(1)

Math 8680 Nov. 11, 2024 (subbing for Gregg M.)

McKay correspondence

I. Older correspondences

II. McKay's addition

III. Proofs

I. Older ~~correspondences~~ (classifications & correspondences) - see Buchweitz, Faber, Ingalls "Magic square..."

∃ bijections between these families of subgroups $H<G$ of groups G, up to G-conjugation:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{fin. subgroups}\\ H<SU_2(\mathbb{C})\\ (\underline{\text{or}}\ SL_2(\mathbb{C}))\\ \text{with } |H| \text{ even}\end{array}\right\} \arrow[d] & \left\{\begin{array}{c}\text{fin. ref'n gps}\\ H<U_2(\mathbb{C})\\ (\underline{\text{or}}\ GL_2(\mathbb{C}))\\ \text{containing } -I_2=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\end{array}\right\} \arrow[l, "(-)\cap SL_2(\mathbb{C})"'] \arrow[d, "\begin{array}{l}\text{direct explanations}\\ \text{newer (2002, 2024)}\end{array}"] \\
\left\{\begin{array}{c}\text{fin. } H<SO_3(\mathbb{R})\\ (\underline{\text{or}}\ SL_3(\mathbb{R}))\end{array}\right\} & \left\{\begin{array}{c}\text{fin. ref'n gps}\\ H<O_3(\mathbb{R})\\ (\underline{\text{or}}\ GL_3(\mathbb{R}))\end{array}\right\} \arrow[l, "(-)\cap SL_2(\mathbb{R})"']
\end{tikzcd}$$

Recall $GL_2(\mathbb{C})=\left\{A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathbb{C}^{2\times2}:\det A\neq0\right\} > U_2(\mathbb{C})=\left\{A^T\overline{A}=I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}\right\}$ | $O_3(\mathbb{R})=\{A\in\mathbb{R}^{3\times3}:A^TA=I_3\}$

$\vee$ $\qquad$ $\vee$

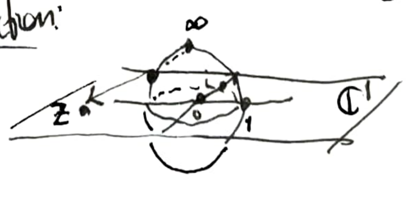
$SL_2(\mathbb{C})=\{\det A=1\} > SU_2(\mathbb{C})=\{\det A=1\}$ | $SO_3(\mathbb{R})=\{\det A=1\}$

= rotations in $\mathbb{R}^3$ or on $\mathbb{S}^2$

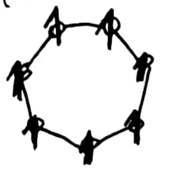
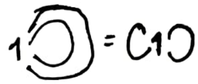
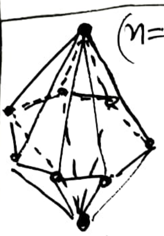
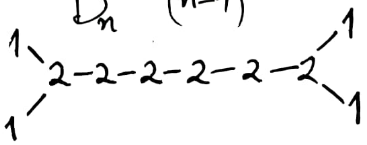
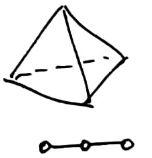
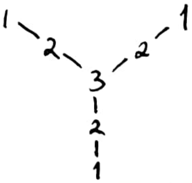
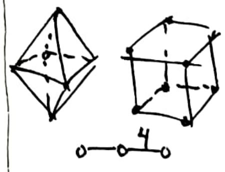
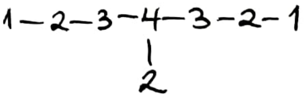
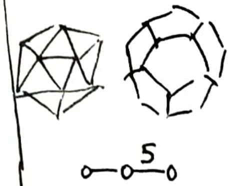
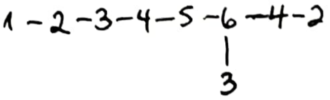
The left vertical map comes from $SU_2(\mathbb{C})<GL_2(\mathbb{C})$ acting on $\mathbb{S}^2=\mathbb{C}^1\sqcup\{\infty\}$ via lin. fractional transformations $z\longmapsto\frac{az+b}{cz+d}$, sending to $SU_2(\mathbb{C})\xrightarrow{2:1}\!\!\!\!\rightarrow SO_3(\mathbb{R})$

