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FINITE SUBGROUPS OF SU_2, DYNKIN DIAGRAMS AND AFFINE COXETER ELEMENTS

ROBERT STEINBERG

Dedicated to the memory of my friend Ernst Straus

Using, among other things, some properties of affine Coxeter elements, for which we also present normal forms, we offer an explanation of the McKay correspondence, which associates to each finite subgroup of SU_2 an affine Dynkin diagram.

J. McKay [**M**] has observed that for each finite (Kleinian) subgroup G of SU_2 the columns of the character table of G, one column for each conjugacy class, form a complete set of eigenvectors for the corresponding affine Cartan matrix (of type A_n, D_n or E_n), the one that arises in connection with the resolution of the singularity of $\mathbf{C}^2/G$ at the origin (see 1(9) below). As he has observed, this follows at once from: if ρ is the two-dimensional representation by which G is defined, $\{\rho_i\}$ is the set of (complex) irreducible representations of G, and $\sum n_{ij}\rho_j$ denotes the decomposition of $\rho \otimes \rho_i$, then $C \equiv [c_{ij}] \equiv [2\delta_{ij} - n_{ij}]$ is the relevant Cartan matrix. Partial explanations have been given by several authors (see [**G**], [**H**], [**K**], [**S**$_1$, Appendix III]). Here we shall give our own explanation of this and some related facts, including two normal forms for affine Coxeter elements which enter into our considerations. Section 1 details mainly with McKay's correspondence, Section 2 mainly with affine Coxeter elements. As general references for Kleinian groups, Kleinian singularities and root systems, we cite [**C**, Chapters 7, 11], [**S**$_1$, Section 6], [**B**], and the survey article [**S**$_2$].

1. In this section G is a finite group, ρ is a faithful (complex) representation of G of finite dimension d, $\{\rho_i\}$ is the set of all irreducible representations of G with ρ_0 the trivial one, $\sum n_{ij}\rho_j$ denotes the decomposition of $\rho \otimes \rho_i$, and C is the matrix $[d\delta_{ij} - n_{ij}]$.

(1) The column $[\chi_j(x)$ (x in G fixed, $j = 1, 2, \ldots$)$]$ of the character table of G is an eigenvector of C with $d - \chi(x) = \chi(1) - \chi(x)$ as the corresponding eigenvalue. In particular $[d_1, d_2, \ldots]$ ($d_i = \dim \rho_i$) is an eigenvector corresponding to the eigenvalue 0.

We have $\chi(x)\chi_i(x) = \sum n_{ij}\chi_j(x)$, whence the first statement. Then $x = 1$ yields the second.

(2) The following equations hold.

(a) $n_{ij} = n_{\bar{j}\bar{i}}$ (bar denotes dual)

(b) $dd_i = \sum n_{ij}d_j$

(c) $dd_i = \sum n_{ji}d_j$

(d) $n_{ij} = n_{ji}$ for all i and j if and only if ρ is self-dual.

(a) This follows from $n_{ij} = (\chi\chi_i, \chi_j) = \text{Average}\, \chi\chi_i\bar{\chi}_j$.

(b) This is the second statement of (1). (c) $dd_i = dd_{\bar{i}} = \sum n_{\bar{i}j}d_j = \sum n_{\bar{j}i}d_{\bar{j}} = \sum n_{ji}d_j$. (d) If $n_{ij} = n_{ji}$ always then $n_{0j} = n_{j0} = n_{0\bar{j}}$ and ρ is self-dual. If ρ is self-dual then $n_{ij} = n_{\bar{i}\bar{j}} = n_{ji}$.

(3) Now form a real vector space V with a basis vector α_i for each ρ_i and a scalar product given by $(\alpha_i, \alpha_j) = c_{ij} \equiv d\delta_{ij} - n_{ij}$. Then the line through $\sum d_i\alpha_i$ is the radical of (,) from the left and from the right, and it is also the radical of the quadratic form (α, α) and this form is positive semidefinite.

