THE MCKAY CORRESPONDENCE

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ABSTRACT. This is a leisurely, very elementary, introduction to the "classical" McKay Correspondence, based on a series of seminar talks I gave at Syracuse University in April 2006. I provide fairly complete proofs until the end, where I ran out of steam. Nothing here is original with me, and much of this is better-presented in (many!) other places.

Since giving the talks, I've reformatted these notes slightly, and corrected a few of the most glaring boners, but there are still no doubt mistakes, falsehoods, and misattributions. I'd be grateful for any corrections or suggestions.

My main benefit from giving the talks was clarifying in my own mind which statements are true in arbitrary dimension and which require dimension two. I hope this is valuable to some others as well.

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1. The Two-dimensional Correspondence: a look ahead

Goal. To state our eventual goal, let

- k be a field;
- $S = k[[x,y]]$ the power series ring in two variables;
- $G \subseteq \mathrm{GL}_2(k)$ a finite group with order invertible in k, acting on S by linear changes of coordinates;
- $R = S^G$ the fixed ring.

We could use the polynomial ring $k[x,y]$ throughout, but it's convenient to work with complete local rings. We consider only finitely generated modules and finite-dimensional representations.

The various parts of the following theorem have many attributions. I will cite them (to the best of my knowledge, which is seriously spotty in places) as we prove the pieces, but here are some dropped names: McKay, Auslander, Reiten, Artin, Verdier, Gonzalez-Sprinberg, Herzog, and undoubtedly others.

The McKay Correspondence. *Assume that G contains no pseudo-reflections (see below for the definition). Then there are one-one correspondences between*

(1) irreducible representations of G, that is, kG-modules, where kG is the group algebra;
(2) indecomposable projective modules over $\mathrm{End}_R(S)$;
(3) indecomposable reflexive R-modules (i.e., those satisfying $M \cong M^{**}$, where $M^* = \mathrm{Hom}_R(M,R)$; note that reflexive $\iff$ depth 2 $\iff$ MCM in this context);
(4) irreducible components of the exceptional fibre $\pi^{-1}(\mathfrak{m})$, where $\pi\colon Z \longrightarrow \mathrm{Spec}\,R$ is a resolution of singularities and $\mathfrak{m}$ is the closed point of $\mathrm{Spec}\,R$.

In fact, these correspondences extend to isomorphisms between

(a) the McKay quiver of G
(b) the Auslander-Reiten quiver of R,

and, in case G is contained in $\mathrm{SL}_2(\mathbb{C})$,

(c) the desingularization graph of $\mathrm{Spec}\,R$.

It turns out that in the case where $G \subset \mathrm{SL}_2(\mathbb{C})$, each of these is (obtained from) an ADE graph.

Various pieces of the theorem are true more generally, and are in fact easiest to see in less restrictive situations. In the end, we will prove $1 \longleftrightarrow 2 \longleftrightarrow 3$ and say some things about $3 \longleftrightarrow 4$.

2. The Skew Group Algebra

To begin, let's set notation that will remain with us for a while. With k a field, put $S = k[[x_1,\dots,x_n]]$ for some $n \geq 1$, and let $G \subseteq \mathrm{GL}_n(k)$ be a finite group with order invertible in k. (Nearly everything said below breaks horribly in the "modular" case, so I'm just going to ignore that it even exists.) Let G act on S by linear changes of variables. I'm going to experiment with using the representation-theoretic notation s^g for the image of a ring element s under the action of the group element g.

Definition. In this setup, let $S\#G$ denote the *twisted (or 'skew') group algebra*. As an S-module, $S\#G$ is free on the elements of G; the product of two elements s_1g_1 and s_2g_2 is

$$(s_1g_1)(s_2g_2) = s_1s_2^{g_1}g_1g_2\,.$$

(So moving g_1 past s_2 "twists" the ring element.)

Remark.

(1) An $S\#G$-module is nothing but an S-module with an action of G compatible with the S-action: $(sm)^g = s^g m^g$. Note that since the action of G on S is defined on the variables and extended linearly, we have $(st)^g = s^g t^g$ for all s,t, and so S itself is an $S\#G$-module.

(2) A $S\#G$-linear map $f : M \longrightarrow N$ of $S\#G$-modules is a homomorphism of the underlying S-modules that respects the action of G: $f(m^g) = f(m)^g$. This allows us to define an $S\#G$-module structure on $\mathrm{Hom}_S(M,N)$ by $f^g(m) = f(m^{g^{-1}})^g$.

(3) It follows immediately that an S-linear map $f : M \longrightarrow N$ between $S\#G$-modules is invariant under the action of G if, and only if, it is $S\#G$-linear: if $f^g = f$, then

$$f(m) = f^g(m) = f(m^{g^{-1}})^g$$

for every $g \in G$, so that $f(m)^{g^{-1}} = f(m^{g^{-1}})$.

