 \\ 0 & x & y^2 & 0 & 0 & 0 \\ y & 0 & x & 0 & 0 & 0 \end{pmatrix}$$

8.24 Remark. We observe that the forms above for the indecomposable matrix factorizations over the two-dimensional ADE singularities make it easy to find the indecomposable matrix factorizations in dimension one. When the matrix (involving only x and y) has the distinctive anti-diagonal block shape, the non-zero blocks constitute (up to a sign) an indecomposable matrix factorization for the one-dimensional ADE polynomial in x and y. When the matrix does not have block form, $(\varphi, -\varphi)$ is an indecomposable matrix factorization. See §3 of Chapter 10.

§4 Characteristics 2, 3, 5

If the characteristic of k is different from 2, Theorem 8.6 reduces the classification of hypersurfaces of finite CM type to the case of dimension one. We quote the following two theorems due to Greuel and Kröning [GK90] (cf. also the paper [KS85] by Kiyek and Steinke):

8.25 Theorem (Characteristic 3)**.** *Let k be an algebraically closed field of characteristic 3, let $d \geqslant 1$, and let $R = k[\![x,y,x_2,\dots,x_d]\!]/(f)$, where $0 \neq f \in (x,y,x_2,\dots,x_d)^2$. Then R has finite CM type if and only if $R \cong k[\![x,y,x_2,\dots,x_d]\!]/(g + x_2^2 + \cdots + x_d^2)$, where $g \in k[x,y]$ is one of the following:*

(A_n): $x^2 + y^{n+1}\,, \qquad n \geqslant 1$

(D_n): $x^2 y + y^{n-1}\,, \qquad n \geqslant 4$

(E_6^0): $x^3 + y^4$

(E_6^1): $x^3 + y^4 + x^2 y^2$

(E_7^0): $x^3 + x y^3$

(E_7^1): $x^3 + x y^3 + x^2 y^2$

(E_8^0): $x^3 + y^5$

(E_8^1): $x^3 + y^5 + x^2 y^3$

(E_8^2): $x^3 + y^5 + x^2 y^2$

8.26 Theorem (Characteristic 5)**.** *Let k be an algebraically closed field of characteristic 5, let $d \geqslant 1$, and let $R = k[\![x,y,x_2,\dots,x_d]\!]/(f)$, where $0 \neq f \in$*

$(x, y, x_2, \ldots, x_d)^2$. *Then R has finite CM type if and only if* $R \cong k[\![x, y, x_2, \ldots, x_d]\!]/(g + x_2^2 + \cdots + x_d^2)$, *where* $g \in k[x, y]$ *is one of the following:*

(A_n): $x^2 + y^{n+1}$, $\quad n \geqslant 1$

(D_n): $x^2 y + y^{n-1}$, $\quad n \geqslant 4$

(E_6): $x^3 + y^4$

(E_7): $x^3 + xy^3$

(E_8^0): $x^3 + y^5$

(E_8^1): $x^3 + y^5 + xy^4$

There is a similar, but longer, list in characteristic two.

In characteristics different from two, notice that $S[\![u, v]\!]/(f + u^2 + v^2) \cong S[\![u, v]\!]/(f + uv)$, via the transformation $u \mapsto \frac{u+v}{2}$, $v \mapsto \frac{u-v}{2\sqrt{-1}}$. Thus, if one does not mind skipping a dimension, one can transfer finite CM type up and down along the iterated double branched cover $R^{\sharp\sharp} = S[\![u, v]\!]/(f + uv)$, where $R = S/(f)$. Remarkably, this works in characteristic two as well.

8.27 Theorem (Solberg, Greuel and Kroning [Sol89, GK90]). *Let k be an algebraically closed field of arbitrary characteristic, let $d \geqslant 3$, and let* $R = k[\![x_0, \ldots, x_d]\!]/(f)$, *where* $0 \neq f \in (x_0, \ldots, x_d)^2$. *Then R has finite CM type if and only if there exists a non-zero non-unit* $g \in k[\![x_0, \ldots, x_{d-2}]\!]$ *such that* $k[\![x_0, \ldots, x_{d-2}]\!]/(g)$ *has finite CM type and* $R \cong k[\![x_0, \ldots, x_d]\!]/(g + x_{d-1}x_d)$.

Solberg proved the "if" direction in his 1987 dissertation [Sol89]. He showed, in fact, that, for any non-zero non-unit $g \in k[\![x_0, \ldots, x_{d-2}]\!]$, $k[\![x_0, \ldots, x_{d-2}]\!]/(g)$

has finite CM type if and only if $k[\![x_0,\dots,x_d]\!]/(g+x_{d-1}x_d)$ has finite CM type. The proof, which uses the theory of AR sequences (cf. Chapter 10, is quite unlike the proof in characteristics different from two, in that there seems to be no nice correspondence between MCM R-modules and MCM $R^{\sharp\sharp}$-modules (such as in Theorem 7.30). In 1988 Greuel and Kröning [GK90] used deformation theory to show that if R as in the theorem has finite CM type, then $R \cong k[\![x_0,\dots,x_d]\!]/(g+x_{d-1}x_d)$ for a suitable non-zero non-unit element $g \in k[\![x_0,\dots,x_{d-2}]\!]$, thereby establishing the converse of the theorem.

In order to finish the classification of complete hypersurface singularities of finite CM type in characteristic two, one needs to classify those singularities in dimensions one and two. The normal forms are itemized in Section 5 of [Sol89] and in [GK90] and depend on earlier work of Artin [Art77], Artin and Verdier [AV85], and Kiyek and Steinke [KS85].

§5 Exercises

8.28 Exercise. Let $(S,\mathfrak{n})$ be a regular local ring, and $f \in \mathfrak{n}^r \setminus \mathfrak{n}^{r+1}$. Show that the hypersurface ring $S/(f)$ has multiplicity r.

9

Auslander–Buchweitz Theory

As we saw back in Chapter 2, trying to understand the whole category of finitely generated modules over a local ring is impractical, so we restrict to maximal Cohen–Macaulay modules. In fact, this is not as restrictive as it seems at first: any finitely generated module over a CM local ring with canonical module can be "approximated" by a MCM module, in a precise sense due originally to Auslander and Buchweitz [AB89]. The theory as originally constructed in [AB89] is quite abstract, and has since been further generalized. In keeping with our general strategy, we adopt a stubbornly concrete point of view. We deal exclusively with CM local rings, finitely generated modules, and approximations by MCM modules. We also use the more limited terminology of *MCM approximations* and *FID hulls,* rather than the general notions of (pre)covers and (pre)envelopes.

In the first section we recall some basics on finitely generated modules of finite injective dimension, and particularly canonical modules, which occupy the central locus in the theory. We then detail the theory of MCM approximations and FID hulls, following (a de-categorified version of) Auslander and Buchweitz's original construction closely. Finally, we give some applications in terms of Auslander's δ-invariant. Other applications will appear in later chapters.

§1 Canonical modules

Here we give a mostly self-contained, if hasty, primer on finitely generated modules of finite injective dimension over local rings and the most distinguished of such modules, the canonical module. Of course we focus on those aspects most relevant to the study of MCM modules.

As usual, we mulishly stick to the case of finitely generated modules, ignoring generalizations such as dualizing complexes and, in another direction, semidualizing modules.

The one fact we state without proof is not hard, but would take us away from our planned route. See the exercises or [BH93, Theorems 3.1.14 and 3.1.17] for a proof.

9.1 Lemma. *Let $(R,\mathfrak{m},k)$ be a local ring and N a non-zero finitely generated R-module. Then*

$$\operatorname{injdim}_R N = \sup\left\{ i \;\middle|\; \operatorname{Ext}^i_R(k,N) \neq 0 \right\}.$$

If the injective dimension of N is finite, then it is equal to $\operatorname{depth} R$.

As an aside, we should point out here the conjecture of H. Bass [Bas63]: "It seems conceivable that, say for A local, there exist finitely generated $M \neq 0$ with finite injective dimension only if A is a Cohen–Macaulay ring." As Bass points out in the next sentence, the converse is true.

9.2 Proposition. *Let $(R,\mathfrak{m},k)$ be a CM local ring. Then R admits a non-zero finitely generated module of finite injective dimension.*

Proof. Let $\mathbf{x}$ be a system of parameters for R and $\overline{R}$ the quotient $R/(\mathbf{x})$. The injective hull $E = E_{\overline{R}}(k)$ of the residue field of $\overline{R}$ has finite length over $\overline{R}$ and hence over R. It follows that $M = \operatorname{Hom}_R(\overline{R}, E)$ is finitely generated over R, and dualizing the Koszul resolution of $\overline{R}$ into E displays $\operatorname{injdim} M < \infty$. □

Bass' conjecture, that the converse of Proposition 9.2 holds, was established for local rings of prime characteristic or essentially of finite type over a field of characteristic zero by C. Peskine and L. Szpiro [PS73] using their Intersection Theorem. Since P. Roberts has proved the Intersection Theorem for all local rings [Rob87], Bass' Conjecture holds in general.

The first hint of a connection between modules of finite injective dimension and MCM modules comes in the next result, also due to Peskine and Szpiro [PS73, I, 4.15].

9.3 Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring and M, N non-zero finitely generated R-modules with $\operatorname{injdim}_R N < \infty$. Then*

$$\operatorname{depth} R - \operatorname{depth} M = \sup\left\{ i \;\middle|\; \operatorname{Ext}^i_R(M, N) \neq 0 \right\}.$$

Proof. By Lemma 9.1, we know that $t := \operatorname{injdim}_R N$, since finite, is equal to $\operatorname{depth} R$. Induct on $\operatorname{depth} M$. We have $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > t$, giving one inequality when $\operatorname{depth} M = 0$. For the other, observe that $\operatorname{depth} M = 0$ means that $\mathfrak{m}$ is an associated prime of M, so the residue field k embeds into M, giving a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow C \longrightarrow 0.$$

Apply $\operatorname{Hom}_R(-, N)$ to obtain an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^t_R(M, N) \longrightarrow \operatorname{Ext}^t_R(k, N) \longrightarrow \operatorname{Ext}^{t+1}_R(C, N).$$

The right-most term vanishes as $t = \operatorname{injdim}_R N$, while the middle term is non-zero by Lemma 9.1. The left-hand term is thus non-zero as well, giving the equality in this case.

Supposing now $\operatorname{depth} M > 0$, take $x \in \mathfrak{m}$ a non-zerodivisor on M, and use the long exact sequence of Ext and induction to finish the proof. □

9.4 Proposition. *Let $(R, \mathfrak{m}, k)$ be a CM local ring and M, N non-zero finitely generated R-modules. Then*

(i) M is MCM if and only if $\operatorname{Ext}^i_R(M, Y) = 0$ for all $i > 0$ and all finitely generated R-modules Y of finite injective dimension, and

(ii) N has finite injective dimension if and only if $\operatorname{Ext}^i_R(X, N) = 0$ for all $i > 0$ and all MCM R-modules X.

Proof. The forward direction of each statement is immediate from the Theorem, as is the converse in (i) (since we know that there is at least one Y to test against, by Proposition 9.2). The only assertion remaining is to show that $\operatorname{injdim}_R N < \infty$ if $\operatorname{Ext}^i_R(X, N) = 0$ for all $i > 0$ and all MCM X. Take for X a sufficiently high syzygy of the residue field k and use standard index-shifting to see that $\operatorname{Ext}^i_R(k, N) = 0$ for $i \gg 0$, so N has finite injective dimension by Lemma 9.1. □

Colloquially, we interpret Prop. 9.4 as expressing that MCM modules and finitely generated modules of finite injective dimension are "orthogonal." It will transpire that the intersection is "spanned" by a single module, namely the canonical module, to which we now turn.

9.5 Definition. Let $(R, \mathfrak{m}, k)$ be a CM local ring of dimension d. A finitely generated R-module ω is a *canonical module* for R if ω is MCM, has finite injective dimension, and satisfies

$$\dim_k \operatorname{Ext}^d_R(k, \omega) = 1 .$$

The condition on $\operatorname{Ext}^d_R(k,\omega)$ is a sort of rank-one normalizing assumption: taking into account the calculation of both depth and injective dimension in terms of $\operatorname{Ext}^i_R(k,-)$, we see that the only non-vanishing such Ext is for $i = d$, where it must be a finite-dimensional vector space. In particular, we can write Definition 9.5 compactly as

$$\operatorname{Ext}^i_R(k,\omega) \cong \begin{cases} k & \text{if } i = \dim R, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We need a laundry list of properties of canonical modules. First, here is a standard lemma. See Exercise 9.47 for a proof.

9.6 Lemma. *Let $(R, \mathfrak{m})$ be a local ring and $\varphi \colon M \longrightarrow N$ a homomorphism of finitely generated R-modules. Let x be an N-regular element in $\mathfrak{m}$. If $\overline{\varphi} \colon M/xM \longrightarrow N/xN$ is an isomorphism, then φ is an isomorphism.*

For an Artinian local ring $(R, \mathfrak{m}, k)$, a canonical module ω is injective with one-dimensional socle, and therefore $\omega \cong E_R(k)$, the injective hull of the residue field. In general, suppose that R is CM local, $\mathbf{x}$ is a system of parameters, and ω is a canonical module for R. Then standard index-shifting reveals that $\overline{\omega} = \omega/\mathbf{x}\omega$ is a canonical module for $\overline{R} = R/(\mathbf{x})$, so $\overline{\omega} \cong E_{\overline{R}}(k)$. Since $\operatorname{Ext}^i_R(\omega,\omega) = 0$ for $i > 0$ by Theorem 9.3, it's easy to see that

$$\operatorname{Hom}_R(\omega,\omega) \otimes_R \overline{R} \cong \operatorname{Hom}_{\overline{R}}(\overline{\omega},\overline{\omega}) \cong \overline{R}$$

and so $\operatorname{Hom}_R(\omega,\omega)$ is cyclic by NAK. It is also MCM: apply $\operatorname{Hom}_R(-,\omega)$ to a free resolution of ω and use the Depth Lemma on the resulting exact sequence. Choosing a generator $\varphi \in \operatorname{Hom}_R(\omega,\omega)$, we apply Lemma 9.6 to the induced homomorphism $R \longrightarrow \operatorname{Hom}_R(\omega,\omega)$, $1 \mapsto \varphi$, which shows that $\operatorname{Hom}_R(\omega,\omega) \cong R$. Repeating the argument with putatively different canonical modules ω and ω', and applying Lemma 9.6 to φ itself, shows that ω is unique up to isomorphism.

If the canonical module ω has finite projective dimension, then it is free by the Auslander–Buchsbaum formula, and $\operatorname{End}_R(\omega) \cong R$ forces $\omega \cong R$. In this case R is Gorenstein (either by definition or by observation). Conversely, if R is Gorenstein, then the regular module R is its own canonical module.

A canonical module is also sometimes called a *dualizing module,* as we now explain. Let ω be a canonical module for R and M a MCM module. Then Theorem 9.3 yields $\operatorname{Ext}^i_R(M,\omega) = 0$ for $i > 0$, so $M^\vee := \operatorname{Hom}_R(M,\omega)$ is again a MCM module (as can be seen by dualizing a free resolution of M and applying the Depth Lemma). There is a natural biduality homomorphism

$$M \longrightarrow M^{\vee\vee} = \operatorname{Hom}_R(\operatorname{Hom}_R(M,\omega),\omega),$$

sending $z \in M$ to "evaluation at z," which one shows to be an isomorphism by reducing to the Artinian case and applying Lemma 9.6 again. More generally, if M is just CM of codepth $t = \dim R - \operatorname{depth} M$, then $\operatorname{Ext}^i_R(M,\omega) = 0$ for all $i \neq t$; for $i > t$ this is Theorem 9.3 again, and for $i < t$ it follows upon contemplating a maximal regular sequence in $\operatorname{Ann} M$. The single non-vanishing $M^\vee = \operatorname{Ext}^t_R(M,\omega)$ is again CM of codepth t by induction on

$\dim M$, and again one checks that $M \longrightarrow M^{\vee\vee} = \operatorname{Ext}^t_R(\operatorname{Ext}^t_R(M,\omega),\omega)$ is an isomorphism.

We encapsulate the preceding discussion in a single statement.

9.7 Theorem. *Let $(R,\mathfrak{m},k)$ be a CM local ring and ω a canonical module for R. Then*

(i) ω is unique up to isomorphism, and R is Gorenstein if and only if $\omega \cong R$;

(ii) $\operatorname{End}_R(\omega) \cong R$.

(iii) Let M be a CM R-module of codepth t, and set $M^\vee = \operatorname{Ext}^t_R(M,\omega)$. Then

(a) $M^\vee$ is also CM of codepth t;

(b) $\operatorname{Ext}^t_R(M,\omega) = 0$ *for $i \neq t$; and*

(c) $M^{\vee\vee}$ is naturally isomorphic to M.

It's straightforward to check that in addition to behaving well with respect to factoring out a regular sequence, the canonical module ω_R of a CM local ring R behaves well with respect to completion and localization:

$$\omega_{\widehat{R}} \cong \widehat{\omega_R} \qquad \text{and} \qquad \omega_{R_\mathfrak{p}} \cong (\omega_R)_\mathfrak{p}\,.$$

In particular, a local ring R is Gorenstein if and only if the completion $hatR$ is Gorenstein, and localizations of Gorenstein rings are again Gorenstein.

Let S and R be CM local rings and $\varphi\colon S \longrightarrow R$ now a module-finite ring homomorphism. Then R is a CM S-module of codepth $t = \dim S - \dim R$.

If ω_S is a canonical module for S, then, we have $\operatorname{Ext}^i_S(R,\omega_S) = 0$ for $i \neq t$, and one checks easily by reducing to the case $t = 0$ that $\operatorname{Ext}^t_S(R,\omega_S)$ is a canonical module for R. In particular, if R is a homomorphic image of a Gorenstein local ring S, then R has a canonical module. This was first observed by Sharp [Sha71]. In particular, a complete CM local ring is a homomorphic image of a regular local ring by Cohen's structure theorems, so has a canonical module.

The converse of Sharp's result also holds, as proved by Foxby [Fox72] and Reiten [Rei72] independently, so that a CM local ring R has a canonical module if and only if R is a homomorphic image of a Gorenstein local ring.

The stipulation that $\operatorname{Ext}^{\dim R}_R(k,\omega_R) \cong k$ is, as we observed, a kind of rank-one condition. Indeed, under a mild additional condition it forces ω_R to be isomorphic to an ideal of R. We say that R is *generically Gorenstein* if $R_\mathfrak{p}$ is Gorenstein for each minimal prime $\mathfrak{p}$ of R.

9.8 Proposition. *Let R be a CM local ring and ω a canonical module for R. If R is generically Gorenstein, then ω is isomorphic to an ideal of R, and conversely. In this case, ω is an ideal of pure height one (that is, every associated prime of ω has height one), and R/ω is a Gorenstein ring of dimension* $\dim R - 1$.

Proof. As $R_\mathfrak{p}$ is Gorenstein for every minimal $\mathfrak{p}$, we conclude that $\omega_\mathfrak{p}$ is free of rank one for those primes. In particular if we denote by K the total quotient ring, obtained by inverting the complement of the union of those minimal primes, then $\omega \otimes_R K$ is a rank-one projective module over the semilocal ring K. Thus $\omega \otimes_R K \cong K$. Fixing an isomorphism and composing with the natural map gives an R-homomorphism $\omega \longrightarrow K$, which is injective as ω is torsion-free. Multiplying the image by a carefully chosen non-zerodivisor clears the denominators and knocks the image down into R, where it is an ideal. Being locally free at the minimal primes, it has height at least one.

Since ω is MCM, the short exact sequence

$$0 \longrightarrow \omega \longrightarrow R \longrightarrow R/\omega \longrightarrow 0$$

forces $\operatorname{depth}(R/\omega) \geqslant \dim R - 1$, and since $\operatorname{height}\omega \geqslant 1$ we have $\dim R/\omega \leqslant \dim R - 1$. Thus R/ω is a CM ring, in particular, unmixed, so ω has pure height one. Furthermore, R/ω is a CM R-module of codepth 1. Applying $\operatorname{Hom}_R(-,\omega)$ thus gives an exact sequence

$$\operatorname{Hom}_R(R/\omega,\omega) \longrightarrow \omega \longrightarrow R \longrightarrow \operatorname{Ext}^1_R(R/\omega,\omega) \longrightarrow 0$$

and $\operatorname{Ext}^1_R(R/\omega,\omega) = (R/\omega)^\vee$ is the canonical module for R/ω by the discussion after Theorem 9.7. Since the leftmost term in the exact sequence vanishes, $(R/\omega)^\vee$ is clearly isomorphic to R/ω itself, so R/ω is Gorenstein.

For the converse, assume that ω is embedded into R as an ideal. Then as before we see that $\operatorname{height}\omega \geqslant 1$, so ω is not contained in any minimal prime and $R_\mathfrak{p}$ is Gorenstein for every minimal $\mathfrak{p}$. □

We quickly observe, using this result, that there does indeed exist a CM local ring which is not a homomorphic image of a Gorenstein local ring, and hence does not admit a canonical module. This was first constructed by Ferrand and Raynaud [FR70]. Specifically, they construct a one-dimensional local domain $(R,\mathfrak{m})$ such that the completion $\widehat{R}$ is not generically Gorenstein. If R were to have a canonical module ω_R, it would be embeddable as an $\mathfrak{m}$-primary ideal of R. The completion $\widehat{\omega_R}$ is then a canonical module for $\widehat{R}$, and is an ideal of $\widehat{R}$. But this contradicts the criterion above.

We finish the section with the promised identification of the intersection of the class of MCM modules with that of modules of finite injective dimension.

9.9 Proposition. *Let R be a CM local ring with canonical module ω and let M be a finitely generated R-module. If M is both MCM and of finite injective dimension, then M is isomorphic to a direct sum of copies of ω.*

Proof. Let F be a free module mapping onto the dual $M^\vee = \operatorname{Hom}_R(M,\omega)$ with kernel K. Dualizing gives a short exact sequence

$$0 \longrightarrow M \longrightarrow F^\vee \longrightarrow K^\vee \longrightarrow 0$$

where $K^\vee$ is MCM as K is. Proposition 9.4(ii) implies that the sequence splits as $\operatorname{injdim}_R M < \infty$, making M a direct summand of $F^\vee$. Dualizing again displays $M^\vee$ as a direct summand of the free module $F \cong F^{\vee\vee}$, whence $M^\vee$ is free and M is a direct sum of copies of ω. □

If R is not assumed to have a canonical module, the MCM modules of finite injective dimension are called *Gorenstein modules*. Should any exist, there is one of minimal rank and all others are direct sums of copies of the minimal one. See Corollary A.18 for an application of Gorenstein modules.

§2 MCM approximations and FID hulls

Throughout this section, $(R,\mathfrak{m},k)$ denotes a CM local ring with canonical module ω.

Propositions 9.4 and 9.9 suggest that we view the MCM modules and modules of finite injective dimension over R as orthogonal subspaces of the space of all finitely generated modules, with intersection spanned by the canonical module ω. Guided by this intuition and memories of basic linear algebra, we expect to be able to project any R-module onto these subspaces.

9.10 Definition. Let M be a non-zero finitely generated R-module. An exact sequence of finitely generated R-modules

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$$

is a *MCM approximation* of M if X is MCM and $\operatorname{injdim}_R Y < \infty$. Dually, an exact sequence

$$0 \longrightarrow M \longrightarrow Y' \longrightarrow X' \longrightarrow 0$$

is a *hull of finite injective dimension* or *FID hull* if $\operatorname{injdim} Y' < \infty$ and either X' is MCM or $X' = 0$.[1]

We sometimes abuse language and refer in a synecdoche to the modules X and Y' as the MCM approximation and FID hull of M, rather than the whole extensions.

The orthogonality relations between MCM modules and modules of finite injective dimension translate into lifting properties for the MCM approximations and FID hulls.

9.11 Proposition. *Let* $0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$ *be a MCM approximation of* M *and let* $\varphi\colon Z \longrightarrow M$ *be a homomorphism with* Z *MCM. Then* φ *factors through* X*. Any two liftings of* φ *are homotopic, i.e. their difference factors through* Y*.*

[1]The definition is made slightly unwieldy by the possibility $X' = 0$, but this is far preferable to the alternative, which would lead to considering the zero module to be MCM.

Proof. Applying $\operatorname{Hom}_R(Z,-)$ to the approximation gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(Z,Y) \longrightarrow \operatorname{Hom}_R(Z,X) \longrightarrow \operatorname{Hom}_R(Z,M) \longrightarrow \operatorname{Ext}^1_R(Z,Y),$$

the rightmost term of which vanishes by Proposition 9.4(ii). Thus $\varphi \in \operatorname{Hom}_R(Z,M)$ lifts to an element of $\operatorname{Hom}_R(Z,X)$. The final assertion follows as well from exactness. □

We leave it as an exercise for the reader to state and prove the dual statement for FID hulls.

The lifting property of Proposition 9.11 allows a Schanuel-type result: if $0 \longrightarrow Y_1 \longrightarrow X_1 \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow Y_2 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$ are two MCM approximations of the same module M, then $X_1 \oplus Y_2 \cong X_2 \oplus Y_1$. We leave the details to the reader. (One can also proceed directly, via the orthogonality relation $\operatorname{Ext}^1_R(X_i, Y_j) = 0$; compare with Lemma A.8.) Just as for free resolutions, this motivates a notion of minimality for MCM approximations.

9.12 Definition. Let $s\colon 0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{p} M \longrightarrow 0$ be a MCM approximation of a non-zero finitely generated R-module M. We say that s is *minimal* provided Y and X have no non-zero direct summand in common via i. In other words, for any direct-sum decomposition $X = X_0 \oplus X_1$ with $X_0 \subseteq \operatorname{im} i$, we must have $X_0 = 0$.

Observe that any common direct summand of Y and X is both MCM and of finite injective dimension, so by Proposition 9.9 is a direct sum of copies of the canonical module ω.

While the definition of minimality above is quite natural, in practice a more technical notion is useful.

9.13 Definition. Let $s\colon 0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{p} M \longrightarrow 0$ be a MCM approximation of a non-zero finitely generated R-module M. We say that s is *right minimal* if whenever $\varphi\colon X \longrightarrow X$ is an endomorphism such that $p\varphi = p$, in fact φ is an automorphism.

The equivalence of minimality and right minimality is "well-known to experts"; the proof we give here is due to M. Hashimoto and A. Shida [HS97] (see also [Yos93]). It turns out that passing to the completion is essential to the argument.

9.14 Lemma. *Let $s\colon 0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{p} M \longrightarrow 0$ be a MCM approximation of a non-zero R-module M. Let $\widehat{s}\colon 0 \longrightarrow \widehat{Y} \xrightarrow{\widehat{i}} \widehat{X} \xrightarrow{\widehat{p}} \widehat{M} \longrightarrow 0$ be the completion of s. Then $\widehat{s}$ is a MCM approximation of $\widehat{M}$, and the following are equivalent.*

(i) $\widehat{s}$ is right minimal;

(ii) s is right minimal;

(iii) s is minimal;

(iv) $\widehat{s}$ is minimal.

Proof. That $\widehat{s}$ is a MCM approximation of $\widehat{M}$ is trivial; the real matter is the equivalence.

(i) $\implies$ (ii) Assume that $\widehat{s}$ is right minimal, and $\varphi \in \operatorname{End}_R(X)$ satisfies $p\varphi = p$. Then $\widehat{p}\widehat{\varphi} = \widehat{p}$, so $\widehat{\varphi}$ is an automorphism by hypothesis, whence φ is an automorphism as well.

(ii) $\implies$ (iii) If $X = X_0 \oplus X_1$ is a direct sum decomposition of X with $X_0 \subseteq \operatorname{im} i$, then the idempotent $\varphi\colon X \twoheadrightarrow X_0 \hookrightarrow X$ obtained from the projection onto X_0 satisfies $p\varphi = p$. Thus $X_0 \neq 0$ implies that s is not right minimal.

(iii) $\implies$ (iv) Assume that $\widehat{s}$ is not minimal, so that $\widehat{Y}$ and $\widehat{X}$ have a common non-zero direct summand via i. We have already observed that such a direct summand must be a direct sum of copies of the canonical module $\widehat{\omega}$, so there exist homomorphisms $\sigma\colon \widehat{X} \longrightarrow \widehat{\omega}$ and $\tau\colon \widehat{\omega} \longrightarrow \widehat{Y}$ such that

$$\sigma \widehat{i} \tau\colon \widehat{\omega} \longrightarrow \widehat{Y} \longrightarrow \widehat{X} \longrightarrow \widehat{\omega}$$

is the identity on $\widehat{\omega}$. Write $\sigma = \sum_j a_j \widehat{\sigma}_j$ and $\tau = \sum_k b_k \widehat{\tau}_k$, where $\sigma_j \in \operatorname{Hom}_R(X,\omega)$, $\tau_k \in \operatorname{Hom}_R(\omega, Y)$, and $a_j, b_k \in \widehat{R}$. Then

$$\sum_{j,k} a_j b_k \widehat{\sigma}_j \widehat{i} \widehat{\tau}_k = 1 \quad \in \quad \operatorname{End}_{\widehat{R}}(\widehat{\omega}) \cong \widehat{R}\,.$$

Since $\widehat{R}$ is local, at least one of the summands $a_j b_k \widehat{\sigma}_j \widehat{i} \widehat{\tau}_k$ is a unit of $\widehat{R}$. It follows that $\sigma_j i \tau_k$ is a unit of R, that is, $\sigma_k i \tau_k\colon \omega \longrightarrow \omega$ is an isomorphism. Thus s is not minimal.

(iv) $\implies$ (i) We assume that $R = \widehat{R}$ is complete. Let $\varphi\colon X \longrightarrow X$ be a non-isomorphism satisfying $p\varphi = p$. Let $\Lambda \subset \operatorname{End}_R(X)$ be the subring generated by R and φ, and observe that Λ is commutative and is a finitely generated R-module.

As φ carries the kernel of p into itself, s is naturally a short exact sequence of (finitely generated) Λ-modules. In particular, multiplication by $\varphi \in \Lambda$ is the identity on the non-zero module M, so by Nakayama's Lemma φ is not contained in the radical of Λ. On the other hand, φ is not an isomorphism on X, so is not a unit of Λ. Thus Λ is not an nc-local ring. Since

R is Henselian (Corollary 1.8), it follows that Λ contains a non-trivial idempotent $e \neq 0, 1$.

Now $\varphi \in R + (1-\varphi)\Lambda$, so $R + (1-\varphi)\Lambda = \Lambda$. In particular, $\overline{\Lambda} := \Lambda/(1-\varphi)\Lambda$ is a quotient of R, so is a local ring. Replacing e by $1-e$ if necessary, we may assume that $\overline{e} = \overline{1}$ in $\overline{\Lambda}$. Since φ acts as the identity on M, we see that M is naturally a $\overline{\Lambda}$-module, and in particular e also acts as the identity on M.

Set $X_0 = \operatorname{im}(1-e) = \ker(e) \subseteq X$. Then X_0 is a non-zero direct summand of X, and $p(X_0) = 0$ since e acts trivially on M. Thus s is not minimal. □

9.15 Proposition. *If a finitely generated module M admits a MCM approximation, then there is a minimal one, which moreover is unique up to isomorphism of exact sequences inducing the identity on M.*

Proof. Removing any direct summands isomorphic to ω common to Y and X via i in a given MCM approximation of M, we arrive at a minimal one. For uniqueness, suppose we have two minimal approximations $s\colon 0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{p} M \longrightarrow 0$ and $s'\colon 0 \longrightarrow Y' \xrightarrow{i'} X' \xrightarrow{p'} M \longrightarrow 0$. The lifting property delivers a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{i} & X & \xrightarrow{p} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \| & & \\
0 & \longrightarrow & Y' & \xrightarrow{i'} & X' & \xrightarrow{p'} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{i} & X & \xrightarrow{p} & M & \longrightarrow & 0
\end{array}$$

in which, in particular, $p\beta\alpha = p$. Since minimality implies right minimality, $\beta\alpha$ is an isomorphism. A similar diagram shows that $\alpha\beta$ is an isomorphism as well, so that s and s' are isomorphic exact sequences via an isomorphism which is the identity on M. $\square$

Here is yet a third notion of minimality for MCM approximations, introduced by Hashimoto and Shida [HS97] and used to good effect by Simon and Strooker [SS02]. Set $d = \dim R$. It's immediate from the definition that a MCM approximation $0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$ induces isomorphisms

$$\operatorname{Ext}^i_R(k,M) \cong \begin{cases} \operatorname{Ext}^{i+1}_R(k,Y) & \text{for } 0 \leqslant i \leqslant d-2 \text{ and} \\ \operatorname{Ext}^i_R(k,X) & \text{for } i \geqslant d+1, \end{cases}$$

and a 4-term exact sequence

$$0 \longrightarrow \operatorname{Ext}^{d-1}_R(k,M) \longrightarrow \operatorname{Ext}^d_R(k,Y) \longrightarrow \operatorname{Ext}^d_R(k,X) \longrightarrow \operatorname{Ext}^d_R(k,M) \longrightarrow 0.$$

We will call the approximation Ext-*minimal* if the induced map of k-vector spaces $\operatorname{Ext}^d_R(k,Y) \longrightarrow \operatorname{Ext}^d_R(k,X)$ in the middle of this exact sequence is the zero map. Equivalently, one of the natural maps $\operatorname{Ext}^{d-1}_R(k,M) \longrightarrow \operatorname{Ext}^d_R(k,Y)$ and $\operatorname{Ext}^d_R(k,X) \longrightarrow \operatorname{Ext}^d_R(k,M)$ is an isomorphism (and hence both are). This means in particular that the Bass numbers of M are completely determined by X and Y.

If in an approximation of M there is a non-zero indecomposable direct summand of Y carried isomorphically to a summand of X, then we've already seen that the summand must be isomorphic to ω, and so $\operatorname{Ext}^d_R(k,Y) \longrightarrow \operatorname{Ext}^d_R(k,X)$ has as a summand the identity map on $k = \operatorname{Ext}^d_R(k,\omega)$. Thus Ext-minimality implies minimality as defined above. In fact, all three notions of minimality are equivalent. As the proof of this fact uses some local cohomology, we relegate it to the Exercises.

9.16 Proposition. *Let $(R,\mathfrak{m})$ be a CM local ring with canonical module, and let M be a non-zero finitely generated R-module. For a given MCM approximation of M, minimality, right minimality, and* Ext*-minimality are equivalent.* □

The considerations above are exactly paralleled on the FID hull side. A FID hull $0 \longrightarrow M \xrightarrow{j} Y \xrightarrow{q} X \longrightarrow 0$ is *minimal* if Y and X have no non-zero direct summand in common via q, is *left minimal* if every endomorphism $\psi \in \operatorname{End}_R(Y)$ such that $\psi j = j$ is in fact an automorphism, and is Ext-*minimal* if the induced linear map $\operatorname{Ext}^d_R(k,Y) \longrightarrow \operatorname{Ext}^d_R(k,X)$ is zero. The three notions are equivalent by arguments exactly similar to those above.

We turn now to existence. The construction of MCM approximations is most transparent when the approximated module is CM, so we state that case separately. In particular, the construction below applies when M is an R-module of finite length, for example $M = R/\mathfrak{m}^n$ for some $n \geqslant 1$. We will return to this example in §4.

9.17 Proposition. *Let $(R,\mathfrak{m})$ be a CM local ring with canonical module ω, and let M be a CM R-module. Then M has a minimal MCM approximation.*

Proof. Let $t = \operatorname{codepth} M$. By Theorem 9.7, $M^\vee = \operatorname{Ext}^t_R(M,\omega)$ is again CM of codepth t. In a truncated minimal free resolution of $M^\vee$

$$0 \longrightarrow \operatorname{syz}^R_t(M^\vee) \longrightarrow F_{t-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M^\vee \longrightarrow 0$$

the t^{th} syzygy $\operatorname{syz}^R_t(M^\vee)$ is MCM. Apply $\operatorname{Hom}_R(-,\omega)$ to get a complex

$$0 \longrightarrow F_0^\vee \longrightarrow F_1^\vee \longrightarrow \cdots \longrightarrow F_{t-1}^\vee \longrightarrow \operatorname{syz}^R_t(M^\vee)^\vee \longrightarrow 0$$

with homology $\operatorname{Ext}^i_R(M^\vee,\omega)$, which is $M^{\vee\vee} \cong M$ for $i = t$ and trivial otherwise. Inserting the homology at the rightmost end, and defining K to be the kernel, we get a short exact sequence

$$0 \longrightarrow K \longrightarrow \operatorname{syz}^R_t(M^\vee)^\vee \longrightarrow M \longrightarrow 0\,, \tag{9.17.1}$$

in which the middle term is MCM. Since K has a finite resolution by direct sums of copies of $R^\vee = \omega$, it has finite injective dimension, so that (9.17.1) is a MCM approximation of M.

It is easy to see that our initial choice of a minimal resolution forces the obtained approximation to be minimal as well. □

For the general case, we give an independent construction of a MCM approximation of a finitely generated module, which simultaneously produces an FID hull as well. This argument is essentially that of [AB89], though in a more concrete setting. (There are [at least] two other constructions: the *pitchfork construction*, originally due also to Auslander and Buchweitz, and the *gluing construction* of Herzog and Martsinkovsky [HM93].)

9.18 Theorem. *Let $(R,\mathfrak{m},k)$ be a CM local ring with canonical module ω, and let M be a finitely generated R-module. Then M admits a MCM approximation and a FID hull.*

Proof. We construct the approximation and hull by induction on $t := \operatorname{codepth} M$. When M is MCM itself, the MCM approximation is trivial. For a FID hull, take a free module F mapping onto the dual $M^\vee = \operatorname{Hom}_R(M,\omega)$ as in the proof of Prop. 9.17. In the short exact sequence

$$0 \longrightarrow \operatorname{syz}^R_1(M^\vee) \longrightarrow F \longrightarrow M^\vee \longrightarrow 0\,,$$

the syzygy module $\operatorname{syz}_1^R(M^\vee)$ is again MCM, so applying $\operatorname{Hom}_R(-,\omega)$ gives another exact sequence

$$0 \longrightarrow M \longrightarrow F^\vee \longrightarrow \operatorname{syz}_1^R(M^\vee)^\vee \longrightarrow 0$$

in which $F^\vee \cong \omega^n$ has finite injective dimension and $\operatorname{syz}_1^R(M^\vee)^\vee$ is MCM.

Suppose now that $\operatorname{codepth} M = t \geqslant 1$. Taking a syzygy of M in a minimal free resolution

$$0 \longrightarrow \operatorname{syz}_1^R(M) \longrightarrow F \longrightarrow M \longrightarrow 0$$

we have by induction a FID hull of $\operatorname{syz}_1^R(M)$

$$0 \longrightarrow \operatorname{syz}_1^R(M) \longrightarrow Y' \longrightarrow X' \longrightarrow 0\,.$$

Construct the pushout diagram from these two sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{syz}_1^R(M) & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y' & \longrightarrow & W & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & X' & = & X' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

As X' is MCM and F is free, the exact middle column forces W to be MCM, so that the middle row is a MCM approximation of M.

A FID hull for W exists by the base case of the induction:

$$0 \longrightarrow W \longrightarrow Y'' \longrightarrow X'' \longrightarrow 0$$

and constructing another pushout

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y' & = & Y' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & W & \longrightarrow & Y'' & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & Z & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

we see from the middle column that Z has finite injective dimension, so the bottom row is a FID hull for M. □

9.19 Notation. Having now established both existence and uniqueness of minimal MCM approximations and FID hulls, we introduce some notation for them. The minimal MCM approximation of M is denoted by

$$0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow 0\,,$$

while the minimal FID hull of M is denoted

$$0 \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0\,.$$

To show off the new notation, here is the final diagram of the proof of

Theorem 9.18.

(9.19.1)
$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & Y_M & = & Y_M & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & X_M & \longrightarrow & \omega^{(n)} & \longrightarrow & X^M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & Y^M & \longrightarrow & X^M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Here $n = \mu_R(X_M^\vee)$ as the middle row is an FID hull for X_M.

We also record a few curiosities that arose in the proof of Theorem 9.18.

9.20 Proposition. *Up to adding or deleting direct summands isomorphic to ω, we have*

(i) $Y_M \cong Y^{\mathrm{syz}_1^R(M)}$*;*

(ii) $X^M \cong X^{X_M}$ *; and*

(iii) X_M *is an extension of a free module by* $X^{\mathrm{syz}_1^R(M)}$*, that is, there is a short exact sequence* $0 \longrightarrow F \longrightarrow X_M \longrightarrow X^{\mathrm{syz}_1^R(M)} \longrightarrow 0$ *with* F *free.*

In particular, if R *is Gorenstein then we have as well*

(iv) $X_M \cong X^{\mathrm{syz}_1^R(M)}$*;*

(v) $X_M \cong \mathrm{syz}_1^R(X^M)$*; and*

(vi) $Y_M \cong \mathrm{syz}_1^R(Y^M)$.

We see already that the case of a Gorenstein local ring is special. In this case, finite injective dimension coincides with finite projective dimension, making the theory more tractable. We will see more advantages of the Gorenstein condition in §4; see also Exercises 9.48 and 9.49.

We record here for later reference the case of codepth 1.

9.21 Proposition. *Let R be a CM local ring with canonical module and let M be an R-module of codepth 1, that is* $\operatorname{depth} M = \dim R - 1$*. Let $\xi_1, \dots, \xi_t$ be a minimal set of generators for the (nonzero) module $\operatorname{Ext}^1_R(M, \omega)$, and let E be the extension of M by $\omega^{(t)}$ corresponding to the element $\xi = (\xi_1, \dots, \xi_t) \in \operatorname{Ext}^1_R(M, \omega^{(t)}) \cong \operatorname{Ext}^1_R(M, \omega)^{(t)}$. Then E is a MCM module and*

$$\xi : 0 \longrightarrow \omega^t \longrightarrow E \longrightarrow M \longrightarrow 0$$

is the minimal MCM approximation of M. In particular, this construction coincides with that of Proposition 9.17 if M is CM, i.e. if $\operatorname{Hom}_R(M, \omega) = 0$.

To close out this section, we have a few more words to say about uniqueness. Since every MCM module is its own MCM approximation, the function $M \rightsquigarrow X_M$ is in general neither injective nor surjective. However, we may restrict to CM modules of a fixed codepth and ask whether every MCM module X is a MCM approximation of a CM module of codepth r. For $r = 1$ and $r = 2$, these questions have essentially been answered by Yoshino–Isogawa [YI00] and Kato [Kat07]. Here is the criterion for $r = 1$.

9.22 Proposition. *Let R be a CM local ring with a canonical module, and assume that R is generically Gorenstein. Let X be a MCM R-module. Then*

X is a MCM approximation of some CM module M of codepth 1 if and only if X has constant rank.

Proof. First assume that X has constant rank s. Then there is a short exact sequence

$$0 \longrightarrow R^{(s)} \longrightarrow X \longrightarrow N \longrightarrow 0$$

in which N is a torsion module. In particular, N has dimension at most $\dim R - 1$. However, the Depth Lemma ensures that N has depth at least $\dim R - 1$, so N is CM of codepth 1. As R is generically Gorenstein, the canonical module ω embeds into R as an ideal of pure height one (Prop. 9.8). We therefore have embeddings $\omega^{(s)} \hookrightarrow R^{(s)}$ and $R^{(s)} \hookrightarrow X$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega^{(s)} & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & R^{(s)} & \longrightarrow & X & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

The Snake Lemma delivers an isomorphism from the kernel of $M \longrightarrow N$ onto $(R/\omega)^{(s)}$, and hence an exact sequence

$$0 \longrightarrow (R/\omega)^{(s)} \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Therefore M is also CM of codepth 1, and the top row of the diagram is a MCM approximation of M.

For the converse, suppose that M is CM of codepth 1 and that X is a MCM approximation of M. Then $X \cong X_M \oplus \omega^{(t)}$ for some $t \geqslant 0$. In the minimal MCM approximation

$$0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow 0,$$

we see that M is torsion, whence of rank zero, and Y_M is isomorphic to a direct sum of copies of ω. As R is generically Gorenstein, Y_M has constant rank, and so X_M and X do as well. □

It's clear that a local ring R is a domain if and only if every finitely generated R-module has constant rank. If in addition R is CM, then it follows that R is a domain if and only if every MCM module has constant rank. (Take a high syzygy of an arbitrary finitely generated module M and compute the rank of M as an alternating sum.) These observations prove the following corollary.

9.23 Corollary. *Let R be a CM local ring with a canonical module and assume that R is generically Gorenstein. The following statements are equivalent.*

(i) For every MCM R-module X, there exists a CM module M of codepth 1 such that X is MCM approximation of M.

(ii) R is a domain.

The question of the injectivity of the function $M \rightsquigarrow X_M$ for modules M of a fixed codepth is, as far as we can tell, still open. The corresponding question for FID hulls, however, has a positive answer when R is Gorenstein, due to Kato [Kat99].

§3 Numerical invariants

Since the minimal MCM approximation and minimal FID hull of a module M are uniquely determined up to isomorphism by M, any numerical information we derive from X_M, Y_M, X^M, and Y^M are invariants of M. For example, if R is Henselian we might consider the number of indecomposable direct summands appearing in a direct sum decomposition of X_M or Y^M as a kind of measure of the complexity of M, or if R is generically Gorenstein we might consider $\operatorname{rank} Y^M$. All these possibilities were pointed out by Buchweitz [Buc86], but seem not to have gotten much attention. In this section we introduce two other numerical invariants of M, namely $\delta(M)$, first defined by Auslander; and $\gamma(M)$, defined by Herzog and Martsinkovsky. We also introduce a mysterious new invariant, $\epsilon(M)$, about which we can say little.

Throughout, $(R,\mathfrak{m})$ is still a CM local ring with canonical module ω. For an arbitrary finitely generated R-module Z, we define the *free rank* of Z, denoted $\text{f-rank}\, Z$, to be the rank of a maximal free direct summand of Z. In other words, $Z \cong \underline{Z} \oplus R^{(\text{f-rank}\, Z)}$ with $\underline{Z}$ *stable,* i.e. having no non-trivial free direct summands. Dually, the *canonical rank* of Z, $\omega\text{-rank}\, Z$, is the largest integer n such that $\omega^{(n)}$ is a direct summand of Z.

9.24 Definition. Let M be a finitely generated R-module with minimal MCM approximation $0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow 0$ and minimal FID hull $0 \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0$. Then we define

$$\begin{aligned} \delta(M) &= \text{f-rank}\, X_M\,; \\ \gamma(M) &= \omega\text{-rank}\, X_M\,; \text{ and} \\ \epsilon(M) &= \omega\text{-rank}\, Y^M\,. \end{aligned}$$

For the rest of the section, we fix once and for all the minimal MCM approximation

$$0 \longrightarrow Y_M \xrightarrow{\ i\ } X_M \xrightarrow{\ p\ } M \longrightarrow 0$$

and minimal FID hull

$$0 \longrightarrow M \xrightarrow{j} Y^M \xrightarrow{q} X^M \longrightarrow 0$$

of a chosen R-module M. Note first that since we chose our approximation and hull to be (Ext-)minimal, we have

$$\operatorname{Ext}^d_R(k, X_M) \cong \operatorname{Ext}^d_R(k, M) \cong \operatorname{Ext}^d_R(k, Y^M),$$

where $d = \dim R$. This, together with the fact (see Exercise 9.52) that $\operatorname{Ext}^d_R(k, Z) \neq 0$ for every non-zero finitely generated R-module Z, immediately gives the following crude bounds.

9.25 Proposition. *Set $s = \dim_k \operatorname{Ext}^d_R(k, M)$. Then*

(i) $\delta(M) \cdot \dim_k \operatorname{Ext}^d_R(k, R) \leqslant s$*, with equality if and only if X_M is free. In particular, if $\dim_k \operatorname{Ext}^d_R(k, M) < \dim_k \operatorname{Ext}^d_R(k, R)$, then $\delta(M) = 0$.*

(ii) $\gamma(M) \leqslant s$*, with equality if and only if M has finite injective dimension.*

(iii) $\epsilon(M) \leqslant s$*, with equality if and only if M is MCM.*

Note that the question of which modules M satisfy "X_M is free" is quite subtle. One situation in which it holds is when R is Gorenstein and M has finite projective dimension; see Exercise 9.48. However, it may hold in other cases as well, for example $M = R/\omega$, where ω is embedded as an ideal of height one as in Prop. 9.8.

To obtain sharper bounds, as well as a better understanding of what exactly each invariant measures, we consider them separately. Of the three, $\delta(M)$ has received the most attention, and we begin there.

9.26 Lemma. *Let M be a finitely generated R-module. Write $X_M = \underline{X} \oplus F$, where F is a free module of rank $\delta(M)$ and $\underline{X}$ is stable. Then*

$$\delta(M) = \mu_R\left(M/p\left(\underline{X}\right)\right).$$

Proof. The commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(p|_{\underline{X}}) & \longrightarrow & Y_M & \longrightarrow & \ker \overline{p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{X} & \longrightarrow & X_M & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \overline{p}} & & \\
0 & \longrightarrow & p(\underline{X}) & \longrightarrow & M & \longrightarrow & M/p(\underline{X}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

shows that $\delta(M) = \operatorname{rank} F \geqslant \mu_R(M/p(\underline{X}))$. If $\operatorname{rank} F > \mu_R(M/p(\underline{X}))$, then $\ker \overline{p}$ has a non-zero free direct summand. Since Y_M maps onto $\ker \overline{p}$, Y_M also has a free summand, which we easily see is a common direct summand of Y_M and X_M. As our approximation was chosen minimal, we must have equality. □

The lemma allows us to characterize $\delta(M)$ without referring to the MCM approximation of M.

9.27 Proposition. *Let M be a finitely generated R-module. The delta-invariant $\delta(M)$ is the minimum free rank of all MCM modules Z admitting a surjective homomorphism onto M.*

Proof. Denote the minimum by $\delta' = \delta'(M)$, and set $\delta = \delta(M)$. Then evidently $\delta' \leqslant \delta$. For the other inequality, let $\varphi\colon Z \longrightarrow M$ be a surjection with Z MCM and $\operatorname{f-rank} Z = \delta'$. Write $Z = \underline{Z} \oplus R^{(\delta')}$ and $X_M = \underline{X} \oplus R^{(\delta)}$. The lifting property applied to $\varphi|_{\underline{Z}}$ gives a homomorphism $\alpha\colon \underline{Z} \longrightarrow \underline{X} \oplus R^{(\delta)}$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\varphi|_{\underline{Z}} & \longrightarrow & \underline{Z} & \xrightarrow{\varphi|_{\underline{Z}}} & M & & \\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 0 & \longrightarrow & Y_M & \longrightarrow & \underline{X} \oplus R^{(\delta)} & \xrightarrow[p]{} & M & \longrightarrow & 0 \end{array}$$

As $\underline{Z}$ has no free direct summands, the image of the composition $\underline{Z} \longrightarrow \underline{X} \oplus R^{(\delta)} \twoheadrightarrow R^{(\delta)}$ is contained in $\mathfrak{m}R^{(\delta)}$. Thus $\alpha\left(\underline{Z}\right)$ contributes no minimal generators to $M/p(\underline{X})$, and therefore $\delta = \mu_R\left(M/p\left(\underline{X}\right)\right) \leqslant \mu_R\left(M/p\alpha\left(\underline{Z}\right)\right) \leqslant \delta'$. □

In particular, Prop. 9.27 implies that for a MCM module X, we have $\delta(X) = \operatorname{f-rank} X$, and for M arbitrary, $\delta(M) = 0$ if and only if M is a homomorphic image of a stable MCM module. We also obtain some basic properties of δ.

9.28 Corollary. *Let M and N be finitely generated R-modules.*

(i) $\delta(M \oplus N) = \delta(M) + \delta(N)$.

(ii) $\delta(N) \leqslant \delta(M)$ *if there is a surjection* $M \twoheadrightarrow N$.

(iii) $\delta(M) \leqslant \mu_R(M)$.

Proof. Since minimality is equivalent to Ext-minimality, the direct sum of minimal MCM approximations of M and N is again minimal. Thus

$X_{M\oplus N} \cong X_M \oplus X_N$. The free rank of $X_M \oplus X_N$ is the sum of those of X_M and X_N, since a direct sum has a free summand if and only if one summand does. The second and third statements are clear from the Proposition. □

9.29 Remark. We point out a historically significant consequence of Cor. 9.28. Suppose that R is Gorenstein and M is an R-module equipped with a surjection onto a non-zero module N of finite projective dimension. Since the minimal MCM approximation of N is simply a free cover (Ex. 9.48), we have $\delta(N) > 0$, and hence $\delta(M) > 0$. It was at first conjectured that $\delta(M) > 0$ if and only if M has a non-zero quotient module of finite projective dimension, but a counterexample was given by S. Ding [Din94]. Ding proves a formula for $\delta(R/I)$, where R is a one-dimensional Gorenstein local ring and I is an ideal of R containing a non-zerodivisor:

$$\delta(R/I) = 1 + \lambda(\operatorname{soc}(R/I)) - \mu_R\left(I^*\right).$$

He then takes $R = k[[t^3, t^4]]$, where k is a field, and $I = (t^8 + t^9, t^{10})$. He shows that $\delta(R/I) = 1$ and that I is not contained in any proper principal ideal of R, so R/I cannot map onto a non-zero module of finite projective dimension.

We also mention here in passing a remarkable application of the δ-invariant, due to A. Martsinkovsky [Mar90, Mar91]. Let $S = k[[x_1, \dots, x_n]]$ be a power series ring over an algebraically closed field of characteristic zero. Let $f \in S$ be a polynomial such that the hypersurface ring $R = S/(f)$ is an isolated singularity. The Jacobian ideal $j(f)$, generated by the partial derivatives of f, and its image $\overline{j(f)}$ in R, are thus primary to the respective maximal ideals. Martsinkovsky shows that $\delta\left(R/\overline{j(f)}\right) = 0$ if and only if

$f \in j(f)$. In fact, these are equivalent to $f \in (x_1,\dots,x_n)j(f)$, which by a foundational result of Saito [Sai71] occurs if and only if f is *quasi-homogeneous*, i.e. there is an integral weighting of the variables $x_1,\dots,x_n$ under which f is homogeneous.

Turning now to $\gamma(M) = \omega\text{-rank}\, X_M$, we have an analogue of Lemma 9.26, the proof of which is similar enough that we skip it.

9.30 Lemma. *Let M be a finitely generated R-module, and write $X_M = \overline{X} \oplus \omega^{(\gamma(M))}$, where $\overline{X}$ has no direct summand isomorphic to ω. Then*

$$\gamma(M) \cdot \mu_R(\omega) = \mu_R\left(M/p(\overline{X})\right).$$

As a consequence, we find an unexpected restriction on the R-modules of finite injective dimension.

9.31 Proposition. *Let M be a finitely generated R-module of finite injective dimension. Then $\gamma(M) \cdot \mu_R(\omega) = \mu_R(M)$. In particular, $\mu_R(M)$ is an integer multiple of $\mu_R(\omega)$.*

There is obviously no direct analogue of Prop. 9.27 for $\gamma(M)$; as long as R is not Gorenstein, every M is a homomorphic image of a MCM module without ω-summands, namely, a free module. Still, we do retain additivity, and in certain cases the other assertions of Cor. 9.28.

9.32 Proposition. *Let M and N be R-modules. Then $\gamma(M \oplus N) = \gamma(M) + \gamma(N)$.*

The next result fails without the assumption of finite injective dimension. For example, consider a non-Gorenstein ring R and a free module F mapping onto the canonical module ω. We have $\gamma(F) = 0$ and $\gamma(\omega) = 1$.

9.33 Proposition. *Let $N \subseteq M$ be R-modules, both of finite injective dimension. Then $\gamma(M/N) \leqslant \gamma(M) - \gamma(N)$.*

Proof. Since each of M, N, and M/N has finite injective dimension, Prop. 9.25 allows us to compute $\gamma(-)$ as $\dim_k \operatorname{Ext}^d_R(k,-)$. The long exact sequence of Ext ends with

$$\operatorname{Ext}^d_R(k,N) \longrightarrow \operatorname{Ext}^d_R(k,M) \longrightarrow \operatorname{Ext}^d_R(k,M/N) \longrightarrow 0\,,$$

and a dimension count gives the inequality. □

In case M has codepth 1, the explicit construction of MCM approximations in Prop. 9.21 allows us to compute $\gamma(M)$ directly. We leave the proof as yet another exercise.

9.34 Proposition. *Let M be an R-module of codepth 1 (not necessarily Cohen–Macaulay). Then we have $\gamma(M) = \mu_R(\operatorname{Ext}^1_R(M,\omega))$.*

For CM modules, the δ- and γ-invariants are dual. This follows easily from the construction of MCM approximations in this case.

9.35 Proposition. *Let M be a CM R-module of codepth t, and write $M^\vee = \operatorname{Ext}^t_R(M,\omega)$ as usual. Then $\delta(M^\vee) = \gamma(\operatorname{syz}^R_t(M))$.*

In fact, one can show, using the gluing construction of Herzog and Martsinkovsky [HM93], that $\delta\left(\operatorname{syz}_i(M^\vee)\right) = \gamma\left(\operatorname{syz}_{t-i}(M)\right)$ for $i = 0,\dots,t$.

When R is Gorenstein, δ and γ coincide, allowing us to combine all the above results, and enabling new ones. Here is an example.

9.36 Proposition. *Assume that R is a Gorenstein ring, and let M be a finitely generated R-module. Then*

$$\delta(M) = \mu_R\left(Y^M\right) - \mu_R\left(X^M\right).$$

Proof. Consider the diagram (9.19.1) following the construction of MCM approximations and FID hulls. In the Gorenstein situation, the $\omega^{(n)}$ in the center becomes a free module $R^{(n)}$. Thus $\delta(M) = \text{f-rank}\, X_M = n - \mu_R(X^M)$. The middle column implies $n \geqslant \mu_R(Y^M)$, but in fact we have equality: the image of the vertical arrow $Y_M \longrightarrow R^n$ is contained in $\mathfrak{m}R^{(n)}$ by the minimality of the left-hand column. Combining these gives the formula of the statement. □

Closing out this section, we turn to the ϵ-invariant $\epsilon(M) = \omega\text{-rank}(Y^M)$. This number seems quite mysterious. We record a few basic observations, but much remains to be learned. The first assertion follows from the construction of Y^M, and the second from the definition.

9.37 Proposition. *Let M be a finitely generated R-module.*

(i) If M is MCM, then $\epsilon(M) = \mu_R(M^\vee)$.

(ii) If M has finite injective dimension, then $\epsilon(M) = \omega\text{-rank}(M)$.

§4 The index and applications to finite CM type

Once again, in this section $(R, \mathfrak{m})$ is a CM local ring with canonical module ω. As a warm-up exercise, here is a straightforward result attributed to

Auslander.

9.38 Proposition. *The following conditions are equivalent.*

(i) R *is a regular local ring.*

(ii) $\delta(\mathrm{syz}_n^R(k)) > 0$ *for all* $n \geqslant 0$*.*

(iii) $\delta(k) = 1$*, i.e.* k *is not a homomorphic image of a stable MCM module.*

(iv) $\gamma(\mathrm{syz}_d^R(k)) > 0$*, where* $d = \dim R$*.*

Proof. If R is a regular local ring, then every MCM module is free, so $\delta(M) > 0$ for every module M in particular (ii) holds. Statement (ii) implies (iii) trivially. If R is non-regular, then there is at least one MCM R-module M without free summands, and the composition $M \longrightarrow M/\mathfrak{m}M \cong k^{(\mu_R(M))} \twoheadrightarrow k$ shows $\delta(k) = 0$. Thus the first three statements are equivalent. Finally, the construction of minimal MCM approximations for CM modules in Prop. 9.17 shows that $\delta(k) = \text{f-rank}(\mathrm{syz}_d^R(k^\vee)^\vee) = \omega\text{-rank}(\mathrm{syz}_d^R(k)) = \gamma(\mathrm{syz}_d^R(k))$, whence (iii) $\iff$ (iv). □

For a moment, let us set $\delta_n = \delta(R/\mathfrak{m}^n)$ for each $n \geqslant 0$. Then the Proposition says simply that if R is not regular, then $\delta_0 = 0$. The surjection $R/\mathfrak{m}^{n+1} \twoheadrightarrow R/\mathfrak{m}^n$ gives $\delta_{n+1} \geqslant \delta_n$, and every δ_n is at most 1 by Cor. 9.28. Thus the sequence $\{\delta_n\}$ is non-decreasing, with

$$0 = \delta_0 \leqslant \delta_1 \leqslant \cdots \leqslant \delta_n \leqslant \delta_{n+1} \leqslant \cdots \leqslant 1\,.$$

If ever $\delta_n = 1$, the sequence stabilizes there. Let us define the *index of* R to be the point at which that stabilization occurs, that is,

$$\mathrm{index}(R) = \min\left\{\, n \;\middle|\; \delta\left(R/\mathfrak{m}^n\right) = 1 \right\}$$

and set $\operatorname{index}(R) = \infty$ if $\delta(R/\mathfrak{m}^n) = 0$ for every n. Equivalently, $\operatorname{index}(R)$ is the least integer n such that any MCM R-module X mapping onto $R/\mathfrak{m}^n$ has a free direct summand. In these terms, the Proposition says that R is regular if and only if $\operatorname{index}(R) = 1$.

Next we point out that the index of R is finite if R is Gorenstein. Let $\mathbf{x}$ be a system of parameters in the maximal ideal $\mathfrak{m}$. Then $R/(\mathbf{x})$ has finite projective dimension, so $\delta(R/(\mathbf{x})) > 0$ since the MCM approximation is just a free cover (Exercise 9.48). The ideal generated by $\mathbf{x}$ being $\mathfrak{m}$-primary, we have $\mathfrak{m}^n \subseteq (\mathbf{x})$ for some n, and the surjection $R/\mathfrak{m}^n \longrightarrow R/(\mathbf{x})$ gives $\delta_n \geqslant \delta(R/(\mathbf{x})) > 0$. Thus $\operatorname{index}(R) \leqslant n$. In fact, we see that the index of R is bounded above by the *generalized Loewy length* of R,

$$\ell\ell(R) = \inf\left\{ n \;\middle|\; \text{there exists a s.o.p. } \mathbf{x} \text{ with } \mathfrak{m}^n \subseteq (\mathbf{x}) \right\}.$$

It has been conjectured by Ding that in fact $\operatorname{index}(R) = \ell\ell(R)$; as long as the residue field of R is infinite [HS97], this is still open, despite partial results by Ding [Din92, Din93, Din94] and Herzog [Her94], who proved it in case R is homogeneous graded over a field.

In this section we will give Ding's proof that the index of R is finite if and only if R is Gorenstein on the punctured spectrum; moreover, in this case the index is bounded by the Loewy length. This will be Theorem 9.42, to which we come after some preliminaries.

9.39 Lemma. *Let $(R,\mathfrak{m})$ be a CM local ring with canonical module ω and let $x \in \mathfrak{m}$ be a non-zerodivisor. Then $\delta(R/(x)) > 0$ if and only if $\operatorname{syz}_1^R(\omega/x\omega)$ has a direct summand isomorphic to ω.*

Proof. The minimal MCM approximation of a module of codepth 1 is computed in Prop. 9.21; in the case of $R/(x)$ we see that it is obtained by dualizing a free resolution of $(R/(x))^\vee = \operatorname{Ext}^1_R(R/(x),\omega) \cong \omega_{R/(x)} \cong \omega/x\omega$. It therefore takes the form

$$0 \longrightarrow F^\vee \longrightarrow \operatorname{syz}_1^R(\omega/x\omega)^\vee \longrightarrow R/(x) \longrightarrow 0$$

where F is a free module. Thus $\delta(R/(x)) = \text{f-rank}\left(\operatorname{syz}_1^R(\omega/x\omega)^\vee\right)$ is equal to $\omega\text{-rank}\left(\operatorname{syz}_1^R(\omega/x\omega)\right)$. □

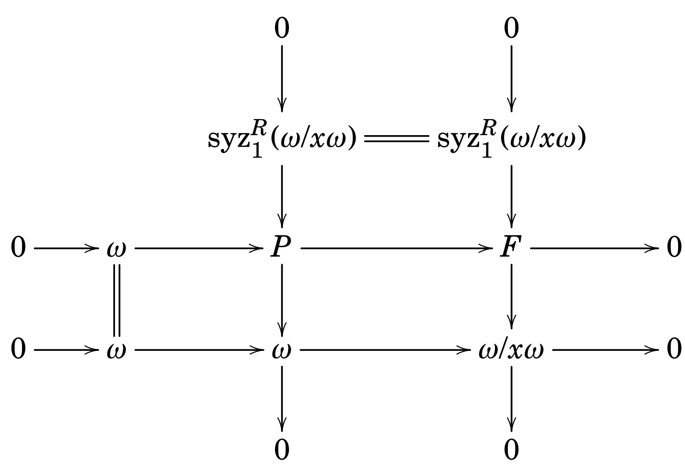
9.40 Lemma. *The following are equivalent for a non-zerodivisor $x \in \mathfrak{m}$:*

(i) $\operatorname{syz}_1^R(\omega/x\omega)$ *has a direct summand isomorphic to* ω*;*

(ii) $\operatorname{syz}_1^R(\omega/x\omega) \cong \omega \oplus \operatorname{syz}_1^R(\omega)$*;*

(iii) the multiplication map $\omega \xrightarrow{x} \omega$ *factors through a free module.*

Proof. (i) $\implies$ (ii) Form the pullback of a free cover $F \longrightarrow \omega/x\omega$ and the surjection $\omega \longrightarrow \omega/x\omega$ to obtain a diagram as below.