By (2b) and (2c) the given line belongs to these radicals. It will be enough to prove the converse for the quadratic form since its radical contains the others. With $\alpha = \sum x_i\alpha_i$ arbitrary in V we have

$$\begin{aligned} 2(\alpha, \alpha) &= 2\sum c_{ij}x_ix_j \\ &= 2\sum (d - n_{ii})x_i^2 - 2\sum n_{ij}x_ix_j \qquad (i \neq j) \\ &= \sum (n_{ij} + n_{ji})d_i^{-1}d_jx_i^2 - 2\sum n_{ij}x_ix_j \qquad (i \neq j) \end{aligned}$$

by (2b) and (2c). For $i < j$ the pairs i, j and j, i together contribute

$$\begin{aligned} &(n_{ij} + n_{ji})\left(d_i^{-1}d_jx_i^2 + d_j^{-1}d_ix_j^2 - 2x_ix_j\right) \\ &\qquad = (n_{ij} + n_{ji})d_i^{-1}d_j^{-1}\left(d_jx_i - d_ix_j\right)^2 \geq 0. \end{aligned}$$

Thus (α, α) is positive semidefinite. Now for each i there exists a sequence $i_1, i_2, \ldots, i_n$ with $i_1 = 0$ corresponding to the trivial representation, $i_n = i$, and $n(i_p, i_{p+1}) = 0$ for all p; this is because ρ_i is necessarily contained in some tensor power of the faithful representation ρ. It follows from this and the above inequalities that if $(\alpha, \alpha) = 0$ then $x_i = (x_0/d_0)d_i$ for all i, so that α is in the line of $\sum d_i\alpha_i$.

We now specialize to the case in which ρ imbeds G into SU_2. We assume that $G \neq \{1\}$.

(4) (a) $c_{ij} = c_{ji}$ (i.e. $n_{ij} = n_{ji}$) always.

(b) $c_{ii} = 2$ (i.e. $n_{ii} = 0$) always.

(c) If $G \neq \{\pm 1\}$ and $i \neq j$ then $c_{ij} = 0$ or -1 (i.e. $n_{ij} = 0$ or 1).

In other words (a) C is symmetric, (b) ρ_i is disjoint from $\rho \otimes \rho_i$, and (c) $\rho \otimes \rho_i$ is multiplicity-free.

(a) This is by (2d): if α, α^{-1} are the eigenvalues of $\rho(x)$ then $\alpha + \alpha^{-1} = \alpha + \bar{\alpha} \in \mathbf{R}$. (b) If ρ is reducible it has the form $\sigma \dotplus \bar{\sigma}$ with $\dim \sigma = 1$. Thus G is cyclic and all ρ_i have dimension 1. So, since G is nontrivial, $\sigma \otimes \rho_i$ and $\bar{\sigma} \otimes \rho_i$ are different from ρ_i and hence disjoint from it. If ρ is irreducible then $\{\pm 1\} \subseteq$ Center (G). For then 2 divides $|G|$ and -1 is the unique element of order 2 of SU_2. Now if -1 in G acts as (multiplication by) 1 (resp. -1) on ρ_i, it acts as -1 (resp. 1) on $\rho \otimes \rho_i$, hence also on its irreducible components, all of which must thus be different from ρ_i. (c) We have

$$\sum_j n_{ij}^2 = (\rho \otimes \rho_i, \rho \otimes \rho_i) \equiv \mathrm{Av}|\chi(x)|^2|\chi_i(x)|^2 \leq \mathrm{Av}\, 4|\chi_i(x)|^2 = 4,$$

since $|\chi(x)| \leq 2$ for all x in G with equality only if $x = \pm 1$. If the strict inequality holds then $\sum_j n_{ij}^2 < 4$ and each n_{ij} is 0 or 1. If equality holds then $\chi_i(x) = 0$ for all $x \neq \pm 1$. If also multiplicity occurs then $\rho \otimes \rho_i = 2\rho_j$ with ρ_j irreducible, and $\rho \otimes \rho_j = 2\rho_i$ because of the values of χ_i and χ_j. Now -1, if it is in G, acts trivially on ρ_i or on ρ_j, say on ρ_j. If H denotes G modulo its intersection with $\{\pm 1\}$, then ρ_j yields an irreducible representation of H with character value d_j at 1, 0 elsewhere, whence $|H| = d_j^2$. However $|H| = \sum' d_j^2$, summed over all irreducible representations of H. It follows that ρ_j is the unique irreducible representation of H, hence that H is trivial. Thus $G \subseteq \{\pm 1\}$, contradicting our assumptions.