(4) We can rewrite the previous item more suggestively:

$$\mathrm{Hom}_{S\#G}(M,N) = \mathrm{Hom}_S(M,N)^G$$

for all S-modules M and N.

Fact. Since $|G|$ is invertible, taking G-invariants of a $S\#G$-module is an exact functor.

Proof. To see this, denote by M^G the G-invariants of an $S\#G$-module M, and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of $S\#G$-modules. It's trivial to show, and true in utter generality, that $-^G$ is left-exact. We'll just show why $B^G \longrightarrow C^G$ is a surjection if $f: B \longrightarrow C$ is. Let $c \in C^G$, and let $b \in B$ be a preimage in B. Then $f(b) = c$, and since f is a homomorphism of $S\#G$-modules, we also have $f(b^g) = f(b)^g = c^g = c$ for every $g \in G$. Then

$$f\left(\frac{1}{|G|}\sum_{g\in G} b^g\right) = c\,,$$

and the element on the left is fixed by every element of g. $\square$

An enormous amount of useful information flows from this observation.

Corollary. *An $S\#G$-module M is projective if, and only if, it is projective (i.e., free) as an S-module.*

Proof. Onlyifity is clear, since $S\#G$ is by definition free over S. For ifity, suppose that M is S-free. Then $\operatorname{Hom}_S(M,-)$ is exact, so $\operatorname{Hom}_S(M,-)^G$ is exact (being the composition of exact functors). But that is $\operatorname{Hom}_{S\#G}(M,-)$, so M is $S\#G$-free. $\square$

Corollary. $\operatorname{Ext}^i_{S\#G}(M,N) = \operatorname{Ext}^i_S(M,N)^G$ *for S-modules M and N, and all $i \geq 0$.*

Corollary. *$S\#G$ has global dimension equal to n.*

Proof. One inequality follows immediately from the corollaries above: $\operatorname{Ext}^i_S(-,-) = 0$ for all $i > n$, so $\operatorname{Ext}^i_{S\#G}(-,-)$ vanishes as well. The other inequality follows upon observing that the residue field k of S is also an $S\#G$-module (with trivial action), and the Koszul complex resolving it is also an $S\#G$-module resolution. We will return to these facts below (so probably should have mentioned them earlier). $\square$

3. Representations and Projectives

Keep all the notation from above. In addition, set $\mathfrak{m} = (x_1,\dots,x_n)$, the maximal ideal of S. Our next goal is to discuss the pair of functors

$$\{\text{projective } S\#G\text{-modules } P\} \longleftrightarrow \{\text{representations } W \text{ of } G\}$$

given by $P \mapsto P/\mathfrak{m}P$ and $W \mapsto S \otimes_k W$. We'll prove that these two functors are inverse on objects. (They are not equivalences, though, since Hom-sets on the right are k-vector spaces and the same is not true of those on the left.)

Definition. Let M be an $S\#G$-module and W a representation of G, that is, a kG-module. Define a $S\#G$-module structure on $M \otimes_k W$ by

$$sg(m \otimes w) = sm^g \otimes w^g\,.$$

The proof of the next lemma follows from the fact that $S\#G$-projectivity is equivalent to S-projectivity.

Lemma. If P is a projective $S\#G$-module, then $P \otimes_k W$ is again $S\#G$-projective for any representation W of G. In particular, $S \otimes_k W$ is $S\#G$-projective for every W.

Proposition. *Let P be a projective $S\#G$-module. Then $P \cong S \otimes_k P/\mathfrak{m}P$. Moreover,if $S \otimes_k W \cong S \otimes_k W'$ for two kG-modules W, W', then $W \cong W'$.*

Proof. The maximal ideal $\mathfrak{m}$ is stable under the action of G, so $\mathfrak{m}(S\#G)$ is a two-sided ideal of $S\#G$. Since $S\#G$ is finitely generated over S, this implies $\mathfrak{m}(S\#G) \subseteq \operatorname{rad}(S\#G)$. But $S\#G/\mathfrak{m}(S\#G) = kG$ is semisimple, so $\mathfrak{m}(S\#G) = \operatorname{rad}(S\#G)$.

It follows that if P is a projective $S\#G$-module, then $\mathfrak{m}P = \operatorname{rad}P$, so $P/\mathfrak{m}P$ is a kG-module, and $P \longrightarrow P/\mathfrak{m}P$ is a projective cover of $S\#G$-modules. (This is a point where local-ness and completeness make our lives easier.)