$\pm\begin{bmatrix}1&0\\0&1\end{bmatrix}\longmapsto I_3$

after identifying $\mathbb{S}^2\leftrightarrow\mathbb{C}^1\sqcup\{\infty\}$ via stereographic projection:



II. McKay (1980): $\left\{\text{Finite subgroups } H < SU_2(\mathbb{C}) \text{ (or } SL_2(\mathbb{C}))\right\} \longleftrightarrow \left\{\tilde{A}, \tilde{D}, \tilde{E} \text{ affine simply laced diagrams}\right\}$

$H < SU_2(\mathbb{C})$		pre-McKay ←	affine diagram (with labeling via dimensions of H-irreducibles)
cyclic group $\mathbb{Z}/n\mathbb{Z}$ $= \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix} \right\rangle$ $\zeta_n = e^{2\pi i/n}$ ↑ cyclic, reducible			$\tilde{A}_{n-1}$ $(n=7)$  $\tilde{A}_1$  ↕ $H = \{\pm I_2\}$ $\tilde{A}_0$  ↕ $H = \{I_2\}$
↓ irreducible, non-abelian "binary polyhedral groups"	presentation $H = \langle a,b,c \mid a^p = b^q = c^r = abc \rangle$ $abc = (-I_2)$ with (p,q,r) having $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$	polytope P with $W = \mathrm{Aut}(P) \subset O_3(\mathbb{R})$ $W^+ \subset SO_3(\mathbb{R})$	
binary dihedral group $\mathbb{BD}_{n-2}$ $= \left\langle \underbrace{\begin{bmatrix} \zeta_{n-2} & 0 \\ 0 & \zeta_{n-2}^{-1} \end{bmatrix}}_{c}, \underbrace{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}_{a} \right\rangle$ $b := ac$	$(2,2,n-2)$	$(n=9)$  suspension of $(n-2)$-gon 	$\tilde{D}_n$ $(n=9)$ 
binary tetrahedral group $\mathbb{BT} = \left\langle \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \frac{1}{1-i}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \right\rangle$	$(2,3,3)$		$\tilde{E}_6$ 
binary octahedral group $\mathbb{BO} = \left\langle \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{bmatrix}, \frac{1}{1-i}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \right\rangle$	$(2,3,4)$		$\tilde{E}_7$ 
binary icosahedral group $\mathbb{BI} = \left\langle \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} \zeta_{10} & 0 \\ 0 & \zeta_{10}^{-1} \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} \zeta_5-\zeta_5^{-1}, & \zeta_5^{2}-\zeta_5^{-2} \\ \zeta_5^{2}-\zeta_5^{-2} & \zeta_5^{-1}-\zeta_5 \end{bmatrix} \right\rangle$	$(2,3,5)$		$\tilde{E}_8$ 

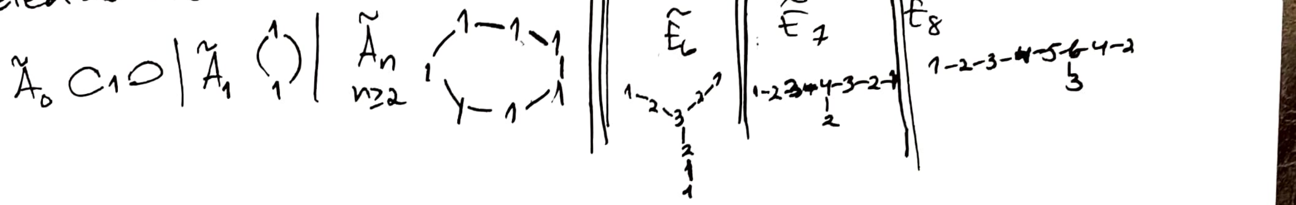
(3)

How does his correspondence work ?