We now introduce a diagram Γ with one vertex corresponding to each basis vector α_i of V (or to each irreducible representation ρ_i of G) and one edge for each pair i, j such that $n_{ij} = 1$ (i.e. $c_{ij} = -1$) in (4c), which is unambiguous by (4a). By (4b) no edge of Γ is a loop.

(5) Γ (resp. C) is the extended Dynkin diagram (resp. matrix) of a reduced, irreducible root system with all roots of one length (which we take to be $\sqrt{2}$) and $\Gamma' \equiv \{\alpha_i | i \neq 0\}$ as a simple system and $\sum_{i \neq 0} d_i \alpha_i$ as the corresponding highest root.

By the argument at the end of (3) Γ is connected. But then so is Γ': We have $\rho \otimes \rho_0 = \rho$. Thus if ρ is irreducible it yields the unique vertex of Γ joined to α_0 and Γ' is connected. If ρ is reducible then Γ is a loop, as may be checked; thus again Γ' is connected. Now by (3), if V' is the subspace of V generated by the α's other than α_0 and L is the line $\mathbf{R}\sum d_i \alpha_i$, then V' projects isometrically onto V/L and there (,) is positive definite. We identify the two spaces. Further $(\alpha_i, \alpha_i) = 2$ and for $i \neq j$ $(\alpha_i, \alpha_j) = -1$ or 0 according as α_i and α_j are or are not joined in Γ.

Thus $\{\alpha_i | i \neq 0\}$ is a simple system for an irreducible root system in which $(\alpha, \alpha) = 2$ for every root and Γ' is its Dynkin diagram. Further $-\alpha_0$ is a root since $(-\alpha_0, -\alpha_0) = 2$, and it is dominant and hence the highest root since also $(-\alpha_0, \alpha_i) = n_{0i} \geq 0$ for $i \neq 0$. Thus $\Gamma = \Gamma' \cup \{\alpha_0\}$ is the corresponding extended Dynkin diagram. On V' we have $-\alpha_0 = -d_0\alpha_0 = \sum_{i\neq 0} d_i\alpha_i$, whence the last point of (5).

(6) G/G' is isomorphic to F, the center of the simply connected complex Lie group L whose extended Cartan matrix is C.

First the orders of the two groups are equal: $|G/G'|$ is the number of 1-dimensional representations of G, i.e., the number of d_i's equal to 1, hence is 1 + the number of coefficients that are 1 in the highest root, which is known to be $|F|$. An isomorphism is given by $x \in G/G' \to \Pi\alpha_i^*(\det \rho_i(x))$. Here α_i^* denotes the coroot of α_i, viewed as a 1-parameter subgroup int L. All of this is relative to a choice of a maximal torus and an ordering of its character group. The proposed isomorphism is injective since if x is in the kernel then $\det \rho_i(x) = 1$ for all i, whence $\rho_i(x) = 1$ for all i with $d_i = 1$, and $x \in G'$. The image is in F since if α_j is any simple root then $\alpha_j(\text{image}) = \Pi(\det \rho_i(x))^{c(i,j)} = 1$, as we see by taking determinants in $\rho \otimes \rho_i = \sum n_{ij}\rho_j$ and using $\det \rho = 1$, $\det \rho_0 = 1$ and $c_{ij} = 2\delta_{ij} - n_{ij}$.

(7) The (unextended) Dynkin diagram for G' can be gotten from that for G by deleting all vertices α_i for which $d_i = 1$.

At present this is only an empirical observation.

Because of (6) and (7) the derived series for G can be written down easily in any given case. For example, $E_7 \supset E_6 \supset D_4 \supset A_1 \supset \{1\}$, with corresponding quotients C_2, C_3, $C_2 \times C_2$, C_2.

If G is reducible on $\mathbf{C}^2$ and hence cyclic, then Γ is a cycle, hence of type A_n, as is mentioned above. Conversely if Γ contains a cycle then by standard arguments Γ must be a cycle, α_0 (corresponding to the trivial representation) has two neighbors and $\rho = \rho \otimes \rho_0$ is reducible. Aside from these cases, the possibilities for G are classified by numbers $p_1 \geq q_1 \geq r_1 \geq 2$ with $p_1^{-1} + q_1^{-1} + r_1^{-1} > 1$, ($2p_1, 2q_1, 2r_1$ are the orders of the maximal cyclic subgroups of G, one subgroup for each conjugacy class), and so are the possibilities for Γ' (ordinary Dynkin diagram) (p, q, r are the branch lengths including the branch point).