The middle row splits, giving a short exact sequence

$$0 \longrightarrow \operatorname{syz}_1^R(\omega/x\omega) \longrightarrow F \oplus \omega \longrightarrow \omega \longrightarrow 0$$

in the middle column. As $\operatorname{Ext}_R^1(\omega,\omega) = 0$, any ω-summand of $\operatorname{syz}_1^R(\omega/x\omega)$ must split out as an isomorphism $\omega \longrightarrow \omega$, leaving $\operatorname{syz}_1^R(\omega)$ behind.

(ii) $\implies$ (iii) Letting $F \longrightarrow \omega$ now be a free cover of ω, another pullback gives the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{syz}_1^R(\omega) & = & \operatorname{syz}_1^R(\omega) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{syz}_1^R(\omega/x\omega) & \longrightarrow & F & \longrightarrow & \omega/x\omega & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \omega & \xrightarrow{x} & \omega & \longrightarrow & \omega/x\omega & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Applying Miyata's theorem (Theorem 6.1), the left-hand column must split, so that $\omega \xrightarrow{x} \omega$ factors through F.

(iii) $\implies$ (i) If we have a factorization of the multiplication homomorphism $\omega \xrightarrow{x} \omega$ through a free module, say $\omega \longrightarrow G \longrightarrow \omega$, we may pull back in two stages:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{syz}_1^R(\omega) & \longrightarrow & Q & \longrightarrow & \omega & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{syz}_1^R(\omega) & \longrightarrow & P & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{syz}_1^R(\omega) & \longrightarrow & F & \longrightarrow & \omega & \longrightarrow & 0
\end{array}$$

The result is the same as if we had pulled back by $\omega \xrightarrow{x} \omega$ directly, by the functoriality of Ext. Doing so in two stages, however, reveals that the middle row splits as G is free, and so the top row splits as well. This gives $Q \cong \omega \oplus \operatorname{syz}_1^R(\omega)$ and the middle column thus presents Q as the first syzygy of $\operatorname{cok}(\omega \xrightarrow{x} \omega) \cong \omega/x\omega$, giving even property (ii) and in particular (i). □

Putting the lemmas together, we see that $\delta(R/(x)) = 0$ for a nonzerodivisor $x \in \mathfrak{m}$ if and only if x is in the ideal of $\operatorname{End}_R(\omega) \cong R$ consisting of those elements for which the corresponding multiplication factors through a free module. Let us identify this ideal explicitly.

9.41 Lemma. *Let R be a CM local ring with canonical module ω. The following three ideals of R coincide.*

(i) $\left\{ x \in R \;\middle|\; \omega \xrightarrow{x} \omega \text{ factors through a free module} \right\}$;

(ii) the trace $\tau_\omega(R)$ *of ω in R, which is generated by all homomorphic images of ω in R;*

(iii) the image of the natural map

$$\alpha \colon \operatorname{Hom}_R(\omega, R) \otimes_R \omega \longrightarrow \operatorname{End}_R(\omega) = R$$

defined by $\alpha(f \otimes a)(b) = f(b) \cdot a$. (Note that this is not *the evaluation map* $\operatorname{ev}(f \otimes a) = f(a)$*.)*

Proof. We prove (i) ⊇ (ii) ⊇ (iii) ⊇ (i).

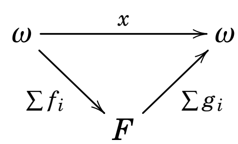
Let $x \in \tau_\omega(R)$, so that there is a linear functional $f \colon \omega \longrightarrow R$ and an element $a \in \omega$ with $f(a) = x$. Defining $g \colon R \longrightarrow \omega$ by $g(1) = a$, we have a factorization $x = g \circ f \colon \omega \longrightarrow \omega$.

Now if $x \in \operatorname{im}\alpha$, then there exist homomorphisms $f_i \colon \omega \longrightarrow R$ and elements $a_i \in \omega$ such that

$$\alpha\left(\sum_i f_i \otimes a_i\right)(b) = xb$$

for every $b \in \omega$. Define homomorphisms $g_i \colon \omega \longrightarrow R$ by $g_i(b) = \alpha(f_i \otimes b)$ for all $b \in \omega$. Then $\sum_i g_i(a_i) = x$, so that x is contained in the sum of the images of the g_i, hence in the trace ideal.

Finally, suppose we have a commutative diagram



with F a free module and $\sum f_i$, $\sum g_i$ the decompositions along an isomorphism $F \cong R^{(n)}$. Then for $a \in \omega$, we have

$$\begin{aligned}\alpha\left(\sum f_i \otimes g_i(1)\right)(a) &= \sum f_i(a) \cdot g_i(1)\\ &= \sum g_i(f_i(a))\\ &= xa\end{aligned}$$

so that $x \in \operatorname{im}\alpha$. □

From either of the first two descriptions above, we see that $1 \in \tau_\omega(R)$ if and only if R is Gorenstein. It follows that $\tau_\omega(R)$ defines the *Gorenstein locus* of R, that is, a localization $R_\mathfrak{p}$ is Gorenstein if and only if $\tau_\omega(R) \not\subseteq \mathfrak{p}$. In particular, R is Gorenstein on the punctured spectrum if and only if $\tau_\omega(R)$ is $\mathfrak{m}$-primary.

9.42 Theorem (Ding)**.** *The index of a CM local ring $(R,\mathfrak{m})$ with canonical module ω is finite if and only if R is Gorenstein on the punctured spectrum.*

Proof. Assume first that R is Gorenstein on the punctured spectrum, so that $\tau_\omega(R)$ is $\mathfrak{m}$-primary. Then there exists a regular sequence $x_1,\dots,x_d$ in $\tau_\omega(R)$, where $d = \dim R$. We claim by induction on d that $\delta(R/(x_1,\dots,x_d)) \neq 0$. The case $d = 1$ is immediate from Lemmas 9.39 and 9.40.

Suppose $d > 1$ and X is a MCM R-module with a surjection $X \longrightarrow R/(x_1,\dots,x_d)$. Tensor with $\overline{R} = R/(x_1)$ to get a surjection $X/x_1X \longrightarrow \overline{R}/(\overline{x_2},\dots,\overline{x_d})$, where overlines indicate passage to $\overline{R}$. Since $\overline{x_2},\dots,\overline{x_d}$ are in $\tau_{\overline{\omega}}(\overline{R})$, the inductive hypothesis says that X/x_1X has an $R/(x_1)$-free direct summand. But then there is a surjection $X \longrightarrow X/x_1X \longrightarrow \overline{R}$, so that $\operatorname{f-rank} X \geqslant \delta(\overline{R}) > 0$, and X has a non-trivial R-free direct summand, showing $\delta(R/(x_1,\dots,x_d)) > 0$.

Now let us assume that $\tau_\omega(R)$ is not $\mathfrak{m}$-primary. For any power $\mathfrak{m}^n$ of the maximal ideal, we may find a non-zerodivisor $z_{(n)} \in \mathfrak{m}^n \setminus \tau_\omega(R)$. By Lemmas 9.39 and 9.40, $\delta(R/(z_{(n)})) = 0$ for every n, and the surjection $R/(z_{(n)}) \longrightarrow R/\mathfrak{m}^n$ gives $\delta(R/\mathfrak{m}^n) = 0$ for all n, so that $\operatorname{index}(R) = \infty$. □

As an application of Ding's theorem, we prove that CM local rings of finite CM type are Gorenstein on the punctured spectrum. Of course this follows trivially from Theorem 6.12, since isolated singularities are Gorenstein on the punctured spectrum. This proof is completely independent, however, and may have other applications. It relies upon Guralnick's results in Chapter 6.

9.43 Theorem. *Let $(R,\mathfrak{m})$ be a CM local ring of finite CM type. Then R has finite index. If in particular R has a canonical module, then R is Gorenstein on the punctured spectrum.*

Proof. Let $\{M_1,\dots,M_r\}$ be a complete set of representatives for the isomorphism classes of non-free indecomposable MCM R-modules. By Corollary 1.13, since R is not a direct summand of any M_i, there exist integers n_i, $i=1,\dots,r$, such that for $s \geqslant n_i$, $R/\mathfrak{m}^s$ is not a direct summand of $M_i/\mathfrak{m}^s M_i$. Then for $s \geqslant n_i$, there exists no surjection $M_i/\mathfrak{m}^s M_i \longrightarrow R/\mathfrak{m}^s$ by Lemma 1.11. Set $N = \max\{n_i\}$. Let X be any stable MCM R-module, and decompose $X \cong M_1^{(a_1)} \oplus \cdots \oplus M_r^{(a_r)}$. If there were a surjection $X \longrightarrow R/\mathfrak{m}^N$, then (since R is local) one of the summands M_i would map onto $R/\mathfrak{m}^N$, contradicting the choice of N. As X was arbitrary, this shows that $\operatorname{index}(R) < \infty$. □

9.44 Remark. The foundation of Ding's theorem is in identifying the nonzerodivisors x such that $\delta(R/(x)) > 0$. One might also ask about $\delta(\omega/x\omega)$, as well as the corresponding values of the γ-invariant. It's easy to see that the minimal MCM approximation of $\omega/x\omega$ is the short exact sequence $0 \longrightarrow \omega \xrightarrow{x} \omega \longrightarrow \omega/x\omega \longrightarrow 0$, which gives $\delta(\omega/x\omega) = 0$ and $\gamma(\omega/x\omega) = 1$. However, $\gamma(R/(x))$ is much more mysterious. We have $X_{R/(x)} \cong \operatorname{syz}_1^R(\omega/x\omega)^\vee$, so $\gamma(R/(x)) > 0$ if and only if $\operatorname{syz}_1^R(\omega/x\omega)$ has a non-zero free direct summand. We know of no effective criterion for this.

9.45 Remark. As a final note, we observe that Auslander's criterion for regularity, Proposition 9.38, can be interpreted via the construction of MCM approximations for CM modules in Proposition 9.17. Assume that R is Gorenstein. Then condition (iv) can be written $\delta(\operatorname{syz}_d^R(k)) > 0$, and since $\operatorname{syz}_d^R(k)$ is MCM, this says simply that R is regular if and only if $\operatorname{syz}_d^R(k)$ has a non-trivial free direct summand. This is a special case of a result of Herzog [Her94], which generalizes a case of Levin's solution of a conjecture of Kaplansky: if there exists a finitely generated R-module M such that $\mathfrak{m}M \neq 0$ and $\mathfrak{m}M$ has finite projective dimension, then R is regular; in particular, if $\operatorname{syz}_d^R(R/\mathfrak{m}^n)$ is free for some n then R is regular. Yoshino has conjectured [Yos98] that for any positive integers t and n, $\delta(\operatorname{syz}_t^R(R/\mathfrak{m}^n)) > 0$ if and only if R is regular local, and has proven the conjecture when R is Gorenstein and the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ has depth at least $d-1$.

§5 Exercises

9.46 Exercise. Finish the proof of Theorem 9.3.

9.47 Exercise. Prove Lemma 9.6, using Nakayama's Lemma for surjectivity and Krull's Intersection Theorem for injectivity.

9.48 Exercise. Assume that R is Gorenstein and M is an R-module of finite projective dimension. Then the minimal MCM approximation of M is just a minimal free cover.

9.49 Exercise. Let R be a CM local ring with canonical module ω, and let M be a finitely generated R-module of finite injective dimension. Show that M has a finite resolution by copies of ω

$$0 \longrightarrow \omega^{n_t} \longrightarrow \cdots \longrightarrow \omega^{n_1} \longrightarrow \omega^{n_0} \longrightarrow M \longrightarrow 0\,.$$

9.50 Exercise. Let $x \in \mathfrak{m}$ be a non-zerodivisor. Prove that $X^{R/(x)} \cong \operatorname{syz}_2^R(\omega/x\omega)^\vee$, and so $\epsilon(R/(x)) = \delta(\operatorname{syz}_2^R(\omega/x\omega))$.

9.51 Exercise. Let R be CM local and M a finitely generated R-module. Define the *stable MCM trace* of M to be the submodule $\tau(M)$ generated by all homomorphic images $f(X)$, where X is a stable MCM module and $f \in \operatorname{Hom}_R(X,M)$. Show that $\delta(M) = \mu_R(M/\tau(M))$.

9.52 Exercise. Let $(R,\mathfrak{m})$ be a local ring. Denote by $\mu^i(\mathfrak{p},M)$ the number of copies of the injective hull of $R/\mathfrak{p}$ appearing at the i^{th} step of a minimal injective resolution of M. This integer is called the i^{th} *Bass number* of M at $\mathfrak{p}$. It is equal to the vector-space dimension of $\operatorname{Ext}^i_R(R/\mathfrak{p},M)_\mathfrak{p}$ over the field $(R/\mathfrak{p})_\mathfrak{p}$.

(i) If $\mu^i(\mathfrak{p},M) > 0$ and height $\mathfrak{q}/\mathfrak{p} = 1$, prove that $\mu^{i+1}(\mathfrak{q},M) > 0$.

(ii) If M has infinite injective dimension, prove that $\mu^i(\mathfrak{m},M) > 0$ for all $i \geqslant \dim M$. (Hint: go by induction on $\dim M$, the base case being easy. For the inductive step, distinguish two cases: (a) $\operatorname{injdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \infty$ for some prime $\mathfrak{p} \neq \mathfrak{m}$, or (b) $\operatorname{injdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for every $\mathfrak{p} \neq \mathfrak{m}$. In the first case, use the previous part of this exercise; in the second, conclude that $\operatorname{injdim}_R(M) < \infty$.)

9.53 Exercise. (This exercise still needs some work.) This exercise gives a proof of the last remaining implication in Proposition 9.16, following [SS02]. Let $(R,\mathfrak{m},k)$ be a CM complete local ring of dimension d with canonical module ω.

(i) Let M be a MCM R-module with minimal injective resolution $I^\bullet$. Prove that $\operatorname{Ext}^d_R(k,M) = \operatorname{socle}(I^d)$ is an essential submodule of the local cohomology $H^d_\mathfrak{m}(M) = H^d(\Gamma(I^\bullet))$.

(ii) Let M and N be finitely generated R-modules with M MCM and N having FID. Let $f : N \longrightarrow M$ be a homomorphism. Prove that the ω-

rank of f (that is, the number of direct summands isomorphic to ω common to N and M via f) is equal to the k-dimension of the image of the homomorphism $\mathrm{Ext}^d_R(k, f)$.

9.54 Exercise. Let R be a Gorenstein local ring (or, more generally, a CM local ring with canonical module ω and satisfying $\tau_\omega(R) \supseteq \mathfrak{m}$) with infinite residue field. Assume that R is not regular. Then

$$e(R) \geqslant \mu_R(\mathfrak{m}) - \dim R - 1 + \operatorname{index}(R).$$

In particular, if R has *minimal multiplicity* $\mathrm{e}(R) = \mu_R(\mathfrak{m}) - \dim R + 1$, then $\operatorname{index}(R) = 2$. (Compare with Corollary 5.34.)

10
Auslander-Reiten Theory

In this chapter we give an introduction to Auslander–Reiten sequences, also known as almost split sequences, and the Auslander–Reiten quiver. AR sequences are certain short exact sequences which were first introduced in the representation theory of Artin algebras, where they have played a central role. They have since been used fruitfully throughout representation theory. The information contained within the AR sequences is conveniently arranged in the AR quiver, which in some sense gives a picture of the whole category of MCM modules. We illustrate with several examples in §3.

§1 AR sequences

For this section, $(R, \mathfrak{m}, k)$ will be a Henselian CM local ring with a canonical module ω.

We begin with the definition.

10.1 Definition. Let M and N be indecomposable MCM R-modules, and let

(10.1.1) $$0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$$

be a short exact sequence of R-modules.

(i) We say that (10.1.1) is an *AR sequence ending in M* if it is non-split, but for every MCM module X and every homomorphism $f : X \longrightarrow M$ which is not a split surjection, f factors through p.

(ii) We say that (10.1.1) is an *AR sequence starting from* N if it is non-split, but for every MCM module Y and every homomorphism $g\colon N \longrightarrow Y$ which is not a split injection, g lifts through i.

We will be concerned almost exclusively with AR sequences ending in a module, and in fact will often call (10.1.1) an AR sequence *for* M. In fact, the two halves of the definition are equivalent; see Exercise 10.41. We will therefore even allow ourselves to call (10.1.1) *an AR sequence* without further qualification if it satisfies either condition.

Observe that if (10.1.1) is an AR sequence, then in particular it is non-split, so that M is not free and N is not isomorphic to the canonical module ω.

As with MCM approximations, we take care of the uniqueness of AR sequences first, then consider existence.

10.2 Proposition. *Suppose that* $0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$ *and* $0 \longrightarrow N' \xrightarrow{i'} E' \xrightarrow{p'} M \longrightarrow 0$ *are two AR sequences for* M*. Then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & E & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \Vert & & \\ 0 & \longrightarrow & N' & \xrightarrow[i']{} & E' & \xrightarrow[p']{} & M & \longrightarrow & 0 \end{array}$$

in which the first and second vertical maps are isomorphisms.

Proof. Since both sequences are AR sequences for M, neither p nor p' is a split surjection. Therefore each factors through the other, giving a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & E & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & N' & \xrightarrow{i'} & E' & \xrightarrow{p'} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\varphi'} & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow[i]{} & E & \xrightarrow[p]{} & M & \longrightarrow & 0 \end{array}$$

with exact rows.

Consider $\psi'\psi \in \operatorname{End}_R(N)$. If $\psi'\psi$ is a unit of this nc-local ring, then $\psi'\psi$ is an isomorphism, so ψ is a split injection. As N and N' are both indecomposable, ψ is an isomorphism, and φ is as well by the Snake Lemma.

If $\psi'\psi$ is not a unit of $\operatorname{End}_R(N)$, then $\sigma := 1_N - \psi'\psi$ is. Define $\tau\colon E \longrightarrow N$ by $\tau(e) = e - \varphi'\varphi(e)$. This has image in N since $p\varphi'\varphi(e) = p(e)$ for all e by the commutativity of the diagram. Now $\tau(i(n)) = \sigma(n)$ for every $n \in N$. Since σ is a unit of $\operatorname{End}_R(N)$, this implies that i is a split surjection, contradicting the assumption that the top row is an AR sequence. □

For existence of AR sequences, we first observe that we will need to impose an additional restriction on M or R.

10.3 Proposition. *Assume that there exists an AR sequence for M. Then M is locally free on the punctured spectrum of R. In particular, if every indecomposable MCM R-module has an AR sequence, then R has at most an isolated singularity.*

Proof. Let $\alpha\colon 0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ be an AR sequence for M. Since α is non-split, M is not free. Let $L = \operatorname{syz}_1^R(M)$, so that there is a short exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F a finitely generated free module. Suppose that $M_{\mathfrak{p}}$ is not free for some prime ideal $\mathfrak{p} \neq \mathfrak{m}$. Then

$$0 \longrightarrow L_{\mathfrak{p}} \longrightarrow F_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow 0$$

is still non-split, so in particular $\operatorname{Ext}^1_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, L_{\mathfrak{p}}) = \operatorname{Ext}^1_R(M,L)_{\mathfrak{p}}$ is non-zero. Choose an indecomposable direct summand K of L such that $\operatorname{Ext}^1_R(M,K)_{\mathfrak{p}}$ is non-zero, and let $\beta \in \operatorname{Ext}^1_R(M,K)$ be such that $\frac{\beta}{1} \neq 0$ in $\operatorname{Ext}^1_R(M,K)_{\mathfrak{p}}$. Then the annihilator of β is contained in $\mathfrak{p}$. Let $r \in \mathfrak{m} \setminus \mathfrak{p}$. Then for every $n \geqslant 0$, $r^n \notin \mathfrak{p}$, so that $r^n\beta \neq 0$. In particular $r^n\beta$ is represented by a non-split short exact sequence for all $n \geqslant 0$. Choosing a representative $0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$ for β, and representatives $0 \longrightarrow K \longrightarrow G_n \longrightarrow M \longrightarrow 0$ for each $r^n\beta$ as well, we obtain a commutative diagram

$$\begin{array}{lccccccccc}
\beta: & 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle r^n} & & \downarrow & & \| & & \\
r^n\beta: & 0 & \longrightarrow & K & \longrightarrow & G_n & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f_n} & & \downarrow & & \| & & \\
\alpha: & 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

with exact rows. The top half of this diagram is the pushout representing $r^n\beta$ as a multiple of β, while the vertical arrows in the bottom half are provided by the lifting property of AR sequences. Let $f_{n*}\colon \operatorname{Ext}^1_R(M,K) \longrightarrow \operatorname{Ext}^1_R(M,N)$ denote the homomorphism induced by f_n. Then $\alpha = f_{n*}(r^n\beta) = r^n f_{n*}(\beta) \in r^n \operatorname{Ext}^1_R(M,N)$ for every $n \geqslant 0$, and so $\alpha = 0$, a contradiction.

The last assertion follows from the first and Lemma 6.9. □

In fact, the converse of Proposition 10.3 holds as well. The proof requires a few technical results and two auxiliary tools, which are useful in other contexts as well: the stable Hom and the Auslander transpose.

10.4 Definition. Let M and N be finitely generated modules over a commutative (Noetherian, as always) ring A. Denote by $\mathfrak{P}(M,N)$ the submodule of A-homomorphisms from M to N that factor through a projective A-module, and put

$$\underline{\mathrm{Hom}}_R(M,N) = \mathrm{Hom}_R(M,N)/\mathfrak{P}(M,N).$$

We call $\underline{\mathrm{Hom}}_R(M,N)$ the *stable* Hom *module.* We also write $\underline{\mathrm{End}}_A(M)$ for $\underline{\mathrm{Hom}}_A(M,M)$ and refer to it as the *stable endomorphism ring.*

Observe that $\mathfrak{P}(M,M)$ is a two-sided ideal of the (non-commutative) ring $\mathrm{End}_A(M)$, so that $\underline{\mathrm{End}}_A(M)$ really is a ring. In particular, it is a quotient of $\mathrm{End}_A(M)$, so the stable endomorphism ring is nc-local if the usual endomorphism ring is.

As with the usual Hom, the stable Hom module $\underline{\mathrm{Hom}}_A(M,N)$ is naturally a left $\underline{\mathrm{End}}_A(M)$-module and a right $\underline{\mathrm{End}}_A(N)$-module. We leave the straightforward check that these actions are well-defined to the reader.

10.5 Remark. Recall that we write M^* for $\mathrm{Hom}_R(M,R)$. Note that $\mathfrak{P}(M,N)$ is the image of the natural homomorphism

$$\rho_M^N : M^* \otimes_A N \longrightarrow \mathrm{Hom}_R(M,N)$$

defined by $\rho(f \otimes y)(x) = f(x)y$ for $f \in M^*$, $y \in N$, and $x \in M$. In particular M is projective if and only if ρ_M^M is surjective.

The other auxiliary tool we need to construct AR sequences is just as easy to define, though we need some more detailed properties from it.

10.6 Definition. Let A be a ring and M a finitely generated A-module with projective presentation

$$P_1 \xrightarrow{\varphi} P_0 \longrightarrow M \longrightarrow 0. \tag{10.6.1}$$

The *Auslander transpose* $\operatorname{Tr} M$ of M is defined by

$$\operatorname{Tr} M = \operatorname{cok}(\varphi^* : P_0^* \longrightarrow P_1^*),$$

where $(-)^* = \operatorname{Hom}_A(-, A)$. In other words, $\operatorname{Tr} M$ is defined by the exactness of the sequence

$$0 \longrightarrow M^* \longrightarrow P_1^* \xrightarrow{\varphi^*} P_0^* \longrightarrow \operatorname{Tr} M \longrightarrow 0.$$

10.7 Remarks. The Auslander transpose depends, up to projective direct summands, only on M. That is, if $\varphi' : P_1' \longrightarrow P_0'$ is another projective presentation of M, then there are projective A-modules Q and Q' such that $\operatorname{cok}\varphi^* \oplus Q \cong \operatorname{cok}(\varphi')^* \oplus Q'$. In particular $\operatorname{Tr} M$ is only well-defined up to "stable equivalence." However, we will work with $\operatorname{Tr} M$ as if it were well-defined, taking care only to apply in it in situations where the ambiguity will not matter, such as the vanishing of $\operatorname{Ext}^i_A(\operatorname{Tr} M, -)$ or $\operatorname{Tor}_i^A(\operatorname{Tr} M, -)$ for $i \geqslant 1$.

It is easy to check that $\operatorname{Tr} P$ is projective if P is, and that $\operatorname{Tr}(M \oplus N) \cong \operatorname{Tr} M \oplus \operatorname{Tr} N$ up to projective direct summands. Furthermore, in (10.6.1) φ^* is a projective presentation of $\operatorname{Tr} M$, and $\varphi^{**} = \varphi$ canonically, so we have $\operatorname{Tr}(\operatorname{Tr} M) = M$ up to projective summands for every finitely generated A-module M.

When A is a local (or graded) ring, we can give a more apparently intrinsic definition of $\operatorname{Tr} M$ by insisting that φ be a minimal presentation, i.e.

all the entries of a matrix representing φ lie in the maximal ideal. However, even then we will not have $\operatorname{Tr}(\operatorname{Tr} M) = M$ on the nose in general, since the Auslander transpose of any free module will be zero.

Finally, one can check that $\operatorname{Tr}(-)$ commutes with arbitrary base change. For example, it commutes (up to projective summands, as always) with localization and passing to $A/(x)$ for an arbitrary element $x \in A$.

The Auslander transpose is intimately related to the canonical biduality homomorphism $\sigma_M : M \longrightarrow M^{**}$, defined by

$$\sigma_M(x)(f) = f(x)$$

for $x \in M$ and $f \in M^*$. More generally, we have the following proposition.

10.8 Proposition. *Let M and N be finitely generated A-modules. Then there is an exact sequence*

$$0 \longrightarrow \operatorname{Ext}^1_A(\operatorname{Tr} M, N) \longrightarrow M \otimes_A N \xrightarrow{\sigma_M^N} \operatorname{Hom}_A(M^*, N) \longrightarrow \operatorname{Ext}^2_A(\operatorname{Tr} M, N) \longrightarrow 0$$

in which σ_M^N is defined by $\sigma_M^N(x \otimes y)(f) = f(x)y$ for $x \in M$, $y \in N$, and $f \in M^*$. Moreover we have

$$\operatorname{Ext}^i_A(\operatorname{Tr} M, N) \cong \operatorname{Ext}^{i-2}_A(M^*, N)$$

for all $i \geqslant 3$. In particular, taking $N = A$ gives an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_A(\operatorname{Tr} M, A) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow \operatorname{Ext}^2_A(\operatorname{Tr} M, A) \longrightarrow 0$$

and isomorphisms

$$\operatorname{Ext}^i_A(\operatorname{Tr} M, R) \cong \operatorname{Ext}^{i-2}_A(M^*, R)$$

for $i \geqslant 3$.

We leave the proof as an exercise. The proposition motivates the following definition.

10.9 Definition. A finitely generated A-module M is called *n-torsionless* if $\mathrm{Ext}^i_A(\mathrm{Tr}\, M, A) = 0$ for $i = 1, \dots, n$.

In particular, M is 1-torsionless if and only if $\sigma_M \colon M \longrightarrow M^{**}$ is injective, 2-torsionless if and only if M is reflexive, and n-torsionless for some $n \geqslant 3$ if and only if M is reflexive and $\mathrm{Ext}^i_A(M^*, R) = 0$ for $i = 1, \dots, n-2$.

10.10 Proposition. *Suppose that a finitely generated A-module M is n-torsionless. Then M is a n^{th} syzygy.*

Proof. For $n = 0$ there is nothing to prove. For $n = 1$, let $P \longrightarrow M^*$ be a surjection with P projective; then the composition of the injections $M \longrightarrow M^{**}$ and $M^{**} \longrightarrow P^*$ shows that M is a submodule of a projective, whence a first syzygy. Similarly for $n \geqslant 2$, let $P_{n-1} \longrightarrow \cdots P_0 \longrightarrow M^* \longrightarrow 0$ be a projective resolution of M^*. Dualizing and using the definition of n-torsionlessness, we see that

$$0 \longrightarrow M \longrightarrow P_0^* \longrightarrow \cdots \longrightarrow P_{n-1}^*$$

is exact, so M is a n^{th} syzygy. □

10.11 Proposition. *Let R be a CM local ring of dimension d, and let M be a finitely generated R-module. Assume that R is Gorenstein on the punctured spectrum. Then the following are equivalent:*

(i) M is MCM;

(ii) M is a d^{th} syzygy;

(iii) *M is d-torsionless, i.e.* $\operatorname{Ext}^i_R(\operatorname{Tr} M, R) = 0$ *for* $i = 1, \dots, d$.

Proof. Items (i) and (ii) are equivalent by Corollary A.18, since R is Gorenstein on the punctured spectrum. The implication (iii) $\implies$ (ii) follows from the previous proposition. We have only to prove (i) implies (iii). So assume that M is MCM. The case $d = 0$ is vacuous. For $d = 1$, the four-term exact sequence of Proposition 10.8 and the hypothesis that R is Gorenstein on the punctured spectrum combine to show that $\operatorname{Ext}^1_R(\operatorname{Tr} M, R)$ has finite length. Since $\operatorname{Ext}^1_R(\operatorname{Tr} M, R)$ embeds in M by Proposition 10.8 and M is torsion-free, this implies $\operatorname{Ext}^1_R(\operatorname{Tr} M, R) = 0$.

Now assume that $d \geqslant 2$. Let $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a free presentation of M, so that

$$0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0$$

is exact. Splice this together with a free resolution of M^* to get a resolution of $\operatorname{Tr} M$

$$G_{d+1} \xrightarrow{\varphi_{d+1}} G_d \xrightarrow{\varphi_d} \cdots \xrightarrow{\varphi_3} G_2 \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0.$$

Dualize, obtaining a complex

$$0 \longrightarrow (\operatorname{Tr} M)^* \longrightarrow P_1 \longrightarrow P_0 \longrightarrow G_2^* \xrightarrow{\varphi_3^*} \cdots \xrightarrow{\varphi_d^*} G_d^* \xrightarrow{\varphi_{d+1}^*} G_{d+1}^*$$

in which $\ker \varphi_3^* \cong M$ since M is reflexive. The truncation of this complex at M

$$0 \longrightarrow M \longrightarrow G_2^* \xrightarrow{\varphi_3^*} \cdots \xrightarrow{\varphi_d^*} G_d^* \xrightarrow{\varphi_{d+1}^*} G_{d+1}^* \tag{10.11.1}$$

is a complex of MCM R-modules, and since R is Gorenstein on the punctured spectrum, the homology $\operatorname{Ext}^{i-2}_R(M^*, R)$ has finite length. The Lemme d'Acyclicité (Exercise 10.45) therefore implies that (10.11.1) is exact, so that M is a d^{th} syzygy. □

The most useful consequence of Proposition 10.11 from the point of view of AR theory is the following fact. Recall that we write $\operatorname{redsyz}_n^R(M)$ for the *reduced* n^{th} syzygy module, i.e. the module obtained by deleting any non-trivial free direct summands from the n^{th} syzygy module $\operatorname{syz}_n^R(M)$. In particular $\operatorname{redsyz}_0^R(M)$ is gotten from M by deleting any free direct summands.

10.12 Proposition. *Let R be a CM local ring of dimension d and assume that R is Gorenstein on the punctured spectrum. Let M be an indecomposable non-free MCM R-module which is locally free on the punctured spectrum. Then* $\operatorname{redsyz}_j^R(\operatorname{Tr} M)$ *is indecomposable for every $j = 0, \dots, d$.*

Proof. Fix a free presentation $P_1 \xrightarrow{\varphi} P_0 \longrightarrow M \longrightarrow 0$ of M, so that $\operatorname{Tr} M$ appears in an exact sequence

$$0 \longrightarrow M^* \longrightarrow P_0^* \xrightarrow{\varphi^*} P_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0 .$$

First consider the case $j = 0$. It suffices to prove that if $\operatorname{Tr} M \cong X \oplus Y$ for R-modules X and Y, then one of X or Y is free. If $\operatorname{Tr} M \cong X \oplus Y$, then φ^* can be decomposed as the direct sum of two matrices, that is, φ^* is equivalent to a matrix of the form $\left[\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right]$ with $X \cong \operatorname{cok} \alpha$ and $Y \cong \operatorname{cok} \beta$. But then $M = \operatorname{cok} \varphi \cong \operatorname{cok} \alpha^* \oplus \operatorname{cok} \beta^*$. This forces one of $\operatorname{cok} \alpha$ or $\operatorname{cok} \beta$ to be zero, which means that one of $X \cong \operatorname{cok} \alpha^*$ or $Y \cong \operatorname{cok} \beta^*$ is free.

Next assume that $j = 1$, and let N be the image of $\varphi^* : P_1^* \longrightarrow P_0^*$, so that $N \cong \operatorname{redsyz}_1^R(\operatorname{Tr} M) \oplus G$ for some finitely generated free module G. Again it suffices to prove that if $N \cong X \oplus Y$, then one of X or Y is free. Let F be a finitely generated free module mapping onto M^*, and let $f : F \longrightarrow P_0^*$ be the composition so that we have an exact sequence

$$F \xrightarrow{f} P_0^* \xrightarrow{\varphi^*} P_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0.$$

The dual of this sequence is exact since $\operatorname{Ext}^1_R(\operatorname{Tr} M, R) = 0$ by Proposition 10.11, so we obtain the exact sequence

$$P_1^{**} \xrightarrow{\varphi^{**}} P_0^{**} \xrightarrow{f^*} F^*$$

It follows that $M \cong \operatorname{cok} \varphi^{**} \cong \operatorname{im} f^*$. Now, if $N = \operatorname{cok} f$ decomposes as $N \cong X \oplus Y$, then f can be put in block-diagonal form $\left[\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right]$. It follows that $M \cong \operatorname{im} \alpha^* \oplus \operatorname{im} \beta^*$, so that one of $\operatorname{im} \alpha^*$ or $\operatorname{im} \beta^*$ is zero. This implies that one of $X = \operatorname{cok} \alpha$ or $Y = \operatorname{cok} \beta$ is free.

Now assume that $j \geqslant 2$, and we will show by induction on j that $\operatorname{redsyz}^R_j(\operatorname{Tr} M)$ is indecomposable. Note that since $d \geqslant 2$ and R is Gorenstein in codimension one, M is reflexive by Proposition A.15. Thus the case $j = 2$ is clear: if $\operatorname{redsyz}^R_2(\operatorname{Tr} M) = \operatorname{redsyz}^R_0(M^*)$ decomposes, then so does $M \cong M^{**}$.

Assume $2 < j < d$, and that $\operatorname{redsyz}^R_{j-1}(\operatorname{Tr} M)$ is indecomposable. Note that Proposition A.15 again implies that $\operatorname{redsyz}^R_{j-1}(\operatorname{Tr} M)$ and $\operatorname{redsyz}^R_j(\operatorname{Tr} M)$ are reflexive. We have an exact sequence

$$0 \longrightarrow \operatorname{redsyz}^R_j(\operatorname{Tr} M) \oplus G \longrightarrow F \longrightarrow \operatorname{redsyz}^R_{j-1}(\operatorname{Tr} M) \longrightarrow 0,$$

with F and G finitely generated free modules. By Proposition 10.11, we have

$$\operatorname{Ext}^1_R(\operatorname{redsyz}^R_{j-1}(\operatorname{Tr} M), R) = \operatorname{Ext}^j_R(\operatorname{Tr} M, R) = 0,$$

so that the dual sequence

$$0 \longrightarrow (\operatorname{redsyz}^R_{j-1}(\operatorname{Tr} M))^* \longrightarrow F^* \longrightarrow (\operatorname{redsyz}^R_j(\operatorname{Tr} M))^* \oplus G^* \longrightarrow 0$$

is also exact. If $\operatorname{redsyz}^R_j(\operatorname{Tr} M)$ decomposes as $X \oplus Y$ with neither X nor Y free, then $\operatorname{syz}^R_1(X^*)$ and $\operatorname{syz}^R_1(Y^*)$ are direct summands of $(\operatorname{redsyz}^R_{j-1}(\operatorname{Tr} M))^*$. We know that X^* and Y^* are non-zero since both X and Y embed in a free module, and neither X^* nor Y^* is free by the reflexivity of $\operatorname{redsyz}^R_j(\operatorname{Tr} M)$. Thus $(\operatorname{redsyz}^R_{j-1}(\operatorname{Tr} M))^*$ is decomposed non-trivially, so that $\operatorname{redsyz}^R_{j-1}(\operatorname{Tr} M)$ is as well, a contradiction. □

Next we see how the Auslander transpose and stable Hom interact. Notice that for any A-module M, $\operatorname{Tr} M$ is naturally a module over $\operatorname{End}_A(M)$, since any endomorphism of M lifts to an endomorphism of its projective presentation, thus inducing an endomorphism of $\operatorname{Tr} M$.

10.13 Proposition. *Let A be a commutative ring and M, N two finitely generated A-modules. Then*

$$\underline{\operatorname{Hom}}_A(M,N) \cong \operatorname{Tor}^A_1(\operatorname{Tr} M, N).$$

Furthermore, this isomorphism is natural in both M and N, and is even an isomorphism of $\underline{\operatorname{End}}_A(M)$- and $\underline{\operatorname{End}}_A(N)$-modules.

Proof. Let $P_1 \xrightarrow{\varphi} P_0 \longrightarrow M \longrightarrow 0$ be our chosen projective presentation of M. Then we have the exact sequence

$$0 \longrightarrow M^* \longrightarrow P_0^* \xrightarrow{\varphi^*} P_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0.$$

Tensoring with N yields the complex

$$M^* \otimes_A N \longrightarrow P_0^* \otimes_A N \xrightarrow{\varphi^* \otimes 1_N} P_1^* \otimes_A N \longrightarrow \operatorname{Tr} M \otimes_A N \longrightarrow 0.$$

The homology of this complex at $P_0^* \otimes_A N$ is identified as $\operatorname{Tor}^A_1(\operatorname{Tr} M, N)$. On the other hand, since the P_i are projective A-modules, the natural homomorphisms $P_i^* \otimes_A N \longrightarrow \operatorname{Hom}_A(P_i^*, N)$ are isomorphisms (Exercise 10.46). It follows that $\ker(\varphi^* \otimes_A 1_N) \cong \mathrm{Hom}_R(M,N)$, and so $\mathrm{Tor}_1^A(\mathrm{Tr}\,M,N)$ is isomorphic to the quotient of $\mathrm{Hom}_A(M,N)$ by the image of $M^* \otimes_A N \longrightarrow \mathrm{Hom}_A(P_0,N)$, namely $\mathrm{Tor}_1^A(\mathrm{Tr}\,M,N) \cong \underline{\mathrm{Hom}}_A(M,N)$.

We leave the "Furthermore" to the reader. □

Our last preparation before showing the existence of AR sequences is a short sequence of technical lemmas. The first one has the appearance of a spectral sequence, but can be proven by hand just as easily, and we leave it to the reader.

10.14 Lemma ([CE99, VI.5.1]). *Let A be a commutative ring and X, Y, Z A-modules. Then the* Hom-⊗ *adjointness isomorphism*

$$\mathrm{Hom}_A(X,\mathrm{Hom}_A(Y,Z)) \longrightarrow \mathrm{Hom}_A(X \otimes_A Y, Z)$$

induces homomorphisms

$$\mathrm{Ext}^i_A(X,\mathrm{Hom}_A(Y,Z)) \longrightarrow \mathrm{Hom}_A(\mathrm{Tor}_i^A(X,Y),Z)$$

for every $i \geqslant 0$, which are isomorphisms if Z is injective.

10.15 Lemma. *Let $(R,\mathfrak{m},k)$ be a CM local ring of dimension d with canonical module ω. Let $E = E_R(k)$ be the injective hull of the residue field of R. For any two R-modules X and Y such that Y is MCM and $\mathrm{Tor}_i^R(X,Y)$ has finite length for all $i > 0$, we have*

$$\mathrm{Ext}^i_R(X,\mathrm{Hom}_R(Y,E)) \cong \mathrm{Ext}^{i+d}_R(X,\mathrm{Hom}_R(Y,\omega)).$$

Proof. Let $0 \longrightarrow \omega \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^d \longrightarrow 0$ be a (finite) injective resolution of ω. Let $\kappa(\mathfrak{p})$ denote the residue field of $R_\mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of

R. Since $\operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}),\omega) = 0$ for $i < \operatorname{height}\mathfrak{p}$, and is isomorphic to $\kappa(\mathfrak{p})$ for $i = \operatorname{height}\mathfrak{p}$, we see first that $I^d \cong E$, and second (by an easy induction) that $\operatorname{Hom}_R(L, I^j) = 0$ for every $j < d$ and every R-module L of finite length.

Apply $\operatorname{Hom}_R(Y,-)$ to $I^\bullet$. Since Y is MCM, $\operatorname{Ext}^i_R(Y,\omega) = 0$ for $i > 0$, so the result is an exact sequence

$$(10.15.1)\quad 0 \longrightarrow \operatorname{Hom}_R(Y,\omega) \longrightarrow \operatorname{Hom}_R(Y,I^0) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_R(Y,I^d) \longrightarrow 0$$

Now from Lemma 10.14, we have

$$\operatorname{Ext}^i_R(X, \operatorname{Hom}_R(Y,I^j)) \cong \operatorname{Hom}_R(\operatorname{Tor}^R_i(X,Y), I^j)$$

for every $i, j \geqslant 0$, For $i \geqslant 1$ and $j < d$, however, the right-hand side vanishes since $\operatorname{Tor}^R_i(X,Y)$ has finite length. Thus applying $\operatorname{Hom}_R(X,-)$ to (10.15.1), we may use the long exact sequence of Ext to find that

$$\operatorname{Ext}^i_R(X, \operatorname{Hom}_R(Y,I^d)) \cong \operatorname{Ext}^i_R(X, \operatorname{Hom}_R(Y,\omega)).$$

□

10.16 Proposition. *Let $(R,\mathfrak{m},k)$ be a CM local ring of dimension d with canonical module ω. Let M and N be finitely generated R-modules with M locally free on the punctured spectrum and N MCM. Then there is an isomorphism*

$$\operatorname{Hom}_R(\underline{\operatorname{Hom}}_R(M,N), E_R(k)) \cong \operatorname{Ext}^1_R(N, (\operatorname{syz}^R_d(\operatorname{Tr} M))^\vee),$$

where $-^\vee$ as usual denotes $\operatorname{Hom}_R(-,\omega)$. This isomorphism is natural in M and N, and is even an isomorphism of $\underline{\operatorname{End}}_R(M)$- and $\underline{\operatorname{End}}_R(N)$-modules.

Proof. By Proposition 10.13, we have $\underline{\operatorname{Hom}}_R(M,N) \cong \operatorname{Tor}_1^R(\operatorname{Tr} M, N)$. Making that substitution in the left-hand side and applying Lemma 10.14, we see

$$\begin{aligned}\operatorname{Hom}_R(\underline{\operatorname{Hom}}_R(M,N), E_R(k)) &\cong \operatorname{Hom}_R(\operatorname{Tor}_1^R(\operatorname{Tr} M, N)), E_R(k) \\ &\cong \operatorname{Ext}^1_R(\operatorname{Tr} M, \operatorname{Hom}_R(N, E_R(k))).\end{aligned}$$

By Lemma 10.15, this last is isomorphic to $\operatorname{Ext}_R^{d+1}(\operatorname{Tr} M, \operatorname{Hom}_R(N,\omega))$ since $\ell(\operatorname{Tor}_i^R(\operatorname{Tr} M, N)) < \infty$ for all $i \geqslant 1$. Take a reduced d^{th} syzygy of $\operatorname{Tr} M$, as foreshadowed by Proposition 10.12, to get $\operatorname{Ext}^1_R(\operatorname{redsyz}_d^R(\operatorname{Tr} M), N^\vee)$. Finally, canonical duality for the MCM modules $\operatorname{syz}_d^R \operatorname{Tr} M$ and $N^\vee$ shows that this last module is naturally isomorphic to $\operatorname{Ext}^1_R(N, (\operatorname{redsyz}_d^R \operatorname{Tr} M)^\vee)$.

Again we leave the assertion about naturality to the reader. □

For brevity, from now on we write

$$\tau(M) = \operatorname{Hom}_R(\operatorname{redsyz}_d^R \operatorname{Tr} M, \omega)$$

and call it the *Auslander translate* of M.

10.17 Theorem. *Let $(R,\mathfrak{m},k)$ be a Henselian CM local ring of dimension d and let M be an indecomposable MCM R-module which is locally free on the punctured spectrum. Then there exists an AR sequence for M*

$$\alpha\colon 0 \longrightarrow \tau(M) \longrightarrow E \longrightarrow M \longrightarrow 0\,.$$

Precisely, the $\underline{\operatorname{End}}_R(M)$*-module* $\operatorname{Ext}^1_R(M,\tau(M))$ *has one-dimensional socle, and any representative for a generator for that socle is an AR sequence for M.*

Proof. First observe that $\underline{\mathrm{End}}_R(M)$ is a quotient of the nc-local ring $\mathrm{End}_R(M)$, so is again nc-local. Thus the Matlis dual $\mathrm{Hom}_R(\underline{\mathrm{End}}_R(M), E_R(k))$ has a one-dimensional socle. By Proposition 10.16, this Matlis dual is isomorphic to $\mathrm{Ext}^1_R(M, \tau(M))$. Let $\alpha\colon 0 \longrightarrow \tau(M) \longrightarrow E \longrightarrow M \longrightarrow 0$ be a generator for the socle of $\mathrm{Ext}^1_R(M, \tau(M))$.

We know from Proposition 10.12 that $\mathrm{redsyz}^R_d \operatorname{Tr} M$ is indecomposable, so its canonical dual $\tau(M)$ is indecomposable as well. It therefore suffices to check the lifting property. Let $f\colon X \longrightarrow M$ be a homomorphism of MCM R-modules. Then pullback along f induces a homomorphism $f^*\colon \mathrm{Ext}^1_R(M, \tau(M)) \longrightarrow \mathrm{Ext}^1_R(X, \tau(M))$. If f does not factor through E, then the image of α in $\mathrm{Ext}^1_R(X, \tau(M))$ is non-zero. Since α generates the socle and α does not go to zero, we see that in fact f^* must be injective. By Proposition 10.16, this injective homomorphism is the same as the one

$$\mathrm{Hom}_R(\underline{\mathrm{End}}_R(M), E_R(k)) \longrightarrow \mathrm{Hom}_R(\underline{\mathrm{Hom}}(X, M), E_R(k))$$

induced by $f\colon X \longrightarrow M$. Since f^* is injective, Matlis duality implies that

$$\underline{\mathrm{Hom}}_R(X, M) \longrightarrow \underline{\mathrm{End}}_R(M)$$

is surjective. In particular, the map $\mathrm{Hom}_R(X, M) \longrightarrow \mathrm{End}_R(M)$ induced by f is surjective. It follows that f is a split surjection, so we are done. □

10.18 Corollary. *Let R be a Henselian CM local ring with canonical module, and assume that R is an isolated singularity. Then every indecomposable non-free MCM R-module has an AR sequence.*

§2 AR quivers

The Auslander–Reiten quiver is a convenient scheme for packaging AR sequences. Up to first approximation, we could define it already: The AR quiver of a Henselian CM local ring with isolated singularity is the directed graph having a vertex $[M]$ for each indecomposable non-free MCM module M, a dotted line joining $[M]$ to $[\tau(M)]$, and an arrow $[X] \longrightarrow [M]$ for each occurrence of X in a direct-sum decomposition of the middle term of the AR sequence for M.

Unfortunately, this first approximation omits the indecomposable free module R. It is also manifestly asymmetrical: it takes into account only the AR sequences ending in a module, and omits those starting from a module. To remedy these defects, as well as for later use (particularly in Chapter 14), we introduce now irreducible homomorphisms between MCM modules, and use them to define the AR quiver. We then reconcile this definition with the naive one above, and check to see what additional information we've gained.

In this section, $(R, \mathfrak{m}, k)$ is a Henselian CM local ring with canonical module ω, and we assume that R has an isolated singularity.

10.19 Definition. Let M and N be MCM R-modules. A homomorphism $\varphi\colon M \longrightarrow N$ is called *irreducible* if it is neither a split injection nor a split surjection, and in any factorization

$$\begin{array}{ccc} M & \xrightarrow{\quad\varphi\quad} & N \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & X & \end{array}$$

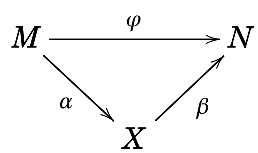
with X a MCM R-module, either g is a split injection or h is a split surjection.

Observe that the set of irreducible homomorphisms is not a submodule of $\operatorname{Hom}_R(M,N)$. We can, however, describe it more precisely.

10.20 Definition. Let M and N be MCM R-modules.

(i) Let $\operatorname{rad}(M,N) \subseteq \operatorname{Hom}_R(M,N)$ be the submodule consisting of those homomorphisms $\varphi\colon M \longrightarrow N$ such that, when we decompose $M = \bigoplus_j M_j$ and $N = \bigoplus_i N_i$ into indecomposable modules, and accordingly decompose $\varphi = (\varphi_{ij}\colon M_j \longrightarrow N_i)_{ij}$, no φ_{ij} is an isomorphism.

(ii) Let $\operatorname{rad}^2(M,N) \subseteq \operatorname{Hom}_R(M,N)$ be the submodule of those homomorphisms $\varphi\colon M \longrightarrow N$ for which there is a factorization



with X MCM, $\alpha \in \operatorname{rad}(M,X)$ and $\beta \in \operatorname{rad}(X,N)$.

10.21 Remark. Suppose that M and N are indecomposable. If M and N are not isomorphic, then $\operatorname{rad}(M,N)$ is simply $\operatorname{Hom}_R(M,N)$. If, on the other hand, $M \cong N$, then $\operatorname{rad}(M,N)$ is the Jacobson radical of the nc-local ring $\operatorname{End}_R(M)$, whence the name. In particular $\mathfrak{m}\operatorname{End}_R(M) \subseteq \operatorname{rad}(M,M)$ by Lemma 1.6.

For any M and N, not necessarily indecomposable, it's clear that the set of irreducible homomorphisms from M to N coincides with $\operatorname{rad}(M,N) \setminus \operatorname{rad}^2(M,N)$. Furthermore we have $\mathfrak{m}\operatorname{rad}(M,N) \subseteq \operatorname{rad}^2(M,N)$ (Exercise 10.48), so that the following definition makes sense.

10.22 Definition. Let M and N be MCM R-modules, and put

$$\operatorname{Irr}(M,N) = \operatorname{rad}(M,N)\big/\operatorname{rad}^2(M,N).$$

Denote by $\operatorname{irr}(M,N)$ the k-vector space dimension of $\operatorname{Irr}(M,N)$.

Now we are ready to define the AR quiver of R. We impose an additional hypothesis on R, that the residue field k be algebraically closed.

10.23 Definition. Let $(R,\mathfrak{m},k)$ be a Henselian CM local ring with a canonical module. Assume that R has an isolated singularity and that k is algebraically closed. The *Auslander–Reiten (AR) quiver* for R is the graph Γ with

- vertices $[M]$ for each indecomposable MCM R-module M;
- r arrows from $[M]$ to $[N]$ if $\operatorname{irr}(M,N) = r$; and
- a dotted (undirected) line between $[M]$ and its AR translate $[\tau(M)]$ for every M.

Without the assumption that k be algebraically closed, we would need to define the AR quiver as a *valued* quiver, as follows. Suppose $[M]$ and $[N]$ are vertices in Γ, and that there is an irreducible homomorphism $M \longrightarrow N$. The abelian group $\operatorname{Irr}(M,N)$ is naturally a $\operatorname{End}_R(N)$-$\operatorname{End}_R(M)$ bimodule, with the left and right actions inherited from those on $\operatorname{Hom}_R(M,N)$. As such, it is annihilated by the radical of each endomorphism ring (see again Exercise 10.48). Let m be the dimension of $\operatorname{Irr}(M,N)$ as a right vector space over $\operatorname{End}_R(M)\big/\operatorname{rad}(M,M)$, and symmetrically let n be the dimension of $\operatorname{Irr}(M,N)$ over $\operatorname{End}_R(N)\big/\operatorname{rad}(N,N)$. Then we would draw an arrow from

$[M]$ to $[N]$ in Γ, and decorate it with the ordered pair (m,n). In the special case of an algebraically closed field k, $\operatorname{End}_R(M)/\operatorname{rad}(M,M)$ is in fact isomorphic to k for every indecomposable M, so we always have $m = n$.

We now reconcile the definition of the AR quiver with our earlier naive version, which included only the non-free indecomposable MCM modules.

10.24 Proposition. *Let* $0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$ *be an AR sequence. Then* i *and* p *are irreducible homomorphisms.*

Proof. We prove only the assertion about p, since the other is exactly dual. First we claim that p is *right minimal*, that is (see Definition 9.13), that whenever $\varphi\colon E \longrightarrow E$ is an endomorphism such that $p\varphi = p$, in fact φ is an automorphism. The proof of this is similar to that of Proposition 10.2: the existence of $\varphi \in \operatorname{End}_R(E)$ such that $p\varphi = p$ defines a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & E & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow[i]{} & E & \xrightarrow[p]{} & M & \longrightarrow & 0 \end{array}$$

of exact sequences, where ψ is the restriction of φ to N. To see that φ is an isomorphism, it suffices by the Snake Lemma to show that ψ is an isomorphism. If not, then since N is indecomposable and $\operatorname{End}_R(N)$ is therefore nc-local, $1_N - \psi$ is an isomorphism. Then $(1_E - \varphi)\colon E \longrightarrow N$ restricts to an isomorphism on N, so splits the AR sequence. This contradiction proves the claim.

We now show p is irreducible. Assume that we have a factorization

$$\begin{array}{ccc} E & \xrightarrow{\quad p \quad} & M \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & X & \end{array}$$

in which g is not a split surjection. The lifting property of AR sequences delivers a homomorphism $u: X \longrightarrow E$ such that $g = pu$. Thus we obtain a larger commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{f} & X & \xrightarrow{u} & E \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle p} & \\ & & M\,. & & \end{array}$$

Since p is right minimal by the claim, uf is an automorphism of E. In particular, f is a split injection. □

Recall that we write $A \mid B$ to mean that A is isomorphic to a direct summand of B.

10.25 Proposition. *Let* $0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$ *be an AR sequence.*

(i) A homomorphism $\varphi: X \longrightarrow M$ *is irreducible if and only if* $X \mid E$ *and* φ *factors through the inclusion* j *of* X *as a direct summand of* E*, that is,* $\varphi = pj$ *for a split injection* j*.*

(ii) A homomorphism $\psi: N \longrightarrow Y$ *is irreducible if and only if* $Y \mid E$ *and* ψ *lifts over the projection* π *of* E *onto* Y*, that is,* $\psi = \pi i$ *for a split surjection* π*.*

Proof. Again we prove only the first part and leave the dual to the reader.

Assume first that $\varphi: X \longrightarrow M$ is irreducible. The lifting property of AR sequences gives a factorization $\varphi = pj$ for some $j: X \longrightarrow E$. Since φ is irreducible and p is not a split surjection, j is a split injection.

For the converse, assume that $E \cong X \oplus X'$, and write $p = [\alpha\ \beta]: X \oplus X' \longrightarrow M$ along this decomposition. We must show that α is irreducible. First observe that neither α nor β is a split surjection, since p is not. If, now, we have a factorization

$$\begin{array}{ccc} X & \xrightarrow{\ \alpha\ } & M \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & Z & \end{array}$$

with Z MCM and h not a split surjection, then we obtain a diagram

$$\begin{array}{ccc} X \oplus X' & \xrightarrow{\ [\alpha\ \beta]\ } & M \\ & {\scriptstyle \left[\begin{smallmatrix} g & 0 \\ 0 & 1_{X'} \end{smallmatrix}\right]}\searrow \quad \nearrow{\scriptstyle [h\ \beta]} & \\ & Z \oplus X' . & \end{array}$$

As $p = [\alpha\ \ \beta]$ is irreducible by Proposition 10.24, and $[h\ \ \beta]$ is not a split surjection by Exercise 1.23, we find that g is a split injection. □

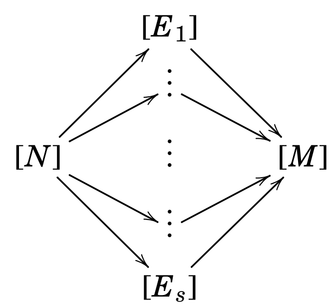
10.26 Corollary. *Let* $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ *be an AR sequence. Then for any indecomposable MCM R-module X,* $\operatorname{irr}(N, X) = \operatorname{irr}(X, M)$ *is the multiplicity of X in the decomposition of E as a direct sum of indecomposables.*

Now we deal with $[R]$.

10.27 Proposition. *Let $(R, \mathfrak{m})$ be a Henselian local ring with a canonical module, and assume that R has an isolated singularity. Let* $0 \longrightarrow Y \longrightarrow X \xrightarrow{q} \mathfrak{m} \longrightarrow 0$ *be the minimal MCM approximation of the maximal ideal $\mathfrak{m}$. (If* $\dim R \leqslant 1$*, we take $X = \mathfrak{m}$ and $Y = 0$.) Then a homomorphism $\varphi \colon M \longrightarrow R$ with M MCM is irreducible if and only if $M \mid X$ and φ factors through the inclusion of M as a direct summand of X, that is, $\varphi = qj$ for some split injection j.*

Proof. Assume that $\varphi \colon M \longrightarrow R$ is irreducible. Since φ is not a split surjection, the image of φ is contained in $\mathfrak{m}$. We can therefore lift φ to factor through q, obtaining a factorization $M \xrightarrow{j} X \xrightarrow{q} \mathfrak{m}$. This factorization composes with the inclusion of $\mathfrak{m}$ into R to give a factorization of $\varphi : M \xrightarrow{j} X \longrightarrow R$. Since φ is irreducible and $X \longrightarrow R$ is not surjective, j is a split injection. □

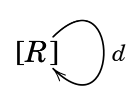
10.28 Remark. Putting Propositions 10.25 and 10.27 together, we find in particular that the AR quiver is *locally finite*, i.e. each vertex has only finitely many arrows incident to it. The local structure of the quiver is



where $N = \tau(M)$ and $E = \bigoplus_{i=1}^{s} E_i$ is the middle term of the AR sequence ending in M.

§3 Examples

10.29 Example. We can compute the AR quiver for a power series ring $R = k[[x_1, \dots, x_d]]$ directly. It has a single vertex, $[R]$, and the irreducible homomorphisms $R \longrightarrow R$ are by Proposition 10.27 and Exercise 9.48 the direct summands of $R^{(d)} \xrightarrow{[x_1 \ \dots \ x_d]} R$, the beginning of the Koszul resolution of $\mathfrak{m} = (x_1, \dots, x_d)$. Thus $\operatorname{irr}(R,R) = d$ and

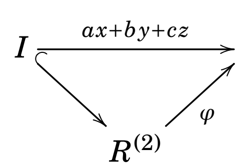


is the AR quiver. Note alternatively that $\mathfrak{m} = \mathrm{rad}(R,R)$, while $\mathfrak{m}^2 = \mathrm{rad}^2(R,R)$, and $d = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

10.30 Example. We can also compute directly the AR quiver for the two-dimensional (A_1) singularity $k[\![x,y,z]\!]/(xz-y^2)$, though this one is less trivial. By Example 4.23, there is a single non-free indecomposable MCM module, namely the ideal

$$I = (x,y)R \cong \mathrm{cok}\left(\begin{bmatrix} y & -x \\ -z & y \end{bmatrix}, \begin{bmatrix} y & x \\ z & y \end{bmatrix}\right).$$

We compute $\mathrm{Irr}(I,I)$ from the definition: we have $\mathrm{Hom}_R(I,I) \cong R$ since R is integrally closed, so that $\mathrm{rad}(I,I) = \mathfrak{m}$, the maximal ideal (x,y,z). Furthermore, for any element $f \in \mathfrak{m}$, the endomorphism of I given by multiplication by f factors through $R^{(2)}$. Indeed, I is isomorphic to the submodule of $R^{(2)}$ generated by the column vectors $\binom{y}{x}$ and $\binom{z}{y}$. If $f = ax + by + cz$, then the diagram



commutes, where φ is defined by $\varphi(e_1) = \binom{ax+cz}{cy}$ and $\varphi(e_2) = \binom{bz}{ax+by}$. Therefore $\mathrm{rad}^2(I,I) = \mathfrak{m} = \mathrm{rad}(I,I)$ and $\mathrm{Irr}(I,I) = 0$.

It follows that in the AR sequence ending in I,

$$0 \longrightarrow \tau(I) \longrightarrow E \longrightarrow I \longrightarrow 0,$$

E has no direct summands isomorphic to I, so is necessarily free. Since $\tau(I) = (\mathrm{redsyz}_2^R(\mathrm{Tr}\, I))^\vee = (I^*)^\vee = I$, the AR sequence is of the form

$$0 \longrightarrow I \longrightarrow R^{(2)} \longrightarrow I \longrightarrow 0,$$

and is the beginning of the free resolution of I. We conclude that the AR quiver of R is

$$[R] \rightleftarrows [I].$$

The direct approach of Example 10.30 is impractical in general, but we can use the material of Chapters 4 and 5 to compute the AR quivers of the complete Kleinian singularities (A_n), (D_n), (E_6), (E_7), and (E_8) of Table 5.2. They are isomorphic to the McKay–Gabriel quivers of the associated finite subgroups of $\mathrm{SL}(2,k)$.

Recall the setup and definition of the McKay–Gabriel quiver in dimension two. Let k be a field and $V = ku + kv$ a two-dimensional k-vector space. Let $G \subseteq \mathrm{GL}(V) \cong \mathrm{GL}(2,k)$ be a finite group with order invertible in k, and assume that G acts on V with no non-trivial pseudo-reflections. In this situation the k-representations of G, the projective modules over the twisted group ring $S\#G$, and the MCM R-modules are equivalent as categories by Corollaries 4.18 and 5.4 and Theorem 5.3. Explicitly, the functor defined by $W \mapsto S \otimes_k W$ is an equivalence between the finite-dimensional representations of G and the finitely generated projective $S\#G$-modules, while the functor given by $P \mapsto P^G$ gives an equivalence between the latter category and $\mathrm{add}_R(S)$, the R-direct summands of S. Since $\dim V = 2$, these are all the MCM R-modules by Theorem 5.3.

Writing $V_0 = k, V_1, \ldots, V_d$ for a complete set of non-isomorphic irreducible representations of G, we set

$$P_j = S \otimes_k V_j \qquad \text{and} \qquad M_j = (S \otimes_k V_j)^G$$

for $j = 0, \ldots, d$. Then $P_0 = S, P_1, \ldots, P_d$ are the indecomposable finitely generated projective $S\#G$-modules, and $M_0 = R, M_1, \dots, M_d$ are the indecomposable MCM R-modules.

The McKay–Gabriel quiver Γ for G (see Definitions 4.19 and 4.20 and Theorem 4.21) has for vertices the indecomposable projective $S\#G$-modules $P_0, \dots, P_d$. For each i and j, we draw m_{ij} arrows $P_i \longrightarrow P_j$ if V_i appears with multiplicity m_{ij} in the irreducible decomposition of $V \otimes_k V_j$.

10.31 Proposition. *With notation as above, the McKay–Gabriel quiver is isomorphic to the AR quiver of $R = S^G$. (We ignore the Auslander translate τ.)*

Proof. First observe that R is a two-dimensional normal domain, whence an isolated singularity, so that AR quiver of R is defined.

It follows from Corollaries 4.18 and 5.4 and Theorem 5.3, as in the discussion above, that the equivalence of categories defined by

$$P_j = S \otimes_k V_j \quad \mapsto \quad M_j = (S \otimes_k V_j)^G$$

induces a bijection between the vertices of the McKay–Gabriel quiver and those of the AR quiver. It remains to determine the arrows.