Since projective covers are unique, this implies that two projective $S\#G$-modules P and Q are isomorphic if, and only if, $P/\mathfrak{m}P \cong Q/\mathfrak{m}Q$. Furthermore, if W is any kG-module, then $S \otimes_k W$ is projective and $(S \otimes_k W)/\mathfrak{m}(S \otimes_k W) \cong S/\mathfrak{m}S \otimes_k W \cong W$, so $S \otimes_k W$ is a $S\#G$-projective cover of W.[1] □

Corollary. *Let $V_0, \dots, V_d$ be a complete set of nonisomorphic simple kG-modules. Then*

$$S \otimes_k V_0, \dots, S \otimes_k V_d$$

is a complete set of nonisomorphic indecomposable projective $S\#G$-modules.

4. The McKay quiver and the Gabriel quiver

The one-one correspondence between projectives and representations described above extends to an isomorphism of two graphs naturally associated to these data.

Keeping all notation as above, let in addition V be the n-dimensional kG-module (= representation of G) coming from the given embedding $G \subseteq \mathrm{GL}_n(k)$. Let $V_0, V_1, \dots, V_d$ be a complete set of the nonisomorphic simple kG-modules, with V_0 the trivial module k.

[1] This proof could be smoother.

Definition. The *McKay quiver* of $G \subseteq \mathrm{GL}_n(k)$ has

- vertices $V_0, \dots, V_d$, and
- m arrows $V_i \longrightarrow V_j$ if the multiplicity of V_i in $V \otimes_k V_j$ is equal to m.

For $i = 0, \dots, d$ let $P_i = S \otimes_k V_i$ be a complete set of the indecomposable projective $S\#G$-modules, with $P_0 = S \otimes_k V_0 = S$. Then we know that $V_i = P_i/\mathfrak{m}P_i$ is also a $S\#G$-module, so has a resolution of length at most n

$$0 \longrightarrow Q_n^{(j)} \longrightarrow Q_{n-1}^{(j)} \longrightarrow \cdots \longrightarrow Q_1^{(j)} \longrightarrow P_j \longrightarrow (V_j \longrightarrow)0$$

by projective $S\#G$-modules $Q_i^{(j)}$ for $i = 1, \dots, n$ and $j = 0, \dots, d$.

Definition. The *Gabriel quiver* of $G \subseteq \mathrm{GL}_n(k)$ has

- vertices $P_0, \dots, P_d$, and
- m arrows $P_i \longrightarrow P_j$ if the multiplicity of P_i in $Q_1^{(j)}$ is equal to m.

Theorem (Auslander '86)**.** *These two quivers are isomorphic.*

Proof. First consider the trivial module $V_0 = k$. It is also the S-module $S/\mathfrak{m}$, so is resolved over S by the Koszul complex, which we write as

$$0 \longrightarrow S \otimes_k \textstyle\bigwedge^n V \longrightarrow S \otimes_k \bigwedge^{n-1} V \longrightarrow \cdots \longrightarrow S \otimes_k V \longrightarrow S \longrightarrow 0$$

where again V is the given representation of G, and the maps are natural contractions. But these maps are easily checked to be G-linear as well, so this is a minimal $S\#G$-projective resolution of V_0.

The minimal $S\#G$-projective resolution of V_j is then obtained by tensoring with V_j to get

$$0 \longrightarrow S \otimes_k (\textstyle\bigwedge^n V \otimes_k V_j) \longrightarrow \cdots \longrightarrow S \otimes_k (V \otimes_k V_j) \longrightarrow S \otimes_k V_j \longrightarrow 0,$$

which gives $Q_1^{(j)} \cong S \otimes_k (V \otimes_k V_j)$, so the multiplicity of $P_i = S \otimes_k V_i$ in $Q_1^{(j)}$ is equal to the multiplicity of V_i in $V \otimes_k V_j$. □

5. Invariants, and the Skew Algebra as Endomorphism Ring

The notation remains the same, with one addition.

Set $R = S^G = k[[x_1, \dots, x_n]]^G$. Then we enjoy the following catalog of properties. (Asserted here without proofs – I haven't thought about the best path to take among these statements.)

(1) R is a complete local domain;

(2) R has dimension n;

(3) R is integrally closed (since $Q(R) = R \cap Q(S)$);

(4) S is integral over R, and even a finitely generated R-module;

(5) There is a "Reynolds operator" $\rho \colon S \longrightarrow R$, sending s to the average of its orbit, which makes R a direct summand of S;

(6) R is Cohen–Macaulay (by the Hochster–Roberts theorem, which is often stated assuming that k is algebraically closed, but which applies in this case since we have a Reynolds operator);

(7) $IS \cap R = I$ for every ideal I of R (also follows from the existence of ρ);

(8) S is a maximal Cohen–Macaulay R-module, so in particular reflexive (specifically, $x_1^{|G|}, \dots, x_n^{|G|}$ is an R-regular sequence).