(Old) **THEOREM 1:** Γ is a finite connected (undirected) multigraph with a vertex-labeling $f: V \to \{1,2,\ldots\}$ that is additive in the sense

$$2f(v) = \sum_{\substack{\text{edges } v-w \\ \text{incident to } v \text{ in } \Gamma}} f(w)$$

and $f^{-1}(1) \neq \emptyset$, $\Longleftrightarrow$ Γ is an $\tilde{A}$-$\tilde{D}$-$\tilde{E}$ simply laced diagram labeled as shown:

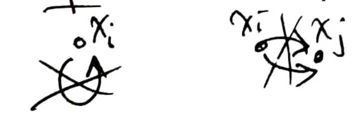


(New) **THEOREM 2:** For any finite subgroup $G < SL_2(\mathbb{C})$ acting on $V = \mathbb{C}^2$, its McKay quiver Γ with vertices $\{\underset{\substack{\| \\ \mathbb{1} \\ \text{trivial}}}{\chi_0}, \chi_1, \chi_2, \_\, \chi_\ell\}$ = the irreducible (complex) G-characters

and m_{ij} arcs $\chi_i \overset{m_{ij}}{\rightrightarrows} \chi_j$ whenever $\chi_i \otimes \underbrace{\chi_V}_{\substack{G \text{ acting on} \\ V = \mathbb{C}^2}} = \sum_{j=0}^{\ell} m_{ij}\chi_j$ $\left[\begin{array}{l} \text{i.e.} \\ m_{ij} = \langle \chi_i \chi_V, \chi_j \rangle_G \\ = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_V(g)\overline{\chi_j(g)} \end{array}\right]$

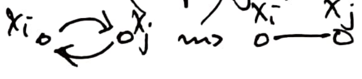
(a) is connected

(b) is loopless and simple i.e. $m_{ij} \in \{0,1\}$ and $m_{ii} = 0$



if $G \not\leq \{\pm I_2\}$

(c) has $m_{ij} = m_{ji}$, so it corresponds to a (simple) undirected graph Γ



(d) has an additive labeling by $f(\chi_i) = \deg(\chi_i) = \chi_i(e)$ with $f(\chi_0) = f(\mathbb{1}) = 1$.

[so Γ is $\hat{A}$-$\tilde{D}$-$\tilde{E}$ as in THM 1]

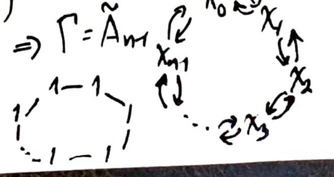
EXAMPLE: $G = \mathbb{Z}/n\mathbb{Z} \hookrightarrow SL_2(\mathbb{C})$ has $\{\chi_0, \chi_1, \ldots, \chi_{n-1}\}$ with $\chi_i(g) = \zeta_n^i$

$\| \quad \{e, g, g^2, \ldots, g^{n-1}\}$ $\qquad \chi_0 = \mathbb{1}$

$$g \longmapsto \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix}$$

so $\chi_V = \chi_1 + \chi_{n-1}$, $\chi_i \cdot \chi_j = \chi_{\underset{\text{mod } n}{i+j}}$

and $\chi_i \chi_V = \chi_i(\chi_1 + \chi_{n-1})$
$= \chi_{i+1} + \chi_{i-1}$ $\Rightarrow \Gamma = \tilde{A}_{n-1}$



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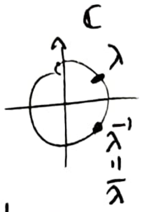
III. Proofs

Let's work on THM 2 first

PROPOSITION: $\overline{\chi_V} = \chi_V$, and as a consequence, $m_{ij} = m_{ji}$.



proof: Since $g \in SL(\underset{\mathbb{C}^2}{\underset{\|}{V}})$ and has finite order, when we diagonalize it to some $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, then $\det(g) = 1 \Rightarrow \lambda = \lambda^{-1} = \bar{\lambda}$

Since λ is a root of unity

Hence $\chi_V(g) = \lambda + \lambda^{-1} = \lambda + \bar{\lambda} = \overline{\chi_V(g)}$.