Consider the Koszul complex over S

$$0 \longrightarrow S \otimes_k \bigwedge^2 V \longrightarrow S \otimes_k V \longrightarrow S \longrightarrow k \longrightarrow 0,$$

which is also an exact sequence of $S\#G$-modules, and tensor with V_j to obtain

$$0 \longrightarrow S \otimes_k \left(\bigwedge^2 V \otimes_k V_j \right) \longrightarrow S \otimes_k (V \otimes_k V_j) \longrightarrow P_j \longrightarrow V_j \longrightarrow 0. \tag{10.31.1}$$

Since $\bigwedge^2 V$ has k-dimension 1, we see that $\bigwedge^2 V \otimes_k V_j$ is a simple $k[G]$-module, so $S \otimes_k (\bigwedge^2 V \otimes_k V_j)$ is an indecomposable projective $S\#G$-module. Take fixed points; since each V_j is simple, we have ${V_j}^G = 0$ for all $j \neq 0$, and ${V_0}^G = k^G = k$. We obtain exact sequences of R-modules

$$(10.31.2) \qquad 0 \longrightarrow \left(S \otimes_k \left(\bigwedge^2 V \otimes_k V_j\right)\right)^G \longrightarrow \left(S \otimes_k (V \otimes_k V_j)\right)^G \xrightarrow{p_j} M_j \longrightarrow 0$$

for each $j \neq 0$, and

$$(10.31.3) \qquad 0 \longrightarrow \left(S \otimes_k \bigwedge^2 V\right)^G \longrightarrow (S \otimes_k V)^G \xrightarrow{p_0} R \longrightarrow k \longrightarrow 0$$

for $j = 0$.

We now claim that (10.31.2) is the AR sequence ending in M_j for all $j = 1, \ldots, d$, while the map p_0 in (10.31.3) is the minimal MCM approximation of the maximal ideal of R. It will then follow from Propositions 10.25 and 10.27 that the number of arrows $[M_i] \longrightarrow [M_j]$ in the AR quiver is equal to the multiplicity of M_i in a direct-sum decomposition of $\left(S \otimes_k (V \otimes_k V_j)\right)^G$, which is equal to the multiplicity of V_i in the direct-sum decomposition of $V \otimes_k V_j$.

First assume that $j \neq 0$. We observed already that $S \otimes_k (\bigwedge^2 V \otimes_k V_j)$ is an indecomposable projective $S\#G$-module, whence its fixed submodule $\left(S \otimes_k (\bigwedge^2 V \otimes_k V_j)\right)^G$ is an indecomposable MCM R-module. Since (10.31.1) is not split, p_j is non-split as well. Assume that X is a MCM R-module and $f : X \longrightarrow M_j$ is a homomorphism that is not a split surjection. There then exists a homomorphism of projective $S\#G$-modules $\widetilde{f} : \widetilde{X} \longrightarrow P_j$, also not a split surjection, such that $\widetilde{X}^G = X$ and $\widetilde{f}^G = f$. This fits into a diagram

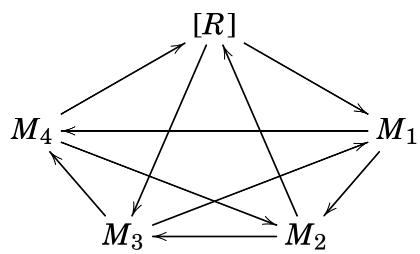
$$\begin{array}{ccccccc} & & \widetilde{X} & & & & \\ & & \big\downarrow{\scriptstyle \widetilde{f}} & & & & \\ S \otimes_k \left(V \otimes_k V_j\right) & \xrightarrow{\widetilde{p}_j} & S \otimes_k V_j & \longrightarrow & V_j & \longrightarrow & 0\,. \end{array}$$

Since the image of $f\colon X \longrightarrow M_j$ is contained in that of $p_j\colon \left(S \otimes_k \left(V \otimes_k V_j\right)\right)^G \longrightarrow M_j$, the image of $\widetilde{f}$ is contained in that of $\widetilde{p}_j$. But $\widetilde{X}$ is projective, so there exists $\widetilde{g}\colon \widetilde{X} \longrightarrow S \otimes_k \left(V \otimes_k V_j\right)$ such that $\widetilde{f} = \widetilde{p}_j \widetilde{g}$. Set $g = \widetilde{g}^G$; then $f = p_j g$, proving the claim in this case.

For $j = 0$, the argument is essentially the same; if $f\colon X \longrightarrow \mathfrak{m}$ is any homomorphism from a MCM R-module X to the maximal ideal of R, then the composition $X \longrightarrow \mathfrak{m} \longrightarrow R$ lifts to a homomorphism $\widetilde{f}\colon \widetilde{X} \longrightarrow S$ of projective $S\#G$-modules. The image of $\widetilde{f}$ is contained in the image of $\widetilde{p}_0\colon S \otimes_k V \longrightarrow S$, so again there exists $\widetilde{g}\colon \widetilde{X} \longrightarrow S \otimes_k V$ making the obvious diagram commute, and f factors through p_0. □

It follows from Proposition 10.31 and §3 of Chapter 5 that the AR quivers for the Kleinian singularities (A_n), (D_n), (E_6), (E_7), and (E_8) are (after replacing pairs of opposing arrows by undirected edges) the corresponding extended ADE diagrams listed in Table 5.2. Indeed, we need not even worry about the Auslander translate τ: since R is Gorenstein of dimension two, $\tau(X) = (\operatorname{syz}^d_R(\operatorname{Tr} X))^\vee \cong X$ for every MCM X.

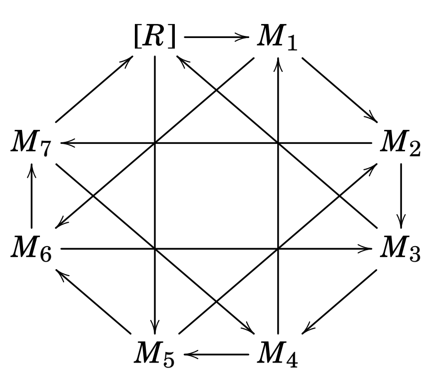
Glancing back at Example 4.23, we can write down a few more AR quivers. For instance, let $R = k[\![u^5, u^2v, uv^3, v^5]\!]$, the fixed ring of the cyclic group of order 5 generated by diag(ζ_5, ζ_5^3). The AR quiver looks like



where

$$M_1 = R(u^4, uv, v^3) \cong (u^5, u^2v, uv^3)$$
$$M_2 = R(u^3, v) \cong (u^5, u^2v)$$
$$M_3 = R(u^2, uv^2, v^4) \cong (u^5, u^4v^2, u^3v^4)$$
$$M_4 = R(u, v^2) \cong (u^5, u^4v^2).$$

For another example, let $R = k[[u^8, u^3v, uv^3, v^8]]$. The AR quiver is



where this time

$$M_1 = R(u^7, u^2v, v^3) \cong (u^8, u^3v, uv^3)$$
$$M_2 = R(u^6, uv, v^6) \cong (u^8, u^3v, u^2v^6)$$
$$M_3 = R(u^5, v) \cong (u^8, u^3v)$$
$$M_4 = R(u^4, u^2v^2, v^4) \cong (u^8, u^6v^2, u^4v^4)$$
$$M_5 = R(u^3, uv^2, v^7) \cong (u^8, u^6v^2, u^5v^7)$$
$$M_6 = R(u^2, u^5v, v^2) \cong (u^2v^6, u^5v^7, v^8)$$
$$M_7 = R(u, v^5) \cong (uv^3, v^8).$$

Before leaving the case of dimension two, we briefly describe how to compute the AR quiver for an arbitrary two-dimensional normal domain which is not necessarily a ring of invariants. The short exact sequence (10.31.3)

$$0 \longrightarrow \left(S \otimes_k \bigwedge^2 V\right)^G \longrightarrow (S \otimes_k V)^G \xrightarrow{p_0} R \longrightarrow k \longrightarrow 0$$

appearing in the proof of Proposition 10.31 is called the *fundamental sequence for R*, and contains within it all the information carried by the entire AR quiver, as the proof of Proposition 10.31 shows. There is an analog of this sequence for general two-dimensional normal domains.

Assume that $(R, \mathfrak{m}, k)$ is a complete local normal domain of dimension 2. Let ω be the canonical module for R. Then we know that $\operatorname{Ext}^2_R(k, \omega) = k$, so there is up to isomorphism a unique four-term exact sequence of the form

$$0 \longrightarrow \omega \xrightarrow{a} E \xrightarrow{b} R \longrightarrow k \longrightarrow 0$$

representing a non-zero element of $\operatorname{Ext}^2_R(k, \omega)$. Call this the *fundamental sequence for R*. The module E is easily seen to be MCM of rank 2.

Let $f : X \longrightarrow R$ be a homomorphism of MCM R-modules which is not a split surjection. Then the image of f is contained in $\mathfrak{m} = \operatorname{im} b$, and since $\operatorname{Ext}^1_R(X, \omega) = 0$, the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega & \longrightarrow & Q & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \omega & \xrightarrow[a]{} & E & \xrightarrow[b]{} & R & & \end{array}$$

has split-exact top row. It follows that f factors through $b : E \longrightarrow R$, so that b is a minimal MCM approximation of the maximal ideal $\mathfrak{m}$.

More is true. Recall from Exercise 5.48 that for reflexive (MCM) R-modules A and B, the reflexive product $A \cdot B$ is defined by $A \cdot B = (A \otimes_R B)^{**}$.

10.32 Theorem ([Aus86b])**.** *Let $(R, \mathfrak{m}, k)$ be a two-dimensional complete local normal domain with canonical module ω. Let*

$$0 \longrightarrow \omega \longrightarrow E \longrightarrow R \longrightarrow k \longrightarrow 0$$

be the fundamental sequence for R, and let M be an indecomposable non-free MCM R-module. Then the induced sequence

$$0 \longrightarrow \omega \cdot M \longrightarrow E \cdot M \longrightarrow M \longrightarrow 0 \tag{10.32.1}$$

is exact. If (10.32.1) is non-split, then it is the AR sequence ending in M. In particular, if $\operatorname{rank} M$ is a unit in R, then (10.32.1) is non-split, so is an AR sequence. The converse is true if k is algebraically closed.

Let us return to the ADE singularities. The AR quivers for the one-dimensional ADE hypersurface singularities can also be obtained from those in dimension two, together with the explicit matrix factorizations for the indecomposable MCM modules listed in §3 of Chapter 5.

For example, consider the one-dimensional (E_6) singularity $R = k[[x,y]](x^3 + y^4)$, where k is a field of characteristic not 2, 3, or 5. Let $R^\# = k[[x,y,z]]/(x^3 + y^4 + z^2)$ be the double branched cover. The matrix factorizations for the indecomposable MCM $R^\sharp$-modules are all of the form $(zI_n - \varphi, zI_n + \varphi)$, where φ is one of the matrices φ_1, φ_2, φ_3, $\overline{\varphi}_3$, φ_4, or $\overline{\varphi}_4$ of 8.21. Flatting those matrix factorizations, i.e. killing z, amounts to ignoring z entirely and focusing simply on the φ_j. When we do this, certain of the matrix factorizations split into non-isomorphic pairs (as indicated by the block format of the matrices), while certain other pairs of matrix factorizations collapse into a single isomorphism class.

Specifically, we can see that φ_1 splits into two non-equivalent matrices

$$\left(\begin{bmatrix} x & y^3 \\ y & -x^2 \end{bmatrix}, \begin{bmatrix} x^2 & y^3 \\ y & -x \end{bmatrix}\right)$$

forming a matrix factorization, and φ_2 splits similarly into the matrix factorization

$$\left(\begin{bmatrix} x & 0 & y^2 \\ y & x & 0 \\ 0 & 0 & x \end{bmatrix}, \begin{bmatrix} x^2 & y^3 & -xy^2 \\ -xy & x^2 & y^3 \\ y^2 & -xy & x^2 \end{bmatrix}\right).$$

On the other hand, over R,

$$\varphi_3 = \begin{bmatrix} iy^2 & 0 & -x^2 & 0 \\ 0 & iy^2 & -xy & -x^2 \\ x & 0 & -iy^2 & 0 \\ -y & x & 0 & -iy^2 \end{bmatrix} \quad \text{and} \quad \overline{\varphi}_3 = \begin{bmatrix} -iy^2 & 0 & -x^2 & 0 \\ 0 & -iy^2 & -xy & -x^2 \\ x & 0 & iy^2 & 0 \\ -y & x & 0 & iy^2 \end{bmatrix}$$

have isomorphic cokernels, as do

$$\varphi_4 = \begin{bmatrix} iy^2 & -x^2 \\ x & -iy^2 \end{bmatrix} \quad \text{and} \quad \overline{\varphi}_4 = \begin{bmatrix} -iy^2 & -x^2 \\ x & iy^2 \end{bmatrix}.$$

Therefore R has 6 non-isomorphic non-free indecomposable MCM modules, namely

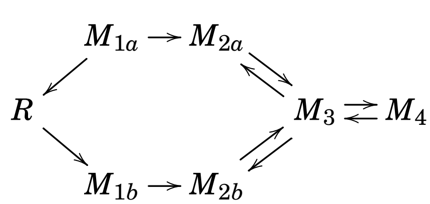
$$M_{1a} = \operatorname{cok} \begin{bmatrix} x & y^3 \\ y & -x^2 \end{bmatrix}, \qquad M_{1b} = \operatorname{cok} \begin{bmatrix} x^2 & y^3 \\ y & -x \end{bmatrix},$$

$$M_{2a} = \operatorname{cok} \begin{bmatrix} x & 0 & y^2 \\ y & x & 0 \\ 0 & 0 & x \end{bmatrix} \qquad M_{2b} = \operatorname{cok} \begin{bmatrix} x^2 & y^3 & -xy^2 \\ -xy & x^2 & y^3 \\ y^2 & -xy & x^2 \end{bmatrix}$$

$$M_3 = \operatorname{cok} \varphi_3 = \operatorname{cok} \overline{\varphi}_3$$

$$M_4 = \operatorname{cok} \varphi_4 = \operatorname{cok} \overline{\varphi}_4 .$$

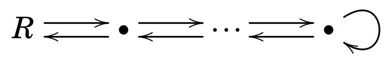
Since each of these modules is self-dual and the Auslander translate τ is given by $(\operatorname{syz}_1^R(-^*))^*$, we have $\tau(M_{1a}) = M_{1b}$, $\tau(M_{2a}) = M_{2b}$, and vice versa, while τ fixes M_3 and M_4. One can compute the irreducible homomorphisms among these modules and obtain the AR quiver



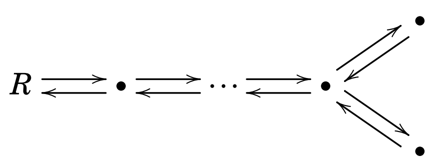
where τ is given by reflection across the horizontal axis.

For completeness, we list the AR quivers for all the one-dimensional ADE singularities below.

10.33. The extended (A_n) Coxeter-Dynkin diagram has $n+1$ nodes. The splitting/collapsing behavior of the matrix factorizations depends on the parity of n. When $n = 2m$ is even, we find

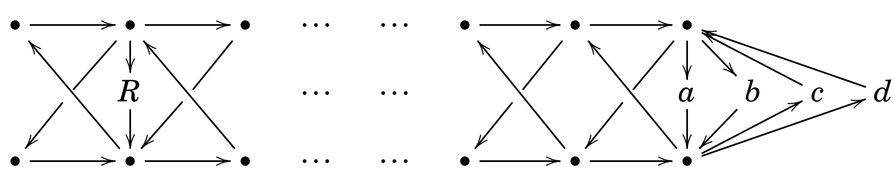


with $m+1$ vertices. The Auslander translate τ is the identity. When $n = 2m+1$ is odd, the quiver is

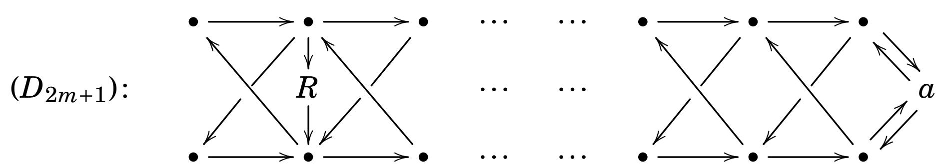


with $m+2$ vertices. Here τ is reflection across the horizontal axis.

10.34. The extended (D_n) diagram also has $n+1$ nodes, and again the quiver depends on the parity of n. When $n = 2m$ is even, every non-free MCM module splits, and the quiver looks like

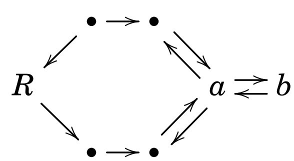
(D_{2m}): 

with $4m+1$ vertices. The translate τ is given by reflection in the horizontal axis for those vertices not on the axis, swaps a and d, and swaps b and c. When $n = 2m+1$ is odd, the two "legs" at the opposite end of the (D_n) diagram from the free module collapse into a single module, giving the quiver



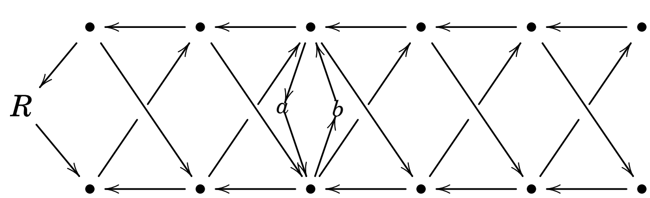
with $4m$ vertices. Again, τ is reflection across the horizontal axis.

10.35. We saw above the the quiver for the one-dimensional (E_6) singularity has the form



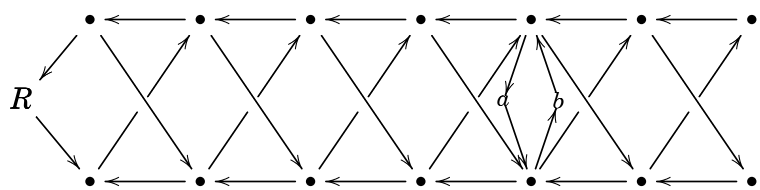
with 7 vertices and τ given by reflection across the horizontal axis.

10.36. For the (E_7) singularity, every non-free indecomposable splits, giving 15 vertices in the AR quiver for the one-dimensional singularity.



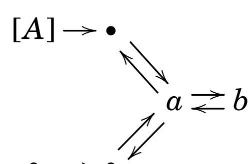
The translate is reflection across the horizontal axis for every vertex except a and b, which are interchanged by τ.

10.37. For the (E_8) singularity, once again every non-free indecomposable splits when flatted.



Here there are 17 vertices; the translate is reflection across the horizontal axis and interchanges a and b.

10.38 Example. Let $A = k[\![t^3, t^4, t^5]\!]$. Then A is a finite birational extension of the (E_6) singularity $R = k[\![x, y]\!]/(x^3 + y^4) \cong k[\![t^3, t^4]\!]$, so has finite CM type by Theorem 3.13. In fact, A is isomorphic to the endomorphism ring of the maximal ideal of R. By Lemma 3.9 every indecomposable MCM R-module other than R itself is actually a MCM A-module, and $\operatorname{Hom}_R(M,N) = \operatorname{Hom}_A(M,N)$ for all non-free MCM R-modules M and N. Thus the AR quiver for A is obtained from the one for R by erasing $[R]$ and all the arrows into and out of $[R]$. As R-modules, $A \cong (t^4, t^6)$, so the quiver is the one below.



§4 Exercises

10.39 Exercise. Prove that a short exact sequence $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ is split if and only if every homomorphism $X \longrightarrow M$ factors through E.

10.40 Exercise. Let $R = D/(t^n)$, where (D,t) is a complete DVR. Then the indecomposable finitely generated R-modules are $D/(t), D/(t^2), \dots, D/(t^n) = R$. Compute the AR sequences for each of the indecomposables, directly from the definition. (Hint: start with $n = 2$.)

10.41 Exercise. Prove, by mimicking the proof of Proposition 10.2, that (10.1.1) is an AR sequence ending in M if and only if it is an AR sequence starting from N. (Hint: Given $\psi\colon N \longrightarrow Y$, it suffices to show that the short exact sequence obtained from the pushout is split. If not, use the lifting property to obtain an endomorphism α of N such that either α is an isomorphism and splits ψ, or $\alpha - 1_N$ is an isomorphism and splits (10.1.1).)

10.42 Exercise. Assume that $0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$ is a non-split short exact sequence of MCM modules satisfying the lifting property to be an AR sequence ending in M. Prove that M is indecomposable.

10.43 Exercise. Prove Remark 10.5: there is an exact sequence

$$M^* \otimes_A N \xrightarrow{\rho} \operatorname{Hom}_A(M,N) \longrightarrow \underline{\operatorname{Hom}}_A(M,N) \longrightarrow 0\,,$$

where ρ sends $f \otimes y$ to the homomorphism $x \mapsto f(x)y$.

10.44 Exercise. Let R be an abstract hypersurface and M, N two MCM R-modules. Prove that $\operatorname{Ext}_R^{2i}(M,N) \cong \underline{\operatorname{Hom}}_R(M,N)$ for all $i \geqslant 1$.

10.45 Exercise (Lemme d'Acyclicité, [PS73]). Let $(A,\mathfrak{m})$ be a local ring and $M_\bullet\colon 0 \longrightarrow M_s \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0$ a complex of finitely generated A-modules. Assume that $\operatorname{depth} M_i \geqslant i$ for each i, and that every homology module $H_i(M_\bullet)$ either has finite length or is zero. Then $M_\bullet$ is exact.

10.46 Exercise. Prove that the natural map $\rho_M^N\colon M^* \otimes_A N$ to $\operatorname{Hom}_A(M,N)$, defined by $\rho(f\otimes y)(x) = f(x)y$, is an isomorphism if either M or N is projective. In particular ρ_M^M is an isomorphism if and only if M is projective.

10.47 Exercise. This exercise shows that if R is an Artinian local ring and M is an indecomposable R-module with an AR sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0\,,$$

then $N \cong (\operatorname{Tr} M)^\vee$.

(a) Let $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be an exact sequence with P_0, P_1 finitely generated projective, and let Z be an arbitrary finitely generated R-module. Use the proof of Proposition 10.13 to show the existence of an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(X,Z) \longrightarrow \operatorname{Hom}_R(P_0,Z) \longrightarrow \operatorname{Hom}_R(P_1,Z) \longrightarrow \operatorname{Tr} X \otimes_R Z \longrightarrow 0$$

and conclude that we have an equality of lengths

$$\ell(\operatorname{Hom}_R(X,Z)) - \ell(\operatorname{Hom}_R(Z,(\operatorname{Tr} X)^\vee)) = \ell(\operatorname{Hom}_R(P_0,Z)) - \ell(\operatorname{Hom}_R(P_1,Z))\,.$$

(b) Let $\sigma\colon 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence of finitely generated R-modules, and define the *defects* of σ on an R-module X by

$$\sigma_*(X) = \operatorname{cok}[\operatorname{Hom}_R(B,X) \longrightarrow \operatorname{Hom}_R(A,X)]$$
$$\sigma^*(X) = \operatorname{cok}[\operatorname{Hom}_R(X,B) \longrightarrow \operatorname{Hom}_R(X,C)]\,.$$

Show that $\ell(\sigma^*(X)) = \ell(\sigma_*((\operatorname{Tr} X)^\vee))$ for every X. Conclude that the following two conditions are equivalent:

(i) every homomorphism $X \longrightarrow C$ factors through g;

(ii) every homomorphism $A \longrightarrow (\operatorname{Tr} X)^\vee$ factors through f.

(c) Prove that if $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ is an AR sequence for M, then $N \cong (\operatorname{Tr} M)^\vee$. (Hint: let $h\colon N \longrightarrow Y$ be given with Y indecomposable and not isomorphic to $(\operatorname{Tr} M)^\vee$. Apply the previous part to $X = \operatorname{Tr}(Y^\vee)$.)

10.48 Exercise. Prove that $\operatorname{rad}(M,N)/\operatorname{rad}^2(M,N)$ is annihilated by the maximal ideal $\mathfrak{m}$, so is a finite-dimensional k-vector space. Your proof will actually show that the quotient is annihilated by the radical of $\operatorname{End}_R(M)$ (acting on the right) and the radical of $\operatorname{End}_R(N)$ (acting on the left).

10.49 Exercise ([Eis95, A.3.22])**.** If $\sigma\colon A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence, prove that (there exists a choice of $\operatorname{Tr} M$ such that) the sequence

$$0 \longrightarrow \operatorname{Hom}_R(\operatorname{Tr} M, A) \longrightarrow \operatorname{Hom}_R(\operatorname{Tr} M, B) \longrightarrow \operatorname{Hom}_R(\operatorname{Tr} M, C) \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$$

is exact. In other words, Tr can be thought of as measuring the non-exactness of $M \otimes_R -$ and, if we set $N = \operatorname{Tr} M$, of $\operatorname{Hom}_R(N, -)$.

10.50 Exercise. Say that an inclusion of modules $A \subset B$ is *pure* if $M \otimes_R A \longrightarrow M \otimes_R B$ is injective for all R-modules M. If σ is as in the previous exercise with $A \longrightarrow B$ pure, then prove that

$$0 \longrightarrow \operatorname{Hom}_R(N, A) \longrightarrow \operatorname{Hom}_R(N, B) \longrightarrow \operatorname{Hom}_R(N, C) \longrightarrow 0$$

is exact for every finitely presented module N. Conclude that if C is finitely presented, then σ splits. (See Exercise 6.23 for a different proof.)

11

Ascent and Descent

We have seen in Chapter 8 that the hypersurface rings $(R,\mathfrak{m},k)$ of finite Cohen-Macaulay type have a particularly nice description when R is complete, k is algebraically closed and R contains a field of characteristic different from 2, 3, and 5. In this section we will see to what extent finite Cohen-Macaulay type ascends to and descends from faithfully flat extensions such as the completion or Henselization, and how it behaves with respect to residue field extension. In 1987 F.-O. Schreyer [Sch87] conjectured that a local ring $(R,\mathfrak{m},k)$ has finite Cohen-Macaulay type if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ has finite Cohen-Macaulay type. We have already seen that Schreyer's conjecture is true in dimension one (Corollary 3.17). We shall see that the "if" direction holds in general, and the "only if" direction holds when R is excellent and Cohen-Macaulay. For rings that are neither excellent nor CM, there are counterexamples (cf. 11.14). Schreyer also conjectured ascent and descent of finite CM type along extensions of the residue field (cf. Theorem 11.16 below). We shall prove descent in general, and ascent in the separable case. Inseparable extensions, however, can cause problems (cf. Example 11.18). We will revisit some of these issues in Chapter 15, where we consider ascent and descent of *bounded* CM type.

§1 Descent

Here is the main result of this section ([Wie98, Theorem 1.5]).

11.1 Theorem. *Let $(R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ be a flat local homomorphism such that $S/\mathfrak{m}S$ is Cohen-Macaulay. If S has finite Cohen-Macaulay type, then so has R.*

The proof requires some preparation.

11.2 Notation. Given a ring A and a finitely generated A-module M, we let $\operatorname{add}(M) = \operatorname{add}_A(M)$ denote the full subcategory of A-mod consisting of finitely generated modules that are isomorphic to direct summands of direct sums of copies of M. Thus $N \in \operatorname{add}(M)$ if and only if $N \mid M^t$ for some positive integer t. We let $+(M) = +_A(M)$ denote the set of isomorphism classes $[N]$ of modules $N \in \operatorname{add}(M)$.

11.3 Proposition ([Wie98, Theorem 1.1])**.** *Let $A \longrightarrow B$ be a faithfully flat homomorphism of commutative rings, and let U and V be finitely presented A-modules. Then $U \in \operatorname{add}_A V$ if and only if $B \otimes_A U \in \operatorname{add}_B(B \otimes_A V)$.*

Proof. The "only if" direction is clear. For the converse, we may assume, by replacing V by a direct sum of copies of V, that $B \otimes_A U \mid B \otimes_A V$. Choose B-homomorphisms $B \otimes_A U \xrightarrow{\alpha} B \otimes_A V$ and $B \otimes_A V \xrightarrow{\beta} B \otimes_A U$ such that $\beta\alpha = 1_{B\otimes_A U}$. Since V is finitely presented and B is flat over A, the natural map $B \otimes_A \operatorname{Hom}_A(V,U) \longrightarrow \operatorname{Hom}_B(B \otimes_A V, B \otimes_A U)$ is an isomorphism. Therefore we can write $\beta = b_1 \otimes \sigma_1 + \cdots + b_r \otimes \sigma_r$, with $b_i \in B$ and $\sigma_i \in \operatorname{Hom}_A(V,U)$ for each i. Put $\sigma = [\sigma_1 \cdots \sigma_r]\colon V^{(r)} \longrightarrow U$. We will show that σ is a split surjection. Since

$$(1_B \otimes \sigma) \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \alpha \quad = \quad 1_{B\otimes_A U},$$

we see that $1_B \otimes \sigma \colon B \otimes_A V^{(r)} \longrightarrow B \otimes_A U$ is a split surjection. Therefore the induced map $(1_B \otimes \sigma)_* \colon \operatorname{Hom}_B(B \otimes_A U, B \otimes_A V^{(r)}) \longrightarrow \operatorname{Hom}_B(B \otimes_A U, B \otimes_A U)$ is surjective. Since U too is finitely presented, the vertical maps in the following commutative square are isomorphisms.

(11.3.1)
$$\begin{array}{ccc} B \otimes_A \operatorname{Hom}_A(U, V^{(r)}) & \xrightarrow{1_B \otimes \sigma_*} & B \otimes_A \operatorname{Hom}_A(U,U) \\ \cong \downarrow & & \downarrow \cong \\ \operatorname{Hom}_B(B \otimes_A U, B \otimes_A V^{(r)}) & \xrightarrow[(1_B \otimes \sigma)_*]{} & \operatorname{Hom}_B(B \otimes_A U, B \otimes_A U) \end{array}$$

Therefore $1_B \otimes_A \sigma_*$ is surjective as well. By faithful flatness, σ_* is surjective, and hence σ is a split surjection. □

The next theorem appears as Theorem 1.1 in [Wie99], with a slightly non-commutative proof. We will give a commutative proof here.

11.4 Theorem. *Let $(R, \mathfrak{m})$ be a local ring, and let M be a finitely generated R-module. Then there are only finitely many isomorphism classes of indecomposable modules in* $\operatorname{add}(M)$*.*

Proof. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R, and write $\widehat{R} \otimes_R M = V_1^{(n_1)} \oplus \cdots \oplus V_t^{(n_t)}$, where each V_i is an indecomposable $\widehat{R}$-module and each $n_i > 0$. If $L \in \operatorname{add}(M)$, then $\widehat{R} \otimes_R L \cong V_1^{(a_1)} \oplus \cdots \oplus V_t^{(a_t)}$ for suitable non-negative integers a_i; moreover, the integers a_i are uniquely determined by the isomorphism class $[L]$, by Corollary 1.8. Thus we have a well-defined map $j \colon +(M) \longrightarrow \mathbb{N}_0^t$, taking $[L]$ to $(a_1, \dots, a_t)$. Moreover, this map is one-to-one, by faithfully flat descent (Corollary 1.14).

If $[L] \in +(M)$ and $j([L])$ is a minimal non-zero element of $j(+(M))$, then L is clearly indecomposable. Conversely, if $[L] \in \operatorname{add}(M)$ and L is indecomposable, we claim that $j([L])$ is a minimal non-zero element of $j(+(M))$. For, suppose that $j([X]) < j([L])$, where $[X] \in +(M)$ is non-zero. Then $\widehat{R} \otimes_R X \mid \widehat{R} \otimes_R L$, so $X \mid L$ by Corollary 1.14. But $X \neq 0$ and $X \ncong L$ (else $j([X] = j([L]))$, and we have a contradiction to the indecomposability of L.

By Dickson's Lemma (Exercise 3.26), $j(+(M))$ has only finite many minimal non-zero elements, and, by what we have just shown, $\operatorname{add}(M)$ has only finitely many isomorphism classes of indecomposable modules. □

With Proposition 11.3 and Theorem 11.4 at our disposal, we can now prove Theorem 11.1.

Proof of Theorem 11.1. The hypothesis that the closed fiber $S/\mathfrak{m}S$ is CM guarantees that $S \otimes_R M$ is a MCM S-module whenever M is a MCM R-module (cf. Exercise 11.21). Let $\mathscr{U}$ be the class of MCM S-modules that occur in direct-sum decompositions of extended MCM modules; thus $Z \in \mathscr{U}$ if and only if there is a MCM R-module X such that Z is isomorphic to an S-direct-summand of $S \otimes_R X$. Let $Z_1, \dots, Z_t$ be a complete set of representatives for isomorphism classes of indecomposable modules in $\mathscr{U}$. Choose, for each i, a MCM R-module X_i such that $Z_i \mid S \otimes_R X_i$, and put $Y = X_1 \oplus \cdots \oplus X_t$.

Suppose now that L is an indecomposable MCM R-module. Then $S \otimes_R L \cong Z_1^{(a_1)} \oplus \cdots \oplus Z_t^{(a_t)}$ for suitable non-negative integers a_i, and it follows that $S \otimes_R L$ is isomorphic to a direct summand of $S \otimes_R Y^{(a)}$, where $a = \max\{a_1, \dots, a_t\}$. By Proposition 11.3, L is a direct summand of a direct sum of copies of Y. Then, by Theorem 11.4, there are only finitely many possibilities for L, up to isomorphism. □

By the way, the class $\mathscr{U}$ in the proof of Theorem 11.1 is *not* necessarily the class of all MCM S-modules. For example, consider the extension

$R = k[\![t^2]\!] \longrightarrow k[\![t^2,t^3]\!] = S$; in this case, the only extended modules are the free ones. (Cf. also Exercise 13.31). The first order of business in the next section will be to find situations where this unfortunate behavior cannot occur, that is, where *every* MCM S-module is a direct summand of an extended MCM module.

§2 Ascent to the completion

It's a long way to the completion of a local ring, so we will make a stop at the Henselization. In this section and the next, we will need to understand the behavior of finite CM type under direct limits of étale and, more generally, unramified extensions. We will recall the basic definitions here and refer to Appendix B for details, in particular, for reconciling our definitions with others in the literature.

11.5 Definition. A local homomorphism of local rings $(R,\mathfrak{m},k) \longrightarrow (S,\mathfrak{n},\ell)$ is *unramified* provided S is essentially of finite type over R (that is, S is a localization of some finitely generated R-algebra) and the following properties hold.

(i) $\mathfrak{m}S = \mathfrak{n}$, and

(ii) $S/\mathfrak{m}S$ is a finite separable field extension of $R/\mathfrak{m}$.

If, in addition, φ is flat, then we say φ is *étale*. (We say also that S is an *unramified*, respectively, *étale* extension of R.) Finally, a *pointed étale* extension is an étale extension $(R,\mathfrak{m},k) \longrightarrow (S,\mathfrak{n},\ell)$ inducing an isomorphism on residue fields.

By Proposition B.9, properties (i) and (**??**) are equivalent to the single requirement that the *diagonal* map $\mu\colon S\otimes_R S \longrightarrow R$ (taking $s_1\otimes s_2$ to s_1s_2) splits as $S\otimes_R S$-modules (equivalently, $\ker(\mu)$ is generated by an idempotent).

It turns out (see [Ive73] for details) that the isomorphism classes of pointed étale extensions of a local ring $(R,\mathfrak{m})$ form a direct system. The remarkable fact that makes this work is that if $R \longrightarrow S$ and $R \longrightarrow T$ are pointed étale extensions then there is *at most one* homomorphism $S \longrightarrow T$ making the obvious diagram commute.

11.6 Definition. The *Henselization* R^{h} of R is the direct limit of a set of representatives of the isomorphism classes of pointed étale extensions of R.

The Henselization is, conveniently, a Henselian ring (cf. Chapter 1 **??** and Appendix B).

Suppose $R \hookrightarrow S$ is a flat local homomorphism. By analogy with the terminology "weakly liftable" of [ADS93], we say that a finitely generated S-module M is *weakly extended* (from R) provided there is a finitely generated R-module N such that $M \mid S\otimes_R N$. If N can be chosen to be a MCM R-module, we say that M is *weakly extended from* $\mathrm{CM}(R)$.

have we ever used this notation before?

Our immediate goal is to show that if R has finite CM type then R^{h} does too. We show in Proposition 11.7 that it will suffice to show that every MCM R^{h}-module is weakly extended from $\mathrm{CM}(R)$. In Lemma 11.8 we show that every finitely generated R^{h}-module is weakly extended. Then in Proposition 11.9 we show, assuming R has finite CM type, that MCM

R^{h}-modules are weakly extended from $\mathrm{CM}(R)$. The proof depends on the fact (Theorem 6.12) that rings of finite CM type have isolated singularities.

11.7 Proposition. *Let* $(R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ *be a local homomorphism. Assume that every MCM S-module is weakly extended from* $\mathrm{CM}(R)$*. If R has finite CM type, so has S.*

Proof. Let $L_1,\dots,L_t$ be a complete list of representatives for the isomorphism classes of indecomposable MCM R-modules. Let $L = L_1 \oplus \cdots \oplus L_t$, and put $V = S \otimes_R L$. Given a MCM S-module M, we choose a MCM R-module N such that $M \mid S \otimes_R N$. Writing $N = L_1^{(a_1)} \oplus \cdots \oplus L_t^{(a_t)}$, we see that $N \in \mathrm{add}_R(L)$ and hence that $M \in \mathrm{add}_S(V)$. Thus $\mathrm{CM}(S) \subseteq \mathrm{add}_S(V)$; now Theorem 11.4 completes the proof. □

11.8 Lemma ([HW09, Theorem 5.2])**.** *Let* $\varphi\colon (R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ *be a flat local homomorphism, and assume that S is a direct limit of étale extensions of R. Then every finitely generated S-module is weakly extended from R.*

Proof. Let M be a finitely generated S-module, and choose a matrix A whose cokernel is M. Since all of the entries of A live in some étale extension T of R, we see that $M = S \otimes_T N$ for some finitely generated T-module N. Refreshing notation, we may assume that $\varphi\colon R \longrightarrow S$ is étale. We apply $- \otimes_S M$ to the diagonal map $\mu\colon S \otimes_R S$, getting a commutative diagram

(11.8.1)
$$\begin{array}{ccc} S \otimes_R S \otimes_S M & \xrightarrow{\delta \otimes 1_M} & S \otimes_S M \\ \cong\downarrow & & \downarrow\cong \\ S \otimes_R M & \longrightarrow & M \end{array}$$

in which the horizontal maps are split surjections of S-modules. The S-module structure on $S \otimes_R M$ comes from the S-action on S, not on M. (The distinction is important; cf. Exercise 11.24.) Thus we have a split injection of S-modules $j\colon M \longrightarrow S\otimes_R M$. Now write ${}_RM$ as a directed union of finitely generated R-modules N_α. The flatness of φ implies that $S\otimes_R M$ is the directed union of the modules $S\otimes_R N_\alpha$. Since $j(M)$ is a finitely generated S-module, there is an index α_0 such that $j(M)\subseteq S\otimes_R N_{\alpha_0}$. We put ${}_RN = {}_RN_{\alpha_0}$. Since $j(M)$ is a direct summand of $S\otimes_R M$, it must be a direct summand of the smaller module $S\otimes_R N$. □

Even if we start with a MCM S-module, there is no reason to believe that the R-module N in the proof of Lemma 11.8 is MCM. The next proposition refines the lemma and will be used both here and in the next section, where we prove ascent along separable extensions of the residue field.

11.9 Proposition. *Let $(R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ be a flat local homomorphism of CM local rings. Assume that the closed fiber $S/\mathfrak{m}S$ is Artinian and that $S_\mathfrak{q}$ is Gorenstein for each prime ideal $\mathfrak{q}\neq\mathfrak{n}$. If every finitely generated S-module is weakly extended from R, then every MCM S-module is weakly extended from* CM(R)*. In particular, if R has finite CM type, so has S.*

Proof. Note that $\dim(R) = \dim(S) =: d$ by [BH93, (A.11)]. Let M be a MCM S-module. Corollary A.18 implies that M is a d^{th} syzygy of some finitely generated S-module U. We choose a finitely generated R-module V such that $U \mid S\otimes_R V$, say, $U\oplus X \cong S\otimes_R V$. Let W be a d^{th} syzygy of V. Then W is MCM by Corollary A.18. Since $R\longrightarrow S$ is flat, $S\otimes_R W$ is a d^{th} syzygy of $S\otimes_R V$, as is $M\oplus L$, where L is a d^{th} syzygy of X. By Schanuel's Lemma (A.8) there are finitely generated free S-modules G_1 and G_2 such that $(S\otimes_R W)\oplus G_1 \cong (L\oplus M)\oplus G_2$. Of course G_1 is extended from a free

R-module F. Putting $N = W \oplus F$, we see that $M \mid S \otimes_R N$. This proves the first assertion, and the second follows from Proposition 11.7. □

11.10 Theorem. *Let $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local homomorphism of CM local rings. Assume that R has finite CM type and that S is the direct limit of a system $\{(S_\alpha, \mathfrak{n}_\alpha)\}_{\alpha \in \Lambda}$ of étale extensions of $(R, \mathfrak{m})$. Then S has finite CM type. In particular, the Henselization R^h has finite CM type.*

Proof. By Lemma 11.8 and Propositions 11.7 and 11.9, it will suffice to show that $S_\mathfrak{q}$ is Gorenstein for each prime ideal $\mathfrak{q} \neq \mathfrak{n}$.

Given an arbitrary non-maximal prime ideal $\mathfrak{q}$ of S, put $\mathfrak{q}_\alpha = \mathfrak{q} \cap S_\alpha$ for $\alpha \in \Lambda$, and let $\mathfrak{p} = \mathfrak{q} \cap R$. Since by Exercise 11.23 $\mathfrak{m} S_\alpha = \mathfrak{n}_\alpha$ for each $\alpha \in \Lambda$, we see that $\mathfrak{m} S = \mathfrak{n}$, and it follows that $\mathfrak{p}$ is a non-maximal prime ideal of R. By Theorem 6.12, $R_\mathfrak{p}$ is a regular local ring. Each extension $R_\mathfrak{p} \longrightarrow S_{\mathfrak{q}_\alpha}$ is étale by Exercise 11.23, and it follows (again from the exercise) that $\mathfrak{p} S_{\mathfrak{q}_\alpha} = \mathfrak{q}_\alpha S_{\mathfrak{q}_\alpha}$ for each α. Therefore $\mathfrak{p} S_\mathfrak{q} = \mathfrak{q} S_\mathfrak{q}$, so the closed fiber $S_\mathfrak{q}/\mathfrak{p} S_\mathfrak{q}$ is a field. Since $R_\mathfrak{p}$ and $S_\mathfrak{q}/\mathfrak{p} S_\mathfrak{q}$ are Gorenstein and $R_\mathfrak{p} \longrightarrow S_\mathfrak{q}$ is flat, [BH93, (3.3.15)] implies that $S_\mathfrak{q}$ is Gorenstein. □

Finally, we prove ascent of finite CM type to the completion for excellent rings. Actually, a condition weaker than excellence suffices. Recall that a Noetherian ring A is *regular* provided $A_\mathfrak{m}$ is a regular local ring for each maximal ideal $\mathfrak{m}$ of A. A Noetherian ring A containing a field k is *geometrically regular over k* provided $\ell \otimes_k A$ is a regular ring for every finite algebraic extension ℓ of k. A homomorphism $\varphi \colon A \longrightarrow B$ of Noetherian rings is *regular* provided φ is flat, and for each $\mathfrak{p} \in \operatorname{Spec}(A)$ the fiber $B_\mathfrak{p}/\mathfrak{p} B_\mathfrak{p}$ is geometrically regular over the field $A_\mathfrak{p}/\mathfrak{p} A_\mathfrak{p}$. Part of the definition of A being *excellent* is that $A \longrightarrow \widehat{A}$ is a regular homomorphism. (The other parts are that A is universally catenary and that the non-singular locus of B is open in Spec(B) for every finitely generated A-algebra B.)

We will need the following consequences of regularity of a ring homomorphism. The first assertion is clear from the definition, while the second follows from the first and from [Mat86, (32.2)].

11.11 Proposition. *Let $A \longrightarrow B$ be a regular homomorphism, $\mathfrak{q} \in \operatorname{Spec}(B)$, and put $\mathfrak{p} = \mathfrak{q} \cap A$.*

(i) The homomorphism $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$ is regular.

(ii) If $A_{\mathfrak{p}}$ is a regular local ring, so is $B_{\mathfrak{q}}$. □

We'll also need the following remarkable theorem due to R. Elkik (cf. [Elk73]).

11.12 Theorem (Elkik)**.** *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated $\widehat{R}$-module. If $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for each non-maximal prime ideal $\mathfrak{p}$ of $\widehat{R}$, then M is extended from the Henselization R^h.* □

11.13 Corollary. *Let $(R, \mathfrak{m})$ be a CM local ring with $\mathfrak{m}$-adic completion $\widehat{R}$. If $\widehat{R}$ has finite CM type, so has R. The converse holds if $R \longrightarrow \widehat{R}$ is regular, in particular if R is excellent.*

Proof. The first assertion is a special case of Theorem 11.1. Suppose now that $R \longrightarrow \widehat{R}$ is regular and that R has finite Cohen-Macaulay type. Let $\mathfrak{q}$ be an arbitrary non-maximal prime ideal of $\widehat{R}$, and set $\mathfrak{p} = \mathfrak{q} \cap R$. Then $R_{\mathfrak{p}}$ is a regular local ring by Theorem 6.12, and Proposition 11.11 implies that $\widehat{R}_{\mathfrak{q}}$ is a regular local ring too. Thus $\widehat{R}$ has an isolated singularity.

Now let M be an arbitrary MCM $\widehat{R}$-module. Then $M_{\mathfrak{q}}$ is a free $\widehat{R}_{\mathfrak{q}}$-module for each non-maximal prime ideal $\mathfrak{q}$ of $\widehat{R}$. By Theorem 11.12, M is extended from the Henselization, that is, there is an R^{h}-module N such that $M \cong N \otimes_{R^{\mathrm{h}}} \widehat{R}$; moreover, N is a MCM R^{h}-module by [BH93, (1.2.16) and (A.11)]. Since R^{h} has finite CM type (Theorem 11.10), Proposition 11.7 implies that $\widehat{R}$ has finite CM type. □

It is unknown whether or not Corollary 11.13 would be true without the hypothesis that R be CM, or without the hypothesis that $R \longrightarrow \widehat{R}$ be regular. The following example, from [LW00], shows, however, that we can't omit *both* hypotheses:

11.14 Example. Let $T = k[[x,y,z]]/\big((x^3-y^7)\cap(y,z)\big)$, where k is any field. We claim that T has infinite CM type. To see this, set $R = k[[x,y]]/(x^3-y^7) \cong k[[t^3,t^7]]$. Then R has infinite CM type by Theorem 3.10, since (DR2) fails for this ring. Further, $R[[z]]$ has infinite CM type: the map $R \longrightarrow R[[z]]$ is flat with CM closed fiber, and Theorem 11.1 applies. Now $R[[z]] \cong T/(x^3-y^7)$. By item (v) of Proposition A.2, every MCM $T/(x^3-y^7)$-module is MCM over T. since $T/(x^3-y^7)$ has infinite CM type, the claim follows.

It is easy to check that the image of x is a nonzerodivisor in T. By [Lec86, Theorem 1], T is the completion of some local integral domain A. Then A has finite CM type; in fact, it has no MCM modules at all! For if A had a MCM module, then A would be universally catenary [Hoc73, Section 1]. But this would imply, by [Mat86, Theorem 31.7], that A is formally equidimensional, that is, all minimal primes of $\widehat{A}$ $(=T)$ have the same dimension. But the two minimal primes of T obviously have dimensions two and one, contradiction.

Another example of this behavior, using a very different construction, can be found in [LW00].

§3 Ascent along separable field extensions

Let $(R,\mathfrak{m},k)$ be a local ring and ℓ/k a field extension. We want to lift the extension $k \hookrightarrow \ell$ to a flat local homomorphism $(R,\mathfrak{m},k) \longrightarrow (S,\mathfrak{n},\ell)$ with certain nice properties. The type of ring extension we seek is dubbed a *gonflement* by Bourbaki [Bou06, Appendice]. Translations of the term range from the innocuous "inflation" to the provocative "swelling" or "intumescence". To avoid choosing one, we have decided to stick with the French word.

11.15 Definition. Let $(R,\mathfrak{m},k)$ be a local ring.

(i) An *elementary gonflement* of R is either

  a) a purely transcendental extension $R \longrightarrow (R[x])_{\mathfrak{m}R[x]}$ (where x is a single indeterminate), or

  b) an extension $R \longrightarrow R[x]/(f)$, where f is a monic polynomial whose reduction modulo $\mathfrak{m}$ is irreducible in $k[x]$.

(ii) A *gonflement* is an extension $(R,\mathfrak{m},k) \longrightarrow S$ with the following property: There is a well-ordered family $\{R_\alpha\}_{0\leqslant\alpha\leqslant\lambda}$ of local extensions $(R,\mathfrak{m},k) \hookrightarrow (R_\alpha,\mathfrak{m}_\alpha,k_\alpha)$ such that

  a) $R_0 = R$ and $R_\lambda = S$,

  b) $R_\beta = \bigcup_{\alpha<\beta} R_\alpha$ if β is a limit ordinal, and

c) $R_{\beta+1}$ is an elementary gonflement of R_β if $\beta \neq \lambda$.

Elementary gonflements of type (ia) are often used to pass to a local ring with infinite residue field. (See Proposition 3.4 for an application.) In this section we will need gonflements that are iterations of elementary gonflements of type (ib).

The following theorem (cf. [Bou06, Appendice, Proposition 2 and Théorème 1, pp. 39–40]) summarizes the basic properties of gonflements.

11.16 Theorem. *Let $(R,\mathfrak{m},k)$ be a local ring.*

(i) Let $(R,\mathfrak{m},k) \longrightarrow S$ be a gonflement.

a) S is local with, say, maximal ideal $\mathfrak{n}$. In particular, S is Noetherian.

b) $\mathfrak{m}B = \mathfrak{n}$.

c) $R \longrightarrow S$ is a flat local homomorphism.

d) With the notation as in the definition, if $\alpha \leqslant \beta \leqslant \lambda$, then $R_\alpha \longrightarrow R_\beta$ is a gonflement.

(ii) If $k \longrightarrow \ell$ is an arbitrary field extension, there is a gonflement $(R,\mathfrak{m},k) \longrightarrow (S,\mathfrak{n},\ell)$ lifting $k \longrightarrow \ell$.

We now prove ascent of finite CM type along gonflements with separable residue field growth.

11.17 Theorem. *Let $(R,\mathfrak{m},k) \longrightarrow (S,\mathfrak{n},\ell)$ be a gonflement.*

(i) If S has finite CM type, then R has finite CM type.

(ii) *Assume R is CM and $k \longrightarrow \ell$ is a separable algebraic extension. If R has finite CM type, so has S.*

Proof. Item (i) is, again, a special case of Theorem 11.1. For the proof of (ii), we keep the notation of Definition 11.15. Each elementary gonflement $R_\alpha \longrightarrow R_{\alpha+1}$ is of type (ib). Moreover, since the induced map on residue fields is a finite separable extension, we see from Exercise 11.23 that $R_\alpha \longrightarrow R_{\alpha+1}$ is étale. By Exercise 11.21 and [BH93, A.11], S is Cohen-Macaulay and $\dim(S) = \dim(R)$. We will show by transfinite induction that each ring R_β is Gorenstein on the punctured spectrum. Theorem 6.12 says R is in fact regular on the punctured spectrum, so we have the case $\alpha = 0$. Now assume $0 < \beta \leqslant \lambda$ and that R_α is Gorenstein on the punctured spectrum for each $\alpha < \beta$. Let $\mathfrak{q}_\beta$ be a non-maximal prime ideal of R_β, and put $\mathfrak{q}_\alpha = \mathfrak{q}_\beta \cap R_\alpha$, for each $\alpha < \beta$. In either case, whether β is a limit ordinal or $\beta = \gamma + 1$ for some γ, the proof of Theorem 11.10 shows that $(R_\beta)_{\mathfrak{q}_\beta}$ is Gorenstein. Now, setting $\beta = \lambda$, we see that S is Gorenstein on the punctured spectrum.

By Propositions 11.7 and 11.9, we need only show that every finitely generated S-module is weakly extended from R. We shall show that, for every $\beta \leqslant \lambda$, every finitely generated R_β-module M is weakly extended from R. We proceed by transfinite induction again, the case $\beta = 0$ being trivial. Suppose $0 < \beta \leqslant \lambda$. If $\beta = \alpha + 1$ for some α, then Lemma 11.8 provides a finitely generated R_α-module N such that $M \mid \left(R_\beta \otimes_{R_\alpha} N\right)$. By induction, N is weakly extended from R, and it follows that M too is weakly extended from R. If β is a limit ordinal, then M is extended from R_α for some $\alpha < \beta$, and again the inductive hypothesis shows that M is weakly extended from

R. □

The separability condition in 11.17 cannot be omitted. Indeed, here is an example of a local ring R with finite CM type and an elementary gonflement $R \longrightarrow S$ such that S has infinite CM type.

11.18 Example ([Wie98, Example 3.4])**.** Let k be an imperfect field of characteristic 2, and let $\alpha \in k - k^2$. Put $R = k[[x,y]]/(x^2 + \alpha y^2)$. Then R is a one-dimensional local domain with multiplicity two, so by Theorem 3.18 R has finite Cohen-Macaulay type. However, by Proposition 3.15, $S = R \otimes_k k(\sqrt{\alpha}) = k(\sqrt{\alpha})[[x,y]]/(x + \sqrt{\alpha}y)^2$ does not have finite Cohen-Macaulay type, since it is Cohen-Macaulay but not reduced.

Recall that we did not give a self-contained proof of Theorem 3.10. Here we describe a proof, independent of the matrix decompositions in [GR78], in an important special case.

11.19 Theorem. *Let* $(R, \mathfrak{m}, k)$ *be an analytically unramified local ring of dimension one. Assume R contains a field and that* $\operatorname{char}(k) \neq 2, 3$ *or* 5. *Then R has finite CM type if and only if R satisfies the Drozd-Roĭter conditions (DR1) and (DR2) of Chapter 3.*

Proof. A complete proof of the "only if" direction is in Chapter 3. For the converse, we may assume, by Theorems 11.16 and 11.17, that k is algebraically closed. Corollary 3.17 (whose proof did not depend on Theorem 3.10!) allows us to assume that R is complete. Then $\overline{R} = k[[t_1]] \times \cdots \times k[[t_s]]$, where $s \leqslant 3$ and the t_i are analytic indeterminates. An elementary but tedious computation (cf. [Yos90, pages 72–73]) now shows that R is a finite birational extension of an ADE singularity A. Since A has finite CM type (Corollary 7.19), Proposition 3.14 implies that R has finite CM type too. □

§4 Equicharacteristic Gorenstein singularities

We now assemble the pieces and obtain a nice characterization of the equicharacteristic Gorenstein singularities of finite CM type.

11.20 Corollary. *Let $(R,\mathfrak{m},k)$ be an excellent, Gorenstein ring containing a field of characteristic different from 2, 3, 5, and let K be an algebraic closure of k. Assume $d := \dim(R) \geqslant 1$ and that k is perfect. Then R has finite CM type if and only if there is a non-zero non-unit $f \in k[[x_0,\dots x_d]]$ such that $\widehat{R} \cong k[[x_0,\dots,x_d]]/(f)$ and $K[[x_0,\dots,x_d]]/(f)$ is a simple singularity (cf. Chapter 8).*

Proof. Using [Mat86, Theorem 22.5], we see that $K[[x_0,\dots,x_d]]/(f)$ is flat over $k[[x_0,\dots,x_d]]/(f)$ for any non-unit $f \in k[[x_0,\dots,x_d]]$. The "if" direction now follows from Theorem 11.1 and the fact (cf. Chapter 8) that simple singularities have finite CM type.

For the converse, suppose R has finite CM type. Since R is CM and excellent, the completion $\widehat{R}$ has finite CM type by Corollary 11.13. Moreover, Theorem **??** implies, since R is Gorenstein, that $\widehat{R}$ is a hypersurface, that is, $\widehat{R} \cong k[[x_0,\dots,x_d]]/(f)$ for some non-zero non-unit f.

We next pass to the ring $A := K \otimes_k \widehat{R}$, which we claim is a direct limit of finite étale extensions of $\widehat{R}$. To see this, write $K = \bigcup_{\alpha\in\Lambda} F_\alpha$, where each F_α is a finite extension of k. For each $\alpha \in \Lambda$, the field extension $k \longrightarrow F_\alpha$ is unramified (since F_α is separable over k), and it follows easily that $\widehat{R} \longrightarrow F_\alpha \otimes_k \widehat{R}$ is unramified as well. Since each $F_\alpha \otimes_k \widehat{R}$ is a finitely generated free $\widehat{R}$-module and $K \otimes_k \widehat{R} = \bigcup_\alpha (F_\alpha \otimes_k \widehat{R})$, the claim follows.

Since $\widehat{R}$ is excellent (being complete) and since A is a direct limit of étale extensions of $\widehat{R}$, a theorem of S. Greco [Gre76, Theorem 5.3] implies that A is excellent. Since K is a gonflement of k, Exercise 11.25 implies that A is a gonflement of $\widehat{R}$, and that K is a coefficient field for A. Therefore A has finite CM type, by Theorem 11.17.

Now $A = T/(f)$, where $T = K \otimes_k k[\![x_0, \dots, x_d]\!] = \bigcup_\alpha F_\alpha[\![x_0, \dots, x_d]\!]$, where, as before, the F_α are finite extensions of k contained in K. Clearly $\widehat{T} = K[\![x_0, \dots, x_d]\!]$, so $\widehat{A} = K[\![x_0, \dots, x_d]\!]/(f)$. By Corollary 11.13, we see that $\widehat{A}$ has finite CM type. Therefore, by **??** $K[\![x_0, \dots, x_d]\!]/(f) = \widehat{A}$ is a simple singularity. □

§5 Exercises

11.21 Exercise. Let $(R, \mathfrak{m}, k) \longrightarrow (S, \mathfrak{n}, \ell)$ be a flat local homomorphism, and let M be a finitely generated R-module. Prove that $S \otimes_R M$ is a MCM S-module if and only if M is MCM and the closed fiber $S/\mathfrak{m}S$ is a CM ring. (Cf. [BH93, (1.2.16) and (A.11)].)

11.22 Exercise. Let $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local homomorphism. Prove that the following two conditions are equivalent:

(i) The induced map $R/\mathfrak{m} \longrightarrow S/\mathfrak{m}S$ is an isomorphism.

(ii) The induced map $R/\mathfrak{m} \longrightarrow S/\mathfrak{n}$ is an isomorphism and $\mathfrak{m}S = \mathfrak{n}$.

11.23 Exercise. Let $\varphi\colon (R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ be a local homomorphism that is essentially of finite type (that is, S is a localization of a finitely generated R-algebra).

`problems here`

(i) Prove that $S/\mathfrak{m}S$ is Artinian.

(ii) Prove that $R \longrightarrow S$ is unramified if and only if

a) $\mathfrak{m}S = \mathfrak{n}$, and

b) $S/\mathfrak{n}$ is a finite, separable field extension of $R/\mathfrak{m}$.

(iii) Let $\mathfrak{q}$ be a prime ideal of S, and put $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. If $R \longrightarrow S$ is unramified, prove that $R_\mathfrak{p} \longrightarrow S_\mathfrak{q}$ is unramified.

11.24 Exercise. Find an example of an étale local homomorphism $R \longrightarrow S$ and a finitely generated S-module M such that the two S-actions on $S \otimes_R M$ (one via the action on S, the other via the action on M) give non-isomorphic S-modules.

11.25 Exercise. Let $(R,\mathfrak{m})$ be a local ring with a coefficient field k, and let K/k be an algebraic field extension. Prove that $K \otimes_k R$ is a gonflement of R and that K is a coefficient field for R. (First do the case where $k \longrightarrow K$ is an elementary gonflement of type (ib) in Definition 11.15.)

11.26 Exercise. Let $(R,\mathfrak{m},k)$ be a one-dimensional local ring satisfying the Drozd-Roĭter conditions (DR1) and (DR2) of Chapter 3, and let $R \longrightarrow$

$(S, \mathfrak{n}, \ell)$ be a gonflement. Prove, without reference to finite CM type, that S satisfies (DR1) and (DR2).

12
Semigroups of modules

In this chapter we study the different ways in which a finitely generated module over a local ring can be decomposed as a direct sum of indecomposable modules. Put another way, we are interested in exactly how badly the Krull-Remak-Schmidt uniqueness property can fail.

Let $(R, \mathfrak{m}, k)$ be a local ring and choose a set $\mathrm{V}(R)$ of representatives for the isomorphism classes $[M]$ of finitely generated R-modules. We make $\mathrm{V}(R)$ into an additive semigroup in the obvious way: $[M] + [N] = [M \oplus N]$. This monoid encodes information about the direct-sum decompositions of finitely generated R-modules. (In what follows, we use the terms "semigroup" and "monoid" interchangeably, though technically a semigroup need not have an identity element.)

In the special case where R is a complete local ring, it follows from the Krull-Remak-Schmidt Theorem 1.8 that $\mathrm{V}(R)$ is a *free monoid*, that is, $V(R) \cong \mathbb{N}_0^{(I)}$, where $\mathbb{N}_0$ is the additive semigroup of non-negative integers and the index set I is the set of atoms of $\mathrm{V}(R)$, that is, the set of representatives $[N]$ for the indecomposable finitely generated R-modules. Furthermore, if M is a finitely generated R-module, then the semigroup $+(M)$ of isomorphism classes $[N]$ of modules $N \in \mathrm{add}(M)$ is free as well.

For a general local ring R, the semigroup $\mathrm{V}(R)$ is naturally a subsemigroup of $\mathrm{V}(\widehat{R})$, and similarly $+(M)$ is a subsemigroup of $+(\widehat{M})$ for an R-module M. This forces various structural restrictions on which semigroups can arise as $\mathrm{V}(R)$ for a local ring R, or as $+(M)$ for a finitely generated R-module M. In short, the semigroup must be a *finitely generated Krull* *monoid.* In §1 we detail these restrictions, and in the rest of the chapter we prove two realization theorems, which show that every finitely generated Krull monoid can be realized in the form $+(M)$ for a suitable local ring R and MCM R-module M. Both these theorems actually realize a semigroup Λ together with a given embedding $\Lambda \subseteq \mathbb{N}_0^{(n)}$. The first construction (Theorem 12.11) gives a one-dimensional domain R and a finitely generated torsion-free module M realizing an *expanded* subsemigroup Λ as $+(M)$, while the second (Theorem 12.16) gives a two-dimensional unique factorization domain R and a finitely generated reflexive module M realizing Λ as $+(M)$, assuming only that Λ is a *full* subsemigroup of $\mathbb{N}_0^{(t)}$. (See Proposition 12.3 for the terminology.)

§1 Krull monoids

In this section, let $(R, \mathfrak{m}, k)$ be a local ring with completion $(\widehat{R}, \widehat{\mathfrak{m}}, k)$. Let $\mathrm{V}(R)$ and $\mathrm{V}(\widehat{R})$ denote the (commutative) semigroups, with respect to direct sum, of finitely generated modules over R and $\widehat{R}$, respectively. We write all our semigroups additively, though we will keep the "multiplicative" notation inspired by direct sums, $x \mid y$, meaning that there exists z such that $x + z = y$. We write 0 for the neutral element [0] corresponding to the zero module.

There is a natural homomorphism of semigroups

$$j\colon \mathrm{V}(R) \longrightarrow \mathrm{V}(\widehat{R})$$

taking $[M]$ to $[\widehat{R} \otimes_R M]$. This homomorphism is injective by Corollary 1.14, so we consider $\mathrm{V}(R)$ as a subsemigroup of $\mathrm{V}(R)$. It follows that $\mathrm{V}(R)$ is *can-*

cellative: if $x+z=y+z$ for $x,y,z \in \mathrm{V}(R)$, then $x=y$. Since in this chapter we will deal only with local rings, all of our semigroups will be tacitly assumed to be cancellative. We also see that $\mathrm{V}(R)$ is *reduced,* i.e. $x+y=0$ implies $x=y=0$.

The homomorphism $j\colon \mathrm{V}(R) \longrightarrow \mathrm{V}(\widehat{R})$ actually satisfies a much stronger condition than injectivity. A *divisor homomorphism* is a semigroup homomorphism $j\colon \Lambda \longrightarrow \Lambda'$ such that $j(x) \mid j(y)$ implies $x \mid y$ for all x and y in Λ. Corollary 1.14 says that $j\colon \mathrm{V}(R) \longrightarrow \mathrm{V}(\widehat{R})$ is a divisor homomorphism. In fact, this holds more generally.

12.1 Proposition ([HW09, Theorem 1.3])**.** *Let $R \longrightarrow S$ be a flat local homomorphism of Noetherian local rings. Then the map $j\colon \mathrm{V}(R) \longrightarrow \mathrm{V}(S)$ taking $[M]$ to $[S \otimes_R M]$ is a divisor homomorphism.*

Proof. Suppose M and N are finitely generated R-modules and that $S \otimes_R M \mid S \otimes_R N$. We want to show that $M \mid N$. By Theorem 1.12 it will be enough to show that $M/\mathfrak{m}^t M \mid N/\mathfrak{m}^t N$ for all $t \geqslant 1$. By passing to the flat local homomorphism $R/\mathfrak{m}^t \longrightarrow S/\mathfrak{m}^t S$, we may assume that R is Artinian and so satisfies Krull-Remak-Schmidt uniqueness. By Proposition 11.3, we know at least that $M \mid N^{(r)}$ for some $r \geqslant 1$. By Corollary 1.8 (or Theorem 1.3 and Corollary 1.5) M is uniquely a direct sum of indecomposable modules. If M itself is indecomposable, KRS immediately implies that $M \mid N$. An easy induction argument using direct-sum cancellation completes the proof (cf. Exercise 12.17). □

12.2 Definition. A *Krull monoid* is a monoid that admits a divisor homomorphism into a free monoid.

Every finitely generated Krull monoid admits a divisor homomorphism into $\mathbb{N}_0^{(t)}$ for some positive integer t. Conversely, it follows easily from Dickson's Lemma (Exercise 3.26) that a monoid admitting a divisor homomorphism to $\mathbb{N}_0^{(t)}$ must be finitely generated.