We need to throw out one degenerate case.

Definition. An element $g \in G$ is a *pseudo-reflection* (for the given representation V, or equivalently for the given embedding into GL_n) if g fixes a codimension-one subspace of V. Equivalently, $\mathrm{rank}(g - \mathrm{id}_n) \le 1$; again equivalently, g has eigenvalue 1 with multiplicity $n-1$.

The canonical example of a pseudo-reflection is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{GL}_2$). This is of course the action that swaps x_1 and x_2; it's well-known that the invariant ring is generated by the symmetric functions, $R = k[[x+y, xy]]$, which (the so-called Fundamental Theorem of Symmetric Functions) form a regular ring.

Theorem (Shepard-Todd, Chevalley, Serre)**.** *With notation as above, suppose that G is generated by pseudo-reflections. Then R is a regular local ring. The converse is true if k has characteristic zero.*

(The canonical *non*-example is $x \mapsto -x$, $y \mapsto -y$, which looks as though it might be reasonably called a reflection, but isn't a pseudo-reflection. The fixed ring is (the non-regular ring) $k[[x^2, xy, y^2]]$. Instead, a reflection is usually defined to be a pseudo-reflection of order 2.)

Definition. We say that the group G is *small* if it contains no pseudo-reflections.

In some sense (which I'm going to deliberately gloss over), we can always assume that G is small: if H is the subgroup generated by pseudo-reflections, then S^H is regular by the Theorem, and the quotient group G/H acting on S^H acts without pseudo-reflections.

Several issues about R become clearer once we assume G is small. For example, it is a theorem of Watanabe that if G is small, then R is Gorenstein if, and only if, G is contained in $\mathrm{SL}_n(k)$. Here is another, which we will use below.

Fact. The group G is small if and only if height-one primes in S are unramified over R. (Let $\mathfrak{Q} \in \operatorname{Spec} S$, and set $\mathfrak{q} = \mathfrak{Q} \cap S \in \operatorname{Spec} R$. Then Q is unramified over R provided $\mathfrak{q} S_{\mathfrak{Q}} = \mathfrak{Q} S_{\mathfrak{Q}}$ and the extension of residue fields $\kappa(\mathfrak{q}) \longrightarrow \kappa(\mathfrak{Q})$ is separable.) Most references I've seen to this statement cite it as a "standard fact" from ramification theory; I haven't found a good source. The idea seems clear enough, though: if g fixes a hyperplane H in V, then the prime ideal of S corresponding to H will ramify.

Proposition (Auslander 1962, purity of the branch locus)**.** *Define*

$$\delta\colon S\#G \longrightarrow \operatorname{End}_R(S)^{op}$$

by

$$\delta(sg)(t) = st^g\,.$$

If G is small, then δ is an isomorphism.

We need a general fact about normal domains, due to Auslander and Buchsbaum (1959, On ramification theory in noetherian rings).

Lemma. Let A be an integrally closed integral domain, M a reflexive A-module, and N a torsion-free A-module. A homomorphism $f\colon M \longrightarrow N$ is an isomorphism if and only if $f_{\mathfrak{p}}$ is an isomorphism for all primes $\mathfrak{p}$ of height one.

Sketch of Proof. Let $K = \ker f$ and $C = \operatorname{coker} f$. By the Depth Lemma, K has depth 2, so is reflexive. But $K_{(0)} = 0$ means that K is annihilated by some nonzero $a \in A$, so $K^* = 0$, which implies $K = K^{**} = 0$. As for C, $C_{\mathfrak{p}} = 0$ for all primes of height 1 implies that $\operatorname{ann}_R(C)$ has height at least 2. But A is a normal domain, hence satisfies Serre's condition S_2, so $\operatorname{ann}_R(C)$ has *grade* at least two, which implies that $C^* = \operatorname{Ext}^1_R(C,R) = 0$. It follows that $f^*\colon N^* \longrightarrow M^*$ is an isomorphism, so f^{**} is as well. But then one chases a diagram to see that $N \longrightarrow N^{**}$ is an isomorphism, so $f^{**} = f$ and we're done. □

Sketch of Proof of Auslander's theorem. Observe that it suffices to show that δ is an isomorphism in codimension one, since we are in a position to apply the Lemma (Hom modules always being reflexive and $S\#G$ being torsion-free).

We have a commutative diagram

$$\begin{tikzcd} S\#G \arrow[r, "\delta"] \arrow[d] & \operatorname{End}_R(S)^{op} \arrow[d] \\ Q(S)\#G \arrow[r] & \operatorname{End}_{Q(R)}(Q(S))^{op} \end{tikzcd}$$

where $Q(-)$ denotes the fraction field. The bottom row is an isomorphism by Galois Theory, and the columns are both monomorphisms, which implies that δ is injective.