$$\text{Then } m_{ji} := \langle \chi_j \chi_V, \chi_i \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_j(g) \chi_V(g) \overline{\chi_i(g)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_j(g) \overline{\chi_V(g) \chi_i(g)} = \langle \chi_j, \chi_V \chi_i \rangle_G$$

$$= \overline{\langle \chi_V \chi_i, \chi_j \rangle_G} = \overline{m_{ij}} = m_{ij}$$ ▨

PROPOSITION: The McKay matrix $M_V := (m_{ij})_{i,j=0,1,\dots,\ell}$ for any group rep'n $G \to GL(V)$ has each column $\begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$ of the character table for G as a $\chi_V(g)$-eigenvector,

In particular, the dimensions of the irreducibles $\begin{bmatrix} \chi_0(e) \\ \chi_1(e) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$ give a $\dim(V)$-eigenvector for M_V,

so when $G \rightarrow SL_2(\mathbb{C})$ then the labeling of the nodes $\{\chi_i\}$ of Γ by $\chi_i(e)$ is additive:

$$M_V \begin{bmatrix} \chi_0(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix} = 2 \begin{bmatrix} \chi_0(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix}$$

i.e. $\sum_{j=0}^{\ell} m_{ij} \chi_j(e) = 2 \chi_i(e)$ for each $i = 0, 1, \dots, \ell$

proof:

$$\chi_i \chi_V = \sum_{j=0}^{\ell} m_{ij} \chi_j$$

$$\Rightarrow \chi_i(g) \chi_V(g) = \sum_{j=0}^{\ell} m_{ij} \chi_j(g)$$

i.e. i^{th} entry of $\chi_V(g) \begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$ = i^{th} entry of $M_V \begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$ ▨

PROPOSITION: If $G \not\subseteq \{\pm I_2\}$ then $m_{ij} \in \{0,1\}$, i.e. $m_{ij} < 2$.

proof: Note that the only scalar matrices $\lambda I_2 = \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$ in $SL_2(\mathbb{C})$ have $\underset{\lambda^2}{\underset{\|}{\det}} = 1$

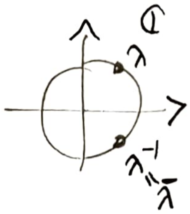
$\Rightarrow \lambda = \pm 1$.

So $G \not\subseteq \{\pm I_2\} \Rightarrow G$ contains at least one non-scalar matrix.

Then $m_{ij} = \|m_{ij}\| = \left\| \frac{1}{|G|} \sum_{g\in G} \chi_i(g)\chi_V(g)\overline{\chi_j(g)} \right\|$

Cauchy-Schwarz: $\|v \cdot w\| \le \|v\|\cdot\|w\|$ for $v,w \in \mathbb{C}^n$

$$\le \frac{1}{|G|}\sqrt{\sum_{g\in G}\|\chi_i(g)\|^2 \underbrace{\|\chi_V(g)\|^2}_{=\|\lambda+\lambda^{-1}\|^2}} \cdot \sqrt{\sum_{g\in G}\|\chi_j(g)\|^2}$$



if g diagonalizes to $\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}$

$\le 2^2$ with strict inequality < 2 if $g \ne \pm I_2$

since G contains some nonscalar matrices

$$< \frac{1}{|G|}\sqrt{\sum_{g\in G}\|\chi_i(g)\|^2} \cdot \sqrt{2^2} \cdot \sqrt{\sum_{g\in G}\|\chi_j(g)\|^2}$$

$$= \frac{2}{|G|}\langle \chi_i, \chi_i \rangle_G \sqrt{|G|} \cdot \langle \chi_j, \chi_j \rangle_G \sqrt{|G|} = 2 \quad ■$$

PROPOSITION: $m_{ii} = 0$ for $\underset{\text{finite}}{G} \hookrightarrow SL_2(\mathbb{C})$ if $G \ne \{I_2\}$.

proof:

CASE 1: G reducible on V, so $g \mapsto \left[\begin{array}{c|c}\lambda(g) & 0\\ \hline 0 & \lambda(g)^{-1}\end{array}\right]$, which forces G ~~to~~ be cyclic, and of type $\tilde{A}_{\ell-1}$ as we saw:

So if $G \ne I_2$ (i.e. $\ell = 1$), then $m_{ii} = 0$.