Finitely generated Krull monoids are called *positive normal affine semigroups* in [BH93]. From [BH93, 6.1.10], we obtain the following characterization of these monoids:

12.3 Proposition. *The following conditions on a semigroup Λ are equivalent:*

(i) *Λ is a finitely generated Krull monoid.*

(ii) *$\Lambda \cong G \cap \mathbb{N}_0^{(t)}$ for some positive integer t and some subgroup G of $\mathbb{Z}^{(t)}$. (That is, Λ is isomorphic to a* full *subsemigroup of $\mathbb{N}_0^{(t)}$.)*

(iii) *$\Lambda \cong W \cap \mathbb{N}_0^{(u)}$ for some positive integer u and some $\mathbb{Q}$-subspace W of $\mathbb{Q}^{(n)}$. (That is, Λ is isomorphic to an* expanded *subsemigroup of $\mathbb{N}_0^{(u)}$.)*

(iv) *There exist positive integers m and n, and an $m \times n$ matrix α over $\mathbb{Z}$, such that $\Lambda \cong \mathbb{N}^{(n)} \cap \ker(\alpha)$.*

Observe that the descriptors "full" and "expanded" refer specifically to a given embedding of a semigroup into a free semigroup, while the definition of a Krull monoid is intrinsic. In addition, note that the group G and the vector space W are not mysterious; they are the group, resp. vector space, generated by Λ.

It's obvious that every expanded subsemigroup of $\mathbb{N}^{(t)}$ is also a full subsemigroup, but the converse can fail. For example, the semigroup

$$\Lambda = \left\{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix} \right] \in \mathbb{N}_0^{(2)} \,\middle|\, x \equiv y \bmod 3 \right\}$$

of is not the restriction to $\mathbb{N}_0^{(2)}$ of the kernel of a matrix, so is not expanded. However, Λ is isomorphic to

$$\Lambda' = \left\{ \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right] \in \mathbb{N}_0^{(3)} \,\middle|\, x + 2y = 3z \right\}.$$

As this example indicates, the number n of (iii) might be larger than the number t of (ii).

Condition (iv) says that a finitely generated Krull monoid can be regarded as the collection of non-negative integer solutions of a homogeneous system of linear equations. For this reason these monoids are sometimes called *Diophantine* monoids.

In order to study uniqueness of direct-sum decompositions in $\mathrm{V}(R)$, it's really enough to examine the local structure around each finitely generated R-module M. Recall that we denote by $\mathrm{add}(M)$ the full subcategory of finitely generated R-modules that are isomorphic to a direct summand of a direct sum of finitely many copies of M, and we let $+(M) \subseteq \mathrm{V}(R)$ be the subset of isomorphism classes $[N]$ of modules $N \in \mathrm{add}(M)$. Then we see that $+(M)$ is also a finitely generated Krull monoid, since the divisor homomorphism $j\colon \mathrm{V}(R) \longrightarrow \mathrm{V}(\widehat{R})$ carries $+(M)$ into $+(\widehat{R} \otimes_R M)$, which is the free submonoid generated by the isomorphism classes of the indecomposable direct summands of $\widehat{R} \otimes_R M$.

The key to understanding the monoids $\mathrm{V}(R)$ and $+(M)$ is knowing which modules over the completion $\widehat{R}$ actually come from R-modules. Recall that if $R \longrightarrow S$ is a ring homomorphism, we say that an S-module N is *extended* (from R) provided there is an R-module M such that $N \cong S \otimes_R M$. In the two remaining sections, we will prove criteria—one in dimension one, and one in dimension two—for identifying which finitely generated modules over the completion $\widehat{R}$ of a local ring R are extended. In both cases, a key ingredient is that modules of finite length are *always* extended. We leave the proof of this fact as an exercise.

12.4 Lemma. *Let R be a local ring with completion $\widehat{R}$, and let L be an $\widehat{R}$-module of finite length. Then L also has finite length as an R-module, and the natural map $L \longrightarrow \widehat{R} \otimes_R L$ is an isomorphism.* □

§2 Realization in dimension one

In dimension one, a beautiful result due to Levy and Odenthal [LO96] tells us exactly which $\widehat{R}$-modules are extended from R. See Corollary 12.7 below. First, we define for any one-dimensional local ring $(R, \mathfrak{m}, k)$ the *Artinian localization* $\mathrm{K}(R)$ by $\mathrm{K}(R) = U^{-1}R$, where U is the complement of the minimal prime ideals (the prime ideals distinct from $\mathfrak{m}$). If R is CM, then $\mathrm{K}(R)$ is just the total quotient ring $\{\text{non-zerodivisors}\}^{-1}R$ as in Chapter 3. If R is not CM, then the natural map $R \longrightarrow \mathrm{K}(R)$ is not injective.

12.5 Proposition. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring, and let N be a finitely generated $\widehat{R}$-module. Then N is extended from R if and only if $\mathrm{K}(\widehat{R}) \otimes_{\widehat{R}} N$ is extended from $\mathrm{K}(R)$.*

Proof. To simplify notation, we set $K = \mathrm{K}(R)$ and $L = \mathrm{K}(\widehat{R})$. (Keep in mind, however, that these are not fields.) If $\mathfrak{q}$ is a minimal prime ideal of $\widehat{R}$, then $\mathfrak{q} \cap R$ is a minimal prime ideal of R, since "going down" holds for flat extensions [BH93, Lemma A.9]. Therefore the inclusion $R \longrightarrow \widehat{R}$ induces a homomorphism $K \longrightarrow L$, and this homomorphism is faithfully flat, since the map $\operatorname{Spec}(\widehat{R}) \longrightarrow \operatorname{Spec}(R)$ is surjective [BH93, Lemma A.10]. The "only if" direction is then clear from $L \otimes_K K \otimes_R M \cong L \otimes_{\widehat{R}} \widehat{R} \otimes_R M$.

For the converse, let X be a finitely generated K-module such that $L \otimes_K X \cong L \otimes_{\widehat{R}} N$. Since K is a localization of R, there is a finitely generated R-module M such that $K \otimes_R M \cong X$. Since $L \otimes_{\widehat{R}} N \cong L \otimes_{\widehat{R}} (\widehat{R} \otimes_R M)$, there is a homomorphism $\varphi\colon N \longrightarrow \widehat{R} \otimes_R M$ inducing an isomorphism from $L \otimes_{\widehat{R}} N$ to $L \otimes_{\widehat{R}} (\widehat{R} \otimes_R M)$. Then the kernel U and cokernel V of φ have finite length and therefore are extended by Lemma 12.4. Now we break the exact sequence

$$0 \longrightarrow U \longrightarrow N \longrightarrow S \otimes_R M \longrightarrow V \longrightarrow 0$$

into two short exact sequences:

$$0 \longrightarrow U \longrightarrow N \longrightarrow W \longrightarrow 0$$

$$0 \longrightarrow W \longrightarrow \widehat{R} \otimes_R M \longrightarrow V \longrightarrow 0.$$

Applying (ii) of Lemma 12.6 below to the second short exact sequence, we see that W is extended. Now we apply (i) of the lemma to the first short exact sequence, to conclude that N is extended. □

12.6 Lemma. *Let $(R, \mathfrak{m})$ be a local ring with completion $\widehat{R}$, and let*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be an exact sequence of finitely generated $\widehat{R}$-modules.

(i) Assume X and Z are extended. If $\operatorname{Ext}^1_{\widehat{R}}(Z,X)$ *has finite length as an R-module (e.g. if Z is locally free on the punctured spectrum of $\widehat{R}$), then Y is extended.*

(ii) Assume Y and Z are extended. If $\operatorname{Hom}_{\widehat{R}}(Y,Z)$ *has finite length as an R-module (e.g. if Z has finite length), then X is extended.*

(iii) Assume X and Y are extended. If $\operatorname{Hom}_{\widehat{R}}(X,Y)$ *has finite length as an R-module (e.g. if X has finite length), then Z is extended.*

Proof. For (i), write $X = \widehat{R} \otimes_R X_0$ and $Z = \widehat{R} \otimes_R Z_0$, where X_0 and Z_0 are finitely generated R-modules. The natural map

$$\widehat{R} \otimes_R \operatorname{Ext}^1_R(Z_0, X_0) \longrightarrow \operatorname{Ext}^1_{\widehat{R}}(Z,X)$$

is an isomorphism since Z_0 is finitely presented, and $\operatorname{Ext}^1_R(Z_0,X_0)$ has finite length by faithful flatness. Therefore the natural map $\operatorname{Ext}^1_R(Z_0,X_0) \longrightarrow \widehat{R} \otimes_R \operatorname{Ext}^1_R(Z_0,X_0)$ is an isomorphism by Lemma 12.4. Combining the two isomorphisms, we see that the given exact sequence, regarded as an element of $\operatorname{Ext}^1_{\widehat{R}}(Z,X)$, comes from a short exact sequence $0 \longrightarrow X_0 \longrightarrow Y_0 \longrightarrow Z_0 \longrightarrow 0$. Clearly, then, $\widehat{R} \otimes_R Y_0 \cong Y$.

To prove (ii), we write $Y = \widehat{R} \otimes_R Y_0$ and $Z = \widehat{R} \otimes_R Z_0$, where Y_0 and Z_0 are finitely generated R-modules. As in the proof of (i) we see that the natural map $\operatorname{Hom}_R(Y_0,Z_0) \longrightarrow \operatorname{Hom}_{\widehat{R}}(Y,Z)$ is an isomorphism. Therefore the given $\widehat{R}$-homomorphism $\beta\colon Y \longrightarrow Z$ comes from a homomorphism $\beta_0\colon Y_0 \longrightarrow Z_0$ in $\operatorname{Hom}_R(Y_0,Z_0)$. Clearly, then, $X \cong \widehat{R} \otimes_R (\ker \beta_0)$. The proof of (iii) is essentially the same: Write $Y = \widehat{R} \otimes_R Y_0$ and $X = \widehat{R} \otimes_R X_0$; show that $\alpha\colon X \longrightarrow Y$ comes from some $\alpha_0 \in \operatorname{Hom}_R(X_0,Y_0)$, and deduce that $Z \cong \widehat{R} \otimes_R (\operatorname{cok} \alpha_0)$. □

12.7 Corollary ([LO96]). *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring whose completion $\widehat{R}$ is reduced, and let N be a finitely generated $\widehat{R}$-module. Then N is extended from R if and only if $\dim_{R_\mathfrak{p}}(N_\mathfrak{p}) = \dim_{R_\mathfrak{q}}(N_\mathfrak{q})$ (vector space dimension) whenever $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of $\widehat{R}$ lying over the same prime ideal of R. In particular, if R is a domain, then N is extended if and only if N has constant rank.*

This gives us a strategy for producing strange direct-sum behavior:

(i) Find a one-dimensional domain R whose completion is reduced but has lots of minimal primes.

(ii) Build indecomposable $\widehat{R}$-modules with highly non-constant ranks.

(iii) Put them together in different ways to get constant-rank modules.

Suppose, to illustrate, that R is a domain whose completion $\widehat{R}$ has two minimal primes $\mathfrak{p}$ and $\mathfrak{q}$. Suppose we can build indecomposable $\widehat{R}$-modules U, V, W and X, with ranks $(\dim_{R_\mathfrak{p}}(-), \dim_{R_\mathfrak{q}}(-)) = (2,0)$, $(0,2)$, $(2,1)$, and $(1,2)$, respectively. Then $U \oplus V$ has constant rank $(2,2)$, so is extended; say, $U \oplus V \cong \widehat{M}$. Similarly, there are R-modules N, F and G such that $V \oplus W \oplus W \cong \widehat{N}$, $W \oplus X \cong \widehat{F}$, and $U \oplus X \oplus X \cong \widehat{G}$. Using the Krull-Remak-Schmidt theorem over $\widehat{R}$, we see easily that no non-zero proper direct summand of any of the modules $\widehat{M}$, $\widehat{N}$, $\widehat{F}$, $\widehat{G}$ has constant rank. It follows from Corollary 12.7 that M, N, F, and G are indecomposable, and of course no two of them are isomorphic since (again by Krull-Remak-Schmidt) their completions are pairwise non-isomorphic. Finally, we see that $M \oplus F \oplus F \cong N \oplus G$, since the two modules have isomorphic completions.

Thus we easily obtain a mild violation of Krull-Remak-Schmidt uniqueness over R.

It's easy to accomplish (i), getting a one-dimensional domain with a lot of splitting but no ramification. In order to facilitate (ii), however, we want to ensure that each analytic branch has infinite Cohen-Macaulay type. The following construction from [Wie01] does the job nicely:

12.8 Construction ([Wie01, (2.3)]). Fix a positive integer s, and let k be any field with $|k| \geqslant s$. Choose distinct elements $t_1, \dots, t_s \in k$. Let Σ be the complement of the union of the maximal ideals $(x - t_i)k[x]$, $i = 1, \dots, s$. We define R by the pullback diagram

(12.8.1)
$$\begin{array}{ccc} R & \longrightarrow & \Sigma^{-1}k[x] \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ k & \longrightarrow & \dfrac{\Sigma^{-1}k[x]}{(x-t_1)^4 \cdots (x-t_s)^4}\,, \end{array}$$

where π is the natural quotient map. Then R is a one-dimensional local domain, (12.8.1) is the conductor square for R (cf. Construction 3.1), and $\widehat{R}$ is reduced with exactly s minimal prime ideals. Indeed, we can rewrite the bottom line R_{art} as $k \hookrightarrow D_1 \times \cdots \times D_s$, where $D_i \cong k[x]/(x^4)$ for each i. The conductor square for the completion is then

$$\begin{array}{ccc} \widehat{R} & \longrightarrow & T_1 \times \cdots \times T_s \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ k & \longrightarrow & D_1 \times \cdots \times D_s\,, \end{array}$$

where each T_i is isomorphic to $k[\![x]\!]$.

We remark that R is the ring of rational functions $f \in k(T)$ such that $f(t_1) = \cdots = f(t_s) \neq \infty$ and the derivatives f', f'' and f''' vanish at each t_i.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of $\widehat{R}$. Recall that the *rank* of a finitely generated $\widehat{R}$-module N is the s-tuple $(r_1, \ldots, r_s)$, where r_i is the dimension of $N_{\mathfrak{p}_i}$ as a vector space over $R_{\mathfrak{p}_i}$.

The next theorem says that even the case $s = 2$ of this example yields the pathology discussed after Corollary 12.7).

12.9 Theorem ([Wie01, (2.4)]). *Fix a positive integer s, and let R be the ring of Construction 12.8. Let $(r_1, \ldots, r_s)$ be any sequence of non-negative integers with not all the r_i equal to zero. Then $\widehat{R}$ has an indecomposable MCM module N with* $\operatorname{rank}(N) = (r_1, \ldots, r_s)$.

Proof. Set $P = T_1^{(r_1)} \times \cdots \times T_s^{(r_s)}$, a projective module over $\widehat{R} \cong T_1 \times \cdots \times T_s$. Lemma 12.10 below, a jazzed-up version of Theorem 2.5, yields an indecomposable $\widehat{R}_{\text{art}}$-module $V \hookrightarrow W$ with $W = D_1^{(r_1)} \times \cdots D_s^{(r_s)}$. Since $P/\mathfrak{c}P \cong W$, Construction 3.1 implies that there exists a torsion-free $\widehat{R}$-module M, unique up to isomorphism, such that $M_{\text{art}} = (V \hookrightarrow W)$. NAK implies that M is indecomposable, and the ranks of M at the minimal primes are precisely $(r_1, \ldots, r_s)$. □

We leave the proof of this lemma as an exercise (Exercise 12.20

12.10 Lemma. *Let k be a field. Fix an integer $s \geqslant 1$, set $D_i = k[x]/(x^4)$ for $i = 1, \ldots, s$, and let $D = D_1 \times \cdots \times D_s$. Let $(r_1, \ldots, r_s)$ be an s-tuple of non-negative integers with at least one positive entry, and assume that $r_1 \geqslant r_i$ for every i. Then the Artinian pair $k \hookrightarrow D$ has an indecomposable module $V \hookrightarrow W$, where $W = D_1^{(r_1)} \times \cdots \times D_s^{(r_s)}$.*

Recalling Condition (iv) of Proposition 12.3, we say that the finitely generated Krull semigroup Λ *can be defined by m equations* provided $\Lambda \cong \mathbb{N}_0^{(n)} \cap \ker(\alpha)$ for some n and some $m \times n$ integer matrix α. Given such an embedding of Λ in $\mathbb{N}_0^{(n)}$, we say a column vector $\lambda \in \Lambda$ is *strictly positive* provided each of its entries is a positive integer. By decreasing n (and removing some columns from α) if necessary, we can harmlessly assume, without changing m, that Λ contains a strictly positive element λ. Specifically, choose an element $\lambda \in \Lambda$ with the largest number of strictly positive coordinates, and throw away all the columns corresponding to zero entries of λ. If any element $\lambda' \in \Lambda$ had a non-zero entry in one of the deleted columns, then $\lambda + \lambda'$ would have more positive entries than λ, a contradiction.

12.11 Theorem ([Wie01, Theorem 2.1])**.** *Fix a non-negative integer m, and let R be the ring R of Construction 12.8 obtained from $s = m+1$. Let Λ be a finitely generated Krull semigroup defined by m equations and containing a strictly positive element λ. Then there exist a maximal Cohen-Macaulay R-module M and a commutative diagram*

$$\begin{array}{ccc}
\Lambda & \hookrightarrow & \mathbb{N}_0^{(n)} \\
\Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \psi} \\
+(M) & \xrightarrow[j]{} & +(\widehat{R} \otimes_R M)
\end{array}$$

in which

(i) j is the natural map taking $[N]$ to $[\widehat{R} \otimes_R N]$,

(ii) φ and ψ are semigroup isomorphisms, and

(iii) $\varphi(\lambda) = [M]$.

Proof. We have $\Lambda = \mathbb{N}_0^{(n)} \cap \ker(\alpha)$, where $\alpha = [a_{ij}]$ is an $m \times n$ matrix over $\mathbb{Z}$. Choose a positive integer h such that $a_{ij} + h \geqslant 0$ for all i, j. For $j = 1, \dots, n$, choose, using Theorem 12.9, a MCM $\widehat{R}$-module L_j such that $\operatorname{rank}(L_j) = (a_{1j} + h, \dots, a_{mj} + h, h)$.

Given any column vector $\beta = [b_1, b_2, \dots, b_n]^{\mathrm{tr}} \in \mathbb{N}_0^{(n)}$, put $N_\beta = L_1^{(b_1)} \oplus \cdots \oplus L_n^{(b_n)}$. The rank of N_β is

$$\left(\sum_{j=1}^{n} (a_{1j} + h) b_j, \dots, \sum_{j=1}^{n} (a_{mj} + h) b_j, \left(\sum_{j=1}^{n} b_j \right) h \right).$$

Since R is a domain, Corollary 12.7 implies that N_β is in the image of $j\colon \mathrm{V}(R) \longrightarrow \mathrm{V}(\widehat{R})$ if and only if $\sum_{j=1}^{n} (a_{ij} + h) b_j = \left(\sum_{j=1}^{n} b_j \right) h$ for each i, that is, if and only if $\beta \in \mathbb{N}_0^{(n)} \cap \ker(\alpha) = \Lambda$. To complete the proof, we let M be the R-module (unique up to isomorphism) such that $\widehat{M} \cong N_\lambda$. □

This corollary makes it very easy to demonstrate spectacular failure of Krull-Remak-Schmidt uniqueness:

12.12 Example. Let

$$\Lambda = \left\{ \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right] \in \mathbb{N}_0^{(3)} \,\middle|\, 72x + y = 73z \right\}.$$

This has three atoms (minimal non-zero elements), namely

$$\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0 \\ 73 \\ 1 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} 73 \\ 0 \\ 72 \end{bmatrix}.$$

Note that $73\alpha = \beta + \gamma$. Taking $s = 2$ in Construction 12.8, we get a local ring R and indecomposable R-modules A, B, C such that $A^{(t)}$ has only the obvious direct-sum decompositions for $t \leqslant 72$, but $A^{(73)} \cong B \oplus C$.

We define the *splitting number* $\operatorname{spl}(R)$ of a one-dimensional local ring R by

$$\operatorname{spl}(R) = \left|\operatorname{Spec}(\widehat{R})\right| - |\operatorname{Spec}(R)|\,.$$

The splitting number of the ring R in Construction 12.8 is $s-1$. Corollary 12.11 says that every finitely generated Krull monoid defined by m equations can be realized as $+(M)$ for some finitely generated module over a one-dimensional local ring (in fact, a domain essentially of finite type over $\mathbb{Q}$) with splitting number m. This is the best possible:

12.13 Proposition. *Let M be a finitely generated module over a one-dimensional local ring R with splitting number m. The embedding $+(M) \hookrightarrow \mathrm{V}(\widehat{R})$ exhibits $+(M)$ as an expanded subsemigroup of the free semigroup $+(\widehat{R}\otimes_R M)$. Moreover, $+(M)$ is defined by m equations.*

Proof. Write $\widehat{R}\otimes_R M = V_1^{(e_1)} \oplus \cdots \oplus V_n^{(e_n)}$, where the V_j are pairwise non-isomorphic indecomposable $\widehat{R}$-modules and the e_i are all positive. We have an embedding $+(M) \hookrightarrow \mathbb{N}_0^{(n)}$ taking $[N]$ to $[b_1,\dots,b_n]^{\mathrm{tr}}$, where $\widehat{R}\otimes_R N \cong V_1^{(b_1)} \oplus \cdots \oplus V_n^{(b_n)}$, and we identify $+(M)$ with its image Λ in $\mathbb{N}_0^{(n)}$. Given a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ with, say, t primes $\mathfrak{q}_1,\dots,\mathfrak{q}_t$ lying over it, there are $t-1$ homogeneous linear equations on the b_j that say that $\widehat{N}$ has constant rank on the fiber over $\mathfrak{p}$ (cf. Corollary 12.7). Letting $\mathfrak{p}$ vary over $\operatorname{Spec}(R)$, we obtain exactly $m = \operatorname{spl}(R)$ equations that must be satisfied by elements of Λ. Conversely, if the b_j satisfy these equations, then $N := V_1^{(b_1)} \oplus \cdots \oplus V_n^{(b_n)}$ has constant rank on each fiber of $\operatorname{Spec}(\widehat{R}) \longrightarrow \operatorname{Spec}(R)$. By Corollary 12.7, N is extended from an R-module, say $N \cong \widehat{R}\otimes_R L$. Clearly $\widehat{R}\otimes_R L \mid \widehat{M^{(u)}}$ if u

is large enough, and it follows from Proposition 12.1 that $L \in +(M)$, whence $[b_1,\dots,b_n]^{\mathrm{tr}} \in \Lambda$. □

In [Kat02] K. Kattchee showed that, for each m, there is a finitely generated Krull monoid Λ that cannot be defined by m equations. Thus no single one-dimensional local ring can realize *every* finitely generated Krull monoid in the form $+(M)$ for a finitely generated module M.

§3 Realization in dimension two

Suppose we have a finitely generated Krull semigroup Λ and a full embedding $\Lambda \subseteq \mathbb{N}_0^{(t)}$., i.e. Λ is the intersection of $\mathbb{N}^{(t)}$ with a subgroup of $\mathbb{Z}^{(t)}$. By Proposition 12.13, we cannot realize this embedding in the form $+(M) \hookrightarrow +(\widehat{R} \otimes_R M)$ for a module M over a one-dimensional local ring R unless Λ is actually an *expanded* subsemigroup of $\mathbb{N}_0^{(t)}$, i.e. the intersection of $\mathbb{N}^{(t)}$ with a subspace of $\mathbb{Q}^{(t)}$. If, however, we go to a two-dimensional ring, then we can realize Λ as $+(M)$, though the ring that does the realizing is less tractable than the one-dimensional rings that realize expanded subsemigroups.

As in the last section, we need a criterion for an $\widehat{R}$-module to be extended from R. For general two-dimensional rings, we know of no such criterion, so we shall restrict to analytically normal domains. (A local domain $(R,\mathfrak{m})$ is *analytically normal* provided its completion $(\widehat{R},\widehat{\mathfrak{m}})$ is also a normal domain.)

We recall two facts from Bourbaki [Bou98, Chapter VII]. Firstly, over a Noetherian normal domain R one can assign to each finitely generated R-module M a *divisor class* $\mathrm{cl}(M) \in \mathrm{Cl}(R)$ in such a way that

1. Taking divisor classes cl(−) is additive on exact sequences, and
2. if J is a fractional ideal of R, then cl(J) is the isomorphism class $[J^{**}]$ of the divisorial (i.e. reflexive) ideal J^{**}, where $-*$ denotes the dual $\operatorname{Hom}_R(-,R)$.

Secondly, each finitely generated torsion-free module M over a Noetherian normal domain R has a "Bourbaki sequence," namely a short exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow J \longrightarrow 0 \tag{12.13.1}$$

wherein F is a free R-module and J is an ideal of R.

The following criterion for a module to be extended is Proposition 3 of [RWW99] (cf. also [Wes88, (1.5)]).

12.14 Proposition. *Let R be a two-dimensional local ring whose $\mathfrak{m}$-adic completion $\widehat{R}$ is a normal domain. Let N be a finitely generated torsion-free $\widehat{R}$-module. Then N is extended from R if and only if* cl(N) *is in the image of the natural homomorphism* $\Phi\colon \operatorname{Cl}(R) \longrightarrow \operatorname{Cl}(\widehat{R})$.

Proof. Suppose $N \cong \widehat{R} \otimes_R M$. Then M is finitely generated and torsion-free, by faithfully flat descent. Choose a Bourbaki sequence (12.13.1) for M; tensoring with $\widehat{R}$ and using the additivity of cl(−) on short exact sequences, we find

$$\operatorname{cl}(N) = \operatorname{cl}(\widehat{R} \otimes_R J) = [(\widehat{R} \otimes_R J)^{**}] = \Phi(\operatorname{cl}(J)).$$

For the converse, choose a Bourbaki sequence

$$0 \longrightarrow G \longrightarrow N \longrightarrow L \longrightarrow 0$$

over $\widehat{R}$, so that G is a free $\widehat{R}$-module and L is an ideal of $\widehat{R}$. Then $\mathrm{cl}(L) = \mathrm{cl}(N)$, and since $\mathrm{cl}(N)$ is in the image of Φ there is a divisorial ideal I of R such that $\widehat{R} \otimes_R I \cong L^{**}$. Set $V = L^{**}/L$. Then V has finite length and hence is extended by Lemma 12.4; it follows from Lemma 12.6(i) and the short exact sequence $0 \longrightarrow L \longrightarrow L^{**} \longrightarrow V \longrightarrow 0$ that L is extended. Moreover, $\widehat{R}_{\mathfrak{p}}$ is a discrete valuation ring for each height-one prime ideal $\mathfrak{p}$, so that $\mathrm{Ext}^1_{\widehat{R}}(I,G)$ has finite length. Now Lemma 12.6(ii) says that N is extended since G and L are. □

As in the last section, we need to guarantee that the complete ring $\widehat{R}$ has a sufficiently rich supply of MCM modules.

12.15 Lemma ([Wie01, Lemma 3.2]). *Let s be any positive integer. There is a complete local normal domain B, containing $\mathbb{C}$, such that* $\dim(B) = 2$ *and* $\mathrm{Cl}(B)$ *contains a copy of* $(\mathbb{R}/\mathbb{Z})^{(s)}$.

Proof. Choose a positive integer d such that $(d-1)(d-2) \geqslant s$, and let V be a smooth projective plane curve of degree d over $\mathbb{C}$. Let A be the homogeneous coordinate ring of V for some embedding $V \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$. Then A is a two-dimensional normal domain, by [Har77, Chap. II, Exercise 8.4(b)]. By [Har77, Appendix B, Sect. 5], $\mathrm{Pic}^0(V) \cong D := (\mathbb{R}/\mathbb{Z})^{2g}$, where $g = \frac{1}{2}(d-1)(d-2)$, the genus of V. Here $\mathrm{Pic}^0(V)$ is the kernel of the degree map $\mathrm{Pic}(V) \longrightarrow \mathbb{Z}$, so $\mathrm{Cl}(V) = \mathrm{Pic}(V) = D \oplus \mathbb{Z}\sigma$, where σ is the class of a divisor of degree 1. There is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(A) \longrightarrow 0,$$

in which $1 \in \mathbb{Z}$ maps to the divisor class $\tau := [H \cdot V]$, where H is a line in $\mathbb{P}^2_{\mathbb{C}}$. (Cf. [Har77, Chap. II, Exercise 6.3].) Thus $\mathrm{Cl}(A) \cong \mathrm{Cl}(V)/\mathbb{Z}\tau$. Since

τ has degree d, we see that $\tau - d\sigma \in D$. Choose an element $\delta \in D$ with $d\delta = \tau - d\sigma$. Recalling that $\mathrm{Cl}(V) = \mathrm{Pic}(V) = D \oplus \mathbb{Z}\sigma$, we define a surjection $f\colon \mathrm{Cl}(V) \longrightarrow D \oplus \mathbb{Z}/(d)$ by sending $x \in D$ to $(x, 0)$ and σ to $(-\delta, 1 + (d))$. Then $\ker(f) = \mathbb{Z}\tau$, so $\mathrm{Cl}(A) \cong D \oplus \mathbb{Z}/d\mathbb{Z}$.

Let $\mathfrak{P}$ be the irrelevant maximal ideal of A. By [Har77, Chap. II, Exercise 6.3(d)], $\mathrm{Cl}(A_{\mathfrak{P}}) \cong \mathrm{Cl}(A)$. The $\mathfrak{P}$-adic completion B of A is an integrally closed domain, by [ZS75, Chap. VIII, Sect. 13]. Moreover $\mathrm{Cl}(A_{\mathfrak{P}}) \longrightarrow \mathrm{Cl}(B)$ is injective by faithfully flat descent, so $\mathrm{Cl}(B)$ contains a copy of $D = (\mathbb{R}/\mathbb{Z})^{(d-1)(d-2)}$, which, in turn, contains a copy of $(\mathbb{R}/\mathbb{Z})^{(s)}$. □

We now have everything we need to prove our realization theorem for full subsemigroups of $\mathbb{N}_0^{(t)}$.

12.16 Theorem. *Let t be a positive integer, and let Λ be a full subsemigroup of $\mathbb{N}_0^{(t)}$. Assume that Λ contains an element λ with strictly positive entries. Then there exist a two-dimensional local unique factorization domain R, a finitely generated reflexive (= MCM) R-module M, and a commutative diagram of semigroups*

$$\begin{array}{ccc} \Lambda & \hookrightarrow & \mathbb{N}_0^{(t)} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ +(M) & \xrightarrow[j]{} & +(\widehat{R} \otimes_R M) \end{array}$$

in which

(i) *j is the natural map taking $[N]$ to $[\widehat{R} \otimes_R N]$,*

(ii) *φ and ψ are isomorphisms, and*

(iii) *$\varphi(\lambda) = [M]$.*

Proof. Let G be the subgroup of $\mathbb{Z}^{(t)}$ generated by Λ, and write $\mathbb{Z}^{(t)}/G = C_1 \oplus \cdots \oplus C_s$, where each C_i is a cyclic group. Then $\mathbb{Z}^{(t)}/G$ can be embedded in $(\mathbb{R}/\mathbb{Z})^{(s)}$.

Let B be the complete local domain provided by Lemma 12.15. Since $\mathbb{Z}^{(t)}/G$ embeds in $\mathrm{Cl}(B)$, there is a group homomorphism $\varpi\colon \mathbb{Z}^{(t)} \longrightarrow \mathrm{Cl}(B)$ with $\ker(\varpi) = G$. Let $\{e_1, \dots, e_t\}$ be the standard basis of $\mathbb{Z}^{(t)}$. For each $i \leqslant t$, write $\varpi(e_i) = [L_i]$, where L_i is a divisorial ideal of B representing the divisor class of $\varpi(e_i)$.

Next we use Heitmann's amazing theorem [Hei93], which implies that B is the completion of some local unique factorization domain R. For each element $m = (m_1, \dots, m_t) \in \mathbb{N}_0^{(t)}$, we let $\psi(m)$ be the isomorphism class of the B-module $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$. The divisor class of this module is $m_1[L_1] + \cdots + m_t[L_t] = \varpi(m_1, \dots, m_t)$. By Proposition 12.14, the module $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$ is the completion of an R-module if and only if its divisor class is trivial, that is, if and only if $m \in G \cap \mathbb{N}_0^{(t)}$. But $m \in G \cap \mathbb{N}_0^{(t)} = \Lambda$, since Λ is a full subsemigroup of $\mathbb{N}_0^{(t)}$. Therefore $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$ is the completion of an R-module if and only if $m \in \Lambda$. If $m \in \Lambda$, we let $\varphi(m)$ be the isomorphism class of a module whose completion is isomorphic to $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$. In particular, choosing a module M such that $[M] = \varphi(\lambda)$, we get the desired commutative diagram. □

§4 Exercises

12.17 Exercise. Complete the proof of Proposition 12.1.

12.18 Exercise. Prove the equivalence of conditions (i)–(iv) of Proposition 12.3.

12.19 Exercise. Prove Lemma 12.4.

12.20 Exercise ([Wie01, Lemma 2.2])**.** Prove the existence of the indecomposable $\widehat{R}_{\mathrm{art}}$-module $V \hookrightarrow W$ in Lemma 12.10, as follows. Let $C = k^{(r_1)}$, viewed as column vectors. Define the "truncated diagonal" $\partial\colon C \longrightarrow W = D_1^{(r_1)} \times \cdots \times D_s^{(r_s)}$ by setting the i^{th} component of $\partial([c_1,\dots,c_{r_1}]^{\text{tr}})$ equal to $[c_1,\dots,c_{r_i}]^{\text{tr}}$. (Here we use $r_1 \geqslant r_i$ for all i.) Let V be the k-subspace of W consisting of all elements

$$\left\{\partial(u) + X\partial(v) + X^3\partial(Hv)\right\},$$

as u and v run over C, where $X = (x,0,\dots,0)$ and H is the nilpotent Jordan block with 1 on the superdiagonal and 0 elsewhere.

(i) Prove that W is generated as a D-module by all elements of the form $\partial(u)$, $u \in C$, so that in particular $DV = W$. (Hint: it suffices to consider elements $w = (w_1,\dots,w_s)$ with only one non-zero entry w_i, and such that $w_i \in D_i^{(r_i)}$ has only one non-zero entry, which is equal to 1.)

(ii) Prove that $V \hookrightarrow W$ is indecomposable along the same lines as the arguments in Chapter 3. (Hint: use the fact that $\{1,x,x^2,x^3\}$ is linearly independent over k.)

13
Countable Cohen–Macaulay type

We shift directions now, and focus on a hitherto unmentioned representation type: countable type.

13.1 Definition. A Cohen-Macaulay local ring $(R, \mathfrak{m})$ is said to have *countable Cohen–Macaulay type* if it admits only countably many isomorphism classes of maximal Cohen–Macaulay modules.

The property of countable type has received much lass attention than finite type, and correspondingly less is known about it. There is however an analogue of Auslander's Theorem (Theorem 13.4), as well as a complete classification of complete hypersurface singularities over $\mathbb{C}$ with countable CM type, due to Buchweitz–Greuel–Schreyer [BGS87]. This has recently been revisited by Burban–Drozd [BD08, BD10]; we present here their approach, which echoes nicely the material in Chapter 3. They use a construction similar to the conductor square to prove that the $A_\infty = k[[x,y,z]]/(xy)$ and $D_\infty = k[[x,y,z]]/(x^2y - z^2)$ singularities have countable type. Apart from these results, there are a few examples due to Schreyer (see Section §4), but much remains to be done.

§1 Structure

The main structural result on CM local rings of countable CM type was conjectured by Schreyer in 1987 [Sch87, Section 7]. He predicted that an analytic local ring R over the complex numbers having countable type has at most a one-dimensional singular locus, that is, $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} > 1$. In this section we prove Schreyer's conjecture more generally for all CM local rings satisfying a souped-up version of prime avoidance due to Burch and Sharp–Vamos. In practice, this means either the ring is complete or the residue field is uncountable. Some assumption of uncountability is necessary to avoid the degenerate case of a countable ring, which has only countably many isomorphism classes of finitely generated modules!

13.2 Lemma ([Bur72, Lemma 3]; see also [SV85]). *] Let A be a Noetherian ring satisfying either of these conditions.*

1. *A is complete local, or*
2. *there is an uncountable set of elements $\{u_\lambda\}_{\lambda\in\Lambda}$ of A such that $u_\lambda - u_\mu$ is a unit of A for every $\lambda \neq \mu$.*

Let $\{\mathfrak{p}_i\}_{i=1}^\infty$ be a countable set of prime ideals of R, and I an ideal with $I \subseteq \bigcup_{i=1}^\infty \mathfrak{p}_i$. Then $I \subseteq \mathfrak{p}_i$ for some i.

Notice that the second condition is satisfied if, for example, $(A,\mathfrak{m})$ is local with $A/\mathfrak{m}$ uncountable. In fact, when 2 is verified, the ideals $\mathfrak{p}_i$ need not even be prime.

We postpone the proof to the end of this section, and move on to a nice application of MCM approximations.

13.3 Lemma. *Let R be a CM local ring of countable CM type, and $\{M_i\}_{i=1}^\infty$ a complete list of all the indecomposable MCM R-modules. Consider the set* *of ideals*

$$\Omega = \left\{ \operatorname{Ann}_R \left(\operatorname{Ext}^i_R(M_j, M_k) \right) \mid i,j,k \geqslant 1 \right\}.$$

Assume that R has a canonical module. Then the following modules have annihilator in Ω:

(i) $\operatorname{Ext}^i_R(M,N)$ *for $i \geqslant 1$, where M is MCM and N is finitely generated;*

(ii) $\operatorname{Ext}^i_R(M,N)$ *for $i \geqslant \dim R + 1$, where M and N are finitely generated.*

Proof. (i) Let N be an arbitrary R-module, and consider a minimal MCM approximation of N

$$0 \longrightarrow Y_N \longrightarrow X_N \longrightarrow N \longrightarrow 0,$$

so $\operatorname{injdim}_R Y_N < \infty$ and X_N is MCM. Applying $\operatorname{Hom}_R(M,-)$ and using the fact that $\operatorname{Ext}^i_R(M,Y_N) = 0$ for all $i > 0$ by Theorem 9.3, we get $\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^i_R(M,X_N)$ for $i \geqslant 1$.

(ii) If $i \geqslant \dim R + 1$, then $\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^{i-\dim R}_R(\operatorname{syz}^R_{\dim R} M, N)$ and the result follows from (i).

□

The set Ω in the statement of Lemma 13.3 is of course at most countable. The subset of non-maximal prime ideals in Ω

$$\Omega' = \Omega \cap (\operatorname{Spec} R \setminus \{\mathfrak{m}\})$$

is then countable as well.

13.4 Theorem. *Let $(R,\mathfrak{m})$ be an excellent CM local ring of countable CM type. Assume that R satisfies countable prime avoidance. Then the singular locus of R has dimension at most one.*

Proof. Set $d = \dim R$, and assume that the singular locus of R has dimension greater than one. Since R is excellent, $\operatorname{Sing} R$ is a closed subset of $\operatorname{Spec} R$, defined by an ideal J such that $\dim R/J \geqslant 2$. Consider the set Ω' of prime Ext-annihilators. Each $\mathfrak{p} \in \Omega'$ contains J. Applying countable prime avoidance to R/J, we find an element $r \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p} \in \Omega'} \mathfrak{p}$. Choose a minimal prime $\mathfrak{q}$ of $J + (r)$; since $\dim R/J \geqslant 2$ we have $\mathfrak{q} \neq \mathfrak{m}$, and over course $\mathfrak{q} \notin \Omega'$. Enlarging $\mathfrak{q}$ if necessary, we may assume $\dim R/\mathfrak{q} = 1$.

Set $M = \operatorname{syz}^R_{d-1} R/\mathfrak{q}$ and $N = \operatorname{syz}^R_d R/\mathfrak{q}$, and consider $\mathfrak{a} = \operatorname{Ann}_R \left(\operatorname{Ext}^1_R(M,N)\right)$. Clearly $\mathfrak{q}$ is contained in $\mathfrak{a}$, as $\operatorname{Ext}^1_R(M,N) \cong \operatorname{Ext}^d_R(R/\mathfrak{q},N)$. Since $\mathfrak{q}$ contains J, the localization $R_\mathfrak{q}$ is not regular, so the residue field $R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}$ has infinite projective dimension and $\operatorname{Ext}^1_R(M,N)_\mathfrak{q} \neq 0$. Therefore $\mathfrak{a} \subseteq q$, and we see that $q \in \Omega'$, a contradiction. □

13.5 Remarks. With a suitable assumption of prime avoidance for sets of cardinality $\aleph_n$, the same proof shows that if R has at most $\aleph_{m-1}$ CM type, then the singular locus of R has dimension at most m.

Theorem 13.4 implies that for an excellent CM local ring of countable CM type, satisfying countable prime avoidance, there are at most finitely many non-maximal prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ such that $R_{\mathfrak{p}_i}$ is not a regular local ring. Each of these localizations has dimension $d-1$. Naturally, one would like to know more about these $R_{\mathfrak{p}_i}$. Peeking ahead at the examples later on in this chapter, we find that in each of them, every $R_{\mathfrak{p}_i}$ has finite CM type! Whether or not this holds in general is still an open question. The next result gives partial information: at least each $R_{\mathfrak{p}_i}$ has countable type.

13.6 Theorem. *Let $(R,\mathfrak{m})$ be a CM local ring with a canonical module. If R has countable CM type, then $R_\mathfrak{p}$ has countable CM type for every $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$ and suppose that $\{M^\alpha\}$ is an uncountable family of finitely generated R-modules such that $\{M^\alpha_\mathfrak{p}\}$ are non-isomorphic MCM $R_\mathfrak{p}$-modules. For each α there is a MCM approximation of M^α

$$(13.6.1) \qquad \chi^\alpha : \qquad 0 \longrightarrow Y^\alpha \longrightarrow X^\alpha \longrightarrow M^\alpha \longrightarrow 0$$

with X^α MCM and $\operatorname{injdim}_R Y^\alpha < \infty$.

Since there are uncountably many MCM modules X^α, there must be uncountably many of some fixed multiplicity e. The fact that there are only countably many non-isomorphic MCM modules of multiplicity e then implies that there are uncountably many short exact sequences

$$(13.6.2) \qquad \chi^\beta : \qquad 0 \longrightarrow Y^\beta \longrightarrow X \longrightarrow M^\beta \longrightarrow 0$$

where X is a fixed MCM module.

Localize at $\mathfrak{p}$; since $M^\beta_\mathfrak{p}$ is MCM over $R_\mathfrak{p}$ and $Y^\beta_\mathfrak{p}$ has finite injective dimension, $\operatorname{Ext}^1_R(M^\beta, Y^\beta)_\mathfrak{p} \cong \operatorname{Ext}^1_{R_\mathfrak{p}}(M^\beta_\mathfrak{p}, Y^\beta_\mathfrak{p}) = 0$ by Prop. 9.4. In particular, the extension χ^β splits when localized at $\mathfrak{p}$. This implies that $M^\beta_\mathfrak{p} \mid X_\mathfrak{p}$ for uncountably many β, which cannot happen by Corollary 1.16 □

The results above, together with the examples in Section §4, suggest a plausible question:

13.7 Question. *Let R be a complete local Cohen-Macaulay ring of dimension at least one, and assume that R has an isolated singularity. If R has countable CM type, must it have finite CM type?*

Here is the proof we omitted earlier.

Proof of Lemma 13.2. Suppose first that $(A,\mathfrak{m})$ is a complete local ring. Suppose that $I \not\subseteq \mathfrak{p}_i$ for each i, but that $I \subseteq \bigcup_i \mathfrak{p}_i$. Obviously $I \subseteq \mathfrak{m}$. Since A is Noetherian, all chains in $\operatorname{Spec} A$ are finite, so we may replace each chain by its maximal element to assume that there are no containments among the $\mathfrak{p}_i$.

Construct a Cauchy sequence in A as follows. Choose $x_1 \in I \setminus \mathfrak{p}_1$, and suppose inductively that we have chosen $x_1,\dots,x_r$ to satisfy

(a) $x_j \notin \mathfrak{p}_i$, and

(b) $x_i - x_j \in I^i \cap \mathfrak{p}_i$

for all $i \leqslant j \leqslant r$. If $x_r \notin \mathfrak{p}_{r+1}$, put $x_{r+1} = x_r$. Otherwise, take $y_{r+1} \in (I^r \cap p_1 \cap \cdots \cap \mathfrak{p}_r) \setminus \mathfrak{p}_{r+1}$ (this is possible since there are no containments among the $\mathfrak{p}_i$) and set $x_{r+1} = x_r + y_{r+1}$. In either case, we have

(a') $x_{r+1} \notin \mathfrak{p}_i$ for $i \leqslant r+1$, and

(b') $x_{r+1} - x_r \in \mathfrak{m}^{r+1} \cap \mathfrak{p}_1 \cap \cdots \cap p_r$, so that if $i < r+1$ then $x_i - x_{r+1} \in \mathfrak{m}^i \cap \mathfrak{p}_i$.

By condition (b), $\{x_1, x_2, \dots\}$ is a Cauchy sequence, so converges to $x \in A$. Since $x_i - x_s \in \mathfrak{p}_i$ for all $i \leqslant s$, and $x_i \notin \mathfrak{p}_i$, we obtain $x_i - x \in \mathfrak{p}_i$ for all i, since $\mathfrak{p}_i$ is closed in the I-adic topology. Therefore $x \notin \mathfrak{p}_i$ for all i, as needed.

Now let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an uncountable family of elements of A as in (2) of Lemma 13.2. Take generators $a_1,\dots,a_k$ for the ideal I, and for each $\lambda \in \Lambda$, set

$$z_\lambda = a_1 + u_\lambda a_2 + u_\lambda^2 a_3 + \cdots + u_\lambda^{k-1} a_k \qquad \in I.$$

Since $\{\mathfrak{p}_i\}$ is countable, and $I \subseteq \bigcup \mathfrak{p}_i$, there exist some $j \geqslant 1$ and uncountably many $\lambda \in \Lambda$ such that $z_\lambda \in \mathfrak{p}_j$. In particular there are distinct elements $\lambda_1, \dots, \lambda_k$ such that $z_{\lambda_i} \in \mathfrak{p}_j$ for $i = 1, \dots, k$.

The $k \times k$ Vandermonde matrix

$$P = \left(u_{\lambda_i}^{j-1}\right)_{i,j}$$

has determinant $\prod_{i \neq j}(u_{\lambda_i} - u_{\lambda_j})$, so is invertible. But

$$P \begin{pmatrix} a_1 & \cdots & a_k \end{pmatrix}^T = \begin{pmatrix} z_{\lambda_1} & \cdots & z_{\lambda_k} \end{pmatrix}^T,$$

so

$$\begin{pmatrix} a_1 & \cdots & a_k \end{pmatrix}^T = P^{-1} \begin{pmatrix} z_{\lambda_1} & \cdots & z_{\lambda_k} \end{pmatrix}^T,$$

which implies $I = (a_1, \dots, a_k) \subseteq \mathfrak{p}_j$. □

§2 Burban–Drozd triples

Our goal in this section and the next is to classify the complete equicharacteristic hypersurfaces of countable CM type in characteristic other than 2. They are the “natural limits” (A_∞) and (D_∞) of the (A_n) and (D_n) singularities. This classification is originally due to Buchweitz, Greuel, and Schreyer [BGS87]; they show that in dimension one, a hypersurface of countable CM type must satisfy an analogue of the simplicity property (see Chapter 8), and then that only the (A_∞) and (D_∞) singularities satisfy this criterion. They then construct all indecomposable MCM modules on these curve singularities. Since there are only countably many in dimension one, Knörrer’s periodicity result Corollary **??** gives the result in all dimensions.

Instead of following this path, we describe a special case of some recent results of Burban and Drozd [BD10], which proceed to the same conclusion by way of the surface singularities rather than the curves. In addition to its satisfying parallels with our treatment of hypersurfaces of finite CM type in Chapters 5 and 8, this method is also pleasantly akin to the “conductor square” construction in Chapter 3. It also allows us to write down, in a manner analogous to §3 of Chapter 8, a complete list of the indecomposable matrix factorizations over the two-dimensional (A_∞) and (D_∞) hypersurfaces.

13.8 Notation. Throughout this section we consider a reduced, CM, complete local ring $(R,\mathfrak{m})$ of dimension 2 which is *not* normal. (The assumption that R is reduced is no imposition, thanks to Theorem 13.4.) We will impose further assumptions later on, cf. 13.13. Since normality is equivalent to both (R_1) and (S_2), this means that R is not regular in codimension one. Let S be the integral closure of R in its total quotient ring. Since R is complete and reduced, S is a finitely generated R-module (Theorem 3.6), which is a direct product of complete local normal domains, each of which is CM.

Let $\mathfrak{c} = (R :_R S) = \operatorname{Hom}_R(S,R)$ be the conductor ideal as in Chapter 3, the largest common ideal of R and S. Set $\overline{R} = R/\mathfrak{c}$ and $\overline{S} = S/\mathfrak{c}$.

13.9 Lemma. *With notation as above we have the following properties.*

(i) The conductor ideal $\mathfrak{c}$ is a MCM module over both R and S.

(ii) The quotients $\overline{R}$ and $\overline{S}$ are (possibly non-reduced) one-dimensional CM rings with $\overline{R} \subseteq \overline{S}$.

(iii) The diagram

$$\begin{array}{ccc} R & \hookrightarrow & S \\ \downarrow & & \downarrow \\ \overline{R} & \hookrightarrow & \overline{S} \end{array}$$

is a pullback diagram of ring homomorphisms.

Proof. Since $\mathfrak{c} = \operatorname{Hom}_R(S,R)$, Exercise 4.25 implies that $\mathfrak{c}$ has depth 2 when considered as an R-module. Since $R \subseteq S$ is a finite extension, $\mathfrak{c}$ is also MCM over S.

The conductor $\mathfrak{c}$ defines the non-normal locus of $\operatorname{Spec} R$. Since for a height-one prime $\mathfrak{p}$ of R, $R_{\mathfrak{p}}$ is normal if and only if it is regular, and R is not regular in codimension one, we see that $\mathfrak{c}$ has height at most one in R. On the other hand, R is reduced, so its localizations at minimal primes are fields, and it follows that $\mathfrak{c}$ has height exactly one in R, hence also in S since $R \subseteq S$ is integral. Therefore $\overline{R}$ and $\overline{S}$ are one-dimensional. Since $\mathfrak{c}$ has depth 2, the quotients $\overline{R}$ and $\overline{S}$ have depth 1 by the Depth Lemma.

The third statement is easy to check. □

Recall from Chapter 5 that the *reflexive product* of two R-modules M and N

$$N \cdot M = (N \otimes_R M)^{\vee\vee}$$

is a MCM R-module, where $-^{\vee} = \operatorname{Hom}_R(-,\omega_R)$. In the special case $N = S$, the reflexive product $S \cdot M$ inherits an S-module structure and so is a MCM S-module. Recall also that for any (not necessarily reflexive) S-module X, there is a short exact sequence (reference Exercises **??** and **??**???)

$$0 \longrightarrow \operatorname{tor}(X) \longrightarrow X \longrightarrow X^{\vee\vee} \longrightarrow L \longrightarrow 0, \tag{13.9.1}$$

where tor(X) denotes the torsion submodule of N and L is an S-module of finite length.

Let M be a MCM R-module. Set $\overline{M} = M/\mathfrak{c}M$ and $\overline{S \cdot M} = (S \cdot M)/\mathfrak{c}(S \cdot M)$, modules over $\overline{R}$ and $\overline{S}$, respectively. By Exercise 13.32, applied to $\overline{R}$ and to $R_\mathfrak{p}$, respectively, we have $\overline{M}^{\vee\vee} \cong \overline{M}/\operatorname{tor}(\overline{M})$ and $(S \cdot M)_\mathfrak{p} \cong (S_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p})/\operatorname{tor}(S_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p})$.

Finally, let A and B be the total quotient rings of $\overline{R}$ and $\overline{S}$, respectively. We are thus faced with a commutative diagram of ring homomorphisms

(13.9.2)
$$\begin{array}{ccc} R & \hookrightarrow & S \\ \downarrow & & \downarrow \\ \overline{R} & \hookrightarrow & \overline{S} \\ \downarrow & & \downarrow \\ A & \hookrightarrow & B \end{array}$$

in which the top square is a pullback. Furthermore, the bottom row is an Artinian pair in the sense of Chapter 2, and a MCM R-module yields a module over the Artinian pair, as we now show.

13.10 Lemma. *Keep the notation established so far, and let M be a MCM R-module.*

(i) We have $B = A \otimes_R S$, that is, if U denotes the set of non-zerodivisors of $\overline{R}$, then $B = U^{-1}\overline{S}$. In particular B is a finitely generated A-module.

(ii) The natural homomorphism of B-modules

$$\theta_M : B \otimes_A (A \otimes_{\overline{R}} \overline{M}) \xrightarrow{\cong} B \otimes_S (S \otimes_R M) \longrightarrow B \otimes_S (S \cdot M)$$

is surjective.

(iii) The natural homomorphism of A-modules

$$A \otimes_{\overline{R}} \overline{M} \longrightarrow B \otimes_A (A \otimes_{\overline{R}} \overline{M}) \xrightarrow{\theta_M} B \otimes_S (S \cdot M)$$

is injective.

Proof. For the first statement, set $C = U^{-1}\overline{S}$. Any $b \in B$ can be written $b = \frac{c}{v}$ where $c \in C$ and v is a non-zerodivisor of S. Since C is Artinian, there is an integer n such that $Cv^n = Cv^{n+1}$, say $v^n = dv^{n+1}$. Then $v^n(1-dv) = 0$ so that $dv = 1$ in B. This shows that $b = dc \in C$.

The exact sequence (13.9.1), with $N = S \otimes_R M$, shows that the cokernel of the natural homomorphism $S \otimes_R M \longrightarrow S \cdot M$ has finite length. Hence that cokernel vanishes upon tensoring with B and θ_M is surjective.

To prove (iii), set $N = (S \otimes_R M)/\operatorname{tor}(S \otimes_R M)$. Then the natural map $M \longrightarrow N$ sending $x \in M$ to $\overline{1 \otimes x}$ is injective. It follows that the restriction $\mathfrak{c}M \longrightarrow \mathfrak{c}N$ is also injective. In fact, it is also surjective: for any $a \in \mathfrak{c}$, $s \in S$, and $x \in M$, we have

$$s(\overline{s \otimes x}) = \overline{as \otimes x} = \overline{1 \otimes asx}$$

in the image of $\mathfrak{c}M$, since $as \in \mathfrak{c}$.

Since N is torsion-free, we have an exact sequence

$$0 \longrightarrow N \longrightarrow N^{\vee\vee} \longrightarrow L \longrightarrow 0$$

where the duals $-^{\vee}$ are computed over S and L is an S-module of finite length. It follows that the cokernel of the restriction $\mathfrak{c}N \hookrightarrow \mathfrak{c}N^{\vee\vee}$ also has finite length. Consider the composition $g \colon M \longrightarrow N \longrightarrow N^{\vee\vee}$ and the induced diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{c}M & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \mathfrak{c}N^{\vee\vee} & \longrightarrow & N^{\vee\vee} & \longrightarrow & \overline{N^{\vee\vee}} & \longrightarrow & 0 \end{array}$$

with exact rows, where f is the restriction of g to $\mathfrak{c}M$. Since g is injective and the cokernel of f has finite length, the Snake Lemma implies that $\ker h$ has finite length as well. Thus $A \otimes_{\overline{R}} h \colon A \otimes_{\overline{R}} \overline{M} \longrightarrow A \otimes_{\overline{R}} \overline{N^{\vee\vee}}$ is injective. Finally we observe that $A \otimes_{\overline{R}} h$ is the natural homomorphism in (iii), since $(S \cdot M)_{\mathfrak{p}} \cong (S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})/\operatorname{tor}(S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$ for all primes $\mathfrak{p}$ minimal over $\mathfrak{c}$. □

13.11 Definition. Keeping all the notation introduced in this section so far, consider the following *category of Burban–Drozd triples* $\mathrm{BD}(R)$. The objects of $\mathrm{BD}(R)$ are triples (N, V, θ), where

- N is a MCM S-module,
- V is a finitely generated A-module, and
- $\theta \colon B \otimes_A V \longrightarrow B \otimes_S N$ is a surjective homomorphism of B-modules such that the composition

  $$V \longrightarrow B \otimes_A V \xrightarrow{\theta} B \otimes_S N$$

  is injective.

The induced map of A-modules $V \longrightarrow B \otimes_S N$ is called a *gluing map.*

A morphism between to triples (N, V, θ) and (N', V', θ') is a pair (f, F) such that $f \colon V \longrightarrow V'$ is a homomorphism of A-modules and $F \colon N \longrightarrow N'$ is a homomorphism of S-modules combining to make the diagram

$$\begin{array}{ccc} B \otimes_A V & \xrightarrow{\theta_V} & B \otimes_S N \\ {\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle 1\otimes F} \\ B \otimes_A V' & \xrightarrow[\theta_{V'}]{} & B \otimes_S N' \end{array}$$

commutative.

The category of Burban–Drozd triples is finer than the category of modules over the Artinian pair $A \hookrightarrow B$, since the homomorphism F above must be defined over S rather than just over B. In particular, an isomorphism of pairs $(f,F)\colon (V,N) \longrightarrow (V',N')$ includes as part of its data an isomorphism of S-modules $F\colon N \longrightarrow N'$, of which there are fewer than isomorphisms of B-modules $B \otimes_S N \longrightarrow B \otimes_S N'$.

13.12 Theorem (Burban–Drozd)**.** *Let R be a reduced CM complete local ring of dimension 2 which is not an isolated singularity. Let $\mathbb{F}$ be the functor from MCM R-modules to* $\mathrm{BD}(R)$ *defined on objects by*

$$\mathbb{F}(M) = (S \cdot M, A \otimes_R M, \theta_M)\,.$$

Then $\mathbb{F}$ is an equivalence of categories.

Lemma 13.10 shows that the functor $\mathbb{F}$ is well-defined. The proof that it is an equivalence is somewhat technical. For the applications we have in mind, a more restricted version suffices.

13.13 Assumptions. We continue to assume that R is a two-dimensional, reduced, CM, complete local ring and that $S \neq R$ is its normalization. Let $\mathfrak{c}$ be the conductor and $\overline{R} = R/\mathfrak{c}$, $\overline{S} = S/\mathfrak{c}$. We impose two additional assumptions.

(i) Assume that S is a *regular ring*. Since R is Henselian, this is equivalent to S being a direct product of regular local rings. Every MCM S-module is thus projective.

(ii) Assume that $\overline{R} = R/\mathfrak{c}$ is also a *regular local ring*, that is, a DVR. It follows that $\overline{S}$ is a free $\overline{R}$-module, and even more, that a finitely generated $\overline{S}$ module is MCM if and only if it is free over $\overline{R}$. Also, the total quotient ring A of $\overline{R}$ is a field.

Under these simplifying assumptions, we may define a category of *modified Burban-Drozd triples* $\mathrm{BD}'(R)$ as follows: it consists of triples $(N, X, \widetilde{\theta})$, where

- N is a finitely generated projective S-module,
- $X \cong \overline{R}^{(n)}$ is a free $\overline{R}$ module of finite rank, and
- $\widetilde{\theta}\colon X \longrightarrow \overline{N} = N \otimes_S \overline{S}$ is a *split* injection of $\overline{R}$-modules such that the induced homomorphism

$$B \otimes_{\overline{R}} X \longrightarrow B \otimes_S N$$

  is a split surjection.

Morphisms of modified triples are defined as in the un-modified case.

Assume the restrictions of 13.13, and let M be a MCM R-module. Since S is a regular ring of dimension 2, the reflexive S-module $S \cdot M$ is in fact projective. Furthermore, the natural homomorphism of R-modules $M \longrightarrow S \cdot M$ is obtained by applying $\mathrm{Hom}_R(\mathfrak{c}, -)$ to the short exact sequence $0 \longrightarrow$

$\mathfrak{c} \longrightarrow R \longrightarrow \overline{R} \longrightarrow 0$. In particular, we have the short exact sequence

$$0 \longrightarrow M \longrightarrow S \cdot M \longrightarrow \operatorname{Ext}^1_R(\overline{R}, M) \longrightarrow 0. \tag{13.13.1}$$

The cokernel $\operatorname{Ext}^1_R(\overline{R}, M)$ is annihilated by $\mathfrak{c}$, so is naturally an $\overline{R}$-module. Moreover, it has depth 1 by the Depth Lemma, so is free over $\overline{R}$ since $\overline{R}$ is a DVR. The induced sequence of $\overline{R}$-modules

$$\overline{M} \xrightarrow{\widetilde{\theta}_M} \overline{S \cdot M} \longrightarrow \operatorname{Ext}^1_R(\overline{R}, M) \longrightarrow 0$$

is thus split exact on the right. The projective $\overline{S}$-module $\overline{S \cdot M}$ is torsion-free as an $\overline{R}$-module, so the torsion submodule of $\overline{M}$ must be in the kernel of the map to $\overline{S \cdot M}$. On the other hand, the kernel is torsion, since it vanishes upon passing to the localization A by Lemma 13.10(ii). Thus we have a short exact sequence

$$0 \longrightarrow (\overline{M})^{\vee\vee} \longrightarrow S \cdot M \longrightarrow \operatorname{Ext}^1_R(\overline{R}, M) \longrightarrow 0$$

which is even split exact over $\overline{R}$. That the induced map $\theta_M : B \otimes_{\overline{R}} (\overline{M})^{\vee\vee} \longrightarrow B \otimes_S (S \cdot M)$ is split surjective follows from Lemma 13.10(iii) and the fact that $S \cdot M$ is S-projective, so $B \otimes_S (S \cdot M)$ is B-projective. These considerations show that the functor $\mathscr{F}$ from MCM R-modules to $\mathrm{BD}'(R)$, given by

$$\mathscr{F}(M) = (S \cdot M, (\overline{M})^{\vee\vee}, \widetilde{\theta}_M)$$

is well-defined.

We now define a quasi-inverse functor $\mathscr{G}$ from $\mathrm{BD}'(R)$ to MCM R-modules, still under the assumptions 13.13. Let $(N, X, \widetilde{\theta})$ be an object of $\mathrm{BD}'(R)$. Let

$\pi\colon N \longrightarrow \overline{N} = \overline{S} \otimes_S N$ be the natural projection, and define M by the pull-back diagram

(13.13.2)
$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\widetilde{\theta}]{} & \overline{N} \end{array}$$

of R-modules. Since $\widetilde{\theta}$ is a split injection of torsion-free modules over the DVR $\overline{R}$, $\operatorname{cok}\widetilde{\theta}$ is an R-module of depth 1. This cokernel is isomorphic to the cokernel of $M \longrightarrow N$, and it follows that $\operatorname{depth}_R M = 2$, so that M is a MCM R-module. Define

$$\mathscr{G}(N,X,\widetilde{\theta}) = M\,.$$

13.14 Theorem. *The functors $\mathscr{F}$ and $\mathscr{G}$ are inverses on objects, namely, for a MCM R-module M and a modified Burban-Drozd triple $(N,X,\widetilde{\theta})$, we have*

$$\mathscr{G}\mathscr{F}(M) \cong M$$

and

$$\mathscr{F}\mathscr{G}(N,X,\widetilde{\theta}) \cong (N,X,\widetilde{\theta})\,.$$

Proof. For the first assertion, it suffices to show that

$$\begin{array}{ccc} M & \longrightarrow & S\cdot M \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ (\overline{M})^{\vee\vee} & \xrightarrow[\widetilde{\theta}]{} & \overline{S\cdot M} \end{array}$$

is a pullback diagram. We have already seen that the homomorphisms $M \longrightarrow S\cdot M$ and $(\overline{M})^{\vee\vee} \longrightarrow \overline{S\cdot M}$ have the same cokernel, namely $\operatorname{Ext}^1_R(\overline{R},M)$. It follows from the Snake Lemma that

$$\ker(M \longrightarrow (\overline{M})^{\vee\vee}) \cong \ker(S\cdot M \longrightarrow \overline{S\cdot M})\,.$$

From this it follows easily that M is the pullback of the diagram above.