Locally in codimension one, S is a DVR, where one can check directly that $\delta_{\mathfrak{p}}$ is an isomorphism. It follows that δ is an isomorphism. □

Note that already at this point we've seen that $S\#G$ is a "non-commutative crepant resolution" of R, whatever that is. See a later section.

At this point, we have established one-one correspondences between

- the irreducible kG-modules
- the indecomposable projective $S\#G$-modules
- the indecomposable projective $\mathrm{End}_R(S)$-modules

From now on, **we always assume that G is small.**

6. Projectivization

There's quite a bit more to be said about this, but all we need for the purposes of these talks is that:

Theorem. *There is an equivalence between the full subcategory* $\mathrm{add}_R(S)$*, of direct summands of R-direct sums of S, and the full subcategory of projective* $\mathrm{End}_R(S)^{op}$*-modules. This equivalence is induced by*

$$X \mapsto \mathrm{Hom}_R(S,X).$$

(When these talks were given, Dan Zacharia had just spoken about this theorem, so I didn't go through the proof. The hypotheses in this situation are of course much stronger than necessary; the theorem holds for essentially any ring R and any R-module S.)

Corollary. *Let n be arbitrary. Then we have one-one correspondences between*

- *the irreducible kG-modules*
- *the indecomposable projective $S\#G$-modules*
- *the indecomposable projective* $\mathrm{End}_R(S)$*-modules*
- *the indecomposable modules in* $\mathrm{add}_R(S)$

7. Herzog's Theorem; Dimension two

We know that, for arbitrary n, S is a MCM R-module, and furthermore that R is a direct summand of S via the "Reynolds operator." In dimension two, something special happens: *every* MCM R-module is a direct summand of S.

Here is the notation in effect:

- k is a field;
- $S = k[[x_1, x_2]]$ is the power series ring in two variables;
- $G \subseteq \mathrm{GL}_2(k)$ is a finite group with order invertible in k;
- $R = S^G$ is the invariant ring.

Theorem (Herzog '76). *Up to isomorphism, the indecomposable reflexive R-modules are precisely the indecomposable R-direct summands of S. In particular, R has only finitely many indecomposable MCM modules.*

Proof. We know that S is MCM over R, so each R-direct summand of S is MCM as well.

Let M be an indecomposable reflexive R-module. Then the split monomorphism $R \longrightarrow S$ induces a split monomorphism $M = \mathrm{Hom}_R(M^*, R) \longrightarrow \mathrm{Hom}_R(M^*, S)$. Now $\mathrm{Hom}_R(M^*, S)$ is an S-module via the action on the codomain; we claim that it has depth two as an S-module. To see this, let

$$F_1 \longrightarrow F_0 \longrightarrow M^* \longrightarrow 0$$

be an R-free presentation of M^*, and apply $\mathrm{Hom}_R(-, S)$ to obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M^*, S) \longrightarrow \mathrm{Hom}_R(F_0, S) \longrightarrow \mathrm{Hom}_R(F_1, S),$$

so (M^*, S) is a second syzygy over S, and hence has depth 2.

In fact, since S is regular of dimension two, this implies that $\mathrm{Hom}_R(M^*, S)$ is *free* as an S-module. Therefore $\mathrm{Hom}_R(M^*, R) = M$ is a direct summand of a free S-module, and so $M \in \mathrm{add}_R(S)$. □

It seems like the following observation should lead to a more general statement, but I don't know of any.

Porism. *If M is a reflexive R-module such that $\mathrm{Ext}_R(M^*, S) = 0$ for $i = 1, \ldots, n$, then $M \in \mathrm{add}_R(S)$.*

Corollary. *Let $n = 2$ and assume that G is small. Then we have one-one correspondences between*

- *the irreducible kG-modules;*
- *the indecomposable projective $S\#G$-modules;*
- *the indecomposable projective $\mathrm{End}_R(S)$-modules;*
- *the indecomposable modules in $\mathrm{add}_R(S)$; and*

- *the indecomposable reflexive R-modules.*

Corollary. *Let $n = 2$ and assume that G is small. Put $M = \bigoplus X$, the direct sum of complete set of representatives X for the indecomposable reflexive R-modules. Put $\Lambda = \mathrm{End}_R(M)$. Then Λ is a reflexive R-module, and has global dimension 2.*

In other language, the endomorphism ring of the direct sum of all MCM R-modules is MCM again and has finite global dimension. This deserves its own digression.