CASE 2: G irreducible on V.

Then an old result of Frobenius says $(2=)\dim_{\mathbb{C}} V$ divides $|G|$, and hence Cauchy's Thm says G contains an element of order 2, and in $SL_2(\mathbb{C})$, this can only be $-I_2$ (it would be g diagonalizing to $\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}$ with $\lambda = -1$).

Since $-I_2 \in Z(G)$, Schur's lemma says it acts as a scalar $\epsilon = \pm 1$ in any irreducible χ_i, so $\chi_i(-g) = \epsilon\,\chi_i(g)\ \forall g \in G$.

(and $\chi_V(-g) = -\chi_V(g)$, of course)

(6) Then one has

$$m_{ii} = \frac{1}{|G|} \sum_{g\in G} \chi_i(g)\,\chi_V(g)\,\overline{\chi_i(g)}$$

$$= \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\cdot \|\chi_i(g)\|^2$$

$$\Rightarrow 2m_{ii} = \underbrace{\frac{1}{|G|} \sum_{g\in G} \Big[\chi_V(g)\cdot\|\chi_i(g)\|^2 + \underbrace{\chi_V(-g)}_{=-\chi_V(g)}\cdot \underbrace{\|\chi_i(-g)\|^2}_{\substack{=\|\epsilon\,\chi_i(g)\|^2\\ =\|\chi_i(g)\|^2}}\Big]}_{=0}$$ ▨

Why is Γ connected?

THEOREM (Burnside 1911) For a faithful rep'n of a finite group $G \hookrightarrow GL_n(\mathbb{C}) = GL(V)$, every irreducible χ_i appears in at least one tensor power $T^m(V) = \underbrace{V\otimes\ldots\otimes V}_{m}$

(so $\exists$ a path $\overset{m \text{ steps}}{\overbrace{\underset{\substack{\shortparallel\\ 1\!\!1_G}}{\chi_0} \to \cdots \to \chi_i}}$ in the McKay quiver for V)

proof: (Brauer 1964) In fact, if $\chi_V : G \to \mathbb{C}$ takes on the distinct character values $\underset{\substack{\shortparallel\\ \dim(V)}}{z_1}, z_2, \ldots, z_r$ with $G = \underset{\substack{\Downarrow\\ e}}{G_1} \sqcup G_2 \sqcup \ldots \sqcup G_r$ having $G_i = \chi_V^{-1}(z_i)$

then we claim χ_i occurs in one of $T^0(V), \ldots, T^{r-1}(V)$.

Otherwise $0 = \langle \chi_{T^m(V)}, \chi_i \rangle_G$ for $m = 0, 1, \ldots, r-1$

$$= \frac{1}{|G|} \sum_{j=1}^{r} \sum_{g\in G_j} \chi_{T^m(V)}(g)\cdot\overline{\chi_i(g)}$$

$$= \frac{1}{|G|} \sum_{j=1}^{r} z_j^m \Big(\sum_{g\in G_j} \overline{\chi_i(g)}\Big)$$

i.e. $$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_r \\ \vdots & & & \vdots \\ z_1^{r-1} & z_2^{r-1} & \cdots & z_r^{r-1} \end{bmatrix}}_{\text{Vandermonde, invertible}} \begin{bmatrix} \sum_{g\in A_1} \overline{\chi_i(g)} \\ \vdots \\ \sum_{g\in A_r} \overline{\chi_i(g)} \end{bmatrix} = \underline{0} \Rightarrow \sum_{g\in A_1} \overline{\chi_i(g)} = 0.$$

But $A_1 = \{e\}$ since $\chi_V(g) = \dim(V)$ $\iff$ g has all eigenvalues 1 on V i.e. $g = e$.

