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GRAPHS, SINGULARITIES, AND FINITE GROUPS

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Introduction. We have seen during the past few years a major assault on the problem of determining all the finite simple groups. We are told that this assault is nearly complete; even if this is so, the story is not an easy one to tell since it is spread over thousands of pages and, apart from being long, it is a story in which almost all the characters play roles only within the theory of finite groups–the impact of developments in other areas of mathematics on the classification problem has been minimal. I want to suggest that there is an immense wealth of connections with other areas which lies ready to be discovered. If I am right, I foresee new proofs of the classification which will owe little or nothing to the current proofs. They will be much shorter and will help us to understand the finite simple groups in a context much wider than finite group theory.

Representation graphs. Let R be a representation of a group G, having irreducible representations $\{R_i\}$, such that

$$R \otimes R_j = \bigoplus_k m_{jk} R_k, \qquad j, k = 1, 2, \ldots, t.$$

The representation graph $\Gamma_R = \Gamma_R(G)$ is the graph with vertex set $\{R_i\}$ and m_{jk} (directed) edges from R_j to R_k. We convene that a pair of opposing directed edges be represented by a single undirected edge.

PROPOSITION 1. *$\Gamma_R(G)$ is connected if and only if R is faithful on G.*

PROPOSITION 2. *$\Gamma_R(G)$ is self-dual (invariant under reversal of edge orientation) if and only if R affords a real-valued character. $\Gamma_R(G)$ is undirected if it is self-dual and has no directed loops.*

An example illuminating both propositions is $G = \Sigma_4$, the symmetric group of degree 4, and R, the unique two-dimensional irreducible representation.

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Finite groups of quaternions. The finite subgroups of real quaternions are abstractly the binary polyhedral groups defined by the relations:

$$\langle a, b, c\rangle: A^a = B^b = C^c = ABC.$$

These are finite for the binary dihedral group $\langle 2, 2, n\rangle$, of order $4n$, the binary tetrahedral group $\langle 2, 3, 3\rangle$, of order 24, the binary octahedral group $\langle 2, 3, 4\rangle$, of order 48, the binary icosahedral group $\langle 2, 3, 5\rangle$, of order 120 and finally the degenerate case of the cyclic group. These groups are described in Coxeter and Moser [5, Preface and §6.5] and in Du Val [7]. Each of these groups with the sole exception of the cyclic group of odd order contains a centre of order two, namely $\{ABC\}$. The classification of finite subgroups of division rings is found in Amitsur [1]. There is a natural embedding of the binary polyhedral groups in $SL_2(\mathbf{C})$, or its compact version $SU_2(\mathbf{C})$, a double cover of $SO_3(\mathbf{R})$, important for the sequel.

Generalized Coxeter graphs. A graph with vertex set V and a weight function $w: V \to \mathbf{R}^{>0}$ is a C_k-graph if

$$\sum_{u \in S(v)} w(u) = k.w(v)$$

where $S(v)$ is the multiset of successors of $v \in V$. Because of their importance, we shall drop the suffix and use 'C-graph' to mean C_2-graph.

In passing it should be remarked that the defining property of a C_k-graph implies that its adjacency matrix (the matrix whose (i, j) entry counts the number of edges from vertex i to adjacent vertex j) has maximum eigenvalue k, see Seneta [11].

PROPOSITION 3. *$\Gamma_R(G)$ is a C_k-graph for $k =$ $\dim(R)$, w: $R_i \to \dim R_i$.*

Lie algebras. The connection between C-graphs and Lie algebras is given by

PROPOSITION 4. *The finite, undirected, connected C-graphs are precisely the Coxeter graphs (Dynkin graphs) for the affine Lie algebras of type $\bar{A}_r$ $(r \geqslant 0)$, $\bar{D}_r$ $(r \geqslant 4)$, $\bar{E}_6$, $\bar{E}_7$, and $\bar{E}_8$. We shall call these the standard types.*

The affine graphs are described (as 'graphes de Dynkin complétés') in Bourbaki [4]. They are constructed by adjoining to the usual graph an extra root, being the negative of the highest root.

PROPOSITION 5. *All circuit-free C-graphs satisfying a 'symmetrisability' condition–if vertices are joined by a directed edge, then they are also joined by an undirected edge–(see Berman, Moody, and Wonenburger* [3]*) are obtained from the undirected C-graphs by 'folding' them. Folding is a weight- and incidence-preserving operation on graphs which maps Γ to the quotient graph Γ/H, by replacing $v \in V$ by its orbit $\{v^H\}$ under a subgroup H of the symmetry group of the graph.*

The graphs of this proposition are found in [3], Dlab and Ringel [6], and Kac [8]. They include the affine graphs for the algebras of type $\bar{B}_r$, $\bar{C}_r$, $\bar{G}_2$, and $\bar{F}_4$.