8. Interlude: Non-commutative crepant resolutions

Definition (Van den Bergh)**.** Let R be a Gorenstein local normal domain. A *non-commutative crepant resolution of R* is an R-algebra Λ of the form $\Lambda = \mathrm{End}_R(M)$, where M is a reflexive R-module, and

(1) $\mathrm{gldim} R = \dim R$
(2) Λ is MCM as an R-module.

With this definition, we can restate the above Corollary as

Corollary. *Let $R = k[[x,y]]^G$, where $G \subseteq GL_2(k)$ has order invertible in k and contains no pseudo-reflections. If R is Gorenstein (so the definition above makes sense) then R has a non-commutative crepant resolution.*

Here's a sketch of the origin of the name. Let $X = \mathrm{Spec} R = \mathbb{C}^2/G$; then X has a canonical minimal resolution $\mathbb{C}^2//G$. Kapranov and Vasserot observed that the derived category of coherent sheaves on $\mathbb{C}^2//G$ is canonically equivalent with the derived category of coherent sheaves over $\mathbb{C}[[x,y]]\#G$. It therefore makes sense to think of $\mathbb{C}[[x,y]]\#G$ as "the" resolution. The "crepant" part is just too silly to talk about.

Bridgeland, King, and Reid have proved a similar result in dimension three. Past that, extending the correspondences above is an area of active research.

Here are two more examples of non-commutative crepant resolutions:

Theorem (GL 2004)**.** *Let R be a CM local ring of finite CM type. Let M be the direct sum of a complete set of representatives for the indecomposable MCM R-modules. Then $\Lambda = \mathrm{End}_R(M)$ has global dimension $\max\{2, \dim R\}$. In particular, if R is Gorenstein of dimension at most 3, then R has a non-commutative crepant resolution.*

The proof of this one is essentially identical to the proof of Auslander's theorem that Artin algebras have finite representation dimension.

Theorem (Buchweitz-GL-Van den Bergh 2006)**.** *Let $S = k[X]$ and $R = S/\det(X)$, where $X = (x_{ij})$ is a $(n \times n)$-matrix of indeterminates. Let $\mathscr{F}, \mathscr{G}$ be free S-modules and think of X as defining a homomorphism $X\colon G \longrightarrow F$. For $a = 0, \dots, n$, set $M_a = \operatorname{coker} \bigwedge^a X\colon \bigwedge^a \mathscr{G} \longrightarrow \bigwedge^a \mathscr{F}$. Let $M = \bigoplus_a M_a$. Then $\Lambda = \operatorname{End}_R(M)$ is a non-commutative crepant resolution of R.*

9. Invariants of Projectives

We know that there is a one-one correspondence between indecomposable projectives over $S\#G$ and R-direct summands of S, but our current description of that correspondence is fairly circuitous. Here is a better one.

Theorem (Auslander '86, Rational singularities and almost split sequences)**.** *There is an equivalence of categories between the category of projective $S\#G$-modules and $\operatorname{add}_R(S)$, induced by $P \mapsto P^G$. In particular, P^G is a MCM R-module for every projective $S\#G$-module P.*

Proof. Define a homomorphism of R-modules $\varphi\colon S \longrightarrow S\#G$ by $phi(s) = \sum_{g \in G} s^g g$. Then the image of φ lands inside the invariants $(S\#G)^G$. In fact, one checks that $\varphi(S) = (S\#G)^G$. Since φ is obviously injective, this gives $(S\#G)^G \cong S$. Call the image S_1 (to keep it separate from S).

It follows that if P is a projective $S\#G$-module, then P^G is an R-direct summand of a direct sum of copies of S_1, so $P^G \in \operatorname{add}_R(S_1) = \operatorname{add}_R(S)$.

To see that $P \mapsto P^G$ is an equivalence, we claim that the map $\alpha\colon \operatorname{End}_{S\#G}(S\#G) \longrightarrow \operatorname{End}_R(S_1)$, defined by $\alpha(f) = f|_{S_1}$, is an R-algebra isomorphism. Why will this do it? Because the projective $S\#G$-modules are obtained from idempotents in $\operatorname{End}_{S\#G}(S\#G)$, while the idempotents in $\operatorname{End}_R(S_1)$ give the R-summands of S.

The claim follows by considering the following sequence of R-algebra homomorphisms:

$$S\#G \xrightarrow[sg \mapsto s^{g^{-1}} g^{-1}]{\gamma} (S\#G)^{op} \xrightarrow[sg \mapsto (th \mapsto thsg)]{\beta} \operatorname{End}_{S\#G}(S\#G) \xrightarrow[f \mapsto f|_{S_1}]{\alpha} \operatorname{End}_R(S_1).$$

One can check (details omitted) that γ and β are bijective, and that the composition is precisely the map $\delta\colon S\#G \longrightarrow \operatorname{End}_R(S)$ defined by $\delta(sg)(t) = st^g$, which we know to be bijective from before. So α is too. $\square$

10. The Auslander-Reiten quiver

Next we show that the isomorphism of quivers between the McKay quiver and the Gabriel quiver, described earlier, extends to a third quiver constructed from the reflexive R-modules.