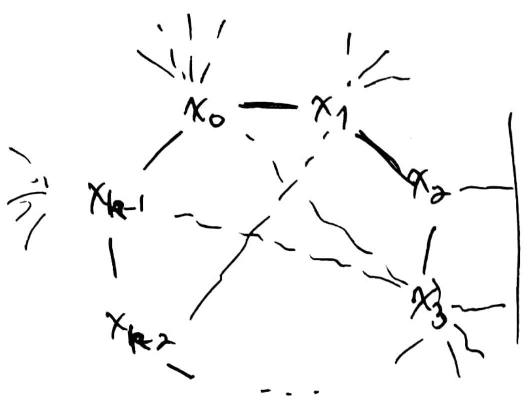
So $0 = \overline{\chi_i(e)} = \dim(V)$, a contradiction. ▨

(7)

Proof of THM 2

Once one shows the graphs Γ are $\tilde{A}$-$\tilde{D}$-$\tilde{E}$, it's not hard to check an additive labeling f is unique up to scaling and then $f^{-1}(1)\neq\emptyset$ pins it down.

- If Γ contains a cycle, then it has no other edges, so $\Gamma=\tilde{A}_{n-1}$:



For $i=0,1,\dots,k-1$,

$$2x_i = x_{i-1}+x_{i+1}+\dots$$

mod k

$$\Rightarrow 2x_i \geq x_{i-1}+x_{i+1}$$

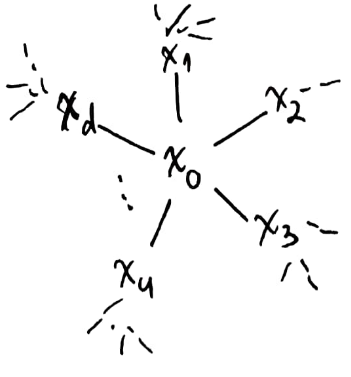
$$\Rightarrow 2\sum_{i=0}^{k-1}x_i \geq \sum_{i=0}^{k-1}x_i + \sum_{i=0}^{k-1}x_i$$

⇒ equality everywhere, so no edges outside the cycle.

So WLOG, Γ is a tree

- Γ has no vertices of deg ≥ 5, and if any have deg$=4$ then $\Gamma = \tilde{D}_4 =$

:



$2x_0 = \sum_{i=1}^{d} x_i + \dots$ and $2x_i = x_0 + \dots$ for $i=1,2,\dots,d$

$\Rightarrow 2x_0 \geq \sum_{i=1}^{d} x_i$ and $2x_i \geq x_0$

$x_i \geq \frac{x_0}{2}$

$\geq \sum_{i=1}^{d}\frac{x_0}{2} = \frac{d}{2}x_0$

$\Rightarrow x_0 \geq \frac{d}{4}x_0 \Rightarrow d \leq 4$, w/ equality only if no other edges

So WLOG Γ has max degree ≤ 3.

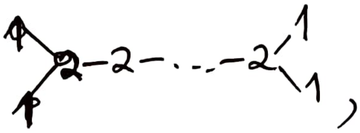
- There exist vertices of deg 3, else starting from a leaf one finds Γ is an infinite path with this labeling:

$$x_0 — 2x_0 — 3x_0 — 4x_0 — \cdots$$

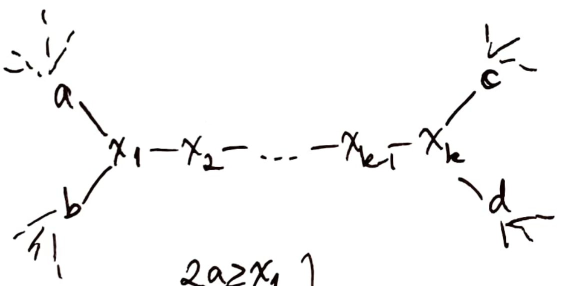
Contradiction

- If there are two or more deg 3 vertices, find two connected by a path, and there are no other edges, i.e.

$\Gamma = \tilde{D}_n =$



, via this calculation:



$$\left.\begin{array}{l} 2a \geq x_1 \\ 2b \geq x_1 \\ 2c \geq x_k \\ 2d \geq x_k \end{array}\right\} \Rightarrow 2(a+b+c+d) \geq 2(x_1+x_k)$$