For the converse, let $(N, X, \widetilde{\theta})$ be an object of $\mathrm{BD}'(R)$ and let M be defined by the pullback (13.13.2). Then $\operatorname{cok}(M \longrightarrow N)$ is isomorphic to $\operatorname{cok}(\widetilde{\theta}\colon X \longrightarrow \overline{N})$, and is in particular an $\overline{R}$-module. The Snake Lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{c}M & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{c}N & \longrightarrow & N & \longrightarrow & \overline{N} & \longrightarrow & 0 \end{array}$$

gives an exact sequence

$$0 \longrightarrow \ker(\overline{M} \longrightarrow \overline{N}) \longrightarrow \operatorname{cok}(\mathfrak{c}M \longrightarrow \mathfrak{c}N) \longrightarrow \operatorname{cok}(M \longrightarrow N).$$

This shows that $\operatorname{cok}(\mathfrak{c}M \longrightarrow \mathfrak{c}N)$ is annihilated by $\mathfrak{c}^2$, so in particular is a torsion R-module.

Now the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{c}M & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{c}N & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

implies that $\overline{M} \longrightarrow X$ is surjective with torsion kernel. Therefore $X \cong \overline{M}/\operatorname{tor}(\overline{M}) \cong (\overline{M})^{\vee\vee}$.

The inclusion $M \hookrightarrow N$ induces a homomorphism $S \cdot M \longrightarrow N$ of reflexive S-modules, so in particular of reflexive R-modules. It suffices by Exercise 13.39 to prove that this is an isomorphism in codimension 1 in R, that is, $(S \cdot M)_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}}$ is an isomorphism for all height-one primes $\mathfrak{p} \in \operatorname{Spec} R$.

Over $R_{\mathfrak{p}}$, the localization of (13.13.2) is still a pullback diagram.

$$\begin{array}{ccccc} M_{\mathfrak{p}} & \longrightarrow & (S \cdot M)_{\mathfrak{p}} & \longrightarrow & N_{\mathfrak{p}} \\ \downarrow & & & & \downarrow \\ (\overline{M})_{\mathfrak{p}} & & \longrightarrow & & \overline{N}_{\mathfrak{p}} \end{array}$$

Since $(S \cdot M)_{\mathfrak{p}} \cong (S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})/\operatorname{tor}(S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$ and the bottom line is a module over the Artinian pair $A \hookrightarrow B$, we can use the machinery of Chapter 3 to see that $(S \cdot M)_{\mathfrak{p}} \cong N_{\mathfrak{p}}$. □

§3 Hypersurfaces of countable CM type

We apply Theorem 13.14 to obtain the complete classification of indecomposable MCM modules over the two-dimensional (A_∞) and (D_∞) complete hypersurface singularities, and derive the Buchweitz–Greuel–Schreyer characterization of hypersurfaces of countable type:

13.15 Theorem (Buchweitz–Greuel–Schreyer)**.** *Let $R = k[\![x, y, z_2, \dots, z_n]\!]/(f)$ be a complete hypersurface singularity with k an algebraically closed uncountable field of characteristic different from 2. Then R has countably infinite CM type if and only if $R \cong k[\![x, y, z_2 \dots, z_n]\!]/(g + z_2^2 + \cdots + z_n^2)$, where $g \in k[x, y]$ is one of the following:*

(A_∞) $g = x^2$, or

(D_∞) $g = x^2 y$.

By Corollary **??**, the proof of Theorem 13.15 reduces to considering hypersurfaces of any fixed dimension. We'll use the results of the previous section to show that the two-dimensional (A_∞) and (D_∞) hypersurfaces have countable type. For the converse, we will use a variant of the notion of simplicity, cf. §2.

Throughout, we assume that k is a field. If the characteristic of k is different from 2, let i be an element with $i^2 = -1$.

13.16 Proposition. *Let $R = k[[x,y,z]]/(x^2+z^2)$ be an (A_∞) hypersurface singularity with k a field of characteristic other than 2. Let M be an indecomposable non-free MCM R-module. Then M is isomorphic to* $\operatorname{cok}(zI-\varphi, zI+\varphi)$*, where φ is one of the following matrices over $k[[x,y]]$.*

- $\begin{pmatrix} ix \end{pmatrix}$ *or* $\begin{pmatrix} -ix \end{pmatrix}$
- $\begin{pmatrix} -ix & y^j \\ & ix \end{pmatrix}$ *for some $j \geqslant 1$*

In particular R has countable CM type.

Observe that the indecomposable matrix factorizations for the (A_∞) singularity are the "limits" of the matrix factorizations for (A_n) (cf. 8.19) as $n \longrightarrow \infty$, since high powers are very small in an adic topology.

Proof. For simplicity in the proof we replace x by ix to assume that

$$R = k[[x,y,z]]/(z^2 - x^2).$$

The integral closure S of R is then

$$S = R/(z-x) \times R/(z+x)$$

with the normalization homomorphism $\nu\colon R \longrightarrow S = S_1 \times S_2$ given by the diagonal embedding $\nu(r) = (\overline{r}, \overline{r})$. In particular, S is a regular ring.

Put another way, S is the R-submodule of the total quotient ring generated by the orthogonal idempotents

$$e_1 = \frac{z+x}{2z} \in S_1 \qquad \text{and} \qquad e_2 = \frac{z-x}{2z} \in S_2\,,$$

which are the identity elements of S_1 and S_2 respectively. In these terms, $\nu(r) = r(e_1 + e_2)$ for $r \in R$.

The conductor of S into R is the ideal $\mathfrak{c} = (x,z)R = (x,z)S$, so that

$$\overline{R} = k[\![x,y,z]\!]/(x,z) \cong k[\![y]\!]$$

is a DVR, and $\overline{S} \cong \overline{R} \times \overline{R}$ is a direct product of two copies of $\overline{R}$. The inclusion $\overline{\nu}\colon \overline{R} \longrightarrow \overline{S}$ is again diagonal, $\overline{\nu}(\overline{r}) = (\overline{r},\overline{r})$. Finally, the quotient field A of $\overline{R}$ is $k((y))$, which embeds diagonally into $B = k((y)) \times k((y))$. Thus all the assumptions of 13.13 are verified, and we may apply Theorem 13.14.

Let $(N, X, \widetilde{\theta})$ be an object of $\mathrm{BD}'(R)$, so that $N \cong S_1^{(p)} \oplus S_2^{(q)}$ for some $p, q \geqslant 0$, while $X \cong \overline{R}^{(n)}$ for some n and $\widetilde{\theta}\colon X \longrightarrow \overline{N}$ is a split injection. The gluing morphism $\theta\colon B \otimes_{\overline{R}} X \longrightarrow B \otimes_S N$ is thus a linear transformation of A-vector spaces $B^{(n)} \longrightarrow B^{(p)} \oplus B^{(q)}$. More precisely, $\widetilde{\theta}$ defines a pair of matrices

$$(\theta_1, \theta_2) \in M_{p\times n}(A) \times M_{q\times n}(A)$$

representing an embedding

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} : A^{(n)} \longrightarrow B^{(p)} \oplus B^{(q)}$$

such that θ is injective (has full column rank) and both θ_1 and θ_2 are surjective (full row rank). Thus in particular we have $\max(p,q) \leqslant n \leqslant p+q$.

Two pairs of matrices (θ_1, θ_2) and (θ'_1, θ'_2) define isomorphic Burban-Drozd triples if and only if there exist isomorphisms

$$f : A^{(n)} \longrightarrow A^{(n)}$$
$$F_1 : S_1^{(p)} \longrightarrow S_1^{(p)}$$
$$F_2 : S_2^{(q)} \longrightarrow S_2^{(q)}$$

such that as homomorphisms $B^{(n)} \longrightarrow B^{(p)}$ and $B^{(n)} \longrightarrow B^{(q)}$ we have

$$\theta'_1 = F_1^{-1} \theta_1 f$$
$$\theta'_2 = F_2^{-1} \theta_2 f\,.$$

See Exercise 13.37 for a guided proof of the next Lemma.

13.17 Lemma. *The indecomposable objects of* $\mathrm{BD}(R)$ *are*

(i) $\left(S_1, \overline{R}, ((1), \emptyset)\right)$ *and* $\left(S_2, \overline{R}, (\emptyset, (1))\right)$

(ii) $\left(S_1 \times S_2, \overline{R}, ((1), (1))\right)$

(iii) $\left(S_1 \times S_2, \overline{R}, \left((1), (y^j)\right)\right)$ *and* $\left(S_1 \times S_2, \overline{R}, \left((y^j), (1)\right)\right)$ *for some* $j \geqslant 1$.

Now we derive the matrix factorizations corresponding to the listed Burban–Drozd triples. The pullback diagram corresponding to the triple $\left(S_1, \overline{R}, ((1), \emptyset)\right)$

$$\begin{array}{ccc} M & \longrightarrow & S_1 \\ \downarrow & & \downarrow \\ \overline{R} & \xrightarrow[\binom{1}{\emptyset}]{} & \overline{S}_1 \end{array}$$

clearly gives $M \cong S_1 = \operatorname{cok}(z - x, z + x)$, the first component of the normalization. Similarly, the triple $\left(S_2, \overline{R}, (\emptyset, (1))\right)$ yields $M \cong S_2 = \operatorname{cok}(z + x, z - x)$.

The diagonal map $((1),(1))\colon \overline{R} \longrightarrow \overline{S}_1 \times \overline{S}_2$ obviously defines the free module R. By symmetry, it suffices now to consider the Burban–Drozd triple $(S_1 \times S_2, \overline{R}, ((1),(y^j)))$. The pullback diagram

$$\begin{array}{ccc} M & \longrightarrow & S_1 \times S_2 \\ \downarrow & & \downarrow \\ \overline{R} & \xrightarrow[\binom{1}{y^j}]{} & \overline{S}_1 \times \overline{S}_2 \end{array}$$

defines M as the module of ordered triples of polynomials

$$(f(y), g_1(x,y,x), g_2(x,y,-x)) \in \overline{R} \times S_1 \times S_2$$

such that $f - g_1 \in \mathfrak{c}S_1$ and $y^j f - g_2 \in \mathfrak{c}S_2$. This is equal to the R-submodule of S generated by $\mathfrak{c} = (x,z) = (z+x, z-x)$ and $e_1 + y^j e_2$, where again $e_1 = (z+x)/2z$ and $e_2 = (z-x)/2z$ are idempotent. Multiplying by the nonzerodivisor $(2z)^j = ((z-x)+(z+x))^j$ to knock the generators down into R, we find

$$\begin{aligned}(x,z,e_1+y^je_2)S &\cong (2z)^j\left(z+x, z-x, \frac{z+x}{2z} + y^j\frac{z-x}{2z}\right)\\ &= \left((z+x)^{j+1}, (z-x)^{j+1}, (2z)^j\left(\frac{z+x}{2z}\right)^j + y^j(2z)^j\left(\frac{z-x}{2z}\right)^j\right)\\ &= \left((z+x)^{j+1}, (z-x)^{j+1}, (z+x)^j + (z-x)^j y^j\right)\\ &= \left((z-x)^j, (z+x)^j + (z-x)^j y^j\right).\end{aligned}$$

The matrix factorization

$$\left(\begin{pmatrix} z+x & y^j \\ & z-x \end{pmatrix}, \begin{pmatrix} z-x & -y^j \\ & z+x \end{pmatrix}\right)$$

provides a minimal free resolution of this ideal and finishes the proof. □

As an aside, we note that the restriction on the characteristic of k could be removed by working instead with the hypersurface defined by xz instead of x^2+z^2. In characteristic not two, of course they are isomorphic, and the former can be shown to have countable type in all characteristics.

13.18 Proposition. *Let $R = k[\![x,y,z]\!]/(x^2y+z^2)$ be a (D_∞) hypersurface singularity, where k is a field of arbitrary characteristic. Let M be an indecomposable non-free MCM R-module. Then M is isomorphic to* $\operatorname{cok}(zI-\varphi, zI+\varphi)$ *for φ one of the following matrices over $k[\![x,y]\!]$.*

- $\begin{pmatrix} & -y \\ x^2 & \end{pmatrix}$
- $\begin{pmatrix} & -xy \\ x & \end{pmatrix}$
- $\begin{pmatrix} & & -xy & \\ & & -y^{j+1} & xy \\ x & & & \\ y^j & -x & & \end{pmatrix}$ *for some* $j \geqslant 1$
- $\begin{pmatrix} & & -xy & \\ & & -y^{j+1} & x \\ x & & & \\ y^{j+1} & -xy & & \end{pmatrix}$ *for some* $j \geqslant 1$

In particular R has countable CM type.

Observe once more that the indecomposable matrix factorizations for the (D_∞) singularity are limits as $n \longrightarrow \infty$ of the matrix factorizations for (D_n), cf. 8.20.

Proof. In this case, the integral closure of R is obtained by adjoining the element $t = \frac{z}{x}$ of the quotient field, so $S = R\left[\frac{z}{x}\right]$. The maximal ideal of R is then $(x,y,z)R = (x,t^2,tx)R$ and that of S is $(x,t)S$. In particular, S is a regular local ring. The conductor is now $\mathfrak{c} = (x,z)R = (x,tx)S = xS$, so that $\overline{R} = R/(x,z) \cong k[[t^2]]$ and $S = S/(x) \cong k[[t]]$ are both DVRs, with $\overline{v}\colon \overline{R} \longrightarrow \overline{S}$ the obvious inclusion. The Artinian pair $A = k((t^2)) \longrightarrow B = k((t)$ is thus a field extension of degree 2.

Let $(N,X,\widetilde{\theta})$ be an object of $\mathrm{BD}'(R)$. The normalization S being regular local, $N \cong S^{(n)}$ is a free S-module, while $X \cong \overline{R}^{(m)}$ is a free $\overline{R}$-module. The gluing map $\theta\colon B\otimes_A V \cong B^{(m)} \longrightarrow B^{(n)} \cong B\otimes_S N$ is thus simply an $n\times m$ matrix over B with full row rank. The condition that the composition $A^{(m)} \longrightarrow B^{(n)}$ be injective amounts to writing $\theta = \theta_0 + t\theta_1$ and requiring $\begin{pmatrix}\theta_0\\ \theta_1\end{pmatrix}\colon A^{(m)} \longrightarrow A^{(2n)} \cong B^{(n)}$ to have full column rank as a matrix over A. In particular we have $n \leqslant m \leqslant 2n$.

Two $n\times m$ matrices θ,θ' over B define isomorphic Burban–Drozd triples if and only if there exist isomorphisms

$$f\colon A^{(m)} \longrightarrow A^{(m)} \qquad \text{and} \qquad F\colon S^{(n)} \longrightarrow S^{(n)}$$

such that, when considered as matrices over B, we have

$$\theta' = F^{-1}\theta f\,.$$

In other words, we are allowed to perform row operations over $\overline{S} = k[[t]]$ and column operations over $A = k((t^2))$.

13.19 Lemma. *The indecomposable objects of* $\mathrm{BD}(R)$ *are*

(i) $\left(S,\overline{R},(1)\right)$

(ii) $\left(S, \overline{R}, (t)\right)$

(iii) $\left(S, \overline{R}^{(2)}, (1\ t)\right)$

(iv) $\left(S^{(2)}, \overline{R}^{(2)}, \left(\begin{smallmatrix} 1 & t \\ t^d & \end{smallmatrix}\right)\right)$ *for some* $d \geqslant 1$.

We leave the proof of Lemma 13.19 as Exercise 13.38.

The MCM R-module corresponding to $\left(S, \overline{R}, (1)\right)$ is given by the pullback

$$\begin{array}{ccc} M & \longrightarrow & S \\ \downarrow & & \downarrow \\ \overline{R} & \longrightarrow & \overline{S} \end{array}$$

where the bottom line is the given inclusion of $A = k((t^2))$ into $B = k((t))$, so is clearly the free module R. In $(S, \overline{R}, (t))$, the natural inclusion is replaced by multiplication by t. The pullback M is the R-submodule of S generated by $\mathfrak{c} = (x, z)$ and $t = \frac{z}{x}$. Multiplying through by the non-zerodivisor x, we find

$$\begin{aligned} M &\cong (x^2, xz, z)R \\ &= (x^2, z)R \\ &\cong \operatorname{cok}\left(\begin{pmatrix} z & y \\ -x^2 & z \end{pmatrix}, \begin{pmatrix} z & -y \\ x^2 & z \end{pmatrix} \right). \end{aligned}$$

The Burban–Drozd triple $\left(S, \overline{R}, (1\ t)\right)$ is defined by the isomorphism $\theta\colon A \xrightarrow{(1\ t)} B$, so corresponds to the normalization S, which has matrix factorization

$$\left(\begin{pmatrix} z & xy \\ -x & z \end{pmatrix}, \begin{pmatrix} z & -xy \\ x & z \end{pmatrix} \right).$$

Finally, let M be the R-module defined by the pullback

$$\begin{array}{ccc} M & \longrightarrow & S^2 \\ \big\downarrow & & \big\downarrow \\ \overline{R}^{(2)} & \xrightarrow[\left(\begin{smallmatrix} 1 & t \\ t^d & 0 \end{smallmatrix}\right)]{} & \overline{S}^{(2)} \end{array}.$$

Then M is the R-submodule of $S^{(2)}$ generated by $\mathfrak{c}S^{(2)}$ and the elements

$$\begin{pmatrix} 1 \\ t^d \end{pmatrix}, \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

Substitute $t = \frac{z}{x}$ to see that the generators are therefore

$$\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ z^d/x^d \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} z/x \\ 0 \end{pmatrix}.$$

Notice that the second generator is a multiple of the last. Multiplication by x on the first component and x^d on the second is injective on S^2, so M_1 is isomorphic to the module generated by

$$\begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^{d+1} \end{pmatrix}, \begin{pmatrix} 0 \\ x^d z \end{pmatrix}, \begin{pmatrix} x \\ z^d \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} z \\ 0 \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} x^2 \\ 0 \end{pmatrix} = x \begin{pmatrix} x \\ z^d \end{pmatrix} - \begin{pmatrix} 0 \\ xz^d \end{pmatrix},$$

so we may replace the first generator by $\left(\begin{smallmatrix} 0 \\ xz^d \end{smallmatrix}\right)$, getting

$$M = \left\langle \begin{pmatrix} 0 \\ xz^d \end{pmatrix}, \begin{pmatrix} 0 \\ x^{d+1} \end{pmatrix}, \begin{pmatrix} 0 \\ x^d z \end{pmatrix}, \begin{pmatrix} x \\ z^d \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix} \right\rangle.$$

At this point we distinguish two cases. If $d = 2m$ is even, then using the relation $xy^2 = -z^2$ in R,

$$xz^d = xz^{2m} = xx^{2m}y^m = x^{d+1}y^m$$

up to sign, so the first generator is a multiple of the second. If $d = 2m+1$ is odd, then

$$xz^d = xz^{2m+1} = xx^{2m}y^m z = x^{d+1}y^m z$$

again up to sign, so that again the first generator is a multiple of the second. In either case, M is generated by

$$\left\langle \begin{pmatrix} x \\ z^d \end{pmatrix}, \begin{pmatrix} 0 \\ x^d z \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x^{d+1} \end{pmatrix}, \right\rangle .$$

Now, it's easy to check that in the case where $d = 2j+1$, $j \geqslant 0$, is odd,

$$M \cong \operatorname{cok}\left(\begin{pmatrix} z & & -xy & \\ & z & -y^{j+1} & x \\ x & & z & \\ y^{j+1} & -xy & & z \end{pmatrix}, \begin{pmatrix} z & & xy & \\ & z & y^{j+1} & -x \\ -x & & z & \\ -y^{j+1} & xy & & z \end{pmatrix} \right)$$

and in case $d = 2j$, $j \geqslant 0$, is even,

$$M \cong \operatorname{cok}\left(\begin{pmatrix} z & & -xy & \\ & z & -y^{j+1} & xy \\ x & & z & \\ y^{j} & -xy & & z \end{pmatrix}, \begin{pmatrix} z & & xy & \\ & z & y^{j+1} & -xy \\ -x & & z & \\ -y^{j} & xy & & z \end{pmatrix} \right)$$

(after a permutation of the generators). □

Now that we have seen that the (A_∞) and (D_∞) hypersurface singularities have countable CM type (in all dimensions, by Corollary **??**), we modify the definition of a *simple singularity* (see Section **??**) to show that these are the only ones.

13.20 Definition. Let $(S,\mathfrak{n})$ be a regular local ring and $R = S/(f)$ a hypersurface singularity, with f a non-zero non-unit. We say R is a *countably simple singularity* (relative to the given presentation) if there are at most countably many ideals $I \subseteq S$ such that $f \in I^2$.

The proof of the next proposition is exactly similar to that of Theorem **??**.

13.21 Proposition. *If a hypersurface ring $R = S/(f)$ as above has countable CM type, then R is a countably simple singularity.*

To prove the converse, we need a weakening of (some Lemma in Chapter 8).

13.22 Lemma. *Let k be an algebraically closed uncountable field of characteristic different from 2, and let $f \in k[[z_0,z_1]]$ be a non-reduced power series such that the quotient $R = k[[z_0,z_1]]/(f)$ is countably simple. Then for all $x, y \in \mathfrak{m}$ we have*

(i) $f \notin (x,y)^4$*, and*

(ii) $f \notin (x^3, x^2y^2, xy^4, y^6) = (x,y^2)^3$.

Proof. If either (i) or (ii) fails, then we have seen already in (something from Chapter 8) that f is not countably simple: in the first case take

$$I_\lambda = (\lambda_0 x + \lambda_1 y) + (x,y)^2, \qquad [\lambda_0 : \lambda_1] \in \mathbb{P}^1_k;$$

and in the second

$$I_\lambda = (x + \lambda y^2, y^3), \qquad \lambda \in k\,.$$

Then $f \in I_\lambda^2$ for all λ in either case. □

13.23 Proposition. *Let k be algebraically closed field of characteristic not equal to 2, and $f \in k[\![u,v]\!]$ a non-reduced power series satisfying (i) and (ii) of Lemma 13.22. Then there is a coordinate system $x_0, y_0 \in (u,v)$ such that either*

$$f = x_0^2 \qquad \text{or} \qquad f = x_0^2 y_0\,.$$

In particular the hypersurface ring $R = k[\![u,v]\!]/(f)$ is isomorphic to one of the (A_∞) or (D_∞) hypersurface singularities.

Proof. As $k[\![u,v]\!]$ is a UFD, we write $f = \alpha f_1^{e_1} \cdots f_r^{e_r}$ with α a unit, each f_i irreducible, and $e_i \geqslant 1$ for $i = 1,\dots,r$. Since f is non-reduced, we have $e_i \geqslant 2$ for at least one i, say e_1. By (ii), we must have $e_1 < 3$ as well, so $e_1 = 2$. We see from (i) that f has multiplicity at most 3, which forces $r \leqslant 2$ and each f_i to have non-zero linear term. Set $x_0 = \sqrt{\alpha} f_1$, so that $f = x_0^2 f_2^{e_2}$ with $e_2 \in \{0,1\}$. Now if $e_2 = 0$, then we have $f = x_0^2$, while if $e_2 = 1$ we take $y_0 = f_2$ so that $f = x_0^2 y_0$. □

Proof of Theorem 13.15. Putting together the pieces, we see from Prop. 13.21 that countable CM type implies countable simplicity, which implies either the (A_∞) or (D_∞) singularity by Lemma 13.22 and Prop. 13.23. Since the (A_∞) and (D_∞) singularities have countable type by Propositions 13.16 and 13.18, the circle closes. □

The equations defining the (A_∞) and (D_∞) hypersurface singularities are natural limiting cases of the (A_n) and (D_n) equations as $n \longrightarrow \infty$. Even more, we saw that the indecomposable matrix factorizations over (A_∞) and (D_∞) are limits of those over (A_n) and (D_n).

13.24 Question. *Are all CM local rings of countable CM type "natural limits" of a "series of singularities" of finite CM type? For those that are, are the indecomposable MCM modules "limits" of MCM modules over singularities in the series?*

To address the question, of course, the first order of business must be to give meaning to the phrases in quotation marks. This is problematic, as Arnold remarked [Arn81]: "Although the series undoubtedly exist, it is not at all clear what a series of singularities is."

§4 Other examples

Besides the hypersurface examples of the last section, very few nontrivial examples of countable CM type are known. In this section we present a few, taken from Schreyer's survey article [Sch87].

In dimension one, we have the following example, which will return triumphantly in Chapter 15.

13.25 Example. Consider the one-dimensional (D_∞) hypersurface singularity $R = k[\![x,y]\!]/(x^2y)$, where k is a field of arbitrary characteristic. Set $E = \operatorname{End}_R(\mathfrak{m})$, where $\mathfrak{m} = (x,y)$ is the maximal ideal. Then we claim that

$$\begin{aligned} E &\cong k[\![x,y,z]\!]/(yz, x^2 - xz, xz - z^2) \\ &\cong k[\![a,b,c]\!]/(ab, ac, c^2). \end{aligned}$$

In particular E is local, so has countable CM type by Lemma 3.9.

That the two alleged presentations of E are isomorphic is a simple matter of a linear change of variables:

$$a = z\,, \qquad b = y\,, \qquad c = x - z\,.$$

To show that in fact E is isomorphic to $A = k[\![x,y,z]\!]/(yz, x^2 - xz, xz - z^2)$, note that the element $x + y$ of R is a non-zerodivisor, and that the fraction $z := \frac{x^2}{x+y}$ is easily checked to be in $\operatorname{End}_R(\mathfrak{m})$ but not in R. Now $E = \operatorname{Hom}_R(\mathfrak{m}, R)$ since $\mathfrak{m}$ does not have a free direct summand, and it follows by duality over the Gorenstein ring R that $E/R \cong \operatorname{Ext}^1_R(R/\mathfrak{m}, R) \cong k$. Therefore $E = R[z]$. Since

$$z^2 = \frac{x^2(x+y)^2}{(x+y)^2} = x^2 \in \mathfrak{m}\,,$$

E is local. One verifies the relations $yz = 0$ and $x^2 = xz = z^2$ in E. Thus we have a surjective homomorphism of R-algebras $A \longrightarrow E$. Since R is a subring of E, and the inclusion $R \hookrightarrow E$ factors through A, we see that R is also a subring of A, and that the surjection $A \longrightarrow E$ fixes R.

The induced homomorphism $A/R \longrightarrow E/R$ is still surjective, and in fact is bijective since A/R is simple as well. It follows from the Five Lemma that $A \longrightarrow E$ is an isomorphism.

By Lemma 3.9, the indecomposable MCM E-modules are precisely the non-free indecomposable MCM R-modules. By Theorem **??** and Proposition 13.18, these are the cokernels of the following matrices over $R = k[\![x,y]\!]/(x^2y)$:

$$(y); \quad (x^2); \quad (x); \quad (xy)$$

$$\begin{pmatrix} x & \\ y^j & -x \end{pmatrix}; \quad \begin{pmatrix} xy & \\ y^{j+1} & -xy \end{pmatrix}; \quad \begin{pmatrix} x & \\ y^{j+1} & -xy \end{pmatrix}; \quad \begin{pmatrix} xy & \\ y^{j+1} & -x \end{pmatrix}$$

for $j \geqslant 1$.

For two-dimensional examples, we note that the proof of Herzog's Theorem 5.2 applies equally well to give the following.

13.26 Proposition. *Quotients of the two-dimensional* (A_∞) *and* (D_∞) *hypersurface singularities by a linearly acting finite group of invertible order have at most countable CM type.* □

13.27 Example. Let R be the two-dimensional (A_∞) hypersurface $R = k[[x,y,z]]/(xy)$, where k is an algebraically closed field of characteristic not 2, and let the cyclic group $\mathbb{Z}/r\mathbb{Z}$ act on R, the generator sending (x,y,z) to $(x,\zeta_r y,\zeta_r z)$, where ζ_r is a primitive r^{th} root of unity. The invariant subring is generated by $x, y^r, y^{r-1}z,\dots,z^r$, and is thus isomorphic to the quotient of $k[[t_0,t_1,\dots,t_r,x]]$ by the 2×2 minors of

$$\begin{pmatrix} t_0 & \cdots & t_{r-1} & 0 \\ t_1 & \cdots & t_r & x \end{pmatrix}.$$

13.28 Example. Let R be the two-dimensional (D_∞) hypersurface $k[[x,y,z]]/(x^2y-z^2)$, where k is an arbitrary field. Let $r = 2m+1$ be an odd positive integer, and let $\mathbb{Z}/r\mathbb{Z}$ act on R by the action sending $(x,y,z)\mapsto(\zeta_r^2 x,\zeta_r^{-1}y,\zeta^{m+2}z)$.

The ring of invariants is complicated to describe in general. If $m=1$, it is generated by x^3, xy^2, y^3, z and hence is isomorphic to

$$k[[a,b,c,z]]/I_2\begin{pmatrix} a & z^2 & b \\ z^2 & b & c \end{pmatrix}.$$

If $m=2$, there are 7 generating invariants

$$x^5, x^3y, x^3z, xy^2, xyz, y^5, y^4z\,,$$

and 15 relations among them. When $m = 4$, the greatest common divisor of $m+2$ and $2m+1$ is no longer 1, and things get really weird.

13.29 Remark. As Schreyer points out [Sch87], the phenomenon observed in Question 13.24 repeats here. The one-dimensional example E is obtained as a limit of the endomorphism rings of the maximal ideals of D_n:

$$\operatorname{End}_{D_n}(\mathfrak{m}) \cong k[[x,y,z]]/I_n\,,$$

where I_n is the ideal of 2×2 minors of $\begin{pmatrix} y & x-z & 0 \\ x-z & y^n & z \end{pmatrix}$.

Similarly, for example 13.27 we may take the quotient of $k[[t_0,t_1,\dots,t_{r+1}]]$ by the 2×2 minors of

$$\begin{pmatrix} t_0 & \cdots & t_{r-1} & t_r^n \\ t_1 & \cdots & t_r & t_{r+1} \end{pmatrix},$$

and for example 13.28 with $m = 1$, we take the quotient of $k[[a,b,c,d]]$ by the maximal minors of

$$\begin{pmatrix} d^2+a^n & c & b \\ b & d^2 & a \end{pmatrix}.$$

As assured by Theorem 6.19, both of these are invariant rings of a finite group acting on power series, the first for a cyclic group action $\mathscr{C}_{nr-n+1,n}$, and the second by a binary dihedral $\mathscr{D}_{2+3n,2+2n}$ (cf. [Sch87, Rie81]).

These examples add some strength to Question 13.24. We also mention the related question, first asked by Schreyer [Sch87]:

13.30 Question. *Is every CM local ring of countable CM type a quotient of one of the* (A_∞) *or* (D_∞) *hypersurface singularities by a finite group action?*

Burban and Drozd have recently announced a negative answer to this question [BD10]. Namely, set

$$A_{m,n} = k[\![x_1, x_2, y_1, y_2, z]\!]/(x_1y_1, x_1y_2, x_2y_1, x_2y_2, x_1z - x_2^n, y_1z - y_2^m).$$

Then $A_{m,n}$ has countable CM type for every $n, m \geqslant 0$. For $n = m$ this ring is isomorphic to a ring of invariants of the (A_∞) hypersurface, but for $m \neq n$ it is not.

§5 Exercises

13.31 Exercise. Let $R = \mathbb{Q}[x, y, z]_{(x,y,z)}/(x^2)$. The completion $\widehat{R} = \mathbb{Q}[\![x, y, z]\!]/(x^2)$ has a two-dimensional singular locus and therefore has uncountable CM type. Since R is countable, only countably many indecomposable $\widehat{R}$-modules are used in direct-sum decompositions of modules of the form $\widehat{R} \otimes_R M$, for MCM R-modules M. Thus the set $\mathscr{U}$ in the proof of Theorem 11.1 is properly contained in the set of all MCM $\widehat{R}$-modules.

13.32 Exercise. Let R be a one-dimensional CM local ring with canonical module ω, and let M be a finitely generated R-module. Prove that $M^{\vee\vee} \cong M/\operatorname{tor}(M)$.

13.33 Exercise. Assume that R is a CM local ring which is Gorenstein in codimension one. Prove that $M^{**} \cong M^{\vee\vee}$.

13.34 Exercise. Prove $S = \operatorname{End}_R(\mathfrak{c})$.

13.35 Exercise. Prove that we can compute double duals over R or S, as we like.

13.36 Exercise. Let R be a CM local ring and M a reflexive R-module which is locally free in codimension one. Show that $M \cdot N \cong \operatorname{Hom}_R(M^*, N)$. Conclude that $S \cdot N \cong \operatorname{Hom}_R(\mathfrak{c}, N)$.

13.37 Exercise. Prove Lemma 13.17, that the listed Burban-Drozd triples are a full set of representatives for the indecomposables of $\mathrm{BD}'(R)$, along the following lines.

- The listed forms are pairwise non-isomorphic and cannot be further decomposed.
- Every object of $\mathrm{BD}'(R)$ splits into direct summands with either $n = p = q$ or $n = p + q$. (Consider the complement of $(\ker\theta_1) + \ker(\theta_2)$ in $A^{(n)}$.)
- In the case $n = p + q$, the object further splits into direct summands with either $n = p$ or $n = q$. Any triple with $n = p$ or $n = q$ can be completely diagonalized, giving one of the factors of the normalization.
- If $n = p = q$, . . . (proof commented out)

13.38 Exercise. Prove Lemma 13.19, that the listed Burban-Drozd triples are a full set of representatives for the indecomposables of $\mathrm{BD}'(R)$, along the following lines.

- The listed forms are pairwise non-isomorphic and cannot be further decomposed.

- The $m \times n$ matrix θ can be reduced (using the rules of Lemma 13.19) to the block form

$$\begin{pmatrix} t^{d_1} I_{s_1} & A_{1,2} & \cdots & A_{1,v} & A_{1,v+1} \\ & t^{d_2} I_{s_2} & \cdots & A_{2,v} & A_{2,v+1} \\ & & \ddots & \vdots & \vdots \\ & & & t^{d_v} I_{s_v} & A_{v,v+1} \end{pmatrix}$$

where

  - $d_1 < d_2 < \cdots < d_v$ and $d_1 = 0$ or 1.
  - Each entry of $A_{i,j}$ has order in t at least $d_i + 1$ for $1 \leqslant i \leqslant v$ and $1 \leqslant j \leqslant v+1$.
  - Each entry of $A_{i,j}$ has order in t at most d_j for $1 \leqslant i \leqslant v$ and $1 \leqslant j \leqslant v$.

- If $A_{1,j} = 0$ for all $j = 2, \ldots, v+1$, then either (1) or (t) is a direct summand of θ and we are done by induction on the number of rows.

- If $A_{1,j} \neq 0$ for some $j \leqslant v$, write $A_{1,j} = t^{d_1} B_{1,j}$ for some matrix $B_{1,j}$ with entries in $k[[t]]$. Show that we may assume $B_{1,j}$ has entries in $k[[t^2]]$, and then diagonalize over $k[[t^2]]$ to assume $B_{1,j} = \left(\begin{smallmatrix} I_{s'} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. If $s' = 0$, return to the previous step, while if $s' > 0$, split out one of

$$\begin{pmatrix} 1 & t \\ & t^{d_j} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} t & t^2 \\ & t^{d_j} \end{pmatrix}.$$

- Consider two cases for each of the above matrices: $d_j = 1$ versus $d_j \neq 1$ in the first matrix, and $d_j = 2$ versus $d_j \neq 2$ in the second. Split out one of the forms listed in Lemma 13.19 in each case.

- Finally, if $A_{1,j} = 0$ for all $j = 2, \dots, \nu$ but $A_{1,\nu+1} \neq 0$, then one of (1), (t), $(1\ t)$, or $(t\ t^2) \sim (1\ t)$ is a direct summand of θ.

13.39 Exercise. Generalize Lemma 4.12 as follows. Let R be a CM reduced two-dimensional complete local ring, not necessarily normal, and let $f : M \longrightarrow N$ be a homomorphism of MCM R-modules. Then f is an isomorphism if and only if $f_{\mathfrak{p}} : M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}}$ is an isomorphism for every height-one prime $\mathfrak{p}$ of R.

14

The Brauer–Thrall Conjectures

In a brief abstract published in the 1941 Bulletin of the AMS [Bra41], R. D. Brauer announced that he had found sufficient conditions for a finite-dimensional algebra A over a field k to have infinitely many non-isomorphic indecomposable finitely generated modules. Some years later, R. M. Thrall [Thr47] claimed similar results: he wrote that Brauer had in fact given three conditions, each sufficient to ensure that A has indecomposable modules of arbitrarily high k-dimension, and he gave a fourth sufficient condition. These were stated in terms of the so-called "Cartan invariants" [ANT44, p. 106] of the rings A, $A/\text{rad}(A)$, $A/\text{rad}(A)^2$, etc. Neither Brauer nor Thrall ever published the details of their work, leaving it to Thrall's student J. P. Jans [Jan57] to publish them. Jans attributes to Brauer and Thrall the following conjectures. Let's say that a finite-dimensional k-algebra A has *bounded representation type* if the k-dimensions of indecomposable finitely generated A-modules are bounded, and *strongly unbounded representation type* if A has infinitely many non-isomorphic modules of k-dimension n for infinitely many n.

14.1 Conjecture (Brauer–Thrall Conjectures)**.** *Let A be a finite-dimensional algebra over a field k.*

I. If A has bounded representation type then A actually has finite representation type.

II. Assume that k is infinite. If A has unbounded representation type, then A has strongly unbounded representation type.

Both conjectures are now theorems. Brauer–Thrall I is due to A. V. Roĭter[Roĭ68], while Brauer–Thrall II was proved (as long as k is perfect) by L. A. Nazarova and Roĭter [NR73]. See [Rin80] or [Gus82] for some history on these results. (It's perhaps interesting to note that Auslander gave a proof of Roĭter's theorem for arbitrary Artinian rings [Aus74]—with length standing in for k-dimension—and that this is where "almost split sequences" made their first appearance.)

We import the definition of bounded type to the context of MCM modules almost verbatim. Recall that the *multiplicity* of a finitely generated module M over a local ring R is denoted $\mathrm{e}(M)$.

14.2 Definition. We say that a CM local ring R has *bounded CM type* provided there is a bound on the multiplicities of the indecomposable MCM R-modules.

If an R-module M has constant rank r, then it is known that $\mathrm{e}(M) = r\,\mathrm{e}(R)$. Thus for modules with constant rank, a bound on multiplicities is equivalent to a bound on ranks.

The first example showing that that bounded and finite type are not equivalent in the context of MCM modules, that is, that Brauer–Thrall I fails, was given by Dieterich in 1980 [Die80]: Let k be a field of characteristic 2, let $A = k[[x]]$, and let G be the two-element group. Then the group ring AG has bounded but infinite CM type. Indeed, note that $AG \cong k[[x,y]]/(y^2)$ (via the map sending the generator of the group to $y-1$). Thus AG has multiplicity 2 but is analytically ramified, whence AG has bounded but infinite CM type by Theorem 3.18. In fact, as we saw in Chapter 13, $k[[x,y]]/(y^2)$ has (countably) infinite CM type for every field k.

Theorem 3.10 says, in part, that if an analytically unramified local ring $(R,\mathfrak{m},k)$ of dimension one with infinite residue field k fails to have finite CM type, then R has $|k|$ indecomposable MCM modules of every rank n. Thus, for these rings, finite CM type and bounded CM type are equivalent, just as for finite-dimensional algebras, and moreover Brauer–Thrall II even holds for these rings. In this chapter we present the proof, due independently to Dieterich [Die87] and Yoshino [Yos87], of Brauer–Thrall I for all complete, equicharacteristic, CM isolated singularities over a perfect field (Theorem 14.21) and show how to use the results of the previous chapters to weaken the hypothesis of completeness to that of excellence. We also give a new proof (independent of the one in Chapter 3) that Brauer–Thrall II holds for complete one-dimensional reduced rings with infinite residue field (Theorem 14.28). The latter result uses Smalø's "inductive step"(Theorem 14.27) for building infinitely many indecomposables in a higher rank from infinitely many in a lower one.

§1 The Harada-Sai Lemma

We will reduce the proof of the first Brauer–Thrall conjecture to a statement about modules of finite length, namely the Harada–Sai Lemma 14.4. In this section we give Eisenbud–de la Peña's proof [EdlP98] of Harada–Sai, and in the next section we show how to extend it to MCM modules. The Lemma gives an upper bound on the lengths of non-zero paths in the Auslander–Reiten quiver. To state it, we make a definition.

14.3 Definition. Let R be a commutative ring and let

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{s-1}} M_s \tag{14.3.1}$$

be a sequence of homomorphisms between R-modules. We say (14.3.1) is a *Harada-Sai sequence* if

(i) each M_i is indecomposable of finite length;

(ii) no f_i is an isomorphism; and

(iii) the composition $f_{s-1}f_{s-2}\cdots f_1$ is non-zero.

Fitting's Lemma (Exercise 1.25) implies that, in the special case where $M_i = M$ and $f_i = f$ are constant for all i, the longest possible Harada–Sai sequence has length $\ell(M)-1$, where as usual $\ell(M)$ denotes the length of M. In general, the Harada–Sai Lemma gives a bound on the length of a Harada-Sai sequence in terms of the lengths of the modules.

14.4 Lemma. *Let (14.3.1) be a Harada-Sai sequence with the length of each M_i bounded above by b. Then $s \leqslant 2^b - 1$.*

In fact we will prove a more precise statement, which determines exactly which sequences of lengths $\ell(M_i)$ are possible in a Harada–Sai sequence.

14.5 Definition. The *length sequence* of a sequence (14.3.1) of modules of finite length is the integer sequence $\underline{\lambda} = (\ell(M_1), \ell(M_2), \ldots, \ell(M_s))$.

We define *special* integer sequences as follows:

$$\lambda^{(1)} = (1)$$

$$\lambda^{(2)} = (2,1,2)$$

$$\lambda^{(3)} = (3,2,3,1,3,2,3)$$

and, in general, $\lambda^{(b)}$ is obtained by inserting b at the beginning, the end, and between every two entries of $\lambda^{(b-1)}$. Alternatively,

$$\lambda^{(b+1)} = (\lambda^{(b)} + \mathbf{1}, 1, \lambda^{(b)} + \mathbf{1}),$$

where **1** is the sequence of all 1s. Notice that $\lambda^{(b)}$ is a list of $2^b - 1$ integers.

We say that one integer sequence λ of length n *embeds* in another integer sequence μ of length m if there is a strictly increasing function $\sigma : \{1,\dots,n\} \longrightarrow \{1,\dots,m\}$ such that $\lambda_i = \mu_{\sigma(i)}$.

Lemma 14.4 follows from the next result.

14.6 Theorem. *There is a Harada–Sai sequence with length sequence $\underline{\lambda}$ if and only if $\underline{\lambda}$ embeds in $\lambda^{(b)}$ for some b.*

Proof. First let

(14.6.1) $$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{s-1}} M_s$$

be a Harada–Sai sequence with length sequence $\underline{\lambda} = (\lambda_1,\dots,\lambda_s)$. Set $b = \max\{\lambda_i\}$. If $b = 1$, then each M_i is simple. As the composition is non-zero and no f_i is an isomorphism, the length of the sequence must be 1. Thus $\underline{\lambda} = (1)$ embeds in $\lambda^{(1)} = (1)$.

Suppose then that $b > 1$. If two consecutive entries of $\underline{\lambda}$ are both equal to b, say $\lambda_i = \lambda_{i+1} = b$, then we may insert some indecomposable summand of $\operatorname{im}(f_i)$ between M_i and M_{i+1}, chosen so that the composition is still nonzero. This gives a new Harada–Sai sequence, one step longer. Thus we may assume that no two consecutive λ_i are equal to b.

Observe that any sub-composition $g = f_j f_{j-1} \cdots f_i \colon M_i \longrightarrow M_{j+1}$, with $i \leqslant j$, is a non-isomorphism. Indeed, if g were an isomorphism, then $f_i \colon M_i \longrightarrow M_{i+1}$ would be injective, so that $\ell(M_{i+1}) > \ell(M_i)$. Then

$$M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_j f_{j-1} \cdots f_{i+1}} M_j \xrightarrow{g^{-1}} M_i$$

is the identity on M_i, so f_i is a split monomorphism. This contradicts the indecomposability of M_{i+1}.

Let $\underline{\lambda}'$ be the integer sequence gotten from $\underline{\lambda}$ by deleting every occurrence of b. Then $\underline{\lambda}'$ is the length sequence of the Harada–Sai sequence obtained by "collapsing" (14.6.1): for each M_i having length equal to b, delete M_i and replace the pair of homomorphisms f_i and f_{i+1} by the composition $f_{i+1} f_i \colon M_{i-1} \longrightarrow M_{i+1}$. By induction $\underline{\lambda}'$ embeds into $\lambda^{(b-1)}$. Since every second element of $\lambda^{(b)}$ is b and the b's in $\underline{\lambda}$ never repeat, this can be extended to an embedding $\underline{\lambda} \longrightarrow \lambda^{(b)}$.

For the other direction, it suffices by the same "collapsing" argument to show that there is a Harada–Sai sequence with length sequence equal to $\lambda^{(b)}$. We state this separately as Example 14.7 below. □

14.7 Example. There is a Harada–Sai sequence with length sequence $\lambda^{(b)}$ for every $b \geqslant 1$. We construct examples over the ring $R = k[x, y]/(xy)$, where k is an arbitrary field, following [EdlP98].

For any (non-commutative) word ω in the symbols x and y, we build an indecomposable R-module $M(\omega)$ of length one more than the length of ω.

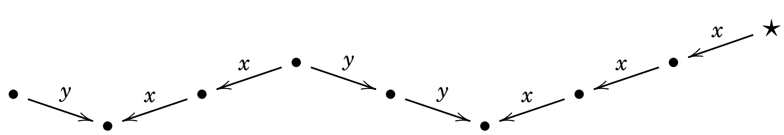
Let $M(\omega)$ be the vector space spanned by basis vectors e_a, one for each letter $a \in \{x,y\}$ in ω, together with an additional distinguished basis element $\star$. Let $s(a)$ denote the successor of a in ω, and $p(a)$ the predecessor; we interpret $\star$ as the last letter of ω, so that $p(\star)$ is the last x or y appearing in ω, and $s(\star)$ is empty. Define the R-module structure on the elements $e_a \in M(\omega)$ by

$$ye_a = \begin{cases} e_{s(a)} & \text{if } a = y, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \qquad xe_a = \begin{cases} e_{p(a)} & \text{if } p(a) = x, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Further define $x\star$ to be the last x or y appearing in ω, and finally set $y\star = 0$. Note two things: that in particular xy and yx both annihilate all basis vectors e_a and $\star$, so that $M(\omega)$ is an R-module, and that $\star \notin xM(\omega)$.

For example, if $\omega = 1$ is the empty word, then $M(\omega)$ is the simple R-module generated by $\star$.

Here is an example to clarify. Suppose $\omega = yx^2y^2x^3$; then $M(\omega)$ has 9 basis vectors, which we represent by bullets and $\star$, and the multiplication table is given by the following "string diagram."



For the example, it is clear that $M(yx^2y^2x^3)$ is indecomposable, since the string diagram is connected. In general, the same observation suffices to see that $M(\omega)$ is indecomposable for all ω.

We will construct inductively Harada–Sai sequences ϵ_b with length sequence $\lambda^{(b)}$. Every homomorphism in these sequences will take basis elements to basis elements; in particular, they will take$\star$ to $\star$, so the composition will be non-zero. For $b = 1$, take ϵ_1 to be the trivial sequence with a single module $M(1)$.

Suppose ϵ_b has the form

$$\epsilon_b : M(\omega_1) \longrightarrow M(\omega_2) \longrightarrow \cdots \longrightarrow M(\omega_{2^b-1})$$

where the lengths of $M(\omega_i)$ are given by the sequence $\lambda^{(b)}$ and each map takes $\star$ to $\star$. Observe that for any ω, the module $M(\omega)$ is naturally a submodule of $M(\omega x)$, where we take the new $\star$ to be the newly added basis element. If $f : M(\eta) \longrightarrow M(\omega)$ is a homomorphism taking $\star$ to $\star$, then f naturally extends to $\widetilde{f} : M(\eta x) \longrightarrow M(\omega x)$, taking the new $\star$ to the new $\star$. Applying this operation to ϵ_b yields

$$\widetilde{\epsilon}_b : M(\omega_1 x) \longrightarrow M(\omega_2 x) \longrightarrow \cdots \longrightarrow M(\omega_{2^{b-1}} x) \longrightarrow \cdots \longrightarrow M(\omega_{2^b-1} x).$$

Since the $(2^{b-1})^{\text{th}}$ entry of $\lambda^{(b)}$ is 1, we see that $\omega_{2^{b-1}}$ was the empty word 1, so that $\omega_{2^{b-1}} x = x$. We truncate $\widetilde{\epsilon}_b$ at $M(\omega_{2^{b-1}} x) = M(x)$, dropping the right-hand half.

Next, observe that R admits a k-algebra automorphism defined by interchanging x and y; this also induces a map on words ω, sending ω to, say, $\widehat{\omega}$. Again, if $f : M(\eta) \longrightarrow M(\omega)$ is a homomorphism preserving $\star$, then we obtain $\widehat{f} : M(\widehat{\omega}) \longrightarrow M(\widehat{\eta})$ with $\widehat{f}(\star) = \star$. Following this inversion with the $\widetilde{?}$ operation described above gives

$$\widehat{\epsilon}_b : M(\widehat{\omega}_{2^b-1} x) \longrightarrow \cdots \longrightarrow M(\widehat{\omega}_{2^{b-1}} x) \longrightarrow \cdots \longrightarrow M(\widehat{\omega}_2 x) \longrightarrow M(\widehat{\omega}_1 x).$$

We again truncate at the $(2^{b-1})^{\text{th}}$ stage, this time dropping the left-hand half, and define α to be the sequence

$$\begin{array}{ccccccccc} M(\omega_1 x) & \longrightarrow & M(\omega_2 x) & \longrightarrow \cdots \longrightarrow & M(\omega_{2^b-1}x) & & & & \\ & & & & \| & & & & \\ & & & & M(x) & & & & \\ & & & & \| & & & & \\ & & & & M(\widehat{\omega}_{2^b-1}x) & \longrightarrow \cdots \longrightarrow & M(\widehat{\omega}_2 x) & \longrightarrow & M(\widehat{\omega}_1 x). \end{array}$$

As each homomorphism in the sequence α takes $\star$ to $\star$, and $\star$ is outside the radical of each module, we may extend α one step to the right, with the map $M(\widehat{\omega}_1 x) \longrightarrow M(1)$ sending $\star$ to $\star$ and killing all the other basis elements.

Finally, the k-vector space dual $-^\vee = \mathrm{Hom}_k(-,k)$ is a functor on R-modules. We take the distinguished element of $\mathrm{Hom}_k(M(\omega),k)$ to be the dual basis element corresponding to the distinguished element $\star$ of $M(\omega)$. We have $M(1)^\vee \cong M(1)$, so we may splice α together with $\alpha^\vee$ to obtain

$$\epsilon_{b+1} \colon \alpha \longrightarrow M(1) \cong M(1)^\vee \longrightarrow \alpha^\vee$$

which has length vector $\left(\lambda^{(b)} + \mathbf{1}, 1, \lambda^{(b)} + \mathbf{1}\right) = \lambda^{(b+1)}$.

§2 Faithful systems of parameters

The goal of this section is to prove an analogue of the Harada–Sai Lemma 14.4 for MCM modules. We will reduce to the case of finite-length modules by killing a particularly nice regular sequence: one that preserves indecomposability, non-isomorphism, and even non-split short exact sequences of MCM modules.

Throughout, $(R,\mathfrak{m},k)$ is a CM local ring of dimension d. We will need to impose additional restrictions later on; see Theorem 14.20 for the full list.

14.8 Definition. Let $\mathbf{x} = x_1,\dots,x_d$ be a system of parameters for R. We say $\mathbf{x}$ is a *faithful* system of parameters if $\mathbf{x}$ annihilates $\operatorname{Ext}^1_R(M,N)$ for every pair of R-modules with M MCM.

In what follows, we write $\mathbf{x}^2$ for the system of parameters $x_1^2,\dots,x_d^2$. Here is the basic property of faithful systems of parameters that makes them well suited to our purposes.

14.9 Proposition. *Let $\mathbf{x} = x_1,\dots x_d$ be a faithful system of parameters, and let M and N be MCM R-modules. For every homomorphism $\varphi\colon M/\mathbf{x}^2M \longrightarrow N/\mathbf{x}^2N$, there exists $\widetilde{\varphi} \in \operatorname{Hom}_R(M,N)$ such that φ and $\widetilde{\varphi}$ induce the same homomorphism $M/\mathbf{x}M \longrightarrow N/\mathbf{x}N$.*

It's interesting to observe the similarity of this statement to Guralnick's Lemma 1.11. The statement could even be given the same form: a commutative rectangle consisting of two squares, the bottom of which also commutes, though the top square might not.

Proof. Our goal is the case $i=0$ of the following statement: there exists a homomorphism

$$\varphi_i\colon M/\left(x_1^2,\dots x_i^2\right)M \longrightarrow N/\left(x_1^2,\dots,x_i^2\right)N$$

such that $\varphi_i \otimes_R R/(\mathbf{x}) = \varphi \otimes_R R/(\mathbf{x})$. We prove this by descending induction on i, taking $\varphi_d = \varphi$ for the base case $i=d$.

Assume that φ_{i+1} has been constructed. Then it suffices to find a homomorphism $\varphi_i : M/(x_1^2,\dots x_i^2)M \longrightarrow N/(x_1^2,\dots,x_i^2)N$ with the following stronger property:

$$\varphi_i \otimes_R R/(x_1^2,\dots,x_i^2,x_{i+1}) = \varphi \otimes_R R/(x_1^2,\dots,x_i^2,x_{i+1}),$$

for then of course killing $x_1,\dots,x_i,x_{i+2},\dots,x_d$ we obtain $\varphi_i \otimes_R R/(\mathbf{x}) = \varphi \otimes_R R/(\mathbf{x})$.

Set $\mathbf{y}_i = x_1^2,\dots,x_i^2$ and $\mathbf{z}_i = x_1^2,\dots,x_i^2,x_{i+1}$. Then we have a commutative diagram with exact rows (as N is MCM and x_{i+1} is an R-regular element)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N/\mathbf{y}_iN & \xrightarrow{x_{i+1}^2} & N/\mathbf{y}_iN & \longrightarrow & N/\mathbf{y}_{i+1}N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_{i+1}} & & \| & & \downarrow & & \\ 0 & \longrightarrow & N/\mathbf{y}_iN & \xrightarrow{x_{i+1}} & N/\mathbf{y}_iN & \longrightarrow & N/\mathbf{z}_iN & \longrightarrow & 0. \end{array}$$

Apply $\operatorname{Hom}_R(M,-)$ to obtain a commutative exact diagram

$$\begin{array}{ccccc} \operatorname{Hom}_R(M,N/\mathbf{y}_iN) & \longrightarrow & \operatorname{Hom}_R(M,N/\mathbf{y}_{i+1}N) & \longrightarrow & \operatorname{Ext}_R^1(M,N/\mathbf{y}_iN) \\ \| & & \downarrow & & \downarrow{\scriptstyle x_{i+1}} \\ \operatorname{Hom}_R(M,N/\mathbf{y}_iN) & \longrightarrow & \operatorname{Hom}_R(M,N/\mathbf{z}_iN) & \longrightarrow & \operatorname{Ext}_R^1(M,N/\mathbf{y}_iN). \end{array}$$

By the definition of a faithful system of parameters, the right-hand vertical map is zero. We have φ_{i+1} living in $\operatorname{Hom}_R(M,N/\mathbf{y}_{i+1}N)$ in the middle of the top row, and an easy diagram chase delivers φ_i in the top-left corner such that $\varphi_i \otimes_R R/(\mathbf{z}_i) \cong \varphi_{i+1} \otimes_R R/(\mathbf{z}_i)$. □

Here are the main consequences of Proposition 14.9. The first and third corollaries are sometimes called "Maranda's Theorem," having first been proven by Maranda [Mar53] in the case of the group ring of a finite group over the ring of p-adic integers, and extended by Higman [Hig60] to arbitrary orders over complete discrete valuation rings.

14.10 Corollary. *Let* $\mathbf{x}$ *be a faithful system of parameters for* R*, and let* M *and* N *be MCM* R*-modules. Suppose that* $\varphi\colon M/\mathbf{x}^2M \longrightarrow N/\mathbf{x}^2N$ *is an isomorphism. Then there exists an isomorphism* $\widetilde{\varphi}\colon M \longrightarrow N$ *such that* $\widetilde{\varphi}\otimes_R R/(\mathbf{x}) = \varphi\otimes_R R/(\mathbf{x})$.

Proof. Proposition 14.9 gives us the homomorphism $\widetilde{\varphi}$; it remains to see that $\widetilde{\varphi}$ is an isomorphism. Since $\widetilde{\varphi}$ is surjective modulo $\mathbf{x}^2$, it is at least surjective by NAK. Similarly, applying the Proposition to φ^{-1}, we find that there is a surjection $\widetilde{\varphi^{-1}}\colon N \longrightarrow M$. By Exercise 3.25, the surjection $\widetilde{\varphi^{-1}}\widetilde{\varphi}\colon M \longrightarrow M$ is an isomorphism, so $\widetilde{\varphi}$ is as well. □

14.11 Corollary. *Let* $\mathbf{x}$ *be a faithful system of parameters for* R*, and let* $s\colon 0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$ *be a short exact sequence of MCM modules. Then* s *is non-split if and only if* $s\otimes_R R/(\mathbf{x}^2)$ *is non-split.*

Proof. Sufficiency is clear: a splitting for s immediately gives a splitting for $s\otimes_R R/(\mathbf{x}^2)$. For the other direction, suppose $\overline{p} = p\otimes_R R/(\mathbf{x}^2)$ is a split epimorphism. Then there exists $\varphi\colon M/\mathbf{x}^2M \longrightarrow E/\mathbf{x}^2E$ such that $\overline{p}\varphi$ is the identity on $M/\mathbf{x}^2M$. Let $\widetilde{\varphi}\colon M \longrightarrow E$ be the lifting guaranteed by Proposition 14.9. Then $(p\widetilde{\varphi})\otimes_R R/(\mathbf{x})$ is the identity on $M/\mathbf{x}M$, so $p\widetilde{\varphi}$ is an isomorphism. Thus s is split. □

14.12 Corollary. *Assume that* R *is Henselian. Let* $\mathbf{x}$ *be a faithful system of parameters for* R*, and let* M *be a MCM* R*-module. Then* M *is indecomposable if and only if* $M/\mathbf{x}^2M$ *is indecomposable.*

Proof. Again, we have only to prove one direction: if M decomposes nontrivially, then so must $M/\mathbf{x}^2M$ by NAK. For the other direction, assume that

M is indecomposable. Then $\mathrm{End}_R(M)$ is a nc-local ring since R is Henselian (see Chapter 1). We have a commutative diagram

$$\begin{array}{ccc} \mathrm{End}_R(M) & \longrightarrow & \mathrm{End}_R(M/\mathbf{x}^2M) \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathrm{End}_R(M/\mathbf{x}M) & \end{array}$$

where each map is the natural one induced by tensoring with $R/(\mathbf{x})$ or $R/(\mathbf{x}^2)$. Let $e \in \mathrm{End}_R(M/\mathbf{x}^2M)$ be an idempotent; we'll show that e is either 0 or 1, so that $M/\mathbf{x}^2M$ is indecomposable. The image $\pi(e)$ of e in $\mathrm{End}_R(M/\mathbf{x}M)$ is still idempotent, and is contained in $\tau(\mathrm{End}_R(M))$ by Proposition 14.9. Since $\mathrm{End}_R(M)$ is nc-local, so is its homomorphic image $\tau(\mathrm{End}_R(M))$, so $\pi(e)$ is either 0 or 1.

If $\pi(e) = 0$, then $e \otimes_R R/(\mathbf{x}) = 0$, so that $e(M/\mathbf{x}^2M) \subseteq \mathbf{x}(M/\mathbf{x}^2M)$. But e is idempotent, so that $\mathrm{im}(e) = \mathrm{im}(e^2) \subseteq \mathrm{im}(\mathbf{x}^2) = 0$ and so $e = 0$. If $\pi(e) = 1$, then the same argument applies to $1 - e$, giving $e = 1$. □

To address the existence of faithful systems of parameters, consider a couple of general lemmas. We leave the proof of the first as an exercise. The second is an easy special case of [Wan94, Lemma 5.10].

14.13 Lemma. *Let Γ be a ring, I an ideal of Γ, and $\Lambda = \Gamma/I$. Then* $\mathrm{Ann}_\Gamma I$ *annihilates* $\mathrm{Ext}^1_\Gamma(\Lambda, K)$ *for every Γ-module K.*

14.14 Lemma. *Let Γ be a ring, I an ideal of Γ, and $\Lambda = \Gamma/I$. Let*

(14.14.1) $$L \xrightarrow{\varphi} M \xrightarrow{\psi} N$$

be an exact sequence of Γ-modules. Then the homology H of the complex

(14.14.2) $$\mathrm{Hom}_\Gamma(\Lambda, L) \xrightarrow{\varphi_*} \mathrm{Hom}_\Gamma(\Lambda, M) \xrightarrow{\psi_*} \mathrm{Hom}_\Gamma(Lambda, N)$$

is annihilated by $\operatorname{Ann}_\Gamma I$.

Proof. Let $K = \ker\varphi$ and $X = \operatorname{im}\varphi$, and let $\eta\colon L \longrightarrow X$ be the surjection induced by φ. Then applying $\operatorname{Hom}_\Gamma(\Lambda,-)$, we see that the cohomology of (14.14.2) is equal to the cokernel of $\operatorname{Hom}_\Gamma(\Lambda,\eta)\colon \operatorname{Hom}_\Gamma(\Lambda,L) \longrightarrow \operatorname{Hom}_\Gamma(\Lambda,X)$. This cokernel is also a submodule of $\operatorname{Ext}^1_\Gamma(\Lambda,K)$, so we are done by the previous lemma. □

We will apply Lemma 14.14 to the *homological different* $\mathfrak{H}_T(R)$ of a homomorphism $T \longrightarrow R$, where R is as above a CM local ring and T is a regular local ring. Recall from Appendix B that if $A \longrightarrow B$ is a homomorphism of commutative rings, we let $\mu\colon B\otimes_A B \longrightarrow B$ be the multiplication map defined by $\mu(b\otimes b') = bb'$, and we set $\mathscr{J} = \ker\mu$. The homological different $\mathfrak{H}_A(B)$ is then defined to be

$$\mathfrak{H}_A(B) = \mu(\operatorname{Ann}_{B\otimes_A B}\mathscr{J}).$$

Notice that for any two B-module M and N, $\operatorname{Hom}_A(M,N)$ is naturally a $B\otimes_A B$-module via the rule $[\varphi(b\otimes b')](m) = \varphi(bm)b'$ for any $\varphi \in \operatorname{Hom}_A(M,N)$, $m \in M$, and $b,b' \in B$. Since for any $B\otimes_A B$-module X, $\operatorname{Hom}_{B\otimes_A B}(R,X)$ is the submodule of X annihilated by $\mathscr{J}$, and $\mathscr{J}$ is generated by elements of the form $b\otimes 1 - 1\otimes b$, we see that

$$\operatorname{Hom}_{B\otimes_A B}(B,\operatorname{Hom}_A(M,N)) = \operatorname{Hom}_B(M,N)$$

for all M, N. Thus in particular $\operatorname{Hom}_B(M,N)$ is a $B\otimes_A B$-module as well, with structure via the map μ.

14.15 Proposition. *Let R be a CM local ring and let $T \subseteq R$ by a regular local ring such that R is a finitely generated T-module. Then $\mathfrak{H}_T(R)$ annihilates $\operatorname{Ext}^1_R(M,N)$ for every MCM R-module M and arbitrary R-module N.*

Proof. Let $0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$ by an injective resolution of N over R. Since M is MCM over R, it is finitely generated and free over T, and the complex

$$\operatorname{Hom}_T(M,I^0) \xrightarrow{\varphi} \operatorname{Hom}_T(M,I^1) \xrightarrow{\psi} \operatorname{Hom}_T(M,I^2)$$

is exact. Apply $\operatorname{Hom}_{R\otimes_T R}(R,-)$; by the discussion above the result is

$$\operatorname{Hom}_R(M,I^0) \xrightarrow{\varphi_*} \operatorname{Hom}_R(M,I^1) \xrightarrow{\psi_*} \operatorname{Hom}_R(M,I^2).$$

The homology H of this complex is naturally $\operatorname{Ext}^1_R(M,N)$, and is by Lemma 14.14 annihilated by $\operatorname{Ann}_{R\otimes_T R}\mathscr{J}$. Since the $R\otimes_T R$-module structure on these Hom modules is via μ, we see that $\mathfrak{H}_T(R) = \mu(\operatorname{Ann}_{R\otimes_T R}\mathscr{J})$ annihilates $\operatorname{Ext}^1_R(M,N)$. □

Put $\mathfrak{H}(R) = \sum_T \mathfrak{H}_T(R)$, where the sum is over all regular local subrings T of R such that R is a finitely generated T-module. It follows immediately from Proposition 14.15 that $\mathfrak{H}(R)$ annihilates $\operatorname{Ext}^1_R(M,N)$ whenever M is MCM.