In the special case $n = 2$, this quiver will be shown to coincide with the Auslander–Reiten quiver coming from the theory of almost split sequences.

For now, let n be arbitrary.

Recall that the simple $S\#G$-modules $V_0,\dots,V_d$ have minimal $S\#G$-projective resolutions

$$\begin{tikzcd} 0 \arrow[r] & S\otimes_k(\bigwedge^n V\otimes_k V_j) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & S\otimes_k(V\otimes_k V_j) \arrow[r] \arrow[d, equal] & S\otimes_k V_j \arrow[r] \arrow[d, equal] & 0 \\ & Q_n^{(j)} & & Q_1^{(j)} & P_j & \end{tikzcd}$$

Since $\dim V = n$, we see that $\bigwedge^n V\otimes_k V_j$ is a simple kG-module, so $Q_n^{(j)}$ is an indecomposable projective $S\#G$-module. Define a permutation on the set $\{P_0,\dots,P_d\}$ by $\tau(P_j) = Q_n^{(j)}$.

This gives the exact sequences

$$0 \longrightarrow \tau(P_j) \longrightarrow Q_{n-1}^{(j)} \longrightarrow \cdots \longrightarrow Q_1^{(j)} \longrightarrow P_j \longrightarrow V_j \longrightarrow 0$$

for each $j = 0,\dots,d$.

Take G-invariants, recalling that $(P_j)^G$ is an indecomposable MCM R-module, denoting it M_j (with $M_0 = R$), and defining $\tau(M_j)$ in an obvious way, to get:

$$0 \longrightarrow \tau(M_j) \longrightarrow E_{n-1}^{(j)} \longrightarrow \cdots \longrightarrow E_1^{(j)} \longrightarrow M_j \longrightarrow V_j^G \longrightarrow 0,$$

an exact sequence of R-modules, each MCM except for $(V_j)^G$. Here we have set $E_i^{(j)} = (Q_i^{(j)})^G$. Note also that $\tau(M_j)$ is indecomposable.

Now,

$$V_j^G = \begin{cases} k & \text{if } j = 0 \text{ (as } V_0 = k \text{ is the trivial representation)} \\ 0 & \text{otherwise (as } V_j \text{ is a nontrivial simple).} \end{cases}$$

So we have exact sequences of MCM R-modules

$$0 \longrightarrow \tau(M_j) \longrightarrow E_{n-1}^{(j)} \longrightarrow \cdots \longrightarrow E_1^{(j)} \longrightarrow M_j \longrightarrow 0$$

for every $j \neq 0$, and the special case

$$0 \longrightarrow \tau(R) \longrightarrow E_{n-1}^{(0)} \longrightarrow \cdots \longrightarrow E_1^{(0)} \longrightarrow R \longrightarrow k \longrightarrow 0$$

for $j = 0$.

Definition. The *Auslander-Reiten quiver*[2] of R has

- vertices $M_0, M_1,\dots,M_d$, and

[2]This is a gross abuse of language, since the quiver defined here coincides with what is usually called the AR quiver only in case $n = 2$. I just couldn't think of another name for it.

- m arrows $M_i \longrightarrow M_j$ if the multiplicity of M_i in $E_1^{(j)}$ is equal to m.

Fact. Assume that $n = 2$. Then the definition above matches the usual definition of the Auslander-Reiten quiver, so that $0 \longrightarrow \tau(M_j) \longrightarrow E_1^{(j)} \longrightarrow M_j \longrightarrow 0$ is the AR sequence ending in M_j for $j \neq 0$, and for $j = 0$ the map $E_0 \longrightarrow R$ contains all the irreducible homomorphisms ending in R.

Theorem (Auslander '86)**.** *Assume $n = 2$. Then the McKay quiver of G, with the trivial representation deleted, is isomorphic to the Auslander-Reiten quiver of R with the vertex $[R]$ deleted.*

11. Obstructions in dimension 3

Lest you get all excited that I keep realizing that so much of the above does not really require $n = 2$, here's one more theorem of Auslander that implies $n = 2$ really is needed somewhere.