$$\boxed{a+b+c+d \underset{(*)}{\geq} x_1+x_k}$$

Also

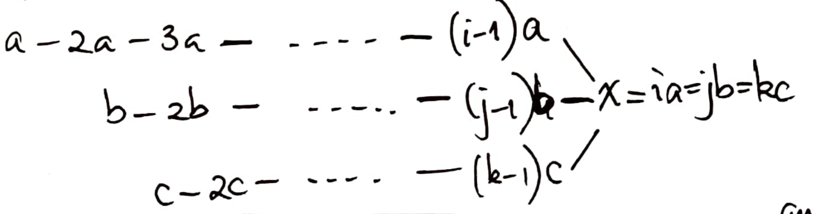
$$\begin{aligned} 2x_1 &= a+b+x_2 \\ 2x_2 &= x_1+x_3 \\ 2x_3 &= x_2+x_4 \\ &\vdots \\ 2x_{k-1} &= x_{k-2}+x_k \\ 2x_k &= x_{k-1}+c+d \end{aligned}$$

$$2\sum_{i=1}^{k} x_i = a+b+c+d + 2\sum_{i=1}^{k} x_i - (x_1+x_k)$$

$\Rightarrow x_1+x_k = a+b+c+d \underset{(*)}{\geq} x_1+x_k$

$\Rightarrow$ equality in $(*)$, so no other edges

So WLOG, Γ has a unique degree 3 vertex, and looks like this:



where after re-indexing, $a \leq b \leq c$

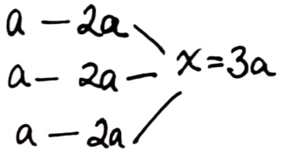
and hence $i \geq j \geq k$

Then $2x = (i-1)a+(j-1)b+(k-1)c = 3x-(a+b+c)$, so $\underset{\substack{\| \\ kc}}{x} = a+b+c \leq 3c \Rightarrow k \leq 3$

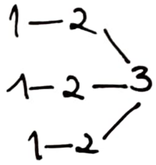
(9)

- If $k=3$, then $x = 3c = \underset{a\le b\le c}{a+b+c} \Rightarrow a=b=c$, $i=j=k$

so $\Gamma =$



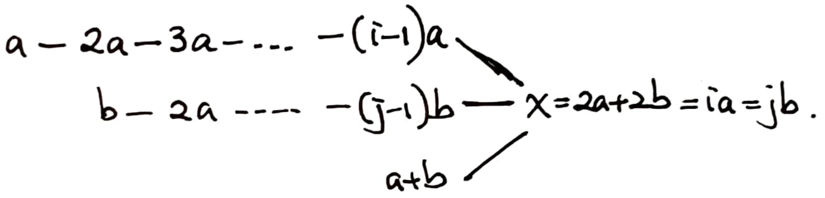
$=$



$= \tilde{E}_6$

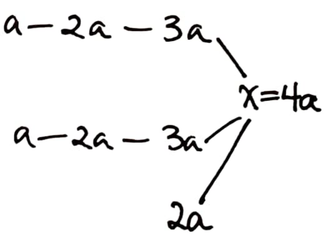
- If $k=2$, then $2c = x = a+b+c$, so $c = a+b$, $x = 2a+2b$

and $\Gamma =$

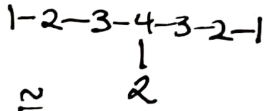


Then $jb = x = 2a+2b \underset{\overset{\uparrow}{a\le b}}{\le} 4b \Rightarrow j\le 4.$

- If $j=4$, then $a=b$, $i=j$ and $\Gamma =$

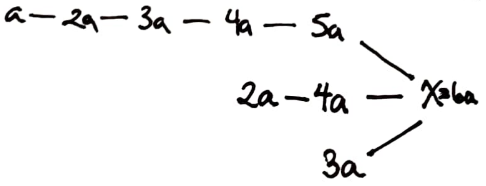


$=$

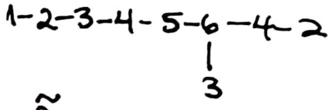


$= \tilde{E}_7$

- If $j=3$, then $3b = jb = x = 2a+2b$, so $b = 2a$, $c = a+b = 3a$, $x = 2a+2b = 6a$



$=$



$= \tilde{E}_8$

- If $j=2$, then $2b = jb = x = 2a+2b$

$\Rightarrow 2a = 0$, so $a=0$, a contradiction.

▨