Let us now introduce a more classical ideal, the Jacobian. Let T be a Noetherian ring and R a finitely generated T-algebra. Then R has a presentation $R = T[x_1,\dots,x_n]/(f_1,\dots,f_m)$ for some n and m. The *Jacobian ideal of R over T* is the ideal $J_T(R)$ in R generated by the $n\times n$ minors of the Jacobian matrix $(\partial f_i/\partial x_j)_{ij}$. We set $J(R) = \sum_T J_T(R)$, where again the sum is over all regular subrings T of R over which R is module-finite.

One can see ([Wan94, Prop. 5.8] or Exercise 14.30) that $J_T(R) \subseteq \mathfrak{H}_T(R)$ for every T, so that $J(R) \subseteq \mathfrak{H}(R)$. Thus we have

14.16 Corollary. *Let R be a CM local ring and let $J(R)$ be the Jacobian ideal of R. Then J annihilates $\operatorname{Ext}^1_R(M,N)$ for every pair of R-modules M, N with M MCM.*

There are two problems with this result. The first is the question of whether any regular local subrings T as in the definition of $J(R)$ actually exist. Luckily, Cohen's structure theorems assure us that when R is complete and contains its residue field k, there exist plenty of regular local rings $T = k[[x_1,\dots,x_d]]$ over which R is module-finite.

The second problem is that $J(R)$ may be trivial if the residue field is not perfect.

14.17 Remark. If $R = T[x]/(f(x))$, then it is easy to see that $J_T(R)$ is the ideal of R generated by the derivative $f'(x)$. Thus in the case when R is a hypersurface $R = k[[x_1,\dots,x_d]]$, $J(R)$ is the ideal of R generated by the partial derivatives $\partial f/\partial x_i$ of f. If k is not perfect, this ideal can be zero.

For example, suppose that k is an imperfect field of characteristic p, and let $\alpha \in k \setminus k^p$. Put $R = k[[x,y]]/(x^p - \alpha y^p)$. Then $J(R) = 0$. Note that R is a one-dimensional domain, so is an isolated singularity. Thus in particular J does not define the singular locus of R.

To address this second problem, we appeal to Nagata's Jacobian Criterion for smoothness of complete local rings [GD64, 22.7.2] (see also [Wan94, Props. 4.4 and 4.5]).

14.18 Theorem. *Let $(R, \mathfrak{m}, k)$ be an equidimensional complete local ring containing its residue field k. Assume that k is perfect. Then the Jacobian ideal $J(R)$ of R defines the singular locus: for a prime ideal $\mathfrak{p}$, $R_{\mathfrak{p}}$ is a regular local ring if and only if $J(R) \not\subseteq \mathfrak{p}$.*

This immediately gives existence of faithful systems of parameters, and our extension of the Harada–Sai Lemma to MCM modules. We leave the details of the proof of existence as an exercise (Exercise 14.31).

14.19 Theorem (Yoshino)**.** *Let $(R, \mathfrak{m}, k)$ be a complete CM local ring containing its residue field k. Assume that k is perfect and that R has an isolated singularity. Then R admits a faithful system of parameters.*

14.20 Theorem (Harada–Sai for MCM modules)**.** *Let R be an equicharacteristic complete CM local ring with perfect residue field and an isolated singularity. Let $\mathbf{x}$ be a faithful system of parameters for R. Let $M_0, M_1, \dots, M_{2^n}$ be indecomposable MCM R-modules, and let $f_i : M_i \longrightarrow M_{i+1}$ be homomorphisms that are not isomorphisms. If $\ell(M_i/\mathbf{x}^2 M_i) \leqslant n$ for all $i = 0, \dots, 2^n$, then $f_{2^n-1} \cdots f_2 f_1 \otimes_R R/(\mathbf{x}^2) = 0$.*

Proof. Set $\widetilde{M_i} = M_i/\mathbf{x}^2 M_i$ and $\widetilde{f_i} = f_i \otimes_R R/(\mathbf{x}^2)$. Then $\widetilde{M_0} \xrightarrow{\widetilde{f_0}} \dots \xrightarrow{\widetilde{f_{2^n-1}}} \widetilde{M_{2^n}}$ is a sequence of indecomposable modules, each with length at most n, in which no $\widetilde{f_i}$ is an isomorphism. It is too long to be a Harada–Sai sequence, however, so we conclude $f_{2^n} \cdots f_2 f_1 \otimes_R R/(\mathbf{x}^2) = 0$. □

§3 Proof of Brauer–Thrall I

We're now ready for the proof of the following theorem, proved independently in the complete case by Dieterich [Die87] and Yoshino [Yos87]. See also [PR90] and [Wan94].

14.21 Theorem. *Let $(R, \mathfrak{m}, k)$ be an excellent equicharacteristic CM local ring with algebraically closed residue field k. Then R has finite CM type if and only if R has bounded CM type and at most an isolated singularity.*

Of course one direction of the theorem follows immediately from Auslander's Theorem 6.12, and requires no hypotheses apart from Cohen–Macaulayness. The content of the theorem is that bounded type and isolated singularity together imply finite type. We need k to be algebraically closed to use the AR quiver.

We begin by considering the complete case, and at the end of the section we show how to relax this restriction. When R is complete and has at most an isolated singularity, we have access to the Auslander-Reiten quiver of R, as well as to faithful systems of parameters. In this case, we will prove

14.22 Theorem. *Let $(R, \mathfrak{m}, k)$ be a complete equicharacteristic CM local ring with algebraically residue field k. Assume that R has at most an isolated singularity. Let Γ be the AR quiver of R and Γ° a non-empty connected component of Γ. If there exists an integer B such that $\mathrm{e}(M) \leqslant B$ for all $[M] \in \Gamma^\circ$, then $\Gamma = \Gamma^\circ$ and Γ is finite. In particular R has finite CM type.*

Here is the strategy of the proof. Assume that Γ° is a connected component of Γ with bounded multiplicities. We want to show that for any

$[M]$ and $[N]$ in Γ, if either of $[M]$ or $[N]$ is in Γ° then there is a path from $[M]$ to $[N]$ in Γ, and furthermore that such a path can be chosen to have bounded length. To do this, we assume no such path exists and derive a contradiction to the Harada–Sai Lemma 14.20.

We fix notation as in Theorem 14.22: $(R, \mathfrak{m}, k)$ is a complete equicharacteristic CM local ring with perfect residue field k and with an isolated singularity. Let Γ be the AR quiver of R. By Theorem 14.19 there exists a faithful system of parameters $\mathbf{x}$ for R. We say that a homomorphism $\varphi \colon M \longrightarrow N$ between R-modules is non-trivial modulo $\mathbf{x}^2$ if $\varphi \otimes_R R/(\mathbf{x}^2) \neq 0$. Abusing notation slightly, we also say that a path in Γ is non-trivial modulo $\mathbf{x}^2$ if the corresponding composition of irreducible maps is non-trivial modulo $\mathbf{x}^2$.

14.23 Lemma. *Fix a non-negative integer n. Let M and N be indecomposable MCM R-modules and $\varphi \colon M \longrightarrow N$ a homomorphism which is non-trivial modulo $\mathbf{x}^2$. Assume that there is no directed path in Γ from $[M]$ to $[N]$ of length $< n$ which is non-trivial modulo $\mathbf{x}^2$. Then the following two statements hold.*

(i) There is a sequence of homomorphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} M_n \xrightarrow{g} N$$

with each M_i indecomposable, each f_i irreducible, and the composition $g f_n \cdots f_1$ non-trivial modulo $\mathbf{x}^2$.

(ii) There is a sequence of homomorphisms

$$M \xrightarrow{h} N_n \xrightarrow{g_n} N_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_1} N_0 = N$$

with each N_i indecomposable, each g_i irreducible, and the composition $g_1 \cdots g_n h$ non-trivial modulo $\mathbf{x}^2$.

Proof. We prove part (ii); the other half is similar.

If $n = 0$, then we may simply take $h = \varphi \colon M \longrightarrow N$. Assume therefore that $n > 0$, there is no directed path of length $< n$ from $[M]$ to $[N]$ which is non-trivial modulo $\mathbf{x}^2$, and that we have constructed a sequence of homomorphisms

$$M \xrightarrow{h} N_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_1} N_0 = N$$

with each N_i indecomposable, each g_i irreducible, and the composition $g_1 \cdots g_{n-1} h$ non-trivial modulo $\mathbf{x}^2$. We wish to insert an indecomposable module N_n into the sequence, extending it by one step. There are two cases, according to whether or not N_{n-1} is free.

If N_{n-1} is not free, then there is an AR sequence $0 \longrightarrow \tau(N_{n-1}) \xrightarrow{i} E \xrightarrow{p} N_{n-1} \longrightarrow 0$ ending in N_{n-1}. Since there is no path from $[M]$ to $[N]$ of length $n-1$, we see that h is not an isomorphism, so is not a split surjection since M and N_{n-1} are both indecomposable. Therefore h factors through E, say as $M \xrightarrow{\alpha} E \xrightarrow{p} N_{n-1}$. Write E as a direct sum of indecomposable MCM modules $E = \bigoplus_{i=1}^r E_i$, and decompose α and q accordingly, $M \xrightarrow{\alpha_i} E \xrightarrow{p_i} N_{n-1}$. Each p_i is irreducible by Proposition 10.25, and there must exist at least one i such that $g_1 \cdots g_{n-1} p_i \alpha_i$ is non-trivial modulo $\mathbf{x}^2$. Set $N_n = E_i$ and $g_n = p_i$, extending the sequence one step.

If N_{n-1} is free, then $N_{n-1} \cong R$, and the image of M is contained in $\mathfrak{m}$ since h is not an isomorphism. Let $0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{p} \mathfrak{m} \longrightarrow 0$ be a minimal MCM approximation of $\mathfrak{m}$. (If $\dim R \leqslant 1$, we take $X = \mathfrak{m}$ and $Y = 0$.) The homomorphism $h \colon M \longrightarrow \mathfrak{m}$ factors through X as $M \xrightarrow{\alpha} X \xrightarrow{p} \mathfrak{m}$. Decompose $X = \bigoplus_{i=1}^{r} X_i$ where each X_i is indecomposable, and write $p = \sum_{i=1}^{r} p_i$, where $p_i \colon X_i \longrightarrow \mathfrak{m}$. By Proposition 10.27, each composition $X_i \xrightarrow{p_i} \mathfrak{m} \hookrightarrow R$ is an irreducible homomorphism, and again we may choose i so that the composition $g_1 \cdots g_{n-1} p_i \alpha_i$ is non-trivial modulo $\mathbf{x}^2$. □

14.24 Lemma. *Let Γ° be a non-empty connected component of the AR quiver Γ of R, and assume that $\ell(M/\mathbf{x}^2 M) \leqslant m$ for every $[M]$ in Γ°. Let $\varphi \colon M \longrightarrow N$ be a homomorphism between indecomposable MCM R-modules which is non-trivial modulo $\mathbf{x}^2$, and assume that either $[M]$ or $[N]$ is in Γ°. Then there is a directed path of length $< 2^m$ from $[M]$ to $[N]$ in Γ which is non-trivial modulo $\mathbf{x}^2$. In particular, both $[M]$ and $[N]$ are in Γ° if either one is.*

Proof. Assume that $[N]$ is in Γ°. If there is no directed path of length $< 2^m$ from $[M]$ to $[N]$, then by Lemma 14.23 there is a sequence of homomorphisms

$$M \xrightarrow{h} N_n \xrightarrow{g_n} N_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_1} N_0 = N$$

with each N_i indecomposable, each g_i irreducible, and the composition $g_1 \cdots g_{2^m} h$ non-trivial modulo $\mathbf{x}^2$. Since Γ° is connected, each $[N_i]$ is in Γ°, so that $\ell(N_i/\mathbf{x}^2 N_i) \leqslant m$ for each i. By the Harada–Sai Lemma 14.20, $g_1 \cdots g_{2^m}$ is trivial modulo $\mathbf{x}^2$, a contradiction.

A symmetric argument using the other half of Lemma 14.23 takes care of the case where $[M]$ is in Γ°. □

We are now ready for the proof of Brauer–Thrall I in the complete case. Keep notation as in the statement of Theorem 14.22.

Proof of Theorem 14.22. We have $e(M) \leqslant B$ for every $[M]$ in Γ°. Choose t large enough that $\mathfrak{m}^t \subseteq (\mathbf{x}^2)$, where $\mathbf{x}$ is the faithful system of parameters guaranteed by Theorem 14.19. Then (see Appendix A) $\ell(M/\mathbf{x}^2 M) \leqslant t^{\dim R} B$ for every $[M]$ in Γ°. Set $m = t^{\dim R} B$.

Let M be any indecomposable MCM module such that $[M]$ is in Γ°. By NAK, there is an element $z \in M \setminus \mathbf{x}^2 M$. Define $\varphi \colon R \longrightarrow M$ by $\varphi(1) = z$; then φ is non-trivial modulo $\mathbf{x}^2$. By Lemma 14.24, $[R]$ is in Γ°, and is connected to $[M]$ by a path of length $< 2^m$ in Γ°.

Now let $[N]$ be arbitrary in Γ. The same argument shows that there is a homomorphism $\psi \colon R \longrightarrow N$ which is non-trivial modulo $\mathbf{x}^2$, whence $[N]$ is in Γ° as well, connected to $[R]$ by a path of length $< 2^m$. Thus $\Gamma = \Gamma^\circ$, and since Γ is a locally finite group of finite diameter, Γ is finite. □

To complete the proof of Theorem 14.21, we need to know that for R an excellent isolated singularity, the hypotheses ascend to the completion $\widehat{R}$, and the conclusion descends back down to R. We have verified most of these details in previous chapters, and all that remains is to assemble the pieces.

Proof of Theorem 14.21. Let R be as in the statement of the theorem, so that R is excellent and has a perfect coefficient field. If R has finite CM type, then R has at most an isolated singularity by Theorem 6.12, and of course R has bounded CM type.

Suppose now that R has bounded CM type and at most an isolated singularity. Since R is excellent, both the Henselization R^h and the completion $\widehat{R}$ have isolated singularities as well (this was verified in the course of the proof of Corollary 11.13). In particular, R^h is Gorenstein on the punctured spectrum, so by Proposition 11.9, every MCM R^h-module M is a direct summand of an extended MCM R-module N. Since R has bounded CM type, we can write N as a direct sum of MCM R-modules N_i of bounded multiplicity. Using KRS over R^h, we deduce that M is a direct summand of some $N_i \otimes_R R^h$, thereby getting a bound on the multiplicity of M. Thus R^h has bounded CM type as well.

Next we must verify that bounded CM type ascends from R^h to $\widehat{R}$. An arbitrary MCM $\widehat{R}$-module M is locally free on the punctured spectrum of $\widehat{R}$, since $\widehat{R}$ has at most an isolated singularity. Thus by Elkik's Theorem 11.12, M is extended from the Henselization. It follows immediately that $\widehat{R}$ has bounded CM type.

By Theorem 14.22, $\widehat{R}$ has finite CM type. This descends to R by Theorem 11.1, completing the proof. □

One cannot completely remove the hypothesis of excellence in Theorem 14.21. For example, let S be any one-dimensional analytically ramified local domain. It is known [Mat73, pp. 138–139] that there is a one-dimensional local domain R between S and its quotient field such that $e(R) = 2$ and $\widehat{R}$ is not reduced. Of course R is not excellent. Then R has bounded but infinite CM type by Theorem 3.18, and of course R has an isolated singularity.

§4 Brauer–Thrall II in dimension one

The second Brauer–Thrall conjecture for MCM modules is known in only a few special cases. See [Die87], [PR90], [PR91] for some results in this direction.

The results of Chapter 3 imply Brauer–Thrall II for one-dimensional reduced rings with algebraically closed residue field. Here we give another proof in the same context. This proof rests on an inductive step, due to Smalø [Sma80], for concluding from the existence of infinitely many indecomposable modules of a given multiplicity, infinitely many of a higher multiplicity. Smalø's result is quite general, and we feel it deserves to be better-known.

We need two lemmas aimed at controlling the growth of multiplicity as one walks through an AR quiver. The first is a general fact about Betti numbers [Avr98, Lemma 4.2.7].

14.25 Lemma. *Let $(R,\mathfrak{m},k)$ be a CM local ring of dimension d and multiplicity e, and let M be a finitely generated R-module. Then*

$$\mu_R(\operatorname{syz}^R_{n+1}(M)) \leqslant (e-1)\mu_R(\operatorname{syz}^R_n(M))$$

for all $n > d - \operatorname{depth} M$.

Proof. We may replace M by $\operatorname{syz}^R_{d-\operatorname{depth} M}(M)$ to assume that M is MCM. We may also assume that the residue field k is infinite, by passing if necessary to an elementary gonflement $R' = R[t]_{\mathfrak{m}[t]}$, which preserves the multiplicity of R and number of generators of syzygies of M. In this case, there exists an R-regular and M-regular sequence $\mathbf{x} = x_1,\dots,x_d$ such that $\mathrm{e}(R) = \mathrm{e}(R/(\mathbf{x})) =$

$\ell(R/(\mathbf{x}))$, and we have $\mu_R(\operatorname{syz}_n^R(M)) = \mu_{R/(\mathbf{x})}(\operatorname{syz}_n^{R/(\mathbf{x})}(M \otimes_R R/(\mathbf{x})))$. We are thus reduced to the case where R is Artinian of length e.

In a minimal free resolution $F_\bullet$ of M, we have $\operatorname{syz}_{n+1}^R(M) \subseteq \mathfrak{m}F_n$, so that

$$(e-1)\mu_R(\operatorname{syz}_n^R(M)) = \ell(\mathfrak{m}F_n) \geqslant \ell(\operatorname{syz}_{n+1}^R(M)) \geqslant \mu_R(\operatorname{syz}_{n+1}^R(M)),$$

for all $n \geqslant 1$. □

14.26 Lemma. *Let $(R,\mathfrak{m})$ be a complete CM local ring with algebraically closed residue field, and assume that R has an isolated singularity. Then there exists a constant $c = c(R)$ such that if $X \longrightarrow Y$ is an irreducible homomorphism of MCM R-modules, then $\mathrm{e}(X) \leqslant c\,\mathrm{e}(Y)$ and $\mathrm{e}(Y) \leqslant c\,\mathrm{e}(X)$.*

Proof. Recall from Chapter 10 that the Auslander-Reiten translate τ is given by $\tau(M) = \operatorname{Hom}_R(\operatorname{syz}_d^R \operatorname{Tr} M, \omega)$, where ω is the canonical module for R. We first claim that

$$\mathrm{e}(\tau(M)) \leqslant e(e-1)^{d+1}\mathrm{e}(M), \tag{14.26.1}$$

where $e = \mathrm{e}(R)$ is the multiplicity of R. To see this, it suffices to prove the inequality for $\mathrm{e}(\operatorname{syz}_d^R \operatorname{Tr} M)$, since dualizing into the canonical module preserves multiplicity. By Lemma 14.25, we have only to prove that $\mathrm{e}(\operatorname{Tr} M) \leqslant e(e-1)\mathrm{e}(M)$. Let $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be a minimal free presentation of M, so that $F_0^* \longrightarrow F_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0$ is a free presentation of $\operatorname{Tr} M$. Then

$$\mathrm{e}(\operatorname{Tr} M) \leqslant \mathrm{e}(F_1^*) = e\ \mu_R(\operatorname{syz}_1^R(M)) \leqslant e(e-1)\mu_R(M) \leqslant e(e-1)\,\mathrm{e}(M),$$

finishing the claim.

Now to the proof of the lemma. We may assume that X and Y are indecomposable. First suppose that Y is not free. Then there is an AR sequence

$$0 \longrightarrow \tau(Y) \longrightarrow E \longrightarrow Y \longrightarrow 0$$

ending in Y, and X is a direct summand of E by Proposition 10.25. Then

$$\begin{aligned} \mathrm{e}(E) &= \mathrm{e}(\tau(Y)) + \mathrm{e}(Y) \\ &\leqslant [e(e-1)^{d+1} + 1]\,\mathrm{e}(Y) \end{aligned}$$

so $\mathrm{e}(X) \leqslant [e(e-1)^{d+1} + 1]\,\mathrm{e}(Y)$.

Now suppose that $Y \cong R$ is free. Then X is a direct summand of the MCM approximation E of the maximal ideal $\mathfrak{m}$ by Proposition 10.25, so $\mathrm{e}(X) \leqslant \mathrm{e}(E)$ is bounded in terms of $\mathrm{e}(R)$.

The other inequality is similar. □

14.27 Theorem (Smalø). *Let $(R, \mathfrak{m})$ be a complete CM local ring with algebraically closed residue field, and assume that R has an isolated singularity. Assume that $\{M_i \mid i \in I\}$ is an infinite family of pairwise non-isomorphic indecomposable MCM R-modules of multiplicity b. Then there exists an integer $b' > b$, a positive integer t, and a subset $J \subseteq I$ with $|J| = |I|$ such that there is a family $\{N_j \mid j \in J\}$ of pairwise non-isomorphic indecomposable MCM R-modules of multiplicity b'. Furthermore there exist non-zero homomorphisms $M_j \longrightarrow N_j$, each of which is a composition of t irreducible maps.*

Proof. Set $s = 2^b - 1$. First observe that since the AR quiver of R is locally finite, there are at most finitely many M_i such that there is a chain of strictly fewer than s irreducible maps starting at M_i and ending at the canonical module ω. Deleting these indices i, we obtain $J' \subseteq I$.

Each M_i remaining is MCM, so $\operatorname{Hom}_R(M_i, \omega)$ is non-zero. By NAK, there exists $\varphi \in \operatorname{Hom}_R(M_i, \omega)$ which is non-trivial modulo $\mathbf{x}^2$. Hence by Lemma 14.23 there is a sequence of homomorphisms

$$M_i = N_{i,0} \xrightarrow{f_{i,1}} N_{i,1} \xrightarrow{f_{i,2}} \cdots \longrightarrow N_{i,s-1} \xrightarrow{f_{i,s}} N_{i,s} \xrightarrow{g} \omega$$

with each $N_{i,j}$ indecomposable, each $f_{i,j}$ irreducible, and the composition $g_i f_{i,s} \cdots f_{i,1}$ non-trivial modulo $\mathbf{x}^2$.

By the Harada–Sai Lemma 14.20, not all the $N_{i,j}$ can have multiplicity less than or equal to b. So there exists $J'' \subseteq J'$, of the same cardinality, and $t \leqslant s$ such that $\mathrm{e}(N_{i,t}) > b$ for all i.

Applying Lemma 14.26 to the irreducible maps connecting M_i to $N_{i,t}$, we find that

$$b < \mathrm{e}(N_{i,t}) \leqslant c^t\, \mathrm{e}(N_{i,0}) = c^t b$$

for some constant c depending only on R. There are thus only finitely many possibilities for $\mathrm{e}(N_{i.t})$ as i ranges over J'', and we take $J''' \subseteq J''$ such that $\mathrm{e}(N_{i,t}) = b' > b$ for all $i \in J'''$.

There may be some repetitions among the isomorphism classes of the $N_{i,t}$. However, for any indecomposable MCM module N, there are only finitely many M with chains of irreducible maps of length t from M to N, so each isomorphism class of $N_{i,t}$ occurs only finitely many times. Pruning away these repetitions, we finally obtain $J = J'''' \subseteq I$ as desired. □

Since we have proved in Chapter 3 that a one-dimensional local ring $(R, \mathfrak{m}, k)$ with reduced completion, infinite residue field, and infinite CM type has at least $|k|$ distinct MCM modules of rank one, we conclude that such a ring has strongly unbounded CM type:

14.28 Theorem. *Let $(R,\mathfrak{m},k)$ be a complete CM local ring of dimension one with algebraically closed residue field k. Suppose that R does not have finite CM type. Then for infinitely many positive integers n, there exist $|k|$ pairwise non-isomorphic indecomposable MCM R-modules of multiplicity n.*

§5 Exercises

14.29 Exercise. Prove Lemma 14.13: For any ring Γ and any quotient ring $\Lambda = \Gamma/I$, the annihilator $\operatorname{Ann}_\Gamma I$ annihilates $\operatorname{Ext}^1_\Gamma(\Lambda,K)$ for every Γ-module K.

14.30 Exercise. Let R be a Noetherian ring and T a subring over which R is finitely generated as an algebra. Prove that $J_T(R) \subseteq \mathfrak{H}_T(R)$.

14.31 Exercise. Fill in the details of the proof of Theorem 14.19: show by induction on j that we may find regular local subrings $T_1,\dots,T_j$ and elements $x_i \in J_{T_i}(R)$ such that $x_1,\dots,x_j$ is part of a system of parameters. For the inductive step, use prime avoidance.

14.32 Exercise. Suppose that $\mathbf{x} = x_1,\dots,x_d$ is an faithful system of parameters in a local ring R. Prove that R has at most an isolated singularity.

15

Bounded Type

In this chapter we classify the complete equicharacteristic hypersurface rings of bounded CM type with residue field of characteristic not equal to 2. It is an astounding coincidence that the answer turns out to be precisely the same as in Chapter 13: The hypersurface rings of bounded but infinite type are the (A_∞) and (D_∞) hypersurface singularities in all positive dimensions. Note that the families of ideals showing countable non-simplicity in Lemma 13.22 for certain classes of hypersurface rings do not give rise to indecomposable modules of large rank; thus there does not seem to be a way to use the results of Chapter 13 to demonstrate unbounded CM type directly.

We also classify the one-dimensional complete CM local rings containing an infinite field and having bounded CM type. There is only one additional isomorphism type, which we have seen already in Example 13.25. The explicit classification, together with the results of Chapter 12, allows us to show that bounded type descends from the completion in dimension one.

§1 Hypersurface rings

To classify the complete hypersurface rings of bounded CM type, we must use Knörrer's results from Chapter 7 to reduce the problem to the case of dimension one. It will be more convenient in what follows to find bounds on the minimal number of generators of MCM modules; luckily, this is the same as bounding their multiplicity. We leave the proof of this fact as an exercise (Exercise 15.13).

15.1 Lemma. *Let A be a local ring. The multiplicities of indecomposable MCM A-modules are bounded if and only if their minimal numbers of generators are bounded.*

15.2 Proposition. *Let $R = S/(f)$ be a complete equicharacteristic hypersurface singularity, where $S = k[\![x_0, \dots x_d]\!]$ and f is a non-zero non-unit of S.*

(i) If $R^\sharp$ has bounded CM type, then R has bounded CM type.

(ii) If the characteristic of k is not 2, then the converse holds as well. More precisely, if $\mu_R(M) \leqslant B$ for each indecomposable MCM R-module, then $\mu_{R^\sharp}(N) \leqslant 2B$ for each indecomposable MCM $R^\sharp$-module N.

Proof. Assume that $R^\sharp$ has bounded CM type, and let B bound the minimal number of generators of MCM $R^\sharp$-modules. Let M be an indecomposable non-free MCM R-module. Then by Proposition 7.15 $M^{\sharp^\flat} \cong M \oplus \operatorname{syz}_1^R(M)$, so M is a direct summand of $M^{\sharp\flat}$. Decompose $M^\sharp$ into indecomposable MCM $R^\sharp$-modules, $M^\sharp \cong N_1 \oplus \cdots \oplus N_t$. Then $M^{\sharp\flat} \cong {N_1}^\flat \oplus \cdots \oplus {N_t}^\flat$, and by Krull-Remak-Schmidt M is a direct summand of some ${N_j}^\flat$. Since $\mu_R({N_j}^\flat) = \mu_{R^\sharp}(N_j)$ for each j, the result follows.

For the converse, assume $\mu_R(M) \leqslant B$ for every indecomposable MCM R-module M, and let N be an indecomposable non-free MCM $R^\sharp$-module. By Proposition 7.17, $N^{\flat\sharp} \cong N \oplus \operatorname{syz}_1^{R^\sharp}(N)$. Decompose $N^\flat$ into indecomposable MCM R-modules, $N^\flat \cong M_1 \oplus \cdots \oplus M_s$. Then $N^{\flat\sharp} \cong {M_1}^\sharp \oplus \cdots \oplus {M_s}^\sharp$. By KRS again, N is a direct summand of some ${M_j}^\sharp$. It will suffice to show that $\mu_{R^\sharp}({M_j}^\sharp) \leqslant 2B$ for each j.

If M_j is not free, we have $\mu_{R^\sharp}(M_j^\sharp) = \mu_R(M_j{}^{\sharp\flat}) = \mu_R(M_j) + \mu_R(\operatorname{syz}_1^R(M))$ by Proposition 7.15. But since M_j is a MCM R-module, all of its Betti numbers are equal to $\mu_R(M_j)$ by Proposition 7.6. Thus $\mu_R(M_j{}^{\sharp\flat}) = 2\mu_R(M_j) \leqslant 2B$. If, on the other hand, $M_j = R$, then $M_j{}^\sharp \cong R^\sharp$, and so $\mu_{R^\sharp}(M_j{}^\sharp) = 1$. □

Our next concern is to show that a hypersurface ring of bounded representation type has multiplicity at most two, as long as the dimension is greater than one. This is a corollary of the following result of Kawasaki [Kaw96, Theorem 4.1], due originally in the graded case to Herzog and Sanders [HS88]. (A similar result was obtained by Dieterich [Die87] using a theorem on the structure of the AR quiver of a complete isolated hypersurface singularity.) Recall that an *abstract hypersurface ring* is a Noetherian local ring $(A,\mathfrak{m})$ such that the $\mathfrak{m}$-adic completion $\widehat{A}$ is isomorphic to $B/(f)$ for some regular local ring B and non-unit f.

15.3 Theorem. *Let $(A,\mathfrak{m})$ be an abstract hypersurface ring of dimension d. Assume that the multiplicity $e = \mathrm{e}(A)$ is greater than 2. Then for each $n > e$, the module $\operatorname{syz}_{d+1}^A(A/\mathfrak{m}^n)$ is indecomposable and*

$$\mu_R\left(\operatorname{syz}_{d+1}^A\left(A/\mathfrak{m}^n\right)\right) \geqslant \binom{d+n-1}{d-1}.$$

We omit the proof, but see Theorem **??** for a stronger result if $d \geqslant 2$ and $\mathrm{e}(A) \geqslant 4$.

Putting Kawasaki's theorem together with Herzog's Theorem 8.14, we have the following result.

15.4 Proposition. *Let $(R,\mathfrak{m},k)$ be a Gorenstein local ring of bounded CM type. Assume $\dim R \geqslant 2$. Then R is an abstract hypersurface ring of multiplicity at most 2.*

When the hypersurface ring in Proposition 15.4 is complete and contains an algebraically closed field of characteristic other then 2, we can show by the same arguments as in Chapter 8 that it is an iterated double branched cover of a one-dimensional hypersurface ring of bounded type.

15.5 Theorem. *Let k be an algebraically closed field of characteristic not equal to 2, and let $R = k[[x_0,\dots,x_d]]/(f)$, where f is a non-zero non-unit of the formal power series ring and $d \geqslant 2$. Then R has bounded CM type if and only if $R \cong k[[x_,\dots,x_d]]/(g+x_2^2+\cdots+x_d^2)$ for some $g \in k[[x_0,x_1]]$ such that $k[[x_0,x_1]]/(g)$ has bounded CM type.*

Actually, the arguments above (and in Chapter 8) do not apply to rings like $k[[x_0,\dots,x_d]]/(x_d^2)$, since they tacitly assume that $g \neq 0$. Indeed, these rings do not have finite or bounded CM type. Here is a proof of a more general result.

15.6 Proposition. *Let $(S,\mathfrak{n},k)$ be a CM local ring of dimension at least two, and let z be an indeterminate. Set $R = S[z]/(z^2)$. Then R has unbounded CM type.*

Proof. We will show that for every $n \geqslant 2$ there is an indecomposable MCM R-module of rank $2n$. In fact, the proof is essentially identical to that of Theorem 2.2.

Fix $n \geqslant 2$, and let W be a free S-module of rank $2n$. Let I be the $n \times n$ identity matrix and H the $n \times n$ nilpotent Jordan block with 1 on the superdiagonal and 0 elsewhere. Let $\{x,y\}$ be part of a minimal generating set for the maximal ideal $\mathfrak{n}$ of S, and put $\Psi = yI + xJ$. Finally, put $\Phi = \left[\begin{smallmatrix} 0 & \Psi \\ 0 & 0 \end{smallmatrix}\right]$. Noting that $\Phi^2 = 0$, we make W into an R-module by letting z act as

$\Phi\colon W \longrightarrow W$. Then W is a MCM R-module, and one shows as in the proof of Theorem 2.2 that W is indecomposable over R. □

§2 Dimension one

The results of the previous section reduce the problem of classifying hypersurface rings of bounded CM type to dimension one. In this section we will deal with those one-dimensional hypersurface rings, as well as the case of non-hypersurface rings of dimension one.

Our problem breaks down according to the multiplicity of the ring. Recall from Theorem 3.18 that over a one-dimensional CM local ring of multiplicity 2 or less, every MCM R-module is isomorphic to a direct sum of ideals of R, whence R has bounded CM type. If on the other hand R has multiplicity 4 or more, then by Proposition 3.4 R has an overring S with $\mu_R(S) \geqslant 4$, and then we may apply Theorem 3.2 to obtain an indecomposable MCM module of constant rank n for every $n \geqslant 1$.

Now we address the troublesome case of multiplicity three for complete equicharacteristic hypersurface rings. Let $R = k[[x,y]]/(f)$, where k is a field and $f \in (x,y)^3 \setminus (x,y)^4$. If R is reduced, we know by Theorem 3.10 that R has bounded CM type if and only if R has finite CM type, that is, if and only if R satisfies the condition

(DR2) $\frac{\mathfrak{m}\overline{R}+R}{R}$ is cyclic as an R-module.

Hence we focus on the case where R is not reduced. Our strategy will be to build finite birational extensions S of R satisfying the hypotheses of Theorem 3.2.

15.7 Theorem. *Let $R = k[[x,y]]/(f)$, where k is a field and f is a non-zero non-unit of the formal power series ring $k[[x,y]]$. Assume that*

(i) $\mathrm{e}(R) = 3$*;*

(ii) R is not reduced; and

(iii) $R \not\cong k[[x,y]]/(x^2y)$.

For each positive integer n, R has an indecomposable MCM module of constant rank n.

Proof. We know f has order 3 and that its factorization into irreducibles has a repeated factor. Thus, up to a unit, we have either $f = g^3$ or $f = g^2h$, where g and h are irreducible elements of $k[[x,y]]$ of order 1, and, in the second case, g and h are relatively prime. After a k-linear change of variables we may assume that $g = x$.

In the second case, if the leading form of h is not a constant multiple of x, then by another change of variable [ZS75, Cor. 2, p. 137] we may assume that $h = y$. This is the case we have ruled out in (iii).

Suppose now that the leading form of h is a constant multiple of x. By a corollary [ZS75, Cor. 1, p.145] of the Weierstrass Preparation Theorem, there exist a unit u and a non-unit power series $q \in k[[y]]$ such that $h = u(x+q)$. Moreover, $q \in y^2k[[y]]$ (since the leading form of h is a constant multiple of x). In summary, there are two cases to consider:

(a) $f = x^3$.

(b) $f = x^2(x+q)$ for some $0 \neq q \in y^2k[[y]]$.

Let $\mathfrak{m} = (x, y)$ be the maximal ideal of R. We must show that R has a finite birational extension S such that $\mu_R(S) = 3$ and $\mathfrak{m}S/\mathfrak{m}$ is not cyclic as an R-module.

In Case (a) we put $S = R\left[\frac{x}{y^2}\right] = R + R\frac{x}{y^2} + R\frac{x^2}{y^4}$. Clearly $\mu_R(S) = 3$, and one checks that $\frac{\mathfrak{m}S}{\mathfrak{m}^2 S + \mathfrak{m}}$ is two-dimensional over $R/\mathfrak{m}$.

Assume now that we are in Case (b). One can argue by descending induction that it suffices to consider the case where q has order 2 in $k[\![y]\!]$. (The case of order 1 is the one we have ruled out.) Put $u = \frac{x}{y^2}$, $v = \frac{x^2+qx}{y^5}$, and $S = R[u, v]$. Once again this can be seen to satisfy the assumptions of Theorem 3.2, and this finishes the proof. □

The argument in the proof of Theorem 15.7 does not apply to the (D_∞) hypersurface ring $R = k[\![x, y]\!]/(x^2 y) \cong k[\![u, v]\!]/(u^2 v - u^3)$. Adjoining the idempotent $\frac{u^2}{v^2}$, one obtains a ring isomorphic to $k[\![v]\!] \times k[\![u, v]\!]/(u^2)$, whose integral closure is $k[\![v]\!] \times \bigcup_{n=1}^{\infty} R\left[\frac{u}{v^n}\right]$. From this information one can easily check that $\mathfrak{m}S/\mathfrak{m}$ is a cyclic R-module for every finite birational extension S of $(R, \mathfrak{m})$, so we cannot apply Theorem 3.2. However, the calculations in Chapter 13 do indeed verify that the one-dimensional (D_∞) and (A_∞) hypersurface rings have bounded type. Combining this with Theorem 15.7, we have a complete classification of the complete one-dimensional equicharacteristic hypersurface rings of bounded CM type.

15.8 Theorem. *Let k be an arbitrary field, and let $R = k[\![x, y]\!]/(f)$ be a complete hypersurface ring of dimension one, where f is a non-zero non-unit of the power series ring. Then R has bounded but infinite CM type if and only if R is isomorphic either to the (A_∞) singularity or to the (D_∞) singularity.*

Further, if R has unbounded CM type, then R has, for each positive integer r, an indecomposable MCM module of constant rank r.

Turning now to the non-hypersurface situation in multiplicity 3, we have the following structural result for the relevant rings.

15.9 Lemma. *Let $(R,\mathfrak{m},k)$ be a one-dimensional local CM ring with k infinite, and suppose $\mathrm{e}(R)=\mu_R(\mathfrak{m})=3$. Let N be the nilradical of R. Then:*

(i) $N^2=0$.

(ii) $\mu_R(N)\leqslant 2$.

(iii) If $\mu_R(N)=2$, then $\mathfrak{m}$ is generated by three elements u, v, w such that $\mathfrak{m}^2=\mathfrak{m}u$ and $N=Rv+Rw$.

(iv) If $\mu_R(N)=1$, then $\mathfrak{m}$ is generated by three elements u, v, w such that $\mathfrak{m}^2=\mathfrak{m}u$, $N=Rw$, and $vw=w^2=0$.

Proof. Since the residue field of R is infinite, we can find a minimal reduction for $\mathfrak{m}$, that is, a non-zerodivisor $u\in\mathfrak{m}$ such that $\mathfrak{m}^{n+1}=u\mathfrak{m}^n$ for all $n\gg 0$. Now, using the formula [Sal78, (1.1)]

(15.9.1) $$\mu_R(J)\leqslant \mathrm{e}(R)-\mathrm{e}(R/J)$$

for an ideal J of height 0 in a one-dimensional CM local ring R, it is straightforward to show (i) and (ii). The other two assertions are easy as well; cf. [LW05] for the details. □

15.10 Theorem. *Let k be an infinite field. The following is a complete list, up to k-isomorphism, of the one-dimensional, complete, equicharacteristic, CM local rings with bounded but infinite CM type and with residue field k:*

(i) the (A_∞) *hypersurface singularity* $k[[x,y]]/(x^2)$*;*

(ii) the (D_∞) *hypersurface singularity* $k[[x,y]]/(x^2y)$*;*

(iii) the endomorphism ring E *of the maximal ideal of the* (D_∞) *singularity, which satisfies*

$$E \cong k[[x,y,z]]/(yz, x^2 - xz, xz - z^2) \cong k[[a,b,c]]/(ab, ac, c^2).$$

Moreover, if $(R,\mathfrak{m},k)$ *is a one-dimensional, complete, equicharacteristic CM local ring and* R *does* not *have bounded CM type, then* R *has, for each positive integer* r*, an indecomposable MCM module of constant rank* r*.*

Proof. The (A_∞) and (D_∞) hypersurface rings have bounded but infinite CM type by the calculations in Chapter 13. In Example 13.25, we showed that E has the presentations asserted above, and that E has countable CM type. More precisely, we used Lemma 3.9 to see that the indecomposable MCM E-modules are precisely the non-free indecomposable MCM modules over the (D_∞) hypersurface ring, whence E has bounded but infinite CM type as well.

To prove that the list is complete and to prove the "Moreover" statement, assume now that $(R,\mathfrak{m},k)$ is a one-dimensional, complete, equicharacteristic CM local ring with k infinite, and that R has infinite CM type but does *not* have indecomposable MCM modules of arbitrarily large constant rank. We will show that R is isomorphic to one of the rings in the statement of the Theorem. As above, we proceed by building finite birational extensions of R to which we may apply Theorem 3.2.

If R is a hypersurface ring, Theorem 15.8 tells us that R is isomorphic to either $k[\![x,y]\!]/(x^2)$ or $k[\![x,y]\!]/(x^2y)$. Thus we assume that $\mu_R(\mathfrak{m}) \geqslant 3$. But $\mathrm{e}(R) \leqslant 3$ by Theorem 3.2 and we know by Exercise 9.54 that $\mathrm{e}(R) \geqslant \mu_R(\mathfrak{m}) - \dim R + 1$. Therefore we may assume that $\mathrm{e}(R) = \mu_R(\mathfrak{m}) = 3$. Thus we are in the situation of Lemma 15.9. Moreover, we may assume that R is not reduced, else we are done by Theorem 3.10, so R has non-trivial nilradical N.

If N requires two generators, then by Lemma 15.9(iii), we can find elements u, v, w in R such that $\mathfrak{m} = Ru + Rv + Rw$, u is a minimal reduction of $\mathfrak{m}$ with $\mathfrak{m}^2 = \mathfrak{m}u$, and $N = Rv + Rw$. Put $S = R\left[\frac{v}{u^2}, \frac{w}{u^2}\right]$. It is easy to verify (by clearing denominators) that $\{1, \frac{v}{u^2}, \frac{w}{u^2}\}$ is a minimal generating set for S as an R-module, and that the images of $\frac{v}{u}$ and $\frac{w}{u}$ form a minimal generating set for $\frac{\mathfrak{m}S}{\mathfrak{m}}$. Thus our basic assumption is violated.

We may therefore assume that N is principal. This is the hard case of the proof; we sketch the argument, and point to [LW05] for the details. Using Lemma 15.9(iv), we once again find elements u, v, w in R such that $\mathfrak{m} = Ru + Rv + Rw$, u is again a minimal reduction of $\mathfrak{m}$ with $\mathfrak{m}^2 = \mathfrak{m}u$, and $N = Rw$ with $vw = w^2 = 0$.

Since $v^2 \in \mathfrak{m}u \subset Ru$, we see that R/Ru is a three-dimensional k-algebra. Further, since $\bigcap_n (Ru^n) = 0$, it follows that R is finitely generated (and free) as a module over the discrete valuation ring $D = k[\![u]\!]$. One checks that $R = D + Dv + Dw$ (and therefore $\{1, v, w\}$ is a basis for R as a D-module).

In order to understand the structure of R we must analyze the equation that puts v^2 into $u\mathfrak{m}$. Thus we write $v^2 = u^r(\alpha u + \beta v + \gamma w)$, where $r \geqslant 1$ and α, β, $\gamma \in D$. Since u is a non-zerodivisor and $vw = w^2 = 0$, we see immediately that $\alpha = 0$. Thus we have

$$v^2 = u^r\left(\beta v + \gamma w\right),$$

with β and γ in D. Moreover, at least one of β and γ must be a unit of D.

If $r \geqslant 2$, put $S = R[\frac{v}{u^2}, \frac{w}{u^2}]$. This finite birational extension contradicts our basic assumption, so we must have $v^2 = u(\beta v + \gamma w)$ with $\beta, \gamma \in D$ and at least one of β, γ a unit of D. We will produce a hypersurface subring $A = D[\![g]\!]$ of R such that $R = \operatorname{End}_A(\mathfrak{m}_A)$. We will then show that $A \cong k[\![x,y]\!]/(x^2y)$ and the proof will be complete.

In the case where γ is not a unit, set $A = D[v+w]$. Then one can show that A is a local ring with maximal ideal $\mathfrak{m}_A = Au + A(v+w)$, and that R is a finite birational extension of A. Since $v(v+w) = (v+w)^2$ and $w(v+w) = 0$, we see that v and w are in $\operatorname{End}_A(\mathfrak{m}_A)$. Since $\operatorname{End}_A(\mathfrak{m}_A)/A$ is simple (as A is Gorenstein), it follows that $R = \operatorname{End}_A(\mathfrak{m}_A)$.

If on the other hand γ is a unit of D, we put $A = D[v] \subseteq R$. Then A is a local ring with maximal ideal $\mathfrak{m}_A = Au + Av$. (The relevant equation this time is $v^3 = u\beta v^2$.) We have $uw = \gamma^{-1}v^2 - \gamma^{-1}\beta uv \in \mathfrak{m}_A$. As in the first case, we conclude that $R = \operatorname{End}_A(\mathfrak{m}_A)$.

By Lemma 3.9, A has infinite CM type but does not have indecomposable MCM modules of arbitrarily large constant rank. Moreover, A cannot have multiplicity 2, since it has a module-finite birational extension of multiplicity greater than 2. By Theorem 15.8, $A \cong k[\![x,y]\!]/(x^2y)$, as desired. □

§3 Descent in dimension one

In this section we use the classification theorem in the previous section, together with the results on extended modules in Chapter 12, to show that bounded CM type passes to and from the completion of an equicharacteristic one-dimensional CM local ring $(R, \mathfrak{m}, k)$ with k infinite. Contrary to the situation in Chapter 11, we do not assume that R is excellent with an isolated singularity; indeed, in dimension one this assumption would make $\widehat{R}$ reduced, in which case finite and bounded CM type are equivalent by Theorem 3.10. We do, however, insist that k be infinite, in order to use the crucial fact from §2 that failure of bounded CM type implies the existence of indecomposable MCM modules of unbounded *constant* rank and also to use the explicit matrices worked out in Proposition 13.18 and Example 13.25 for the indecomposable MCM modules over $k[[x, y]]/(x^2 y)$.

15.11 Theorem. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional equicharacteristic CM local ring with completion $\widehat{R}$. Assume that k is infinite. Then R has bounded CM type if and only if $\widehat{R}$ has bounded CM type. If R has unbounded CM type, then R has, for each r, an indecomposable MCM module of constant rank r.*

Proof. Assume that $\widehat{R}$ does not have bounded CM type. Fix a positive integer r. By Theorem 15.10 we know that $\widehat{R}$ has an indecomposable MCM module M of constant rank r. By Corollary 12.7 there is a finitely generated R-module N, necessarily MCM and with constant rank r, such that $\widehat{N} \cong M$. Obviously N too must be indecomposable.

Assume from now on that $\widehat{R}$ has bounded CM type. If $\widehat{R}$ has *finite* CM type, the same holds for R by Theorem 11.1. Therefore we assume that $\widehat{R}$ has infinite CM type. Then $\widehat{R}$ is isomorphic to one of the three rings of Theorem 15.10.

If $\widehat{R} \cong k[[x,y]]/(x^2)$, then $\mathrm{e}(R) = \mathrm{e}(\widehat{R}) = 2$, and R has bounded CM type by Theorem 3.18. Suppose for the moment that we have verified bounded CM type for any local ring S whose completion is isomorphic to E. If, now, $\widehat{R} \cong k[[x,y]]/(x^2y)$, put $S = \mathrm{End}_R(\mathfrak{m})$. Then $\widehat{S} \cong E$, whence S has bounded CM type. Therefore so has R, by Lemma 3.9. Thus we assume that $\widehat{R} \cong E$.

Our plan is to examine each of the indecomposable non-free E-modules and then use Corollary 12.7 to determine exactly which MCM E-modules are extended from R. As we saw in Example 13.25, those indecomposable MCM modules are the cokernels of the following matrices over $T = k[[x,y]]/(x^2y)$:

$$[y]; \quad [xy]; \quad [x]; \quad [y^2];$$

$$\alpha = \begin{bmatrix} y & x^k \\ 0 & -y \end{bmatrix}; \quad \beta = \begin{bmatrix} xy & x^{k+1} \\ 0 & -xy \end{bmatrix}; \quad \gamma = \begin{bmatrix} xy & x^k \\ 0 & -y \end{bmatrix}; \quad \delta = \begin{bmatrix} y & x^{k+1} \\ 0 & -xy \end{bmatrix}.$$

Let $\mathfrak{P} = (x)$ and $\mathfrak{Q} = (y)$ be the two minimal prime ideals of T. Note that $T_{\mathfrak{P}} \cong k((y))[x]/(x^2)$ and $T_{\mathfrak{Q}} \cong k((x))$. With the exception of $U := \mathrm{cok}[x]$ and $V := \mathrm{cok}[xy]$, each of the E-modules listed above is generically free. The ranks are given in the following table.

φ	$\operatorname{rank}_{\mathfrak{P}} \operatorname{cok} \varphi$	$\operatorname{rank}_{\mathfrak{Q}} \operatorname{cok} \varphi$
$[x^2]$	1	0
$[y]$	0	1
α	1	0
β	1	2
γ	1	1
δ	1	1

Let M be a MCM $\widehat{R}$-module, and write

$$M \cong \left(\bigoplus_{i=1}^{a} A_i\right) \oplus \left(\bigoplus_{j=1}^{b} B_j\right) \oplus \left(\bigoplus_{k=1}^{c} C_k\right) \oplus \left(\bigoplus_{l=1}^{d} D_l\right) \oplus U^{(e)} \oplus V^{(f)}, \tag{15.11.1}$$

where the A_i, B_j, C_k, D_l are indecomposable generically free modules, of ranks $(1,0)$, $(0,1)$, $(1,1)$, $(1,2)$ respectively, and again $U = \operatorname{cok}[x]$ and $V = \operatorname{cok}[xy]$.

Suppose first that R is a domain. Then M is extended if and only if $a = b + d$ and $e = f = 0$. Now the indecomposable MCM R-modules are those whose completions have (a,b,c,d,e,f) minimal and non-trivial with respect to these relations. (We are implicitly using Corollary 1.14 here.) The only possibilities are $(0,0,1,0,0,0)$, $(1,1,0,0,0,0)$, and $(1,0,0,1,0,0)$, and we conclude that the indecomposable MCM R-modules have rank 1 or 2.

Next suppose that R is reduced but not a domain. Then R has exactly two minimal prime ideals, and we see from Corollary 12.7 that every generically free $\widehat{R}$-module is extended from R; however, neither U nor V can be a direct summand of an extended module. In this case, the indecomposable

MCM R-modules are generically free, with ranks $(1,0),(0,1),(1,1)$ and $(1,2)$ at the minimal prime ideals.

Finally, we assume that R is not reduced. We must now consider the two modules U and V that are not generically free. We will see that $U = \operatorname{cok}[x]$ is always extended and that $V = \operatorname{cok}[xy]$ is extended if and only if R has two minimal prime ideals. Note that $U \cong Txy = Exy$ (the nilradical of $E = \widehat{R}$), while $V \cong Tx = Ex$.

The nilradical N of R is of course contained in the nilradical Exy of $\widehat{R}$. Moreover, since $Exy \cong E/(x,z)$ is a faithful module over $E/(x,z) \cong k[[y]]$, every non-zero submodule of Exy is isomorphic to Exy. In particular, $N\widehat{R} \cong Exy$. This shows that U is extended.

Next we deal with V. The kernel of the surjective map $Ex \longrightarrow Exy$, given by multiplication by y, is Ex^2. Thus we have a short exact sequence

$$0 \longrightarrow Ex^2 \longrightarrow V \xrightarrow{y} U \longrightarrow 0\,. \tag{15.11.2}$$

Observe that $Ex^2 = Tx^2 \cong \operatorname{cok}[y]$ is generically free of rank $(0,1)$. Let K be the common total quotient ring of T and $\widehat{R}$. Then $K \otimes_E Ex^2$ is a projective K-module, and as K is Gorenstein, (15.11.2) splits when tensored up to K. In particular, this gives

$$K \otimes_E V \cong \left(K \otimes_E Ex^2\right) \oplus (K \otimes_E U)\,.$$

If, now, R has two minimal primes, then every generically free $\widehat{R}$-module is extended, by Corollary 12.7. In particular Ex^2 is extended, and by Lemma 12.6 so is V. Thus every indecomposable MCM $\widehat{R}$-module is extended, and R has bounded CM type.

If, on the other hand, R has just one minimal prime ideal, then the module M in (15.11.1) is extended if and only if $a = b+d+f$. The $\widehat{R}$-modules corresponding to indecomposable MCM R-modules are therefore U, $V \oplus W$, where W is some generically free module of rank $(0,1)$, and the modules of constant rank 1 and 2 described above. □

Proving descent of bounded CM type in general seems quite difficult. Part of the difficulty lies in the fact that, in general, there is no bound on the number of indecomposable MCM $\widehat{R}$-modules required to decompose the completion of an indecomposable MCM R-module. Thus the argument of Theorem 11.1, while sufficient for showing descent of finite CM type, is not enough for bounded CM type.

Here is an example to illustrate. Recall that for a two-dimensional normal domain, the divisor class group essentially controls which modules are extended to the completion. Precisely (Proposition 12.14), if R and $\widehat{R}$ are both normal domains, then a torsion-free $\widehat{R}$-module N is extended from R if and only if $\mathrm{cl}(N)$ is in the image of the natural map on divisor class groups $\mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(\widehat{R})$.

15.12 Example. Let R be a complete local two-dimensional normal domain containing a field, and assume that the divisor class group $\mathrm{Cl}(R)$ has an element α of infinite order. For example, one might take the ring of Lemma 12.15.

By Heitmann's theorem [Hei93], there is a unique factorization domain A contained in R such that $\widehat{A} = R$. Choose, for each integer n, a divisorial ideal I_n corresponding to $n\alpha \in \mathrm{Cl}(\widehat{A})$. For each $n \geqslant 1$, let $M_n = I_n \oplus N_n$, where N_n is the direct sum of n copies of I_{-1}. Then M_n has trivial divisor class and therefore is extended from A by Proposition 12.14. However, no non-trivial proper direct summand of M_n has trivial divisor class, and it follows that M_n (a direct sum of $n+1$ indecomposable $\widehat{A}$-modules) is extended from an indecomposable MCM A-module.

It is important to note that the example above does not give a counterexample to descent of bounded CM type, but merely points out one difficulty in studying descent.

§4 Exercises

15.13 Exercise. Let A be a local ring. Prove that there is an upper bound on the multiplicities of the indecomposable MCM A-modules if and only if there is a bound on their minimal numbers of generators.

15.14 Exercise. Complete the proof of Theorem 15.7.

15.15 Exercise. Show that the argument of Theorem 15.7 does not apply to $R = k[[u,v]]/(u^2v - v^3)$, since $\mathfrak{m}S/\mathfrak{m}$ is a cyclic R-module for every finite birational extension S of R.

15.16 Exercise. Prove the inequality (15.9.1): $\mu_R(J) \leqslant \mathrm{e}(R) - \mathrm{e}(R/J)$ for an ideal J of height zero in a one-dimensional CM local ring R.

15.17 Exercise. Finish the proof of Lemma 15.9.

16
Tame and Wild Representation Type

In the representation theory of Artin algebras, including modular group representations, the representation types we have considered thus far are a bit unnatural. The more natural representation types are *tame* and *wild*. There are many minor variations on the definitions, but the intent is always the same: the representations of a tame algebra might conceivably be classified, while those of a wild algebra are utterly out of reach. In practice, the latter is taken to mean that classifying the representations of a wild algebra would entail classifying the representations of *every* Artin algebra simultaneously.

§1 Tameness and Wildness

There are several minor variations on the notions of tame and wild representation type, but the intent is always the same: tame representation type allows the possibility of a classification theorem in the style of Jordan canonical form, while for wild type any classification theorem at all is utterly out of reach. The definitions we will use are essentially those of Drozd [Dro77]; they seem to have appeared implicitly first in [DF73]. They make precise the intent mentioned above by invoking the classical unsolved problem of canonical forms for n-tuples of matrices up to simultaneous similarity [GP69] (see Example 16.3 below).

Throughout, we work over a fixed *infinite* field k.

16.1 Definition. Let R be a local k-algebra and let $\mathscr{C}$ be a full subcategory of the finitely-generated R-modules.

(i) We say that $\mathscr{C}$ is *tame,* or *of tame representation type,* if there is one discrete parameter r (such as k-dimension or R-rank) parametrizing the modules in $\mathscr{C}$, such that, for each r, the indecomposables in $\mathscr{C}$ form finitely many one-parameter families and finitely many exceptions. Here a *one-parameter family* is a set of R-modules $\{E/(t-\lambda)E\}_{\lambda\in k}$, where E is a fixed $k[t]$-R-bimodule which is finitely generated and free over $k[t]$.

(ii) We say that $\mathscr{C}$ is *wild,* or *of wild representation type,* if for every finite-dimensional k-algebra Λ (not necessarily commutative!), there exists a *representation embedding* $\mathscr{E}\colon \operatorname{mod}\Lambda \longrightarrow \mathscr{C}$, that is, $\mathscr{E}$ is an exact functor preserving non-isomorphism and indecomposability.

Equivalently ([Kra00]), the algebra R is tame if and only if there is for every n a finite product $S = k[T_1] \times \cdots \times k[T_r]$ of polynomial rings and a k-linear, exact, coherent functor $F\colon \operatorname{Mod} S \longrightarrow \operatorname{Mod} R$ such that each indecomposable in of multiplicity e in R is the image of some indecomposable S-module. On the other side, $\mathscr{C}$ is wild if and only if there exist n-parameter families of indecomposable modules in $\mathscr{C}$ for all $n \gg 0$.

The definitions seem unwieldy at first sight, so we include a couple of simple examples. See [Erd90] for more.

16.2 Example. The "Kronecker algebra" is finite-length tame. Assume k is algebraically closed. Let $R = k[x,y]/(x^2,y^2)$ and $\mathfrak{m}$ its maximal ideal.

Then R is the path algebra of the quiver with two vertices and two parallel arrows x, y from one vertex to the other. Suppose M is a finitely generated R-module with no free summands. Then the socle of R, which is $\mathfrak{m}^2 = (xy)$, annihilates M and $\mathfrak{m}M \subseteq \operatorname{socle} M$. Consider M as a finite-dimensional k-vector space with two linear operators X and Y giving the actions of x and y on M. Choosing a k-basis of M which contains a basis of $\operatorname{socle} M$, we find matrix representations of X and Y as

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}.$$

Finding a canonical form for M is thus equivalent to the problem of classifying pairs of matrices (A,B) up to simultaneous equivalence: $(A,B) \cong (PAQ,PBQ)$. This problem was solved by Kronecker [Kro90]. For k algebraically closed, the complete list of indecomposable M is as follows.

(i) The standard module R and the residue field k;

(ii) For $e = 2n+1$ odd, there are exactly two indecomposable modules of k-dimension e, given by

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & & \\ & 1 & 0 & 0 & \cdots & \\ & & \ddots & & & \\ & & & & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 1 & 0 & \cdots & & \\ & 0 & 1 & 0 & \cdots & \\ & & \ddots & & & \\ & & & & 0 & 1 \end{bmatrix}$$

and by the transposes of these two matrices;

(iii) For each $\lambda \in k$ and each positive n, there is an indecomposable R-module $C_n(\lambda)$ of k-dimension $e = 2n$ on which $A = 1_n$ and $B = J(n,\lambda)$

is the indecomposable Jordan block of size n with eigenvalue λ when $\lambda \in k$. Additionally, there is an indecomposable module $C_n(\infty)$ of k-dimension $2n$ for which $A = J(n,0)$ is the indecomposable Jordan block of size n with eigenvalue 0 and $B = 1_n$.

To show that R is finite-length tame, first notice that we may ignore odd k-dimension. For $e = 2n$ even, take $M = k[T]^e$ and define a right R-action on M by setting

$$X = \begin{bmatrix} 0 & 1_n \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 0 & J(n,T) \\ 0 & 0 \end{bmatrix}.$$

For each $\lambda \in k$, $k[T]/(t-\lambda) \otimes_{k[T]} M$ is isomorphic to $C_n(\lambda)$, so we have a parametrization of the modules of k-dimension $2n$, with one exception.

This example actually gives another proof of Theorem 2.2: Let R be a local k-algebra of finite representation type (with respect to all finitely generated modules). Then $R \cong k[x]/(x^n)$ for some $n \geqslant 0$. Indeed, if the Jacobson radical of R needs two or more generators, then R has $k[x,y]/(x^2,xy,y^2)$ as a homomorphic image. Every finitely generated non-free module over $k[x,y]/(x^2,y^2)$ is also annihilated by xy, so is an R-module of finite length. Thus R is a principal ideal ring $k[x]/I$, and must be Artinian or $R/(x^k)$ is an indecomposable module for every $k \geqslant 0$.

Of course, as soon as one knows a single example of a finite-length tame ring, one knows that any homomorphic image is tame as well. On the other side, as soon as one knows a single example of a wild module subcategory $\mathscr{C}_0$, it is sufficient to find an embedding $\mathscr{C}_0 \longrightarrow \mathscr{C}$ preserving indecomposability and non-isomorphism to know that $\mathscr{C}$ is wild as well. Here is a seed to get us started.

(This eg. needs rewriting, see wildhyps)

16.3 Example ([GP69]). Let k be an infinite field. The non-commutative polynomial ring $k\langle a,b\rangle$ in two variables over k is finite-length wild. To see this, Let $\Lambda = k\langle x_1,\dots,x_n\rangle$ be a finite-dimensional k-algebra (so the x_i are not necessarily algebraically independent) and let V be an Λ-module of finite length. Represent the actions of $x_1,\dots,x_n$ on V by linear operators $X_1,\dots,X_n$. For any n distinct scalars $c_1,\dots,c_n \in k$, define a finite-length $k\langle a,b\rangle$-module M as follows: the underlying vector space is $V^{(n)}$, and we let a and b act by the block-matrices

$$\begin{bmatrix} c_1 1_V & & & \\ & c_2 1_V & & \\ & & \ddots & \\ & & & c_n 1_V \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X_1 & & & \\ 1_V & X_2 & & \\ & \ddots & \ddots & \\ & & 1_V & X_n \end{bmatrix},$$

respectively.

It suffices to prove that this functor is surjective on Hom-sets. A homomorphism $(V,X_1,\dots,X_n) \longrightarrow (V',X'_1,\dots,X'_n)$ is a linear map $\varphi\colon V \longrightarrow V'$ such that $\varphi X_i = X'_i\varphi$ for all i.[1]

Let $\Phi\colon M \longrightarrow M'$ be a homomorphism between $k\langle a,b\rangle$-modules as defined above. Then $\Phi = (\varphi_{ij})$, with each $\varphi_{ij}\colon V \longrightarrow V$; we must show that $\Phi = \operatorname{diag}(\varphi,\dots,\varphi)$ for some φ satisfying $\varphi X_i = X'_i\varphi$.

Observe that $k\langle a,b\rangle$ contains the (commutative) polynomial ring $k[a]$. Forgetting the action of b momentarily, we see that M is a semisimple $k[a]$-module, which immediately implies that $\varphi_{ij} = 0$ for $i \neq j$.

[1] maybe a quick primer on such things?

Now the equation

$$\begin{bmatrix} X_1 & & & \\ 1_V & X_2 & & \\ & \ddots & \ddots & \\ & & 1_V & X_n \end{bmatrix} \Phi = \Phi \begin{bmatrix} X_1 & & & \\ 1_V & X_2 & & \\ & \ddots & \ddots & \\ & & 1_V & X_n \end{bmatrix}$$

becomes

$$\begin{bmatrix} X_1\varphi_{11} & & & & \\ \varphi_{11} & X_2\varphi_{22} & & & \\ & \varphi_{22} & \ddots & & \\ & & \ddots & X_{n-1}\varphi_{n-1,n-1} & \\ & & & \varphi_{n-1,n-1} & X_n\varphi_{nn} \end{bmatrix} = \begin{bmatrix} \varphi_{11}X'_1 & & & & \\ \varphi_{22} & \varphi_{22}X'_2 & & & \\ & \varphi_{33} & \ddots & & \\ & & \ddots & \varphi_{n-1,n-1}X'_{n-1} & \\ & & & \varphi_{nn} & \varphi_{nn}X'_n\varphi_{nn} \end{bmatrix},$$

which implies that $\varphi_{ii} = \varphi_{11}$ for each i. Denote the common value by φ; then the diagonal entries show that $\varphi X_i = X'_i\varphi$.

It's amusing to compare the example to the standard group-theoretic fact that the free non-Abelian group $\mathbb{F}_2$ on two symbols contains as a subgroup every free group $\mathbb{F}_n$. This fact is of course the algebraic basis for the Banach–Tarski paradox [Wag93], together with the identification of our old acquaintance SO(3) (see Chapter 5) as a subgroup of $\mathbb{F}_2$.

The ring $R = k[a,b]/(a^2, ab^2, b^3)$ in the next example is sometimes called the "Drozd algebra." The example follows from the complete characterization of finite-length wild rings due to Klingler and Levy [KL06] (about which more later), but we give a direct proof here.

16.4 Example ([Dro72])**.** Let k be an infinite field, and set $R = k[a,b]/(a^2, ab^2, b^3)$. Then R is finite-length wild. Consequently, the commutative polynomial ring $k[a_1,\ldots,a_n]$ and the commutative power series ring $k[\![a_1,\ldots,a_n]\!]$ are both finite-length wild as soon as $n \geqslant 2$.