Theorem (Auslander '86)**.** *Let $R = k[[x_1, \dots, x_n]]^G$ with $G \subset \mathrm{GL}_n(k)$ a finite group with order invertible in k, and G acting by linear changes of variable. Assume that k is algebraically closed. Then R has finitely many maximal Cohen–Macaulay modules if and only if either*

- *$n \le 2$, or*
- *$n = 3$ and $G = \mathbb{Z}/2\mathbb{Z}$, acting via $x_i \mapsto -x_i$.*

Note that in the exceptional case in this theorem, $R \cong k[[x^2, xy, xz, y^2, yz, z^2]]$ has three indecomposable MCM modules: R and $\omega_R \cong (x, y, z)R$, which are direct summands of $S = k[[x, y, z]]$, and $\mathrm{syz}^1_R(\omega_R)$, which is not.

12. The Gorenstein case

Next I want to restrict to the case $G \subset \mathrm{SL}_2(\mathbb{C})$, where all this connects up with the ADE Coxeter–Dynkin diagrams.

First observe that if we understand the case where $G \subset \mathrm{SL}_n(k)$ and the case where G is cyclic, then in some sense we understand everything. To explain this, let $G \subset \mathrm{GL}_n(k)$ be arbitrary, and let H be the subgroup of G consisting of matrices of determinant 1. Then G/H can be identified with a finite subgroup of $k^\times$ under the determinant map, so is cyclic by a standard exercise.

So now assume that

- k is the complex number field $\mathbb{C}$, and

- $G \subset \mathrm{SL}_2(\mathbb{C})$ is small.

The finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ up to conjugacy have been known for a long time. I don't give proofs here, but the classification uses the facts that (1) every finite subgroup of SL_n is conjugate to a finite subgroup of $SU(n)$, (2) there is a (continuous) two-to-one homomorphism from $SU(2)$ onto $SO(3)$, the orthogonal real 3×3-matrices with determinant one, and (3) the finite subgroups of $SO(3)$ are either cyclic, dihedral, or the group of symmetries of a Platonic solid. In detail, the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are

(C_n) the cyclic group of order $n+1$, generated by

$$\begin{pmatrix} \omega_{n+1} & 0 \\ 0 & \omega_{n+1}^{-1} \end{pmatrix}$$

where ω_{n+1} is a primitive $(n+1)^{\text{th}}$ root of 1.

(D_n) the binary dihedral group of order $4(n-2)$, generated by

$$\begin{pmatrix} 0 & \omega_4 \\ \omega_4 & 0 \end{pmatrix} \qquad \text{and} \qquad C_{2n-5}$$

(T) the binary tetrahedral group of order 24, generated by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \omega_8 & \omega_8^3 \\ \omega_8 & \omega_8^7 \end{pmatrix} \qquad \text{and} \qquad D_4$$

(O) the binary octahedral group of order 48, generated by

$$\begin{pmatrix} \omega_8^3 & 0 \\ 0 & \omega_8^5 \end{pmatrix} \qquad \text{and} \qquad T$$

(I) the binary icosahedral group of order 120, generated by

$$\frac{1}{\sqrt{5}} \begin{pmatrix} \omega_5^4 - \omega_5 & \omega_5^2 - \omega_5^3 \\ \omega_5^2 - \omega_5^3 & \omega_5 - \omega_5^4 \end{pmatrix} \qquad \text{and} \qquad \frac{1}{\sqrt{5}} \begin{pmatrix} \omega_5^2 - \omega_5^4 & \omega_5^4 - 1 \\ 1 - \omega_5 & \omega_5^3 - \omega_5 \end{pmatrix}$$

The modifier "binary" reflects the fact that these subgroups are double covers of the associated symmetry groups.

Furthermore, Klein showed that in each case, R is a hypersurface, defined by a single polynomial in three variables:

- (C_n) aka (A_n)

$$f(x,y,z) = x^{n+1} + y^2 + z^2$$

- (D_n)

$$f(x,y,z) = x^{n-1} + xy^2 + z^2$$

- (T) aka (E_6)

$$f(x,y,z)=x^4+y^3+z^2$$

- (O) aka (E_7)

$$f(x,y,z)=x^3y+y^3+z^2$$

- (I) aka (E_8)

$$f(x,y,z)=x^5+y^3+z^2$$

Here's an amusing way to think of these hypersurfaces, cribbed from John McKay, writing on John Baez's web site:

> we project from the North pole of the sphere escribed to the Platonic solid, through each vertex on to the equatorial plane (which we interpret as the complex plane). Thus we may identify each vertex with a complex number v[i], and we form the (homogeneous) polynomial V(x,y) = prod(x-v[i]y). Similarly we form E(x,y) from the midpoints of the edges, and F(x,y) from the normals through the centre of the faces. These are three functions in two variables and so there is a relation f(V,E,F) = 0.

Someday I hope to revisit this, and explain some of the geometric content (specifically McKay's own contribution, as well as more systematic treatments since then).

[3]

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[3] This is by no means a comprehensive bibliography on the subject; it does include all the sources I used for writing these notes, as well as some places for further reading.