Finite groups and spectral structure. For each finite group of quaternions, G, there is a faithful representation R_Q such that $\Gamma_{R_Q}(G)$ is a graph of standard type. This representation is the two-dimensional one mentioned above and is irreducible except for the cyclic case where it is the direct sum of a faithful irreducible representation and its dual.

PROPOSITION 6. *The eigenvalues of the adjacency matrix of an undirected C-graph are the values of the character afforded by R_Q. The eigenvectors can be taken to be the columns of the character table of the appropriate finite group of quaternions.*

Eigenspaces are all one-dimensional only for $\bar{A}_0$, $\bar{A}_1$, and $\bar{E}_8$. Any linear operator for which the columns of the character table are eigenvectors commutes with the regular representation of the representation algebra mentioned above.

The Cartan matrix of standard type is symmetric and satisfies $C = 2I - A$ where A is the adjacency matrix of the graph. It follows, since R_Q is faithful, that C is positive semidefinite with a one-dimensional kernel spanned by the eigenvector whose components are the dimensions of the irreducible representations of G.

A Cartan matrix of type A, D, or E is the presentation matrix (that is, the entry c_{ij} gives the exponent of generator x_j in the ith relator) of the finite polyhedral group whose character table is obtained from the eigenvectors of the corresponding Cartan matrix of affine type.

The Fischer-Griess simple sporadic group M is generated by a conjugacy class of involutions such that the product of any pair lies in one of 9 conjugacy classes whose periods are given by the weights of the $\bar{E}_8$ graph. The group M contains a subgroup $2.B$ (a central extension of the Baby Monster) which centralizes an involution, and a subgroup $3.F'_{24}$ which centralizes an element of period 3; each of the groups $2.B$ and $3.F'_{24}$ contains elements bearing a similar relation as above to the graphs $\bar{E}_7$ and $\bar{E}_6$ respectively provided the periods are read modulo the centre.

The singularities. We have seen that each C-graph may be interpreted in two ways: firstly as a representation graph of a finite group, and secondly as a Coxeter graph in the classical sense (as a description of a Lie algebra). A connection between these two interpretations has been given by Steinberg in his article in these PROCEEDINGS. This connection is described by Orlik **[10]** in his recent survey article and by Slodowy **[12]**.

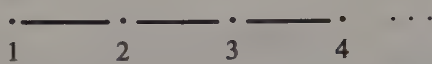
Very briefly, starting with the polynomial invariants of the finite subgroup of $SL_2(\mathbf{C})$, a surface is defined from the single syzygy which relates the three polynomials in two variables. This surface has a singularity (partial derivatives vanish) at the origin; the singularity can be resolved by constructing a smooth surface which is isomorphic to the original one except for a set of component curves which form the pre-image of the origin. The components form a Dynkin curve and the matrix of their intersections (the matrix, indexed by the curves, with (0, 1)-entries indicating intersections of distinct curves and diagonal 'self-intersection numbers' of -2) is the negative of the Cartan matrix for the appropriate Lie algebra. The Dynkin curve is the dual of the Dynkin graph.

There are references to the affine curves in Tate **[14]** and several other references to them in **[2]**.

The universal property. One C-graph we have excluded throughout by our finiteness condition is the important universal C-graph which is the representation graph for $SL_2(\mathbf{C})$ with $R = R_2$, the natural representation. This group occurs in both guises as

$\bar{A}_0$: 

and as

A_∞: 

The representation theory of $SL_2(\mathbf{C})$ is much studied (see, for example, Kirillov [**9**]) and all we need here is the tensor product formula

$$R_2 \otimes R_n = R_{n-1} \oplus R_{n+1}$$

where R_i is the irreducible representation of dimension i. All the undirected C-graphs can be embedded in A_∞ by restricting R_2 to the appropriate subgroup.

By restriction and folding we obtain the Dynkin graphs of all finite rank simple Lie algebras.

The Cartan matrix is again positive semidefinite but now infinite with 2's on its diagonal and -1's on both adjacent diagonals, all other entries being zero.

Conclusion. A paper will appear amplifying this note and containing proofs. I hope that I have been able to indicate that there is much more to be discovered about finite groups and their relation with other areas of mathematics. If this approach is to be successful, its merit will lie in its unifying power and its elegance. Would not the Greeks appreciate the result that the simple Lie algebras may be derived from the Platonic solids?

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