The last sentence follows from the ones before, since any R-module of finite length is also a module of finite length over $k[a,b]$ and $k[\![a,b]\!]$, whence also over $k[a_1,\ldots,a_n]$ and $k[\![a_1,\ldots,a_n]\!]$. Thus by Example 16.3 above, it suffices to construct a representation embedding of the finite-length $\Lambda = k\langle x, y\rangle$-modules into $\operatorname{mod} R$.

Let V be a Λ-module of k-dimension n, with linear operators X and Y representing the Λ-module structure. We define $(32n \times 32n)$-matrices A and B yielding an action on $M = V^{(32)}$. To wit, let

$$A = \begin{bmatrix} 0 & 0 & 1_{V^{(15)}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} B_1 & 0 & B_2 \\ 0 & 0 & B_3 \\ 0 & 0 & B_1 \end{bmatrix},$$

where

$$B_1 = \begin{bmatrix} 0 & 0 & 1_{V^{(5)}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1_{V^{(5)}} & 0 & 0 \\ 0 & C & 0 \end{bmatrix}, \qquad \text{and} \qquad B_3 = \begin{bmatrix} 0 & D & 0 \end{bmatrix},$$

and finally

$$C = \begin{bmatrix} c_1 1_V & & & \\ & c_2 1_V & & \\ & & \ddots & \\ & & & c_5 1_V \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1_V & 0 & 1_V & 1_V & 1_V \\ 0 & 1_V & 1_V & X & Y \end{bmatrix}.$$

Observe that, while all the blocks in B_1, B_2, and B_3 are $(5n \times 5n)$, the blocks in A and B are not of uniform size; their four corner blocks are $(15n \times 15n)$, while the center block is $(2n \times 2n)$.

One verifies easily that $XY = YX$ and $X^2 = XY^2 = Y^3$, so X and Y do indeed define an R-module structure on M.

Let V_1 and V_2 be two n-dimensional Λ-modules, with linear operators A_i, B_i for $i = 1,2$ inducing the actions of a and b, and let M_i be the corresponding R-modules with their endomorphisms X_i, Y_i. Then M_1 and M_2 are R-isomorphic if and only if there is a vector space isomorphism $S\colon V_1^{(32n)} \longrightarrow V_2^{(32n)}$ such that

$$SX_1 = X_2S \qquad \text{and} \qquad SY_1 = Y_2S\,.$$

Relatively straightforward matrix calculations like those in the previous example yield that S is a block-diagonal matrix with constant diagonal block $\sigma\colon V_1 \longrightarrow V_2$. Thus S is an isomorphism if and only if σ is so, and furthermore S is a split surjection if and only if σ is so. Thus the functor $V \rightsquigarrow M$ is a representation embedding (even a representation equivalence), and R is finite-length wild.

We restate one part of this example separately for later use.

16.5 Proposition. *Let $Q = k[a_1,\dots,a_n]$ or $k[[a_1,\dots,a_n]]$, with $n \geqslant 2$. If there is a representation embedding of the finite-length Q-modules into a module category $\mathscr{C}$, then $\mathscr{C}$ is wild.*

§2 Artinian algebras and pairs

The slight awkwardness of Definition 16.1 is more than repaid by the Trichotomy Theorem of Drozd, which confirmed a 1973 conjecture of Donovan and Frieslich.

16.6 Theorem ([Dro77, Dro79, CB88]). *Let Λ be a finite-dimensional algebra over an algebraically closed field k. Then Λ has exactly one of: finite representation type, tame representation type, or wild representation type.*

We omit the proof.

[Talk about Klingler-Levy here.]

For Artinian pairs, we do not know whether a trichotomy result holds. We can at least show that almot all Artinian pairs have wild type.

16.7 Proposition. *Let $A \hookrightarrow B$ be an Artinian pair of k-algebras, with k an algebraically closed field. Suppose that $\mu_A(B) \geqslant 5$. Then the $(A \hookrightarrow B)$-modules are wild.*

Proof. I can jazz up Dade's proof to get it for $\geqslant 6$; surely there's a smarter way. □

§3 Curves

The only trichotomy theorem for MCM modules over a complete local ring, as far as we are aware, is in the case of dimension one.

§4 Hypersurfaces

Here's where GL's joint stuff with Andrew will go. Also maybe some chatter about Popescu et. al.'s theorems.

§5 The generic determinant

key words: indirect proof of wildness

outline:

- define setup
- Bruns' classification of rank-ones
- compute Ext of rank-ones
- extensions give factorizations
- alternating matrices
- proof of wildness

16.8 Notation. Let K be a field, n a positive integer, and $X = (x_{ij})$ the generic $n \times n$ matrix over K. The entries of X thus form a family of n^2 indeterminates; set $S = K[x_{ij}]$, the polynomial ring over S in those variables.

The determinant $\det X$ of X is a homogeneous squarefree polynomial of degree n with coefficients ± 1, and the hypersurface ring $R = S/(\det X)$ is a domain of dimension $n^2 - 1$.

The *classical adjoint* $\operatorname{adj}(X)$ is defined by either of the equalities

$$(16.8.1) \qquad X \operatorname{adj}(X) = (\det X) I_n \qquad \text{and} \qquad \operatorname{adj}(X) X = (\det X) I_n \,.$$

In other terms, the pairs $(X, \operatorname{adj}(X))$ and $(\operatorname{adj}(X), X)$ form matrix factorizations of the hypersurface $\det X$. We thus are given four MCM R-modules for free, as follows.

Put $F = S^{(n)}$, the free module of rank n with canonical ordered basis $(f_1,\dots,f_n)$, and $G = S^{(n)}$ the free S-module of the same rank but with ordered basis $(g_1,\dots,g_n)$. We define R-modules $L = \operatorname{cok}(X,\operatorname{adj}(X))$ and $M = \operatorname{cok}(\operatorname{adj}(X),X)$, or equivalently through the exact sequences of R-modules

(16.8.2)
$$0 \longrightarrow G \xrightarrow{X} F \longrightarrow L \longrightarrow 0$$
$$0 \longrightarrow F \xrightarrow{\operatorname{adj}(X)} G \longrightarrow M \longrightarrow 0.$$

Here we interpret, for example, the matrix X as the homomorphism $G \longrightarrow F$ taking the j^{th} basis vector g_j to $\sum_i x_{ij} f_i$.

For a CM S-module N of codepth t, let's write $N^\vee = \operatorname{Ext}^t_S(N,S)$. If in particular N is a MCM (so free) S-module, then $N^\vee \cong \operatorname{Hom}_S(N,S)$, while if N is a MCM R-module, we have $N^\vee = \operatorname{Ext}^1_S(N,S) \cong \operatorname{Hom}_R(N,R)$. We obtain two more MCM R-modules from this process, defined by the exact sequences of S-modules

(16.8.3)
$$0 \longrightarrow F^\vee \xrightarrow{X^T} G^\vee \longrightarrow L^\vee \longrightarrow 0$$
$$0 \longrightarrow G^\vee \xrightarrow{\operatorname{adj}(X)^T} F^\vee \longrightarrow M^\vee \longrightarrow 0.$$

Write $\overline{N} = N \otimes_S R$ for the reduction of an S-module N modulo the determinant; then the short exact sequences of S-modules above induce short exact sequences of R-modules

(16.8.4)
$$0 \longrightarrow M \longrightarrow \overline{F} \longrightarrow L \longrightarrow 0$$
$$0 \longrightarrow L \longrightarrow \overline{G} \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow M^\vee \longrightarrow \overline{G}^\vee \longrightarrow L^\vee \longrightarrow 0$$
$$0 \longrightarrow L^\vee \longrightarrow \overline{F}^\vee \longrightarrow M^\vee \longrightarrow 0.$$

By 7.8, the modules L and $L^\vee$ have rank one over R, while M and $M^\vee$ have rank $n-1$. In particular, L and $L^\vee$ are indecomposable, and their syzygies M and $M^\vee$ are indecomposable as well by **??** (applied to the localization of R at the obvious maximal ideal).

Observe here that R is a normal domain: its singular locus is defined (in any characteristic) by the partial derivatives $\partial_{ij}(\det X)$ of the determinant, where $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$. These are easily seen to be equal to the entries of the adjoint:

$$\operatorname{adj}(X)_{ij} = \partial_{ji}(\det X).$$

The entries of $\operatorname{adj}(X)$ can also be identified as the $(n-1)\times(n-1)$ minors of X, up to sign. The ideal they generate, $I_{n-1}(X)$, is known to be prime of height 4 in S [BV88, 2.5], so the singular locus of R has codimension three. In particular, R is regular in codimension one, and so is a normal domain.

The MCM modules L and $L^\vee$, having rank one, are isomorphic to ideals of R: if we fix any $n-1$ columns of X, then L is isomorphic to the ideal generated by the $(n-1)\times(n-1)$ minors of those columns [Eis95, Theorem A2.14]. Similarly, $L^\vee$ is obtained by fixing any $n-1$ rows.

More generally, any MCM R-module is reflexive, so the MCM modules of rank one are all isomorphic to divisorial ideals, thus naturally live in the divisor class group $\mathrm{Cl}(R)$. The divisor class group of a generic determinantal ring was computed by Bruns [Bru85] (see also [BH93, 7.3.5]).

16.9 Proposition (Bruns)**.** *The divisor class group of $R = S/(\det X)$ is cyclic, generated by the class of $[L] = -[L^\vee]$. Furthermore, the symbolic powers $L^{(m)}$ representing elements $m[L] \in \mathrm{Cl}(R)$ are equal to the usual powers L^m. Among these, only R, L, and $L^\vee$ are MCM modules.* □

16.10 Corollary. *The generic determinantal hypersurface $R = S/(\det X)$ has "finite CM type in rank one," that is, only finitely many non-isomorphic MCM modules of rank one.*

We turn accordingly to MCM modules of rank two, where the situation is quite different if $n \geqslant 3$. (Observe that for $n = 2$, R is an (A_1) singularity, so has finite CM type.) Specifically, we look for rank-two MCM modules Q which are extensions of the rank-one modules $L, L^\vee$. Even more specifically, we will classify such Q that appear as the middle module in an element of either $\mathrm{Ext}^1_R(L,L)$ or $\mathrm{Ext}^1_R(L,L^\vee)$. The other two combinations of rank-one modules can be recovered from these by applying the K-algebra involution τ of S defined by $\tau(x_{ij}) = x_{ji}$. This τ induces an autoequivalence on R-modules, which we denote τ^*, satisfying $\tau^* L \cong L^\vee$.

Our next task therefore is to compute $\mathrm{Ext}^1_R(L,L)$. This was first done by R. Ile [Ile04]; we give Ile's argument below. In the interest of broader applicability, we will state the result in terms of general matrix factorizations (φ,ψ) over Noetherian rings S, as Ile does, indicating where "specialization to the generic case" simplifies the arguments still further. The proof uses the *Scandinavian complex* $\mathscr{S}c(\varphi)$ attached to a matrix factorization (φ,ψ) by T. Gulliksen and O. Negård [GN72], which we shall have reason to use again in computing $\operatorname{Ext}^1_R(L,L^\vee)$.

16.11 Definition. Let $\varphi: G \longrightarrow F$ be a homomorphism of free modules of the same (finite) rank n over a Noetherian ring S. Assume that $f = \det\varphi$ is an irreducible nonzerodivisor of S and that $\operatorname{Ann}_S \operatorname{cok}\varphi = (f)$, so that $(\varphi, \operatorname{adj}(\varphi), F, G)$ is a matrix factorization of f. The *Scandinavian complex* $\mathscr{S}c(\varphi)$ is

$$0 \longrightarrow S \xrightarrow{?\operatorname{adj}(\varphi)} \operatorname{Hom}_S(F,G) \xrightarrow{(\varphi?,?\varphi)} \mathbb{H} \xrightarrow{?\varphi-\varphi?} \operatorname{Hom}_S(G,F) \xrightarrow{\operatorname{tr}(?\operatorname{adj}(\varphi))} S \longrightarrow 0\,,$$

where $\mathbb{H}$ is the homology in the middle of the short complex

$$S \xrightarrow{\quad\Delta\quad} \operatorname{End}_S(F) \oplus \operatorname{End}_S(G) \xrightarrow{\operatorname{tr}(?)-\operatorname{tr}(?)} S\,,$$

tr(?) denotes the trace function, and Δ is the diagonal map.

The complex $\mathscr{S}c(\varphi)$ is functorial with respect to homomorphisms of matrix factorizations. Here is the main theorem of [GN72].

16.12 Proposition ([GN72]; see also [BV88])**.** *For φ as above, we have*

$$H_0(\mathscr{S}c(\varphi)) \cong S/I_1(\operatorname{adj}(\varphi)) = S/I_{n-1}(\varphi)\,,$$

and

$$\max\{\, q \mid H_q(\mathscr{S}c(\varphi)) \neq 0\} \quad = \quad 4 - \operatorname{grade} I_{n-1}(\varphi)\,.$$

In particular, if the grade of $I_{n-1}(\varphi)$ on S is 4, the maximum possible value, then $\mathscr{S}c(\varphi)$ is a (minimal, in case S is local or graded and no entry of φ is a unit) S-free resolution of $S/I_{n-1}(\varphi)$.

16.13 Remark. It's well-known (see, for example, [HE71] or [BH93, 7.3.1]) that for $\varphi = X$ a generic square matrix of indeterminates, the maximum value, $\operatorname{grade} I_{n-1}(X) = 4$, is achieved.

16.14 Theorem ([Ile04])**.** *Let S be a Noetherian ring and $\varphi : G \longrightarrow F$ a homomorphism between free S-modules of the same rank n, such that $f = \det\varphi$ is an irreducible nonzerodivisor of S. Set $R := S/(f)$ and $L := \operatorname{cok}\varphi$, and assume that $\operatorname{Ann}_S L = (f)$. Then*

$$H_1(\mathscr{S}c(\varphi)) \cong \operatorname{Ext}^1_R(L,L)\,.$$

In particular, if $\operatorname{grade} I_{n-1}(\varphi) = 4$, then $\operatorname{Ext}^1_R(L,L) = 0$.

Proof. To compute the homology of $\mathscr{S}c(\varphi)$ at $\operatorname{Hom}_S(G,F)$, consider the diagram

(16.14.1)
$$\begin{array}{ccccc}
\mathbb{H} & \xrightarrow{?\varphi-\varphi?} & \operatorname{Hom}_S(G,F) & \xrightarrow{\operatorname{tr}(?\operatorname{adj}(\varphi))} & S \\
 & & {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\overline{\cdot 1_L}} \\
 & & \operatorname{Ext}^1_S(L,L) & \xrightarrow[\epsilon]{} & \operatorname{End}_R(M)\,.
\end{array}$$

Here π is the natural surjection and ϵ is defined by pulling back cocycles along $\operatorname{adj}(\varphi)$. That is, for $\chi \in \operatorname{Ext}^1_S(L,L)$, we choose a preimage $U \in \operatorname{Hom}_S(G,F)$ and observe that $(U\operatorname{adj}(\varphi), \operatorname{adj}(\varphi)U)$ is a homomorphism of matrix factorizations

$$(U\operatorname{adj}(\varphi), \operatorname{adj}(\varphi)\,U) : (\varphi, \operatorname{adj}(\varphi)) \longrightarrow (\varphi, \operatorname{adj}(\varphi))\,;$$

put $\epsilon(\chi) = \operatorname{cok}(U\operatorname{adj}(\varphi), \operatorname{adj}(\varphi)\,U) \in \operatorname{End}_R(M)$.

We claim first that the square commutes. The following lemma is the crux of the argument.

16.15 Lemma. *For each* $U \in \mathrm{Hom}_S(G,F)$*, there exists* $V \in \mathrm{Hom}_S(F,G)$ *such that*

$$U\,\mathrm{adj}(\varphi) - \varphi V = \mathrm{tr}(U\,\mathrm{adj}(\varphi)) \cdot 1_F\,.$$

Proof. For the purposes of this proof, we may revert to the generic situation, where $\varphi = X$ is a square matrix of indeterminates. As above, let $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$ be the partial derivative with respect to the variable x_{ij}; then

$$\begin{aligned}\partial_{ij}[(\det X)\cdot 1_F] &= \partial_{ij}[X\,\mathrm{adj}(X)]\\ &= \partial_{ij}(X)\,\mathrm{adj}(X) + X\partial_{ij}(\mathrm{adj}(X)),\end{aligned}$$

where we apply ∂_{ij} to a matrix entry-by-entry. By the identification of the entries of $\mathrm{adj}(X)$, this can be rewritten as

$$E_{ij}\,\mathrm{adj}(X) + X\partial_{ij}(\mathrm{adj}(X)) = \mathrm{adj}(X)_{ji} \cdot 1_F\,. \tag{16.15.1}$$

Write $U = (u_{ij})$; multiplying (16.15.1) by u_{ij} and taking the sum over all (i,j) gives

$$U\,\mathrm{adj}(X) - XV = \left(\sum_{i,j} u_{ij}\,\mathrm{adj}(X)_{ji}\right) \cdot 1_F\,,$$

with $V = -\sum_{i,j} u_{ij}\partial_{ij}(\mathrm{adj}(X))$. The right-hand side of this last equation is equal to $\mathrm{tr}(U\,\mathrm{adj}(X)) \cdot 1_F$. □

Returning to the proof of Theorem 16.14, we must show that

$$\overline{\mathrm{tr}(U\,\mathrm{adj}(\varphi)) \cdot 1_L} = \mathrm{cok}(U\,\mathrm{adj}(\varphi), \mathrm{adj}(\varphi)\,U)\,,$$

as endomorphisms of L, for each $U \in \mathrm{Hom}_S(G,F)$. By the Lemma, there exists $V \in \mathrm{Hom}_S(F,G)$ so that

$$U\,\mathrm{adj}(\varphi) - \varphi V = \mathrm{tr}(U\,\mathrm{adj}(\varphi)) \cdot 1_F\,.$$

In particular, the two sides induce the same endomorphism of L. The term φV factors through F, so gives the zero map on $L = \operatorname{cok}\varphi$; thus $U\operatorname{adj}(\varphi)$ induces $\overline{\operatorname{tr}(U\operatorname{adj}(\varphi))\cdot 1_L}$.

Next we shall show that $\ker\epsilon \cong \operatorname{Ext}^1_R(L,L)$. Indeed, an S-module extension χ, represented by $U \in \operatorname{Hom}_S(G,F)$, is an extension of R-modules if and only if U is part of a homomorphism of matrix factorizations, *i.e.*, there exists $V \in \operatorname{Hom}_S(F,G)$ so that $U\operatorname{adj}(\varphi) = \varphi V$. This is the case precisely when $U\operatorname{adj}(\varphi)$ factors through G, that is, induces the zero endomorphism of L.

Finally, we claim that π induces an isomorphism $H_1(\mathscr{S}c(\varphi)) \longrightarrow \ker\epsilon$. To see this, first let $[U]$ be a homology class. Then the image of U in $\operatorname{End}_R(L)$ is zero, so that $\pi(U) \in \ker\epsilon$. Next, take $U \in \operatorname{Hom}_S(G,F)$ to be a boundary, so that $U = A\varphi - \varphi B$ for some $(A,B) \in \mathbb{H}$. Then the homomorphism of matrix factorizations induced by $\pi(U)$ is equivalent to $(\varphi B\operatorname{adj}(\varphi), \operatorname{adj}(\varphi)A\varphi)$. Since $\varphi B\operatorname{adj}(\varphi)$ factors through G, this is zero in $\operatorname{End}_R(L)$. Lastly, any $\chi \in \operatorname{Ext}^1_R(L,L)$ lifts to $U \in \operatorname{Hom}_S(G,F)$, which must then be a cycle by the commutativity of the square. This finishes the proof. □

Specializing to the case of a generic matrix, we obtain the following calculations.

16.16 Corollary. *Let K be a commutative Noetherian normal domain, $X = (x_{ij})$ the generic $(n\times n)$–matrix over K, $S = K[x_{ij}]$, and $R = S/(\det X)$. Set $L := \operatorname{cok}X$. Then*

(i) $\operatorname{End}_R(L) \cong R$ *;*

(ii) $\operatorname{Ext}^1_R(L,L) = 0$ *;*

(iii) $\operatorname{Ext}^1_S(L,L)$ *is isomorphic to the ideal* $I_{n-1}(X)/(\det X)$ *of* R *; and*

(iv) $\operatorname{Ext}^2_R(L,L) \cong S/I_{n-1}(X)$.

Proof. We have already observed that R is a normal domain. Since L has rank one, the ring $\operatorname{End}_R(L)$ is a finite extension of R contained in its quotient field, so equal to R by normality. Claims (ii), (iii), and (iv) follow from Theorem 16.14 and the diagram (16.14.1): Since grade $I_{n-1}(X) = 4$, we have $\operatorname{Ext}^1_R(L,L) = 0$, and the image of ϵ is equal to the image of tr(? adj(X)), that is, $I_{n-1}(X)$. □

Everything from here on needs jabber surrounding it.

The other Ext-group between rank-one MCM modules, $\operatorname{Ext}^1_R(L,L^\vee)$, does not vanish. To see this, we consider $\operatorname{Hom}_R(M,L^\vee)$, which has $\operatorname{Ext}^1_R(L,L^\vee)$ as a homomorphic image. Computing $\operatorname{Hom}_R(M,L^\vee)$ amounts to factoring the adjoint in a particular way.

16.17 Proposition. *Let* A *be an* $(n \times n)$ alternating *matrix over* K. *There exists then a unique alternating* $(n \times n)$ *matrix* B_A *satisfying*

$$A \operatorname{adj}(X) = X^T B_A .$$

We refer to B_A *as the* companion matrix *for* A.

When n is even, there exist *invertible* alternating matrices A, so that $Y = A^{-1}X^T$, $Z = B_A$ gives an honest factorization of adj(X).

We omit the proof of Proposition 16.17; the matrix B_A is defined by

$$b_{rs} = \sum_{k<l} a_{kl}(-1)^{r+s+k+l}[rs \widehat{\rceil} kl](X),$$

where $A = (a_{kl})_{kl}$ and $[rs \rceil kl](X)$ denotes the (unsigned) determinant of the $(n-2) \times (n-2)$ submatrix of X obtained by deleting rows r and s and columns k and l. We use the convention that $[rs \widehat{|} kl]$ is alternating in both arguments, so in particular vanishes if there is a repetition of indices on either side of the vertical bar.

The equation $A\,\mathrm{adj}(X) = X^T B_A$ defines a commutative diagram of free S-modules

$$\begin{array}{ccccc} G & \xrightarrow{X} & F & \xrightarrow{\mathrm{adj}(X)} & G \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle B_A} & & \\ F & \xrightarrow[\mathrm{adj}(X)^T]{} & G & \xrightarrow[X^T]{} & F\,, \end{array}$$

that is, a homomorphism of matrix factorizations

$$(A, B_A) : (\mathrm{adj}(X), X) \longrightarrow (X^T, \mathrm{adj}(X)^T)$$

and thus a homomorphism of MCM R-modules

$$\mathrm{cok}(A, B_A) : M \longrightarrow L^\vee .$$

In other words, we have a homomorphism $\mathrm{Alt}_n(S) \xrightarrow{A \mapsto \mathrm{cok}(A,B_A)} \mathrm{Hom}_R(M, L^\vee)$. Our next result is that this homomorphism is surjective, so that $\mathrm{Hom}_R(M, L^\vee)$ is generated by the alternating matrices, and moreover that $\mathrm{Hom}_R(M, L^\vee)$ is itself a MCM R-module.

16.18 Theorem. *The R-module $\mathrm{Hom}_R(M, L^\vee)$ is MCM of rank $n-1$, minimally generated by $\binom{n}{2}$ elements. More precisely, it has the following free presentation as an S-module*

$$0 \longrightarrow \mathrm{Alt}_n(S) \xrightarrow{U \mapsto X^T U X} \mathrm{Alt}_n(S) \xrightarrow{A \mapsto \mathrm{cok}(A,B_A)} \mathrm{Hom}_R(M, L^\vee) \longrightarrow 0\,;$$

alternatively, in terms of exterior powers, this exact sequence can be written as

$$0 \longrightarrow \textstyle\bigwedge^2 F^\vee \xrightarrow{\bigwedge^2 X^T} \textstyle\bigwedge^2 G^\vee \longrightarrow \mathrm{Hom}_R(M, L^\vee) \longrightarrow 0\,.$$

Proof. For U an alternating $(n \times n)$–matrix over S, we have

$$\begin{aligned} X^T B_{X^TUX} &= (X^TUX)\operatorname{adj}(X) \\ &= X^TU \cdot (\det X), \end{aligned}$$

so that $B_{X^TUX} = U \cdot (\det X)$. Thus the homomorphism $\operatorname{cok}(X^TUX, B_{X^TUX})$ is zero on the R-module M, and the alleged resolution of $\operatorname{Hom}_R(M, L^\vee)$ is at least a complex.

Put $D := \operatorname{cok}(U \mapsto X^TUX)$. Then D maps to $\operatorname{Hom}_R(M, L^\vee)$ and we must show that this map is an isomorphism. Note first that D is a MCM R-module, with matrix factorization

$$(U \mapsto X^TUX, A \mapsto B_A).$$

Indeed, we have seen that $B_{X^TUX} = U \cdot (\det X)$, and also $X^T B_A X = A \operatorname{adj}(X) X = A \cdot (\det X)$. Thus in particular D is a reflexive R-module, and $U \mapsto X^TUX$ is an injective endomorphism of the module of alternating matrices.

The free module $\operatorname{Alt}_n(S)$ has rank $\binom{n}{2}$, so the determinant of the endomorphism $U \mapsto X^TUX$ is homogeneous of degree $n(n-1)$ in the variables x_{ij}. Since it must also be a unit times $(\det X)^{\operatorname{rank} D}$, we see that D has rank $n-1$ as an R-module, equal to that of $\operatorname{Hom}_R(M, L^\vee)$.

Outside the singular locus $V(I_{n-1}(X))$ of R, at least one maximal minor of X^T is a unit. Thus after elementary transformations and linear changes of variables, $X^T = \operatorname{diag}(0, 1, \ldots, 1)$ and so $\operatorname{adj}(X)^T = E_{11}$, the elementary matrix with 1 at position $(1,1)$ and zeros elsewhere. Now any homomorphism α from the cokernel of E_{11} to the cokernel of $\operatorname{diag}(0, 1, \ldots, 1)$ is induced by an alternating $(n \times n)$–matrix, namely any alternating matrix with first row α. That is, outside the singular locus of R, $\operatorname{Hom}_R(M, L^\vee)$

is indeed generated by homomorphisms $\operatorname{cok}(A,B_A)$ for alternating A. The map $D \longrightarrow \operatorname{Hom}_R(M,L^\vee)$ is thus surjective, and since D and $\operatorname{Hom}_R(M,L^\vee)$ have the same rank, is even an isomorphism, outside $V(I_{n-1}(X))$.

Recall that R is normal and $I_{n-1}(X)$ has codimension 3 in $\operatorname{Spec} R$. The homomorphism $D \longrightarrow \operatorname{Hom}_R(M,L^\vee)$ is thus a homomorphism between reflexive modules over a normal domain, which is an isomorphism in codimension one. It follows that in fact $D \longrightarrow \operatorname{Hom}_R(M,L^\vee)$ is an isomorphism. □

To use this description of $\operatorname{Hom}_R(M,L^\vee)$ to understand $\operatorname{Ext}^1_R(L,L^\vee)$, we need one more fact about matrix factorizations. It is straighforward to verify, but key in what follows.

16.19 Proposition. *Let* $(\alpha,\beta):(\varphi_1,\psi_1,F_1,G_1) \longrightarrow (\varphi_2,\psi_2,F_2,G_2)$ *be a homomorphism of matrix factorizations of* $f \in S$*, set* $R = S/(f)$*, and put* $M_i = \operatorname{cok}(\varphi_i,\psi_i), N_i = \operatorname{cok}(\psi_i,\varphi_i)$ *for* $i = 1,2$*. Then the bottom row of the pushout diagram of* R*-modules*

(16.19.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & \overline{G_1} & \longrightarrow & N_1 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \operatorname{cok}(\alpha,\beta)} & & \Big\downarrow & & \Big\| & & \\ 0 & \longrightarrow & M_2 & \longrightarrow & Q & \longrightarrow & N_1 & \longrightarrow & 0 \end{array}$$

defines an element of $\operatorname{Ext}^1_R(N_1,M_2)$*, which is the image of* $\operatorname{cok}(\alpha,\beta)$ *under the natural surjection* $\operatorname{Hom}_R(M_1,M_2) \longrightarrow \operatorname{Ext}^1_R(N_1,M_2)$*. The module* Q *is again given by a matrix factorization, namely*

$$Q \cong \operatorname{cok}\left(\begin{pmatrix} \varphi_2 & \alpha \\ 0 & \psi_1 \end{pmatrix}, \begin{pmatrix} \psi_2 & -\beta \\ 0 & \varphi_1 \end{pmatrix}\right).$$

Note also that if, in the notation of Proposition 16.19, $\operatorname{cok}(\alpha,\beta)$ factors through a projective R-module, then the bottom row of (16.19.1) splits, and vice versa. In this case, $\operatorname{cok}(\alpha,\beta)$ factors through $\overline{G}_1$, and we have $Q \cong M_2 \oplus N_1$.

Applied to Theorem 16.18, this fact implies the following structure for $\operatorname{Ext}^1_R(L,L^\vee)$.

16.20 Proposition. *For each alternating $(n \times n)$–matrix A over S, there exists an extension*

(16.20.1) $$0 \longrightarrow L^\vee \longrightarrow Q \longrightarrow L \longrightarrow 0\,,$$

which is the image of $\operatorname{cok}(A,B_A)$ *under the natural epimorphism* $\operatorname{Hom}_R(M,L^\vee) \longrightarrow \operatorname{Ext}^1_R(L,L^\vee)$. *In particular, the module Q is a MCM R-module of rank* 2 *given by the matrix factorization*

$$Q = \operatorname{cok}\left(\begin{pmatrix} X^T & A \\ 0 & X \end{pmatrix}, \begin{pmatrix} \operatorname{adj}(X)^T & -B_A \\ 0 & \operatorname{adj}(X) \end{pmatrix}\right).$$

Furthermore, Q is orientable.

The matrix factorization given for Q in Proposition 16.20 may not be of minimal size. Indeed, if A is invertible then one can see that Q requires only n generators. In this case, we have

$$\begin{pmatrix} I_n & 0 \\ -XA^{-1} & I_n \end{pmatrix}\begin{pmatrix} X^T & A \\ 0 & X \end{pmatrix}\begin{pmatrix} I_n & 0 \\ -A^{-1}X^T & I_n \end{pmatrix} = \begin{pmatrix} 0 & A \\ -XA^{-1}X^T & 0 \end{pmatrix},$$

so that the given matrix factorization for Q can be reduced to

$$Q \cong \operatorname{cok}(XA^{-1}X^T, B_A)\,.$$

More generally, if a $(k \times k)$–minor of A is invertible, then the given matrix factorization of Q can be reduced to one of size $2n-k$.

For any n, the graded, orientable rank two MCM R-modules are minimally evenly generated [HK87, 3.1]. In fact, they are presented by yet another alternating matrix over S, as in [BEPP05].

The orientable MCM module Q of Proposition 16.20 is decomposable if and only if $Q \cong L \oplus L^\vee$, equivalently, the sequence (16.20.1) is split exact. To see this, recall that L and $L^\vee$ are up to isomorphism the only MCM R-modules of rank one. As Q is orientable, the only possible direct-sum decomposition for Q is $L \oplus L^\vee$, and by Miyata's theorem, if (16.20.1) is apparently split then it is split.

We state without proof some further facts about $\operatorname{Ext}^1_R(L, L^\vee)$. (Are they used below?)

16.21 Proposition. *As an S-module, E is perfect of grade 4, with support the singular locus $V(I_{n-1}(X))$ of R. More precisely, the annihilator of E is equal to $I_{n-1}(X)$. Its minimal graded free resolution over S is*

$$
\begin{array}{ccccccccc}
 & & & & \operatorname{Sym}_2 G^\vee & & & & \\
 & & & \overset{X^T?+?^TX}{\nearrow} & & \overset{-\operatorname{adj}(X)}{\searrow} & & & \\
0 \longrightarrow \bigwedge^2 F^\vee & \xrightarrow{?X} & F^\vee \otimes_K G^\vee & & & & G^\vee \otimes_K F^\vee & \xrightarrow{?X - X^T?^T} & E \longrightarrow 0 \\
 & & & \underset{?\operatorname{adj}(X)+\operatorname{adj}(X)^T?^T}{\searrow} & & \underset{X^T?}{\nearrow} & & & \\
 & & & & \mathbb{D}_2 F^\vee & & & &
\end{array}
$$

where $\mathbb{D}$ is the kernel of the canonical projection $F \otimes_S F \longrightarrow \bigwedge^2 F$ sending $A \mapsto A - A^T$

As an $S/I_{n-1}(X)$-module, $E = \operatorname{Ext}^1_R(L, L^\vee)$ is a MCM module of rank one, isomorphic to the ideal generated by the maximal minors of $n-2$ fixed rows of X.

The extensions of Proposition 16.20 define rank-two MCM modules Q, and those extensions are classified by the explicit minimal graded free resolution given in Proposition 16.21. Of course inequivalent extensions may have isomorphic middle modules. To describe the MCM modules that appear as middle terms Q, we consider a more general problem.

Let A and B be finitely generated modules over a (commutative, Noetherian) ring R. Fix free resolutions

$$\cdots \longrightarrow P_2 \xrightarrow{X_2} P_1 \xrightarrow{X_1} P_0 \longrightarrow A \longrightarrow 0$$

$$\cdots \longrightarrow Q_2 \xrightarrow{Y_2} Q_1 \xrightarrow{Y_1} Q_0 \longrightarrow B \longrightarrow 0$$

of A and B. An element $\chi \in \operatorname{Ext}^1_R(A,B)$ is an equivalence class of extensions $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ and the isomorphism class of E is determined by χ. The Horseshoe Lemma provides a free resolution of E

$$\cdots \longrightarrow Q_2 \oplus P_2 \xrightarrow{\begin{bmatrix} Y_2 & Z_2 \\ 0 & X_2 \end{bmatrix}} Q_1 \oplus P_1 \xrightarrow{\begin{bmatrix} Y_1 & Z_1 \\ 0 & X_1 \end{bmatrix}} Q_0 \oplus P_0 \longrightarrow E \longrightarrow 0\,.$$

Here the Z_i are homomorphisms in $\operatorname{Hom}_R(P_i, Q_{i-1})$ satisfying $Y_i Z_{i+1} + Z_i X_{i+1} = 0$ for all $i \geqslant 1$.

16.22 Definition. In the situation above, define a sequence of rings

$$R_i := R/(I_1(X_i) + I_1(Y_i))$$

for $i = 1, 2, \ldots$, where as usual $I_1(U)$ is the ideal of R generated by the entries of U. For each i set

$$\mathscr{J}_i(\chi) = \frac{I_1(Z_i) + I_1(X_i) + I_1(Y_i)}{I_1(X_i) + I_1(Y_i)}\,,$$

an ideal of R_i.

It is straightforward to check that the ideals $\mathscr{I}_i(\chi) \subseteq R_i$ are well-defined. In fact, the $\mathscr{I}_i(\chi)$ are invariants of the isomorphism class of the middle term of χ:

16.23 Proposition. *Let $\chi, \chi' \in \operatorname{Ext}^1_R(A,B)$ have middle terms E, E'. If $E \cong E'$, then $\mathscr{I}_i(\chi) = \mathscr{I}_i(\chi')$ for all i.*

The function $\mathscr{I}_i(?)$ thus defines a map from isomorphism classes of modules E appearing as extensions of B by A to ideals of R_i. We can identify which ideals are in the image of $\mathscr{I}_1$.

16.24 Proposition. *Let $Z_1 \colon P_1 \longrightarrow Q_0$ and $Z_2 \colon P_2 \longrightarrow Q_1$ be homomorphisms of free modules such that $Y_1 Z_2 + Z_1 X_2 = 0$. Then there exists $\chi \in \operatorname{Ext}^1_R(A,B)$ such that $\mathscr{I}_1(\chi) = I_1(Z_1) R_1$.*

Proof. Set $E = \operatorname{cok} \begin{bmatrix} Y_1 & Z_1 \\ 0 & X_1 \end{bmatrix}$, so that we have a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & B & \longrightarrow & E & \longrightarrow & A & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Q_0 & \longrightarrow & Q_0 \oplus P_0 & \longrightarrow & P_0 & \longrightarrow & 0 \\
 & & {\scriptstyle Y_1}\uparrow & & \uparrow{\scriptstyle \left[\begin{smallmatrix} Y_1 & Z_1 \\ 0 & X_1 \end{smallmatrix}\right]} & & \uparrow{\scriptstyle X_1} & & \\
0 & \longrightarrow & Q_0 & \longrightarrow & Q_0 \oplus P_0 & \longrightarrow & P_0 & \longrightarrow & 0 \\
 & & {\scriptstyle Y_2}\uparrow & & \uparrow{\scriptstyle \left[\begin{smallmatrix} Y_2 & Z_2 \\ 0 & X_2 \end{smallmatrix}\right]} & & \uparrow{\scriptstyle X_2} & & \\
0 & \longrightarrow & Q_0 & \longrightarrow & Q_0 \oplus P_0 & \longrightarrow & P_0 & \longrightarrow & 0
\end{array}
$$

with exact rows and columns. The map $E \longrightarrow A$ is surjective by commutativity. To see that $B \longrightarrow E$ is injective, it is equivalent by the Snake Lemma to see that the kernel of $\left[\begin{smallmatrix} Y_1 & Z_1 \\ 0 & X_1 \end{smallmatrix}\right]$ maps onto the kernel of X_1. This is a straightforward calculation using $Y_1Z_2 + Z_1X_2 = 0$. □

Assume now that $R = S/(f)$ is a hypersurface ring and A, B are MCM modules over R. The free resolutions of A and B are periodic of period 2, given by matrix factorizations of f. Write $A = \operatorname{cok}(\varphi, \psi)$ and $B = \operatorname{cok}(\varphi', \psi')$. Then the sequence of rings R_i is periodic: we have

$$R_i = \begin{cases} S/(I_1(\varphi) + I_1(\varphi')) & \text{for } i \text{ odd, and} \\ S/(I_1(\psi) + I_1(\psi')) & \text{for } i \text{ even.} \end{cases}$$

For $\chi \in \operatorname{Ext}^1_R(A,B)$, the ideals $\mathscr{I}_1(\chi) \subseteq R_1$ and $\mathscr{I}_2(\chi) \subseteq R_2$ are again invariants of the middle term of χ.

Return now to the generic determinant, with notation as earlier in the section. Consider $\operatorname{Ext}^1_R(L, L^\vee)$. Since $L = \operatorname{cok}(X, \operatorname{adj}(X))$ and $L^\vee = \operatorname{cok}(X^T, \operatorname{adj}(X)^T)$, we have

$$R_i = \begin{cases} S/I_1(X) \cong K & \text{for } i \text{ odd, and} \\ S/I_1(\operatorname{adj}(X) = S/I_{n-1}(X) & \text{for } i \text{ even.} \end{cases}$$

By Proposition 16.21 and Proposition 16.20, every element $\chi \in \operatorname{Ext}^1_R(L, L^\vee)$ is of the form

$$\chi : \qquad 0 \longrightarrow L^\vee \longrightarrow Q \longrightarrow L \longrightarrow 0$$

with

$$Q \cong \operatorname{cok}\left(\begin{pmatrix} X & A \\ 0 & X^T \end{pmatrix}, \begin{pmatrix} \operatorname{adj}(X) & -B_A \\ 0 & \operatorname{adj}(X)^T \end{pmatrix} \right)$$

for some alternating matrix A over S and its companion matrix B_A. We therefore have $\mathscr{I}_1(\chi) = I_1(A)K$ and $\mathscr{I}_2(\chi) = I_1(B_A)S/I_{n-1}(X)$. In particular, for each ideal of $S/I_{n-1}(X)$ of the form $I_1(B_A)$, where B_A is the companion matrix for some alternating matrix A, there exists an orientable MCM R-module Q of rank 2, and distinct ideals yield nonisomorphic modules Q. More precisely, we have the following result.

16.25 Proposition. *There is a surjective function from the isomorphism classes of rank-two MCM R-modules appearing as the middle terms of extensions of L by $L^\vee$ to the set of principal ideals of the polynomial ring in $(n-2)^2$ variables.*

Proof. Let X' be the generic square matrix of size $n-2$, with entries x'_{ij}, $1 \leqslant i,j \leqslant n-2$. Let $S' = K[x'_{ij}]$ be the polynomial ring over K in those indeterminates x'_{ij}, and define $\pi : S \longrightarrow S'$ by $\pi(x_{ij}) = x'_{ij}$ if $i,j \leqslant n-2$ and $\pi(x_{ij}) = 0$ otherwise. The $(n-1)$-minors of X vanish under π, so we obtain an induced epimorphism $\pi : S/I_{n-1}(X) \longrightarrow S'$. Note that all $(n-2)$-minors of X vanish under π as well, save $[n-1,n \,\widehat{|}\, n-1,n]$, which maps to $\det X'$.

Let $\chi \in \operatorname{Ext}^1_R(L,L^\vee)$. Then χ is the image of an alternating matrix A, and the ideal $\mathscr{I}_2(\chi) \subseteq S/I_{n-1}(X)$ is generated by the entries of the companion matrix B_A. Again, $\mathscr{I}_2(\chi)$ depends only on the isomorphism class of the middle term of χ. Recall (Proposition 16.17) that

$$b_{rs} = \sum_{k<l} a_{kl}(-1)^{r+s+k+l}[rs \,\widehat{|}\, kl].$$

The image of $\mathscr{I}_2(\chi)$ in S', then, is generated by the single element $\pi(a_{n-1,n})\cdot \det X'$.

Define $p : \operatorname{Ext}^1_R(L,L^\vee) \longrightarrow \{\text{ideals of } S'\}$ by $p(\chi) = (\pi(a_{n-1,n}))$. Since $\det X'$ is a nonzerodivisor in S', $p(\chi)$ is a well-defined ideal of S'. Letting A vary over all alternating matrices, we see that p is surjective, and by construction $p(\chi)$ depends only on the isomorphism class of the middle term of χ. $\square$

16.26 Corollary. *Let $X = (x_{ij})$ be the generic $(n \times n)$–matrix over the field K, $n \geqslant 3$. Let $R = K[x_{ij}]/(\det X)$ be the generic determinantal hypersurface ring. Then the rank-two orientable MCM R-modules cannot be parametrized by the points of any finite-dimensional algebraic variety over K.*

A

Appendix: Basics

Here we collect some basic definitions and results that are necessary but somewhat peripheral to the main themes of the book. Some of the results are stated without proof; for these, one can find proofs in [Mat86]. We refer to [Mat86] also for any unexplained terminology.

§1 Depth, Serre's conditions and syzygies

Throughout this section we let $(R,\mathfrak{m},k)$ be a local ring.

A.1 Definition. Let M be a finitely generated R-module. The *depth* of M is given by

$$\operatorname{depth}_R(M) = \inf\left\{n \,\middle|\, \operatorname{Ext}^n_R(k,M) \neq 0\right\}.$$

Note that $\operatorname{depth}_R(0) = \inf(\emptyset) = \infty$. Conversely, non-zero modules have finite depth:

A.2 Proposition. *Let M be a non-zero finitely generated R-module.*

(i) $\operatorname{depth}_R(M) < \infty$.

(ii) $\operatorname{depth}_R(M) = \sup\{n \mid$ *there is an M-regular sequence* $(x_1,\dots,x_n)$ *in* $\mathfrak{m}\}$.

(iii) Every maximal M-regular sequence in $\mathfrak{m}$ has length n.

(iv) $\operatorname{depth}_R(M) \leqslant \dim(R/\mathfrak{p})$ *for every* $\mathfrak{p} \in \operatorname{Ass}(M)$. *In particular,* $\operatorname{depth}(M) \leqslant \dim(M) \leqslant \dim(R)$.

(v) *If* $(S,\mathfrak{n}) \longrightarrow (R,\mathfrak{m})$ *is a local homomorphism and* R *is finitely generated as an* S*-module, then* $\operatorname{depth}_S(M) = \operatorname{depth}_R(M)$.

(vi) *If* $\mathfrak{p} \in \operatorname{Spec}(R)$*, then* $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0 \iff \mathfrak{p} \in \operatorname{Ass}(M)$. □

When the base ring R is clear, or when, e.g. as in item (v) it is irrelevant, we often omit the subscript and write "depth(M)".

The next result is called the *Depth Lemma*. It follows easily from the long exact sequence of Ext.

A.3 Lemma. *Let* $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ *be a short exact sequence of finitely generated* R*-modules.*

(i) *If* $\operatorname{depth}(W) < \operatorname{depth}(V)$*, then* $\operatorname{depth}(U) = \operatorname{depth}(W) + 1$.

(ii) $\operatorname{depth}(U) \geqslant \min\{\operatorname{depth}(V), \operatorname{depth}(W)\}$.

(iii) $\operatorname{depth}(V) \geqslant \min\{\operatorname{depth}(U), \operatorname{depth}(W)\}$. □

See [Mat86, Theorem 19.1] for a proof of the next result, the Auslander-Buchsbaum Formula. We write $\operatorname{pd}_R(M)$ for the projective dimension of an R-module M.

A.4 Theorem (Auslander-Buchsbaum Formula)**.** *Let* M *be an* R*-module of finite projective dimension. Then* $\operatorname{depth}(M) + \operatorname{pd}_R(M) = \operatorname{depth}(R)$.

A.5 Definition. Let M be a finitely generated module over a local ring $(R,\mathfrak{m})$, and let n be a non-negative integer. Then M satisfies *Serre's condition* (S_n) provided

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant \min\left\{n, \dim(R_\mathfrak{p})\right\} \text{ for every } \mathfrak{p} \in \operatorname{Spec}(R).$$

(**Warning:** Our terminology differs from that of EGA [GD65, Definition 5.7.2] and Bruns-Herzog [BH93, Section 2.1], where "dim($R_\mathfrak{p}$)" is replaced by "dim($M_\mathfrak{p}$)". Notice, for example, that by the EGA definition every finite-length module would satisfy (S_n) for all n, while this is certainly not the case with the definition we use. Of course, the two conditions agree for the ring itself.)

A.6 Proposition. *These are equivalent for a local ring* $(R, \mathfrak{m})$.

(i) *R is reduced.*

(ii) *R satisfies (S_2), and $R_\mathfrak{p}$ is a field for every minimal prime ideal $\mathfrak{p}$.* □

The next result is called Serre's criterion for normality:

A.7 Proposition. *These are equivalent for a local ring* $(R, \mathfrak{m})$.

(i) *R is a normal domain.*

(ii) *R satisfies (S_2), and $R_\mathfrak{p}$ is a regular local ring for each prime ideal $\mathfrak{p}$ of height at most one.* □

We will say that a finitely generated module M over a ring R is an r^{th} *syzygy* (of N), provided there is an exact sequence

(A.7.1) $$0 \longrightarrow M \longrightarrow F_{r-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow N \longrightarrow 0,$$

where N is a finitely generated module and each F_i is a finitely generated projective module.

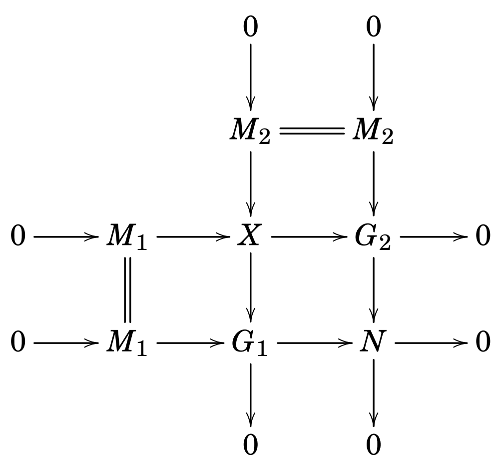
Syzygies are unique up to projective summands, by Schanuel's Lemma:

A.8 Lemma (Schanuel's Lemma)**.** *Let M_1 and M_2 be r^{th} syzygies of N. Then there are finitely generated projective R-modules G_1 and G_2 such that $M_1 \oplus G_2 \cong M_2 \oplus G_1$.*

Proof. By an easy induction argument, it suffices to do the case $r = 1$. Thus we have exact sequences

$$\begin{aligned} &0 \longrightarrow M_1 \longrightarrow G_1 \longrightarrow N \longrightarrow 0 \\ &0 \longrightarrow M_2 \longrightarrow G_2 \longrightarrow N \longrightarrow 0 \end{aligned} \tag{A.8.1}$$

Form the pullback of the two sequences in (A.8.1), getting an exact, commutative diagram.



Since G_1 and G_2 are projective, the top horizontal and left vertical short exact sequences split, and the result follows. □

If R is local and each F_i is chosen minimally, then (reduced) syzygies are unique up to isomorphism, and we let $\operatorname{syz}_r^R(N)$ denote the r^{th} syzygy with respect to a minimal resolution. We define $\operatorname{redsyz}_r^R(N)$ to be the *reduced*

r^{th} *syzygy,* obtained from $\operatorname{syz}_r^R(N)$ by deleting all non-zero free direct summands. The uniqueness is an immediate consequence of the next proposition, which says that minimal resolutions over a local ring are essentially unique.

A.9 Proposition. *Let* $(F_\bullet, d)$ *and* $(G_\bullet, e)$ *be minimal free resolutions of a finitely generated module* M *over a local ring* $(R, \mathfrak{m})$*. Then the matrices* d_i *and* e_i *are equivalent for each* i*.*

Proof. Using projectivity of the F_i, we can fill in vertical arrows, starting with $F_0 \longrightarrow G_0$ and working to the left, in such a way that the following diagram commutes.

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & F_n & \xrightarrow{d_n} & F_{n-1} & \longrightarrow \cdots \longrightarrow & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\cdots \longrightarrow & G_n & \xrightarrow[e_n]{} & G_{n-1} & \longrightarrow \cdots \longrightarrow & G_1 & \xrightarrow[e_1]{} & G_0 & \xrightarrow[e_0]{} & M & \longrightarrow & 0
\end{array}$$

By minimality, $F_0 \cong G_0$, and Nakayama's Lemma implies that the map $F_0 \longrightarrow G_0$ is surjective, hence an isomorphism. Proceed inductively to show that $F_i \cong G_i$ for each i and that each vertical arrow is an isomorphism. □

A.10 Definition. Let R be a Noetherian ring and M a finitely generated R-module. Say that M is *torsion-free* if every non-zerodivisor in R is a non-zerodivisor on M. Equivalently, the natural map $M \longrightarrow K \otimes_R M$, where K is the total quotient ring, is injective.

A.11 Definition. Let R be a Noetherian ring and M a finitely generated R-module. Set $M^* = \operatorname{Hom}_R(M, R)$, the *dual* of M, and $M^{**} = \operatorname{Hom}_R(M^*, R)$, the *bidual.* Define

$$\sigma_M \colon M \longrightarrow M^{**}$$

by $\sigma_M(x)(f) = f(x)$ for $x \in M$ and $f \in M^*$. Say that

(i) M is *torsionless* if σ_M is injective, and

(ii) M is *reflexive* if σ_M is bijective.

There are some implications among torsionlessness, torsion-freeness, and reflexivity.

A.12 Proposition. *Let R be a Noetherian ring and M a finitely generated R-module.*

(i) If M is torsionless, then M is torsion-free.

(ii) M is torsionless if and only if M is a first syzygy.

(iii) If M is reflexive then M is a second syzygy.

See Proposition A.15 below for a converse to (iii).

The properties of torsionlessness and reflexivity are detected by a weakened form of the (S_n) conditions.

A.13 Proposition. *Let R be a Noetherian ring and M a finitely generated R-module.*

(i) M is torsionless if and only if $M_\mathfrak{p}$ is torsionless over $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} R$ and

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geqslant \min\{1, \operatorname{depth} R_\mathfrak{p}\}$$

for every $\mathfrak{p} \in \operatorname{Spec} R$.

(ii) *M is reflexive if and only if $M_\mathfrak{p}$ is reflexive over $R_\mathfrak{p}$ for all primes $\mathfrak{p}$ of height at most one and*

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geqslant \{2, \operatorname{depth} R_\mathfrak{p}\}$$

for every $\mathfrak{p} \in \operatorname{Spec} R$.

A.14 Corollary. *Let R be a normal domain and M a finitely generated R-module. If M is MCM, then M is reflexive. The converse holds if R has dimension two.*

(This material needs to be combined with what comes next.)

Recall that a local ring $(R, \mathfrak{m})$ is *Gorenstein* provided R has finite injective dimension as an R-module. In this case, $\operatorname{injdim}_R(R) = \dim(R)$, by Lemma 9.1. Maximal Cohen-Macaulay modules are reflexive over a Gorenstein ring. (Of course, this is included in the much more general Theorem 9.7.) In fact, it suffices to prove reflexivity of MCM modules for Gorenstein rings of dimension 0 (where it is easy) and 1 (where the result can be found already in Bass' ubiquity paper [Bas63]). We quote the following result from Evans' and Griffith's book [EG85, Theorem 3.6]:

A.15 Proposition. *Let R be a local ring satisfying (S_2), and assume $R_\mathfrak{p}$ is Gorenstein for each prime of height $\leqslant 1$. These conditions are equivalent, for a finitely generated R-module M:*

(i) *M is a second syzygy of some finitely generated R-module.*

(ii) *M is reflexive.*

(iii) *M satisfies (S_2).*

The following result is proved, but not quite stated correctly, in [EG85].

A.16 Theorem. *Let* $(R,\mathfrak{m})$ *be a local ring and* M *a finitely generated* R*-module satisfying Serre's condition* (S_n)*, where* $n \geqslant 1$*. Assume*

(i) R *satisfies* (S_{n-1})*, and*

(ii) $R_{\mathfrak{p}}$ *is Gorenstein for every prime* $\mathfrak{p}$ *with* $\dim(R_{\mathfrak{p}}) \leqslant n-1$*.*

Then there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} F \longrightarrow N \longrightarrow 0, \tag{A.16.1}$$

in which F *is a finitely generated free module and* N *satisfies* (S_{n-1})*.*

Proof. We start with an exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow M^* \longrightarrow 0, \tag{A.16.2}$$

where G is a finitely generated free module and $M^* = \operatorname{Hom}_R(M,R)$. Put $F = G^*$, and dualize (A.16.2), getting an exact sequence

$$0 \longrightarrow M^{**} \xrightarrow{\beta} F \longrightarrow K^* \longrightarrow \operatorname{Ext}^1_R(M^*,R) \longrightarrow 0. \tag{A.16.3}$$

Let $\sigma\colon M \longrightarrow M^{**}$ be the canonical map, let $\alpha = \beta\sigma$, and put $N = \operatorname{cok}\alpha$.

To verify exactness of (A.16.1), we just have to show that σ is one-to-one. Supposing, by way of contradiction, that $L = \ker(\sigma)$ is non-zero, we choose $\mathfrak{p} \in \operatorname{Ass}(L)$. Given any minimal prime $\mathfrak{q}$, we know $R_{\mathfrak{q}}$ is Gorenstein (since $n \geqslant 1$), and hence $\sigma_{\mathfrak{q}}$ is an isomorphism by Proposition A.15. Thus $L_{\mathfrak{q}} = 0$ for each minimal prime $\mathfrak{q}$. In particular, $\dim(R_{\mathfrak{p}}) \geqslant 1$, so $\operatorname{depth}(M_{\mathfrak{p}}) \geqslant 1$. But this contradicts the fact that $\operatorname{depth}(L_{\mathfrak{p}}) = 0$.

Let $\mathfrak{p}$ be a prime of height h. If $h \leqslant n-1$, we need to show that $N_\mathfrak{p}$ is MCM. Since $R_\mathfrak{p}$ is Gorenstein and $M_\mathfrak{p}$ is MCM, the canonical map $\sigma_\mathfrak{p}$ is an isomorphism. Also, $M_\mathfrak{p}^*$ is MCM, so $\operatorname{Ext}^1_{R_\mathfrak{p}}(M_\mathfrak{p}^*, R_\mathfrak{p}) = 0$. The upshot of all of this is that $N_\mathfrak{p} \cong K_\mathfrak{p}^*$. Now (A.16.2) shows that $K_\mathfrak{p}$ is MCM, and therefore so is its dual $K_\mathfrak{p}^*$.

To complete the proof that N satisfies (S_{n-1}), we assume now that $h \geqslant n$. We need to show that $\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \geqslant n-1$. Suppose $\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) < n-1$. Since $\operatorname{depth}_{R_\mathfrak{p}}(F_\mathfrak{p}) \geqslant n-1$, the Depth Lemma A.3, applied to (A.16.1), shows that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 1 + \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) < n$, a contradiction. □

A.17 Corollary. *Let $(R,\mathfrak{m})$ be a local ring, M a finitely generated R-module and n a positive integer. Assume R satisfies Serre's condition (S_n) and $R_\mathfrak{p}$ is Gorenstein for each prime $\mathfrak{p}$ of height at most $n-1$. These are equivalent.*

(i) M is an n^{th} syzygy.

(ii) M satisfies (S_n).

Proof. (i) $\Longrightarrow$ (ii) by the Depth Lemma, and (ii) $\Longrightarrow$ (i) by Theorem A.16. □

A.18 Corollary. *Let $(R,\mathfrak{m})$ be a CM local ring of dimension d, and assume that $R_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$. These are equivalent, for a finitely generated R-module M.*

(i) M is MCM.

(ii) M is a d^{th} syzygy. □

The following special case of Theorem 9.3 follows easily:

A.19 Corollary. *Let M be a MCM module over a CM local ring $(R,\mathfrak{m})$. Assume that $R_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$. Then $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > 0$.*

Proof. Since M is a d^{th} syzygy of some R-module N, and since $\operatorname{injdim}_R(R) = d$, we have $\operatorname{Ext}^i_R(M,R) = \operatorname{Ext}^{i+d}_R(N,R) = 0$. □

A.20 Remark. The hypothesis that R be Gorenstein on the punctured spectrum cannot be weakened, at least when R has a canonical module or, more generally, a *Gorenstein module* [Sha70], that is, a finitely generated module whose completion is a direct sum of copies of the canonical module $\omega_{\widehat{R}}$. Let $(R,\mathfrak{m})$ be a d-dimensional CM local ring having a Gorenstein module G. If G is a d^{th} syzygy, then R is Gorenstein on the punctured spectrum. To see this, we build an exact sequence

(A.20.1) $$0 \longrightarrow G \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is free and M is a $(d-1)^{\text{st}}$ syzygy. Now let $\mathfrak{p}$ be any non-maximal prime ideal. Since $M_\mathfrak{p}$ is MCM and $G_\mathfrak{p}$ is Gorenstein, (A.20.1) splits when localized at $\mathfrak{p}$ (apply Proposition 9.4). But then $G_\mathfrak{p}$ is free, and it follows that $R_\mathfrak{p}$ is Gorenstein. (We thank Bernd Ulrich for showing us this argument (cf. also [LW00, Lemma 1.4]).)

§2 Rank and multiplicity

In this section we recall the definition and basic properties of the multiplicity.

A.21 Definition. Let M be a finitely generated module over a local ring $(R,\mathfrak{m})$ of dimension d. The *multiplicity* of M is defined by

$$\mathrm{e}(M) = \lim_{n\longrightarrow\infty} \frac{d!}{n^d}\ell_R(M/\mathfrak{m}^n M),$$

where $\ell_R(-)$ denotes length as an R-module.

A.22 Theorem. *Let $(R,\mathfrak{m})$ be a local ring of dimension d.*

(i) Let M be a finitely generated R-module. The multiplicity $\mathrm{e}(M)$ is a non-negative integer, and $\mathrm{e}(M) > 0$ if and only if $\dim(M) = d$.

(ii) Multiplicity is additive on exact sequences of finitely generated modules.

A.23 Theorem. *Let $(R,\mathfrak{m})$ be a one-dimensional local ring of dimension one.*

(i) The multiplicity $\mathrm{e}(R)$ of R is the number of generators required for high powers of $\mathfrak{m}$.

(ii) If R is reduced, and M is a finitely generated R-module with constant rank r, then $\mathrm{e}(M) = r\,\mathrm{e}(R)$.

(iii) If R is Cohen-Macaulay, then $\mathrm{e}(R)$ is the sharp bound on the minimal number of generators of ideals of R. It is also the sharp bound on the minimal number of generators for finite birational extensions of R.

(iv) If R is reduced and the integral closure $\overline{R}$ is finitely generated over R, then $\mathrm{e}(R)$ is the number of generators required for $\overline{R}$ as an R-module.

Proof. The first item follows from the definition of multiplicity and $\ell_R(R/\mathfrak{m}^{n+1}) = \ell_R(R/\mathfrak{m}^n)+\mu_R(\mathfrak{m}^n)$. The second follows from the "associativity formula" [Mat86].

We refer to Sally's book [Sal78, Chapter 3, Theorem 1.1] or Greither [Gre82] for the fact the every ideal is generated by at most $\mathrm{e}(R)$ elements. The bound is sharp because high powers of $\mathfrak{m}$ need exactly $\mathrm{e}(R)$ generators by (i). Every finite birational extension of R is isomorphic, as an R-module, to an ideal of R (clear denominators) and therefore is generated by at most $\mathrm{e}(R)$ elements. Proposition 3.4 shows that the bound is sharp.

For (iv) we refer to Greither's paper [Gre82, Theorem 2.1]. □

A.24 Definition. A finitely generated module M over a Noetherian ring R has *constant rank* provided $K \otimes_R M$ is a free K-module, where K is the total quotient ring of R. If $K \otimes_R M \cong K^{(n)}$ (equivalently, $M_\mathfrak{p} \cong R_\mathfrak{p}^{(n)}$ for every $\mathfrak{p} \in \mathrm{Ass}(R)$), we say that M has constant rank n.

We need the following additional facts about multiplicity from [Mat86] for Chapter 14:

14.4 $I \subseteq J \implies \mathrm{e}(I,M) \geqslant \mathrm{e}(J,M)$

14.6 additivity on short exact sequences

14.9 if $\mathbf{x} \subseteq \mathfrak{m}^t$ then $\ell(M/\mathbf{x}M) \geqslant t^d\,\mathrm{e}(M)$

14.11 $\mathbf{x}$ regular $= \ell(M/\mathbf{x}M) = \mathrm{e}(\mathbf{x},M)$.

We also need the fact that $\mathrm{rank}(M) \leqslant \mu_R(M)$, and the existence of minimal reductions if the residue field is infinite.

§3 Henselian rings

Recall the classical criterion for a local ring to be Henselian:

A.25 Definition. Let $(R,\mathfrak{m},k)$ be a local ring. Then R is *Henselian* provided, for every monic polynomial f in $R[x]$ and every factorization $f+\mathfrak{m}[x]=g_1g_2$, where g and h are relatively prime monic polynomials in $k[x]$, there are monic polynomials $g_i \in R[x]$ such that $\widetilde{g}_i+\mathfrak{m}[x]=g_i$ and $f=\widetilde{g}_1\widetilde{g}_2$.

A.26 Theorem. *Let $(R,\mathfrak{m},k)$ be a local ring. These are equivalent:*

(i) R is Henselian.

(ii) Every integral domain which is a module-finite R-algebra is local.

(iii) Every module-finite commutative R-algebra is a direct product of local rings.

(iv) For every module-finite R-algebra Λ (not necessarily commutative), each idempotent of $\Lambda/\mathscr{J}(\Lambda)$ lifts to an idempotent of Λ.

Proof. (i) $\Longrightarrow$ (ii): Let D be a domain that is module-finite over R, and suppose D is not local. Write $\alpha+\beta=1$, where α and β are non-units of D, and therefore non-units of $S:=R[\alpha]$ too. Let $f\in R[x]$ be a monic equation of least degree with $f(\alpha)=0$. By Lemma 1.6, the finite-dimensional k-algebra $S/\mathfrak{m}S=k[\overline{\alpha}]$ is not local, and it follows that the minimal polynomial g for $\overline{\alpha}$ over k is not just a power of a single irreducible polynomial. Since $g \mid \overline{f}$, the factorization of $\overline{f}$ involves at least two distinct monic irreducible factors. Therefore we can write $\overline{f}=g_1g_2$, where g_1 and g_2 are monic polynomials of positive degree. Lifting this factorization to $R[x]$, we have $f=\widetilde{g}_1\widetilde{g}_2$. By minimality of $\deg f$, we have $\widetilde{g}_i(\alpha) \neq 0$, but $g_1(\alpha)g_2(\alpha) = f(\alpha) = 0$, contradiction.

(ii) $\implies$ (iii): Let Λ be a commutative, module-finite R-algebra, with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_t$. By applying (ii) to the domains $R/\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec}(R)$, we see that $\operatorname{Spec}(R)$ is the disjoint union of the sets $X_i = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{m}_i\}$. Moreover, letting $\mathfrak{p}_{ij}$, $j = 1, \dots, s_i$, be the minimal prime ideals contained in $\mathfrak{m}_i$, we see that $X_i = \mathrm{V}(\mathfrak{p}_1) \cup \cdots \cup \mathrm{V}(\mathfrak{p}_{s_i})$, a closed set. Thus we have represented $\operatorname{Spec}(R)$ as a disjoint union of open-and-closed sets, and (iii) follows.

(iii) $\implies$ (iv): Let $e = e^2 \in \Lambda/\mathscr{J}(\Lambda)$, and let ℓ be any lifting of e to Λ. Setting $A = R[\ell]$, one checks that $\mathscr{J}(A) = A \cap \mathscr{J}(\Lambda)$. Let $\overline{\ell}$ be the image of ℓ in $A/\mathscr{J}(A)$. We have an injection $\iota : A/\mathscr{J}(A) \hookrightarrow \Lambda/\mathscr{J}(\Lambda)$, and $\iota(\overline{\ell}) = e$. Therefore $\ell^2 - \ell \in \mathscr{J}(A)$; since A is a direct product of local rings, ℓ clearly lifts to an idempotent of A.

(iv) $\implies$ (i):... □

A.27 Corollary. *Let R be a Henselian local ring, let $\alpha \in R^\times$, and let n be a positive integer prime to* $\operatorname{char}(k)$*. If $\overline{\alpha}$ has an n^{th} root in $k^\times$, then α has an n^{th} root in $R^\times$.*

Proof. Let $f = x^n - \alpha \in R[x]$, and let β be a root of $x^n - \overline{\alpha} \in k[x]$. Write $x^n - \overline{\alpha} = (x - \beta)h(x)$. The hypotheses imply that $x^n - \overline{\alpha}$ has n distinct roots, so $x - \beta$ and $h(x)$ are relatively prime. Since R is Henselian, we get $\widetilde{\beta} \in R^\times$ and $\widetilde{h} \in R[x]$ such that $x^n - \alpha = (x - \widetilde{\beta})\widetilde{h}$. Then $\widetilde{\beta}^n = \alpha$. □

B

Ramification Theory

This appendix contains the basic results we need in the body of the text on unramified and étale ring homomorphisms, as well as the ramification behavior of prime ideals in integral extensions. We also include proofs of the theorem on the purity of the branch locus (Theorem B.12) and results relating ramification to pseudo-reflections in finite groups of linear ring automorphisms.

§1 Unramified homomorphisms

Recall that a ring homomorphism $A \longrightarrow B$ is said to be *of finite type* if B is a finitely generated A-algebra, that is, $B \cong A[x_1,\dots,x_n]/I$ for some polynomial variables $x_1,\dots,x_n$ and an ideal I. We say $A \longrightarrow B$ is *essentially of finite type* if B is a localization (at an arbitrary multiplicatively closed set) of an A-algebra of finite type.

B.1 Definition. Let $(A,\mathfrak{m},k) \longrightarrow (B,\mathfrak{n},\ell)$ be a local homomorphism of local rings. We say that $A \longrightarrow B$ is an *unramified local homomorphism* provided

(i) $\mathfrak{m}B = \mathfrak{n}$,

(ii) $B/\mathfrak{m}B$ is a finite separable field extension of $A/\mathfrak{m}$, and

(iii) B is essentially of finite type over A.

If in addition $A \longrightarrow B$ is flat, we say it is *étale.*

B.2 Remarks. Let $A \longrightarrow B$ be a local homomorphism of local rings as in the definition. Let $\widehat{A}$ and $\widehat{B}$ be the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic completions of A and B, respectively. It is straightforward to check that $A \longrightarrow B$ is unramified or étale if and only if $\widehat{A} \longrightarrow \widehat{B}$ is so.

If $A \longrightarrow B$ is an unramified local homomorphism, then $\widehat{B}$ is a finitely generated $\widehat{A}$-module. Indeed, it follows from the complete version of NAK ([Mat86, Theorem 8.4] or [Eis95, Exercises 7.2 and 7.4]) that any $k = \widehat{A}/\widehat{\mathfrak{m}}$-vector space basis for $\ell = \widehat{B}/\widehat{\mathfrak{n}}$ lifts to a set of $\widehat{A}$-module generators for $\widehat{B}$. If, in particular, there is no residue field growth (for instance, if k is separably or algebraically closed), then $\widehat{A} \longrightarrow \widehat{B}$ is surjective.

If $A \longrightarrow B$ is étale, then $\widehat{B}$ is a finitely generated flat $\widehat{A}$-module, whence $\widehat{B} \cong \widehat{A}^{(n)}$ for some n. If in this case $k = \ell$, then $\widehat{B} = \widehat{A}$.

It's easy to check that if $A \longrightarrow B$ is étale, then A and B share the same Krull dimension and the same depth. Furthermore, A is regular if and only if B is regular. For further permanence results along these lines, we need to globalize the definition.

B.3 Definition. Let A and B be Noetherian rings, and $A \longrightarrow B$ a homomorphism essentially of finite type. Let $\mathfrak{q} \in \operatorname{Spec} B$ and set $\mathfrak{p} = A \cap \mathfrak{q}$. We say that $A \longrightarrow B$ is *unramified at* $\mathfrak{q}$ (or also $\mathfrak{q}$ *is unramified over* A) if and only if the induced map $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$ is an unramified local homomorphism of local rings. Similarly, $A \longrightarrow B$ is *étale at* $\mathfrak{q}$ if and only if $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$ is an étale local homomorphism. Finally, $A \longrightarrow B$ is *unramified*, resp. *étale*, if it is unramified, respectively étale, at every prime ideal $\mathfrak{q} \in \operatorname{Spec} B$.

Here is an easy transitivity property of unramified primes.

B.4 Lemma. *Let $A \longrightarrow B \longrightarrow C$ be homomorphisms, essentially of finite type, of Noetherian rings. Let $\mathfrak{r} \in \operatorname{Spec} C$ and set $\mathfrak{q} = B \cap \mathfrak{r}$.*

(i) If $\mathfrak{r}$ is unramified over B and $\mathfrak{q}$ is unramified over A, then $\mathfrak{r}$ is unramified over A.

(ii) If $\mathfrak{r}$ is unramified over A, then $\mathfrak{r}$ is unramified over B.

It is clear that a local homomorphism $(A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ essentially of finite type is an unramified local homomorphism if and only if $\mathfrak{n}$ is unramified over A. However, it's not at all clear that an unramified local homomorphism is unramified in the sense of Definition B.3. To reconcile these definitions, we must show that being unramified is preserved under localization. The easiest way to do this is to give an alternative description, following [AB59].

B.5 Definition. Let $A \longrightarrow B$ be a homomorphism of Noetherian rings. Define the *diagonal map* $\mu\colon B \otimes_A B \longrightarrow B$ by $\mu(b \otimes b') = bb'$ for all $b, b' \in B$, and set $\mathscr{J} = \ker \mu$. Thus we have a short exact sequence of $B \otimes_A B$-modules

(B.5.1) $$0 \longrightarrow \mathscr{J} \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0\,.$$

B.6 Remarks.

(i) The ideal $\mathscr{J}$ is generated by all elements of the form $b \otimes 1 - 1 \otimes b$, where $b \in B$. Indeed, if $\mu\left(\sum_j b_j \otimes b'_j\right) = 0$, then $\sum_j b_j b'_j = 0$, so that

$$\sum_j b_j \otimes b'_j = \sum_j (1 \otimes b'_j)(b_j \otimes 1 - 1 \otimes b_j)\,.$$

(ii) The ring $B \otimes_A B$, also called the *enveloping algebra* of the A-algebra B, has two A-module structures, one on each side. Thus $\mathscr{J}$ also has two different B-structures. However, these two module structures coincide modulo $\mathscr{J}^2$. The reason is that

$$\mathscr{J}/\mathscr{J}^2 = \big((B \otimes_A B)/\mathscr{J}\big) \otimes_{B \otimes_A B} \mathscr{J}$$

is a $(B \otimes_A B)/\mathscr{J}$-module, and $(B \otimes_A B)/\mathscr{J} = B$. In particular, $\mathscr{J}/\mathscr{J}^2$ has an unambiguous B-module structure.

(iii) The B-module $\mathscr{J}/\mathscr{J}^2$ is also known as the *module of (relative) Kähler differentials of B over A,* denoted $\Omega_{B/A}$ [Eis95, Chapter 16]. It is the universal module of A-linear derivations on B, in the sense that the map $\delta\colon B \longrightarrow \mathscr{J}/\mathscr{J}^2$ sending b to $b \otimes 1 - 1 \otimes b$ is an A-linear derivation (satisfies the Leibniz rule), and given any A-linear derivation $\epsilon\colon B \longrightarrow M$, there exists a unique B-linear homomorphism $\mathscr{J}/\mathscr{J}^2 \longrightarrow M$ making the obvious diagram commute. In particular we have $\mathrm{Der}_A(B,M) \cong \mathrm{Hom}_B(\mathscr{J}/\mathscr{J}^2, M)$ for every B-module M. Though it is very important for a deeper study of unramified maps, will not need this interpretation in this book.

(iv) If $A \longrightarrow B$ is assumed to be essentially of finite type, $\mathscr{J}$ is a finitely generated $B \otimes_A B$-module. To see this, first observe that the question reduces at once to the case where B is of finite type over A. In that case, if $x_1, \ldots, x_n$ are A-algebra generators for B, one checks that the elements $x_i \otimes 1 - 1 \otimes x_i$, for $i = 1, \ldots, n$, generate $\mathscr{J}$. It follows that if $A \longrightarrow B$ is essentially of finite type then $\mathscr{J}/\mathscr{J}^2$ is a finitely generated B-module.

(v) The term "diagonal map" comes from the geometry. If $f\colon A \hookrightarrow B$ is an extension of integral domains which are finitely generated algebras over a field k, then there is a corresponding surjective map of irreducible varieties $f^{\#}\colon Y \longrightarrow X$, where X is the maximal ideal spectrum of A and Y is that of B. In this case, the maximal ideal spectrum of $B \otimes_A B$ is the fiber product

$$Y \times_X Y = \left\{ (y_1, y_2) \in Y \times Y \;\middle|\; f^{\#}(y_1) = f^{\#}(y_2) \right\}.$$

The map $\mu\colon B \otimes_A B \longrightarrow B$ corresponds to the inclusion of the diagonal $\Delta_{Y/X}$ as an irreducible component, $\mu^{\#}\colon Y \longrightarrow Y \times_X Y$. In these terms, $\mathscr{J}$ is the ideal of functions on $Y \times_X Y$ vanishing on the diagonal.

B.7 Lemma. *Let $A \longrightarrow B$ be a homomorphism of Noetherian rings. Then the following conditions are equivalent.*

(i) B is a projective $B \otimes_A B$-module.

(ii) The exact sequence $0 \longrightarrow \mathscr{J} \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$ splits as $B \otimes_A B$-modules.

(iii) $\mu\left(\operatorname{Ann}_{B \otimes_A B}(\mathscr{J})\right) = B$.

If $\mathscr{J}/\mathscr{J}^2$ is a finitely generated B-module (for example, if $A \longrightarrow B$ is essentially of finite type), then these are equivalent to

(iv) $\mathscr{J}$ is generated by an idempotent.

(v) $\mathscr{J}/\mathscr{J}^2 = 0$.

Proof. (i) $\iff$ (ii) is clear.

(ii) $\iff$ (iii): The map $\mu\colon B\otimes_A B \longrightarrow B$ splits over $B\otimes_A B$ if and only if the induced homomorphism

$$\mathrm{Hom}_{B\otimes_A B}(B,\mu)\colon\ \mathrm{Hom}_{B\otimes_A B}(B, B\otimes_A B) \longrightarrow \mathrm{Hom}_{B\otimes_A B}(B,B)$$

is surjective. However, the isomorphism $B \cong (B\otimes_A B)/\mathscr{J}$ shows that $\mathrm{Hom}_{B\otimes_A B}(B, B\otimes_A B) \cong \mathrm{Ann}_{B\otimes_A B}(\mathscr{J})$, so that μ splits if and only if $\mathrm{Hom}_{B\otimes_A B}(B,\mu)$ is surjective, if and only if $\mu\left(\mathrm{Ann}_{B\otimes_A B}\left(\mathscr{J}\right)\right) = B$.

The final two statements are always equivalent for a finitely generated ideal. Assume (iv), so that there exists $z \in \mathscr{J}$ with $xz = x$ for every $x \in \mathscr{J}$. Define $q\colon B\otimes_A B \longrightarrow \mathscr{J}$ by $q(x) = xz$. Then for $x \in \mathscr{J}$, we have $q(x) = x$, so that the sequence splits. Conversely, any splitting q of the map $\mathscr{J} \longrightarrow B\otimes_A B$ yields an idempotent $z = q(1)$, so (ii) and (iv) are equivalent. □

The proof of the next result is too long for us to include here, even though it is the foundation for the theory. See [Eis95, Corollary 16.16].[1]

B.8 Proposition. *Suppose that A is a field and B is an A-algebra essentially of finite type. Then the equivalent conditions of Lemma B.7 hold if and only if B is a direct product of a finite number of fields, each finite and separable over A.*

The condition in the Proposition that B be a direct product of a finite number of fields, each finite and separable over A, is sometimes called a

[1] Sketch: In the special case where A and B are both fields, one can show that if B is projective over $B\otimes_A B$ then $A \longrightarrow B$ is necessarily module-finite. Then a separability idempotent $z \in \mathscr{J}$ is given as follows: let $\alpha \in B$ be a primitive element, with minimal polynomial $f(x) = (x-\alpha)\sum_{i=0}^{n-1} b_i x^i$. Then $z = \left(1 \otimes \frac{1}{f'(\alpha)}\right)\sum_{i=0}^{n-1} \alpha^i \otimes b_i$ is idempotent.

"(classically) separable algebra" in the literature. Equivalently, $K \otimes_A B$ is a reduced ring for every field extension K of A.

We now relate the equivalent conditions of Lemma B.7 to the definitions at the beginning of the Appendix.

B.9 Proposition. *Let $A \longrightarrow B$ be a homomorphism, essentially of finite type, of Noetherian rings. The following statements are equivalent.*

(i) The exact sequence $0 \longrightarrow \mathscr{J} \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$ splits as $B \otimes_A B$-modules.

(ii) B is unramified over A.

(iii) Every maximal ideal of B is unramified over A.

Proof. (i) $\Longrightarrow$ (ii): Let $\mathfrak{q} \in \operatorname{Spec} B$, and let $\mathfrak{p} = A \cap \mathfrak{q}$ be its contraction to A. It is enough to show that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is unramified over the field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, i.e. is a finite direct product of finite separable field extensions. By Proposition B.8, it suffices to show that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is a projective module over $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \otimes_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. Let $p \colon B \longrightarrow B \otimes_A B$ be a splitting for μ, so that $\mu p = 1_B$. Set $y = p(1)$. Then $\mu(y) = 1$ and $y \ker \mu = 0$; in fact, the existence of an element y satisfying these two conditions is easily seen to be equivalent to the existence of a splitting of μ. Consider the diagram

$$\begin{array}{ccccc} B \otimes_A B & \xrightarrow{f} & B_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p} & \xrightarrow{g} & B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu''} \\ B & \longrightarrow & B_\mathfrak{p} & \longrightarrow & B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \end{array}$$

in which the horizontal arrows are the natural ones and the vertical arrows are the respective diagonal maps. Put $y'' = gf(y)$. Then $\mu''(y'') =$

1 and $y\ker(\mu'') = 0$, so that μ'' splits. Since the top-right ring is also $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \otimes_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, this shows that $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \longrightarrow B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is unramified.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i): Since $\mathscr{J}$ is a finitely generated B-module, it suffices to assume that $A \longrightarrow B$ is an unramified local homomorphism of local rings and show that $\mathscr{J} = \mathscr{J}^2$. Once again we reduce to the case where A is a field and B is a separable A-algebra. In this case Proposition B.8 implies that $\mathscr{J} = \mathscr{J}^2$. □

B.10 Remarks. This proposition reconciles the two definitions of unramifiedness given at the beginning of the Appendix, since it implies that unramifiedness is preserved by localization. This has some very satisfactory consequences. One can now use the characterizations of reducedness and normality in terms of the conditions (R_n) and (S_n) to see that if $A \longrightarrow B$ is étale, then A is reduced, resp. normal, if and only if B is so. Note that this fact would be *false* without the hypothesis that $A \longrightarrow B$ is essentially of finite type. Indeed, the natural completion homomorphism $A \longrightarrow \widehat{A}$ satisfies (i) and (ii) of Definition B.1, and is of course flat, but there are many examples of completion not preserving reducedness or normality.

Proposition B.9 also allows us to expand our use of language, saying that a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ is *unramified in B* if the localization $A_\mathfrak{p} \longrightarrow B_\mathfrak{p}$ is unramified, that is, every prime ideal of B lying over $\mathfrak{p}$ is unramified.

We now define the *homological different* of the A-algebra B, which will be used several times in the text. It is the ideal of B

$$\mathfrak{H}_A(B) = \mu\left(\operatorname{Ann}_{B\otimes_A B}\left(\mathscr{J}\right)\right),$$

where $\mu\colon B\otimes_A B \longrightarrow B$ is the diagonal. The homological different defines the *branch locus* of $A \longrightarrow B$, that is, the primes of B which are ramified over A, as we now show.

B.11 Theorem. *Let $A \longrightarrow B$ be a homomorphism, essentially of finite type, of Noetherian rings. A prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ is unramified over A if and only if $\mathfrak{q}$ does not contain $\mathfrak{H}_A(B)$.*

Proof. This follows from Proposition B.9 and condition (iii) of Lemma B.7, together with the observation that formation of $\mathscr{J}$ commutes with localization at $\mathfrak{q}$ and $A\cap\mathfrak{q}$. Precisely, let $\mathfrak{q}\in\operatorname{Spec} B$ and set $\mathfrak{p} = A\cap\mathfrak{q}$. Let S be the multiplicatively closed set of simple tensors $u\otimes v$, where u and v range over $B\setminus\mathfrak{q}$. Then $(B\otimes_A B)_S \cong B_\mathfrak{q}\otimes_A B_\mathfrak{q} \cong B_\mathfrak{q}\otimes_{A_\mathfrak{p}} B_\mathfrak{q}$ and the kernel of the map $\widetilde{\mu}\colon B_\mathfrak{q}\otimes_{A_\mathfrak{p}} B_\mathfrak{q} \longrightarrow B_\mathfrak{q}$ coincides with $(\ker\mu)_S$. □

§2 Purity of the branch locus

Turn now to the theorem on the purity of the branch locus. The proof we give, following Auslander–Buchsbaum [AB59] and Auslander [Aus62], is somewhat lengthy.

For the rest of this Appendix, we will be mainly concerned with finite integral extensions $A \longrightarrow B$ of Noetherian domains. In particular they will be of finite type. Recall that for a finite integral extension, we have the "lying over" and "going up" properties; if in addition A is normal, then we also have "going down" [Mat86, Theorems 9.3 and 9.4]. In particular, in this case we have $\operatorname{height}\mathfrak{q} = \operatorname{height}(A\cap\mathfrak{q})$ for $\mathfrak{q}\in\operatorname{Spec} B$ ([Mat86, 9.8, 9.9]).

Recall also that since a normal domain satisfies Serre's condition (S_2), the associated primes of a principal ideal all have height one (see [Eis95, Theorem 11.5]). In other words, principal ideals have pure height one.

B.12 Theorem. *Let A be a regular ring and $A \hookrightarrow B$ a module-finite ring extension with B normal. Then $\mathfrak{H}_A(B)$ is an ideal of pure codimension one in B. In particular, if $A \longrightarrow B$ is unramified in codimension one, then $A \longrightarrow B$ is unramified.*

First we observe that the condition "unramified in codimension one" can be interpreted in terms of the sequence (B.5.1).

Assume $A \longrightarrow B$ is a module-finite extension of Noetherian normal domains. We write $B \cdot B$ for $(B \otimes_A B)^{**}$, where $-^* = \operatorname{Hom}_B(-, B)$ (see Chapter 5). Since the B-module B is reflexive, and any homomorphism from $B \otimes_A B$ to a reflexive module factors through $B \cdot B$, we see that $\mu \colon B \otimes_A B \longrightarrow B$ factors as $B \otimes_A B \longrightarrow B \cdot B \xrightarrow{\mu^{**}} B$.

B.13 Proposition. *A module-finite extension of Noetherian normal domains $A \longrightarrow B$ is unramified in codimension one if and only if μ^{**} is a split surjection of $B \otimes_A B$-modules.*

Proof. If μ^{**} is a split surjection, then $\mu^{**}_{\mathfrak{p}}$ is a split surjection for all primes $\mathfrak{q}$ of height one in B. For these primes, however, $\mu^{**}_{\mathfrak{q}} = \mu_{\mathfrak{q}}$ since $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} = (B \otimes_A B)_{\mathfrak{q}}$ is a reflexive module over the DVR $B_{\mathfrak{q}}$, where $\mathfrak{p} = A \cap \mathfrak{q}$. Thus μ splits locally at every height-one prime of B, so $A \longrightarrow B$ is unramified in codimension one.

Now assume $A \longrightarrow B$ is unramified in codimension one. Let K be the quotient field of A and L the quotient field of B. Since $A \longrightarrow B$ is unramified at the zero ideal, $K \longrightarrow L$ is unramified, equivalently, a finite separable field extension. In particular, the diagonal map $\eta\colon L \otimes_K L \longrightarrow L$ is a split epimorphism of $L \otimes_K L$-modules.

Since $B \cdot B$ is B-reflexive, it is in particular torsion-free, and so $B \cdot B$ is a submodule of $L \otimes_K L$. We therefore have a commutative diagram of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{L} & \longrightarrow & L \otimes_K L & \xrightarrow{\eta} & L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{J}' & \longrightarrow & B \cdot B & \xrightarrow{\mu^{**}} & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{J} & \longrightarrow & B \otimes_A B & \xrightarrow[\mu]{} & B & \longrightarrow & 0
\end{array}
$$

in which the left-hand modules are by definition the kernels, and in which the top row splits over $L \otimes_K L$ since L/K is separable. Let $\epsilon\colon L \longrightarrow L \otimes_K L$ be a splitting, and let ζ be the restriction of ϵ to B. It will suffice to show that $\zeta(B) \subseteq B \cdot B$, for then ζ will be the splitting of μ^{**} we need. For a height-one prime ideal $\mathfrak{q}$ of B, with $\mathfrak{p} = A \cap \mathfrak{q}$, we do have $\zeta_{\mathfrak{q}}(B_{\mathfrak{q}}) \subseteq (B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}})^{**} = (B \otimes_A B)_{\mathfrak{q}}$, since $A \longrightarrow B$ is unramified in codimension one. But $\operatorname{im}(\zeta) = \bigcap_{\operatorname{height} \mathfrak{q}=1} \operatorname{im}(\zeta_{\mathfrak{q}})$ and $B = \bigcap_{\operatorname{height} \mathfrak{q}=1} B_{\mathfrak{q}}$ as B is normal, so the image of ζ is contained in $B \cdot B$ and ζ is a splitting for μ^{**}. □

Following Auslander and Buchsbaum, we shall first prove Theorem B.12 in the special case where B is a finitely generated projective A-module. In this case the homological different coincides with the Dedekind different from number theory, which we describe now.

Let $A \longrightarrow B$ be a module-finite extension of normal domains. Let K and L be the quotient fields of A and B, respectively. We assume that $K \longrightarrow L$

is a separable extension. (In the situation of Theorem B.12, this follows from the hypothesis.) In this case the trace form $(x,y) \mapsto \mathrm{Tr}_{L/K}(xy)$ is non-degenerate $L \otimes_K L \longrightarrow L$, and since $A \longrightarrow B$ is integral and A is integrally closed in K we have $\mathrm{Tr}_{L/K}(B) \subseteq A$. Set

$$\mathfrak{C}_A(B) = \{x \in L \mid \mathrm{Tr}_{L/K}(xB) \subseteq A\},$$

and call it the *Dedekind complementary module for* B/A. The map $\mathfrak{C}_A(B) \longrightarrow B^* = \mathrm{Hom}_A(B,A)$ defined by sending $x \in \mathfrak{C}_A(B)$ to the map $y \mapsto \mathrm{Tr}_{L/K}(xy)$ is an isomorphism of B-modules. Thus $\mathfrak{C}_A(B)$ is a finitely generated reflexive B-submodule of L, i.e. a divisorial fractional ideal.

We set $\mathfrak{D}_A(B) = (\mathfrak{C}_A(B))^{-1}$, the inverse of the fractional ideal $\mathfrak{C}_A(B)$. This is the *Dedekind different* of B/A. The map $\mathrm{Hom}_B(\mathrm{Hom}_A(B,A),B) \longrightarrow \mathfrak{D}_A(B)$ sending f to $f(\mathrm{Tr}_{L/K})$ is an isomorphism of B-modules. Since $B \subseteq \mathfrak{C}_A(B)$, we have $\mathfrak{D}_A(B) \subseteq B$ and $\mathfrak{D}_A(B)$ is a reflexive ideal of B.

The following theorem is attributed to Noether ([Noe50], posthumous) and Auslander and Buchsbaum.

B.14 Theorem. *Let $A \longrightarrow B$ be a module-finite extension of Noetherian normal domains which induces a separable extension of quotient fields. We have $\mathfrak{H}_A(B) \subseteq \mathfrak{D}_A(B)$, and if B is projective as an A-module then $\mathfrak{H}_A(B) = \mathfrak{D}_A(B)$.*

Proof. Let K and L be the respective quotient fields of A and B as in the discussion above. Set $L^* = \mathrm{Hom}_K(L,K)$ and $B^* = \mathrm{Hom}_A(B,A)$. Define

$$\sigma_L : L \otimes_K L \longrightarrow \mathrm{Hom}_L(L^*,L)$$

by $\sigma_L(x \otimes y)(f) = xf(y)$. Then σ_L restricts to $\sigma_B : B \otimes_A B \longrightarrow \mathrm{Hom}_B(B^*,B)$, defined similarly. It's straightforward to show that σ_B is an isomorphism if B is projective over A; in particular, σ_L is an isomorphism. Its inverse is defined by $(\sigma_L)^{-1}(f) = \sum_j f(x_j^*) \otimes x_j$, where $\{x_j\}$ and $\left\{x_j^*\right\}$ are dual bases for L and L^* over K.

Consider the diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_B(B^*,B) & \xrightarrow{i_B} & \mathrm{Hom}_A(B^*,B) & \xleftarrow{\sigma_B} & B \otimes_A B & \xrightarrow{\mu_B} & B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_L(L^*,L) & \xrightarrow[i_L]{} & \mathrm{Hom}_K(L^*,L) & \xleftarrow[\cong]{\sigma_L} & L \otimes_K L & \xrightarrow[\mu_L]{} & L
\end{array}$$

in which μ_B and μ_L are the respective diagonal maps, i_B and i_L are inclusions, and the vertical arrows are all induced from the inclusion of B into L. Now $\mathrm{Tr}_{L/K}(x) = \sum_j x_j^*(x x_j)$, so if $f \in \mathrm{Hom}_L(L^*,L)$ then we have $f(\mathrm{Tr}_{L/K}) = \sum_j x_j f(x_j^*)$. Thus the composition of the entire bottom row, left to right, is given by

$$\mu_L(\sigma_L)^{-1} i_L(f) = \mu_L\left(\sum_j f(x_j^*) \otimes x\right) = \sum_j f(x_j^*) x_j = f(\mathrm{Tr}_{L/K}).$$

It follows that the image of $\mathrm{Hom}_B(B^*,B)$ in L is $\mathfrak{D}_A(B)$.

The module $\mathrm{Hom}_A(B^*,B)$ is naturally a $B \otimes_A B$-module via $\big((b \otimes b')(f)\big)(g) = bf(g \circ b')$, where the b' on the right represents the map on B given by multiplication by that element. Thus σ_B is a $B \otimes_A B$-module homomorphism. An element $\mathrm{Hom}_A(B^*,B)$ is in the image of i_B if and only if it is a B-module homomorphism, i.e. $(b \otimes 1)(f) = (1 \otimes b)(f)$ for every $b \in B$. This is exactly saying that f annihilates $\mathscr{J} = \ker \mu_B$. Thus implies that $\sigma_B\left(\mathrm{Ann}_{B \otimes_A B}\left(\mathscr{J}\right)\right) \subseteq \mathrm{im}\, i_B$. It follows that $\mathfrak{H}_A(B) = \mu_B\left(\mathrm{Ann}_{B \otimes_A B}\left(\mathscr{J}\right)\right) \subseteq \mathfrak{D}_A(B)$.

Finally, if B is projective as an A-module then σ_B is an isomorphism and $\sigma_B\left(\mathrm{Ann}_{B \otimes_A B}\left(\mathscr{J}\right)\right)$ is equal to the image of i_B. Thus $\mathfrak{H}_A(B) = \mathfrak{D}_A(B)$. □

Next we show that $\mathfrak{D}_A(B)$ has pure height one, so in case they are equal $\mathfrak{H}_A(B)$ does as well. We need a general fact about modules over normal domains.

B.15 Proposition. *Let A be a Noetherian normal domain. Let*

$$0 \longrightarrow M \longrightarrow N \longrightarrow T \longrightarrow 0$$

be a short exact sequence of non-zero finitely generated A-modules wherein M is reflexive and T is torsion. Then $\operatorname{Ann}_A(T)$ is an ideal of pure height one in A.

Proof. This is similar to Lemma 4.12. Let $\mathfrak{p}$ be a prime ideal minimal over the annihilator of T. Then in particular $\mathfrak{p}$ is an associated prime of T, so that $\operatorname{depth} T_{\mathfrak{p}} = 0$. Since M is reflexive, it satisfies (S_2), so that if $\mathfrak{p}$ has height two or more then $\mathfrak{M}_{\mathfrak{p}}$ has depth at least two. This contradicts the Depth Lemma.

□

B.16 Corollary. *Let $A \longrightarrow B$ be a module-finite extension of normal domains. Assume that the induced extension of quotient fields is separable. If $\mathfrak{D}_A(B) \neq B$, then $\mathfrak{D}_A(B)$ is an ideal of pure height one in B. Consequently, $\mathfrak{D}_A(B) = B$ if and only if $A \longrightarrow B$ is unramified in codimension one.*

Proof. For the first statement, take $M = A$ and $N = \mathfrak{C}_A(B)$ in Proposition B.15. In the second statement, necessity follows from $\mathfrak{H}_A(B) \subseteq \mathfrak{D}_A(B)$ and Theorem B.11. Conversely, suppose $\mathfrak{D}_A(B) = B$. Let $\mathfrak{q}$ be a height-one prime of B and set $\mathfrak{p} = A \cap \mathfrak{q}$. Then $A_{\mathfrak{p}}$ is a DVR and B_p is a finitely generated torsion-free $A_{\mathfrak{p}}$-module, whence free. Thus $\mathfrak{H}_{A_{\mathfrak{p}}}(B_{\mathfrak{p}}) = \mathfrak{D}_{A_{\mathfrak{p}}}(B_{\mathfrak{p}}) =$

$(\mathfrak{D}_A(B))_\mathfrak{p} = B_\mathfrak{p}$. By Theorem B.11 $B_\mathfrak{p}$ is unramified over $A_\mathfrak{p}$, so in particular $\mathfrak{q}$ is unramified over $\mathfrak{p}$. □

B.17 Corollary. *If, in the setup of Corollary B.16, B is projective as an A-module, then $A \longrightarrow B$ is unramified if and only if it is unramified in codimension one.*

Now we turn to Auslander's proof of the theorem on the purity of the branch locus. The strategy is to reduce the general case to the situation of Corollary B.17 by proving a purely module-theoretic statement.

B.18 Proposition. *Let $A \longrightarrow B$ be a module-finite extension of Noetherian normal domains which is unramified in codimension one. Assume that A has the following property: If M is a finitely generated reflexive A-module such that $\operatorname{Hom}_A(M,M)$ is isomorphic to a direct sum of copies of M, then M is free. Then $A \longrightarrow B$ is unramified.*

Proof. Let $K \longrightarrow L$ be the extension of quotient fields induced by $A \longrightarrow B$. Then L is a finite separable extension of K. By [Aus62, Prop. 1.1], we may assume in fact that $K \longrightarrow L$ is a Galois extension. (The proof of this result is somewhat technical, so we omit it.)

We are therefore in the situation of Theorem 4.13! Thus $\operatorname{Hom}_A(B,B)$ is isomorphic as a ring to the twisted group ring $B\#G$, where $G = \operatorname{Gal}(L/K)$. As a B-module, and hence as an A-module, $B\#G$ is isomorphic to a direct sum of copies of B. By hypothesis, then, B is a free A-module. Corollary B.17 now says that $A \longrightarrow B$ is unramified. □

Auslander's argument that regular local rings satisfy the condition of Proposition B.18 seems to be unique in the field; we know of nothing else quite like it. We being with three preliminary results.

B.19 Lemma. *Let A be a Noetherian normal domain and M a finitely generated torsion-free A-module. Then* $\operatorname{Hom}_A(M,M)^* \cong \operatorname{Hom}_A(M^*,M^*)$.

Proof. We have the natural map $\rho\colon M^* \otimes_A M \longrightarrow \operatorname{Hom}_A(M,M)$ defined by $\rho(f \otimes y)(x) = f(x)y$, which is an isomorphism if and only if M is free, cf. Exercise 10.46. Dualizing yields $\rho^*\colon \operatorname{Hom}_A(M,M)^* \longrightarrow (M^* \otimes_A M)^* \cong \operatorname{Hom}_A(M^*,M^*)$ by Hom-tensor adjointness. Now ρ^* is a homomorphism between reflexive A-modules, which is an isomorphism in codimension one since A is normal and M is torsion-free. By Lemma 4.12, ρ^* is an isomorphism. □

B.20 Lemma. *Let $(A,\mathfrak{m})$ be a local ring and $f\colon M \longrightarrow N$ a homomorphism of finitely generated A-modules. Assume that $f_\mathfrak{p}\colon M_\mathfrak{p} \longrightarrow N_\mathfrak{p}$ is an isomorphism for every non-maximal prime $\mathfrak{p}$ of A. Then* $\operatorname{Ext}^i_A(f,A)\colon \operatorname{Ext}^i_A(N,A) \longrightarrow \operatorname{Ext}^i_A(M,A)$ *is an isomorphism for* $i = 0,\dots,\operatorname{depth} A - 2$.

Proof. The kernel and cokernel of f both have finite length, so $\operatorname{Ext}^i_A(\ker f,A) = \operatorname{Ext}^i_A(\operatorname{cok} f,A) = 0$ for $i = 0,\dots,\operatorname{depth} A - 1$ [Mat86, Theorem 16.6]. The long exact sequence of Ext now gives the conclusion. □

B.21 Proposition. *Let $(A,\mathfrak{m})$ be a local ring of depth at least 3 and let M be a reflexive A-module such that*

(i) M is locally free on the punctured spectrum of A; and

(ii) $\operatorname{pd}_A M \leqslant 1$.

If M is not free, then

$$\ell\left(\operatorname{Ext}_A^1(\operatorname{Hom}_A(M,M),A)\right) > (\operatorname{rank}_A M)\ \ell\left(\operatorname{Ext}_A^1(M,A)\right).$$

Proof. Assume that M is not free. We have the natural homomorphism $\rho_M\colon M^*\otimes_A M \longrightarrow \operatorname{Hom}_A(M,M)$, defined by $\rho_M(f\otimes x)(y) = f(y)x$, which is an isomorphism if and only if M is free, cf. Remark 10.5. In particular, ρ_M is locally an isomorphism on the punctured spectrum of A, so by Lemma B.20, we have

$$\operatorname{Ext}_A^1(M^*\otimes_A M, A) \cong \operatorname{Ext}_A^1(\operatorname{Hom}_A(M,M),A).$$

Next we claim that there is an injection $\operatorname{Ext}_A^1(M,M)\hookrightarrow \operatorname{Ext}_A^1(M^*\otimes_A M,A)$. Let

(B.21.1) $$0\longrightarrow F_1\longrightarrow F_0\longrightarrow M\longrightarrow 0$$

be a free resolution. Dualizing gives an exact sequence

(B.21.2) $$0\longrightarrow M^*\longrightarrow F_0^*\longrightarrow F_1^*\longrightarrow \operatorname{Ext}_A^1(M,A)\longrightarrow 0,$$

so that $\operatorname{Tor}_{i-2}^A(M^*,M) = \operatorname{Tor}_i^A(\operatorname{Ext}_A^1(M,A),M) = 0$ for all $i\geqslant 3$. In particular, applying $M^*\otimes_A -$ to (B.21.1) results in an exact sequence

$$0\longrightarrow M^*\otimes_A F_1\longrightarrow M^*\otimes_A F_0\longrightarrow M^*\otimes_A M\longrightarrow 0.$$

Dualizing this yields an exact sequence

$$\operatorname{Hom}_A(M^*\otimes_A F_0,A)\xrightarrow{\eta}\operatorname{Hom}_A(M^*\otimes_A F_1,A)\longrightarrow \operatorname{Ext}_A^1(M^*\otimes_A M,A).$$

But the homomorphism η is naturally isomorphic to the homomorphism $\operatorname{Hom}_A(F_0,M^{**})\longrightarrow \operatorname{Hom}_A(F_1,M^{**})$. Since M is reflexive, this implies that the cokernel of η is isomorphic to $\mathrm{Ext}^1_A(M,M)$, whence $\mathrm{Ext}^1_A(M,M) \hookrightarrow \mathrm{Ext}^1_A(M^* \otimes_A M, A)$, as claimed.

Next we claim that $\mathrm{Ext}^1_A(M,M) \cong \mathrm{Ext}^1_A(M,A) \otimes_A M$. This follows immediately from the commutative exact diagram

$$\begin{array}{ccccccc} F_0^* \otimes_A M & \longrightarrow & F_1^* \otimes_A M & \longrightarrow & \mathrm{Ext}^1_A(M,A) \otimes_A M & \longrightarrow & 0 \\ \downarrow \rho^M_{F_0} & & \downarrow \rho^M_{F_1} & & \vdots & & \\ \mathrm{Hom}_A(F_0,M) & \longrightarrow & \mathrm{Hom}_A(F_1,M) & \longrightarrow & \mathrm{Ext}^1_A(M,M) & \longrightarrow & 0 \end{array}$$

in which the rows are the result of applying $-\otimes_A M$ to (B.21.2) and $\mathrm{Hom}_A(-,M)$ to (B.21.1), respectively, the two vertical arrows $\rho^M_{F_i}$ are isomorphisms since each F_i is free, and the third vertical arrow is induced by the other two.

Putting the pieces together so far, we have

$$\begin{aligned} \ell\left(\mathrm{Hom}_A(\mathrm{Ext}^1_A(M,M),A)\right) &= \ell\left(\mathrm{Ext}^1_A(M^* \otimes_A M, A)\right) \\ &\geqslant \ell\left(\mathrm{Ext}^1_A(M,M)\right) \\ &= \ell\left(\mathrm{Ext}^1_A(M,A) \otimes_A M\right) \end{aligned}$$

Set $T = \mathrm{Ext}^1_A(M,A)$. Then $T \neq 0$, since $T = 0$ implies that M^* is free by (B.21.2), whence M is free as well, a contradiction. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Tor}^A_1(T,M) \longrightarrow T \otimes_A F_1 \longrightarrow T \otimes_A F_0 \longrightarrow T \otimes_A M \longrightarrow 0.$$

The rank of M is equal to $\mathrm{rank}_A F_0 - \mathrm{rank}_A F_1$ by (B.21.1), so counting lengths shows that

$$\ell(T \otimes_A M) = (\mathrm{rank}_A M)\ \ell(T) + \ell\left(\mathrm{Tor}^A_1(T,M)\right).$$

But T is a non-zero module of finite length, so $\mathrm{Tor}^A_1(T,M) \neq 0$, which finishes the proof. □

The next proposition serves as a template for Auslander's proof of the theorem on the purity of the branch locus.

B.22 Proposition. *Let $\mathscr{C}$ be a set of pairs (A,M) where A is a local ring and M is a finitely generated reflexive A-module. Assume that*

(i) $(A,M)\in\mathscr{C}$ *implies* $(A_\mathfrak{p},M_\mathfrak{p})\in\mathscr{C}$ *for every* $\mathfrak{p}\in\operatorname{Spec}A$*;*

(ii) $(A,M)\in\mathscr{C}$ *and* $\operatorname{depth}A\leqslant 3$ *imply that M is free; and*

(iii) $(A,M)\in\mathscr{C}$*,* $\operatorname{depth}A>3$*, and M locally free on the punctured spectrum imply that there exists a non-zerodivisor x in the maximal ideal of A such that* $(A/(x),(M/xM)^{**})\in\mathscr{C}$*.*

Then M is free over A for every (A,M) in $\mathscr{C}$.

Proof. If the statement fails, choose a witness $(A,M)\in\mathscr{C}$ with M not A-free and $\dim A$ minimal. By (ii), $\operatorname{depth}A>3$, so that by (iii) we can find a non-zerodivisor x in the maximal ideal of A such that $(\overline{A},\overline{M}^{**})\in\mathscr{C}$, where overlines denote passage modulo x and the duals are taken over $\overline{A}$. Since both $\dim\overline{A}$ and $\dim A_\mathfrak{p}$, for $\mathfrak{p}$ a non-maximal prime, are less than $\dim A$, minimality implies that $\overline{M}^{**}$ is $\overline{A}$-free and $M_\mathfrak{p}$ is $A_\mathfrak{p}$-free for every non-maximal $\mathfrak{p}$. In particular, $\overline{M}_{\overline{\mathfrak{p}}}$ is $\overline{A}_{\overline{\mathfrak{p}}}$-free for every non-maximal prime $\overline{\mathfrak{p}}$ of $\overline{A}$. Thus the natural homomorphism of $\overline{A}$-modules $\overline{M}\longrightarrow\overline{M}^{**}$ is locally an isomorphism on the punctured spectrum of $\overline{A}$. Lemma B.20 then implies

$$\operatorname{Ext}^i_{\overline{A}}(\overline{M}^{**},\overline{A})\cong\operatorname{Ext}^i_{\overline{A}}(\overline{M},\overline{A}) \tag{B.22.1}$$

for $i=0,\dots,\operatorname{depth}\overline{A}-2$. In particular, (B.22.1) holds for $i=0$ and $i=1$ since $\operatorname{depth}\overline{A}-2=\operatorname{depth}A-3>0$. In particular the case $i=1$ says $\operatorname{Ext}^1_{\overline{A}}(\overline{M},\overline{A})=0$ since $\overline{M}^{**}$ is free.

Now, since M is reflexive, the non-zerodivisor x is also a non-zerodivisor on M, so standard index-shifting ([Mat86, p. 140]) gives $\operatorname{Ext}^1_A(M,\overline{A}) = \operatorname{Ext}^1_{\overline{A}}(\overline{M},\overline{A}) = 0$. The short exact sequence $0 \longrightarrow A \xrightarrow{x} A \longrightarrow \overline{A} \longrightarrow 0$ induces the long exact sequence containing

$$\operatorname{Ext}^1_A(M,A) \xrightarrow{x} \operatorname{Ext}^1_A(M,A) \longrightarrow \operatorname{Ext}^1_A(M,\overline{A}) = 0$$

so that $\operatorname{Ext}^1_A(M,A) = 0$ by NAK. In particular $\operatorname{Hom}_A(M,\overline{A}) \cong \overline{M^*}$ from the rest of the long exact sequence. But $\operatorname{Hom}_A(M,\overline{A}) = \operatorname{Hom}_{\overline{A}}(\overline{M},\overline{A})$ as well, so $\overline{M^*} \cong (\overline{M})^*$. Since $\overline{M}$ is $\overline{A}$-free, this shows that $\overline{M^*}$ is free over $\overline{A}$, and since x is a non-zerodivisor on M^* it follows that M^* is A-free. Thus M is A-free, which contradicts the choice of (A,M) and finishes the proof. □

B.23 Proposition. *Let $\mathscr{C}$ be the set of pairs (A,M) where $(A,\mathfrak{m}_A)$ is a regular local ring and M is a reflexive A-module satisfying $\operatorname{End}_A(M) \cong M^{(n)}$ for some n. Then $\mathscr{C}$ satisfies the conditions of Proposition B.22. Thus M is free over A for every such (A,M).*

Proof. The fact that $\mathscr{C}$ satisfies (i) follows from $\operatorname{End}_{R_\mathfrak{p}}(M_\mathfrak{p}) \cong \operatorname{End}_R(M)_\mathfrak{p}$ and the fact that regularity localizes.

For (ii), we note that reflexive modules over a regular local ring of dimension $\leqslant 2$ are automatically free. Therefore M is locally free on the punctured spectrum; also, we may assume that $\dim A = 3$. Finally, the Auslander-Buchsbaum formula gives $\operatorname{pd}_A M \leqslant 1$; we want to show $\operatorname{pd}_A M = 0$. Observe that $n = \operatorname{rank}_A(M)$ (by passing to the quotient field of A), so $\operatorname{Ext}^1_A(\operatorname{Hom}_A(M,M),A) \cong \operatorname{Ext}^1_A(M^{(\operatorname{rank}_A M)},A) \cong \operatorname{Ext}^1_A(M,A)^{(\operatorname{rank}_A M)}$. Thus by Proposition B.21 M is free.

As for (iii), let $(A,M) \in \mathscr{C}$ with $\dim A > 3$ and M locally free on the punctured spectrum. Let $x \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$ be a non-zerodivisor on A, hence on M as well since M is reflexive. Applying $\mathrm{Hom}_A(M,-)$ to the short exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow \overline{M} \longrightarrow 0$ gives

$$0 \longrightarrow \mathrm{Hom}_A(M,M) \xrightarrow{x} \mathrm{Hom}_A(M,M) \longrightarrow \mathrm{Hom}_A(M,\overline{M}) \longrightarrow \mathrm{Ext}^1_A(M,M).$$

As $\mathrm{Hom}_A(M,M) \cong M^{(n)}$, the cokernel of the map $\mathrm{Hom}_A(M,M) \xrightarrow{x} \mathrm{Hom}_A(M,M)$ is $\overline{M}^{(n)}$. This gives an exact sequence

$$0 \longrightarrow \overline{M}^{(n)} \longrightarrow \mathrm{Hom}_A(M,\overline{M}) \longrightarrow \mathrm{Ext}^1_A(M,M).$$

The middle term of this sequence is isomorphic to $\mathrm{Hom}_{\overline{A}}(\overline{M},\overline{M})$, and the rightmost term has finite length as M is locally free. Apply Lemma B.20, with $i = 0$, to the $\overline{A}$-homomorphism $\overline{M}^{(n)} \longrightarrow \mathrm{Hom}_A(\overline{M},\overline{M})$ to find that

$$\mathrm{Hom}_{\overline{A}}(\overline{M},\overline{M})^* \cong \left(\overline{M}^*\right)^{(\mathrm{rank}_A M)},$$

whence

$$\mathrm{Hom}_{\overline{A}}(\overline{M},\overline{M})^{**} \cong \left(\overline{M}^{**}\right)^{(\mathrm{rank}_A M)}.$$

Since A is regular and $x \notin \mathfrak{m}_A^2$, $\overline{A}$ is regular as well. In particular, $\overline{A}$ is a normal domain, so $\mathrm{Hom}_{\overline{A}}(\overline{M},\overline{M})^{**} = \mathrm{Hom}_{\overline{A}}(\overline{M}^{**},\overline{M}^{**})$. Thus $(\overline{A},\overline{M}^{**}) \in \mathscr{C}$. □

B.24 Remark. As Auslander observes, one can use the same strategy to prove that if A is a regular local ring and M is a reflexive A-module such that $\mathrm{End}_A(M)$ is a *free* A-module, then M is free. This is proved by other methods in [AG60], and has been extended to reflexive modules of finite projective dimension over arbitrary local rings [Bra04].

§3 Galois extensions

Let us now investigate ramification in Galois ring extensions. We will see that ramification in codimension one is attributable to the existence of pseudo-reflections in the Galois group, and prove the Chevalley-Shephard-Todd Theorem that finite groups generated by pseudo-reflections have polynomial rings of invariants. We also prove a result due to Prill, which roughly says that for the purposes of this book we may ignore the existence of pseudo-reflections.

B.25 Definition. Let G be a group and V a finite-dimensional faithful representation of G over a field k. Say that $\sigma \in G$ is a *pseudo-reflection* if σ has finite order and the fixed subspace $V^\sigma = \{v \in V \mid \sigma v = v\}$ has codimension one in V. This subspace is called the *reflecting hyperplane* of σ.

A *reflection* is a pseudo-reflection of order 2.

If the V-action of $\sigma \in G$ is diagonalizable, then to say it is a pseudo-reflection is the same as saying $\sigma \sim \operatorname{diag}(1,\dots,1,\lambda)$ where $\lambda \neq 1$ is a root of unity. In any case, the characteristic polynomial of a pseudo-reflection has a root at 1 of multiplicity at least $\dim V - 1$, hence splits into a product of linear factors $(t-1)^{n-1}(t-\lambda)$. If $\lambda = 1$, then the characteristic of k is necessarily $p > 0$; furthermore σ has order p, and is called a *transvection.*

B.26 Notation. Here is the notation we will use for the rest of the Appendix. In contrast to Chapter 4, where we consider the power series case, we will work in the graded polynomial situation, since it clarifies some of the arguments. We leave the translation between the two to the reader. Let k be a field and V an n-dimensional faithful k-representation of a finite group G, so that we may assume $G \subseteq \mathrm{GL}(V) \cong \mathrm{GL}(n,k)$. Set $S = k[V] \cong k[x_1,\dots,x_n]$, viewed as the ring of polynomial functions on V. Then G acts on S by the rule $(\sigma f)(v) = f(\sigma^{-1}v)$, and we set $R = S^G$, the subring of polynomials fixed by this action. Then $R \longrightarrow S$ is a module-finite integral extension of Noetherian normal domains. Let K and L be the quotient fields of R and S, resp.; then L/K is a Galois extension with Galois group G, and S is the integral closure of R in L. Finally, let $\mathfrak{m}$ and $\mathfrak{n}$ denote the obvious homogeneous maximal ideals of R and S.

B.27 Remark. We may identify the spectrum of S with V. Once we do so, the branch locus of $R \longrightarrow S$ is the union of the fixed point subspaces V^σ over all $\sigma \neq 1$ in G. Indeed, set $X = \operatorname{Spec} R$, so that $q\colon V \longrightarrow X = V/G$ is the quotient map. As in Remark B.6(v) and the discussion before Theorem B.11, the branch locus is the intersection of the diagonal $\Delta_{V/X}$ with the non-diagonal components of the fiber product $V \times_X V$. In this case, the fiber product can be written

$$V \times_X V = \{(v_1, v_2) \in V \times V \mid \text{there exists } \sigma \in G \text{ with } \sigma v_1 = v_2\}.$$

This is the union of $|G|$ diagonals $\{(v, \sigma v) \mid v \in V\}$. The branch locus is precisely the set of points $v \in V$ fixed by a non-trivial element of G.

B.28 Theorem (Chevalley-Shephard-Todd). *With notation as in B.26, consider the following conditions.*

(i) $R = S^G$ *is a polynomial ring.*

(ii) S *is free as an* R*-module.*

(iii) $\operatorname{Tor}_1^R(S,k) = 0$.

(iv) G is generated by pseudo-reflections.

We have (i) $\iff$ (ii) $\iff$ (iii) $\implies$ (iv), and all four conditions are equivalent if $|G|$ is invertible in k.

Proof. (i) $\implies$ (ii): Note that S is always a MCM R-module, so if R is a polynomial ring then S is R-free by the Auslander-Buchsbaum formula.

(ii) $\implies$ (i): If S is free over R, then in particular it is flat. For any finitely generated R-module, then, we have $\mathrm{Tor}_i^R(M,k)\otimes_R S = \mathrm{Tor}_i^S(S\otimes_R M, S/\mathfrak{m}S)$. Since S is regular of dimension n and $S/\mathfrak{m}S$ has finite length, the latter Tor vanishes for $i > n$, whence the former does as well. It follows that R is regular, hence a polynomial ring.

(ii) $\iff$ (iii): This is standard.

(i) $\implies$ (iv): Let $H \subseteq G$ be the subgroup of G generated by the pseudo-reflections. Then H is automatically normal. Localize the problem, setting $A = R_\mathfrak{m}$, a regular local ring by hypothesis, and $B = \left(k[V]^H\right)_{k[V]^H\cap\mathfrak{n}}$. Then $A \longrightarrow B$ is a module-finite extension of local normal domains, and $A = B^{G/H}$.

Consider as in Chapter 4 the twisted group ring $B\#(G/H)$. There is, as in that chapter, a natural ring homomorphism $\delta\colon B\#(G/H) \longrightarrow \mathrm{Hom}_A(B,B)$, which considers an element $b\overline{\sigma} \in B\#(G/H)$ as the A-linear endomorphism $b' \mapsto b\overline{\sigma}(b')$ of B. We claim that δ is an isomorphism. Since source and target are reflexive over B, it suffices to check in codimension one. Let $\mathfrak{q}$ and $\mathfrak{p} = A\cap\mathfrak{q}$ be height-one primes of B and A respectively; then $B_\mathfrak{q}$ is a finitely generated free $A_\mathfrak{p}$-module and so $\delta_\mathfrak{q}\colon B_\mathfrak{q}\#(G/H) \longrightarrow \mathrm{Hom}_{A_\mathfrak{p}}(B_\mathfrak{q},B_\mathfrak{q})$ is an isomorphism. This shows that δ is an isomorphism, and in particular $\mathrm{Hom}_A(B,B)$ is isomorphic as an A-module to a direct sum of copies of B. By Proposition B.23, B is free over A.

Since B is A-free, we have $\mathfrak{H}_A(B) = \mathfrak{D}_A(B)$ by Theorem B.14. But $\mathfrak{D}_A(B) = B$ since no non-identity element of G/H fixes a codimension-one subspace of V, i.e. a height-one prime of B. This implies $\mathfrak{H}_A(B) = B$ so that the branch locus is empty. However, if G/H is non-trivial then $A \longrightarrow B$ is ramified at the maximal ideal of B. Thus $G/H = 1$.

Finally, we prove (iv) $\implies$ (iii) under the assumption that $|G|$ is invertible in k. For an arbitrary finitely generated R-module M, set $T(M) = \mathrm{Tor}_1^R(M,k)$. We wish to show $T(S) = 0$. Note that G acts naturally on $T(S)$, which is a finitely generated graded S-module.

Let $\sigma \in G$ be a pseudo-reflection and set $W = V^\sigma$, a linear subspace of codimension one. Let $f \in S$ be a linear form vanishing on W. Then (f) is a prime ideal of S of height one, and σ acts trivially on the quotient $S/(f) \cong k[W]$. For each $g \in S$, then, there exists a unique element $h(g) \in S$ such that $\sigma(g) - g = h(g)f$. The function $g \mapsto h(g)$ is an R-linear endomorphism of S of degree -1, with $\sigma - 1_S = hf$ as functions on S. Applying the functor $T(-)$ gives $T(\sigma) - 1_{T(S)} = T(h)f_{T(S)}$ as functions on $T(S)$. It follows that $\sigma(s) \equiv x \bmod \mathfrak{n}T(S)$ for every $x \in T(S)$. Since G is generated by pseudo-reflections, we conclude that $\sigma(x) \equiv x \bmod \mathfrak{n}T(S)$ for every $\sigma \in G$ and every $x \in T(S)$.

Next we claim that $T(S)^G = 0$. Define $Q : S \longrightarrow S$ by

$$Q(f) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(f),$$

so that in particular $Q(S) = R$. Factor Q as $Q = iQ' : S \longrightarrow R \longrightarrow S$, so that $T(Q) = T(i)T(Q')$. Since $T(R) = 0$, $T(i)$ is the zero map, so $T(Q) = 0$ as well.

Hence
$$0 = T(Q) = \frac{1}{|G|} \sum_{\sigma \in G} T(\sigma),$$
as R-linear maps $T(S) \longrightarrow T(S)$. But that element fixes the G-invariant elements of $T(S)$, so that $T(S)^G = 0$.

Finally, suppose $T(S) \neq 0$. Then there exists a homogeneous element $x \in T(S)$ of minimal positive degree. Since $\sigma(x) \equiv x \bmod \mathfrak{n}T(S)$ for every $\sigma \in G$, x is an invariant of $T(S)$. But then $x = 0$ as $T(S)^G = 0$. This completes the proof. □

It is implicit in the proof of Theorem B.28 that pseudo-reflections are responsible for ramification. Let us now bring that out into the open. Briefly, the situation is this: let W be a codimension-one subspace of V, and $f \in S$ a linear form vanishing on W. Then (f) is a height-one prime of S, and (f) is ramified over R if and only if W is the fixed hyperplane of a pseudo-reflection.

Keep the notation in B.26, so that $R = k[V]^G \subseteq S = k[V]$ is a module-finite extension of normal domains inducing a Galois extension of quotient fields $K \longrightarrow L$. Since $R \longrightarrow S$ is integral, it follows from "going up" and "going down" that a prime ideal $\mathfrak{q}$ of S has height equal to the height of $R \cap \mathfrak{q}$. Furthermore, for a fixed $\mathfrak{p} \in \operatorname{Spec} R$, the primes $\mathfrak{q}$ lying over $\mathfrak{p}$ are all conjugate under the action of G. (If $\mathfrak{q}$ and $\mathfrak{q}'$ lying over $\mathfrak{p}$ are not conjugate, then by "lying over" no conjugate of $\mathfrak{q}$ contains $\mathfrak{q}'$. Use prime avoidance to find an element $s \in \mathfrak{q}'$ so that s avoids all conjugates of $\mathfrak{q}$. Then $\prod_{\sigma \in G} \sigma(s)$ is fixed by G, so in $R \cap \mathfrak{q} = \mathfrak{p}$, but not in $\mathfrak{q}'$.)

Assume now that $\mathfrak{p}$ is a fixed prime of R of height one, and let $\mathfrak{q} \subseteq S$ lie over $\mathfrak{p}$. Then $R_\mathfrak{p} \longrightarrow S_\mathfrak{q}$ is an extension of DVRs, so $\mathfrak{p}S_\mathfrak{q} = \mathfrak{q}^e S_\mathfrak{q}$ for some integer $e = e(\mathfrak{p})$, the *ramification index* of $\mathfrak{q}$ over $\mathfrak{p}$, which is independent of $\mathfrak{q}$ by the previous paragraph. Let $f = f(\mathfrak{p}, \mathfrak{q})$ be the *inertial degree* of $\mathfrak{q}$ over $\mathfrak{p}$, i.e. the degree of the field extension $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \longrightarrow S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$. Then $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is a free $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$-module of rank ef, so $S_\mathfrak{q}$ is a free $R_\mathfrak{p}$-module of rank ef.

Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the distinct primes of S lying over $\mathfrak{p}$, and set $\mathfrak{q} = \mathfrak{q}_1$. Let $D(\mathfrak{q})$ be the *decomposition group* of $\mathfrak{q}$ over $\mathfrak{p}$,

$$D(\mathfrak{q}) = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q}\}\,.$$

By the orbit-stabilizer theorem, $D(\mathfrak{q})$ has index r in G. Furthermore, $S_\mathfrak{q}$ is an extension of $R_\mathfrak{p}$ of rank equal to $D(\mathfrak{q})$, which implies $|D(\mathfrak{q})| = ef$.

Notice that an element of $D(\mathfrak{q})$ induces an automorphism of $S/\mathfrak{q}$. We let $T(\mathfrak{q})$, the *inertia group* of $\mathfrak{q}$ over $\mathfrak{p}$, be the subgroup inducing the identity on $S_\mathfrak{q}$:

$$T(\mathfrak{q}) = \{\sigma \in G \mid \sigma(f) - f \in \mathfrak{q} \text{ for all } f \in S\}\,.$$

Then the quotient $D(\mathfrak{q})/T(\mathfrak{q})$ acts as Galois automorphisms of $S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ fixing $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. It follows that $|D(\mathfrak{q})/T(\mathfrak{q})|$ divides the degree f of this field extension. Combining this with $|D(\mathfrak{q})| = ef$, we see that e divides $|T(\mathfrak{q})|$. In fact $e = |T(\mathfrak{q})|$ as long as $|G|$ is invertible in k:

B.29 Proposition. *Let $\mathfrak{q}$ be a height one prime of S, set $\mathfrak{p} = R \cap \mathfrak{q}$, and suppose that $T(\mathfrak{q}) \neq 1$. Then $\mathfrak{q} = (f)$ for some linear form $f \in S$. If $W \subseteq V$ is the hyperplane on which f vanishes, then $T(\mathfrak{q})$ is the pointwise stabilizer of W, so every non-identity element of $T(\mathfrak{q})$ is a pseudo-reflection. Furthermore if $|G|$ is invertible in k then $e(\mathfrak{p}) = |T(\mathfrak{q})|$.*

Proof. Since $\mathfrak{q}$ is a prime of height one in the UFD S, $\mathfrak{q} = (f)$ for some homogeneous element $f \in S$. If f has degree 2 or more, then every linear form of S survives in $S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}}$, so is acted upon trivially by $T(\mathfrak{q})$. Since $T(\mathfrak{q})$ is non-trivial, we must have $\deg f = 1$, so f is linear. The zero-set of f, $W = \operatorname{Spec} S/\mathfrak{q}$, is the subspace fixed pointwise by $T(\mathfrak{q})$.

For any $\sigma \in T(\mathfrak{q})$, $\sigma(f)$ vanishes on W, so $\sigma(f) = a_\sigma f$ for some scalar $a_\sigma \in k$. Define a linear character $\chi\colon T(\mathfrak{q}) \longrightarrow k^\times$ by $\chi(\sigma) = a_\sigma$. The image of χ is finite, so is cyclic of order prime to the characteristic of k. The kernel of χ consists of the transvections in $T(\mathfrak{q})$ (see the discussion following Definition B.25). Since $|G|$ is not divisible by p, the kernel of χ is trivial, so that $T(\mathfrak{q})$ is cyclic.

Let $\sigma \in T(\mathfrak{q})$ be a generator, and let λ be the unique eigenvalue of σ different from 1. Then λ is an s^{th} root of unity for some $s > 1$. We can find a basis $v_1, \ldots, v_n$ for V such that $v_1, \ldots, v_{n-1}$ span W, so are fixed by σ, and $\sigma v_n = \lambda v_n$. It follows that $k[V]^{T(\mathfrak{q})} \cong k[x_1, \ldots, x_{n-1}, x_n^s]$, and so $\mathfrak{p} = (x_n^s)$ and $e(\mathfrak{p}) = s = |T(\mathfrak{q})|$. □

Recall that we say the group G is *small* if it contains no pseudo-reflections.

B.30 Theorem. *Let $G \subseteq \mathrm{GL}(V)$ be a finite group of linear automorphisms of a finite-dimensional vector space V over a field k. Set $S = k[V]$ and $R = S^G$. Assume that $|G|$ is invertible in k. Then a prime ideal $\mathfrak{q}$ of height one in S is ramified over R if and only if $T(\mathfrak{q}) = 1$. In particular, $R \longrightarrow S$ is unramified in codimension one if and only if G is small.*

Proof. Let $e = e(\mathfrak{p})$ be the ramification degree of $\mathfrak{p} = R \cap \mathfrak{q}$, and $f = f(\mathfrak{p}, \mathfrak{q})$ the degree of the field extension $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}}$. By the discussion before the Proposition, $ef = |D(\mathfrak{q})|$, where $D(\mathfrak{q})$ is the decomposition group of $\mathfrak{q}$ over $\mathfrak{p}$. Since the order of G is prime to the characteristic, we see that f is as well, so the field extension is separable. Therefore $\mathfrak{q}$ is ramified over R if and only if $e > 1$, which occurs if and only if $T(\mathfrak{q}) \neq 1$. □

To close the Appendix, we record a result due to Prill [Pri67].

B.31 Proposition. *Let G be a finite subgroup of* $\mathrm{GL}(V)$*, where V is an n-dimensional vector space over a field k. Set $S = k[V]$ and $R = S^G$. Then there is an n-dimensional vector space V' and a small finite subgroup $G' \subseteq$* $\mathrm{GL}(V')$ *such that $R \cong k[V']^{G'}$.*

Proof. Let H be the normal subgroup of G generated by pseudo-reflections. By the Chevalley-Shephard-Todd theorem B.28, S^H is a polynomial ring on algebraically independent elements, $S^H \cong k[f_1,\dots,f_n]$. The quotient G/H acts naturally on S^H, with $(S^H)^{G/H} = S^G$, so it suffices to show that G/H acts on $V' = \mathrm{span}(f_1,\dots,f_n)$ without pseudo-reflections. Fix $\sigma \in G \setminus H$ and let $\tau \in H$. Since $\sigma\tau \notin H$, the subspace $V^{\sigma\tau}$ fixed by $\sigma\tau$ has codimension at least two. The fixed locus of the action of the coset σH on V' is then the intersection of $V^{\sigma\tau}$ as τ runs over H, so also has codimension at least two. Therefore σH is not a pseudo-reflection. □

In fact the small subgroup G' of the Proposition is unique up to conjugacy in $\mathrm{GL}(n,k)$. We do not prove this; see [Pri67] for a proof in the complex-analytic situation, and [DR69] for a proof in our context.

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