(8) If $\Gamma'(p, q, r)$ is the diagram coming from $G(p_1, q_1, r_1)$ then $(p, q, r) = (p_1, q_1, r_1)$. Thus McKay's correspondence is bijective.

Let x, y, z in G be such that their eigenvalues on $\mathbf{C}^2$ are $\exp(\pm\pi i/p)$, $\exp(\pm\pi i/q)$, $\exp(\pm\pi i/r)$. The elements 1, -1, x^a $(1 \leq a < p_1)$, y^b $(1 \leq b < q_1)$, z^c $(1 \leq c < r_1)$ form a system of representatives of the conjugacy class of G, and on them χ, the character of ρ, has the values 2, -2, $2\cos \pi a/p_1$, etc., resp. By (1) these are the eigenvalues of $2 - C$ with C the corresponding extended Cartan matrix. Thus 2, 2, $2\cos 2\pi a/p_1$, etc., are the eigenvalues of $(2 - C)^2 - 2$. If we can show that this also holds with p_1, q_1, r_1 replaced by p, q, r we will be done. Consider an affine Coxeter element c, the product of the reflections corresponding to the affine simple roots; these are the ordinary simple roots with $1 - \mu$ adjoined (μ is the highest root). Since Γ has no circuits the conjugacy class of c is independent of the order of the factors and the affine simple roots may be so ordered that the first few are mutually orthogonal as are the rest of them. Then in partitioned form we have

$$2 - C = \begin{bmatrix} 0 & N \\ N' & 0 \end{bmatrix}, \quad \text{whence } (2 - C)^2 - 2 = \begin{bmatrix} NN' - 2 & 0 \\ 0 & N'N - 2 \end{bmatrix}.$$

On the other hand if $c = c_1 c_2$ in accordance with this partition of the roots, then

$$c_1 = \begin{bmatrix} -1 & N \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad c_2 = \begin{bmatrix} 1 & 0 \\ N' & -1 \end{bmatrix}$$

in matrix form. It follows that $c + c^{-1} = (2 - C)^2 - 2$. Thus by the above formulas the eigenvalues of c are 1, 1, $\exp(2\pi i a/p_1)$ $(1 \le a < p_1)$, etc., and those of c', the linear part of c, are the same with the first 1 deleted. We now invoke a result which will be proved in the next section (after (10) there).

(*) c', the linear part of c, is conjugate in the Weyl group to c'', the product of the ordinary simple reflections with the one at the branch point excluded. From (*) it follows that c'' has the same eigenvalues as c' as given above. However c'' is the product of three Coxeter elements of types A_n $(n = p - 1, q - 1, r - 1)$ corresponding to the mutually orthogonal subsystems along the branches of Γ', and these, together with the branch root, contribute the eigenvalues 1, $\exp(2\pi i a/p)$ $(1 \le a < p)$, etc. Thus $(p, q, r) = (p_1, q_1, r_1)$, as required.

(9) Consider the minimal resolution of the singular surface $\mathbf{C}^2/G$. The singular fiber is a union of projective lines, and if we form a diagram by taking one node for each line and joining two nodes, by a simple bond, just when the corresponding lines intersect, we get an ordinary Dynkin diagram, of type A_n, D_n or E_n (see [$\mathbf{S}_1$]). It remains to show that this correspondence agrees with McKay's. Let p, q, r be the branch lengths of the diagram just obtained. Type A_n may be included by taking $q = r = 1$ in what follows. Let $C' = [c(i, j)]$ be the ordinary Cartan matrix. Then

the group G is isomorphic to the abstract group defined by n generators and the n relations $\Pi x_i^{c(i,j)} = 1$ $(j = 1, 2, \ldots)$. (Thus G/G', the Abelianized group, is just F, as given in (6) above.) This result is due to Mumford [**Mu**]. The relations yields, via an application of Van Kampen's Theorem, a presentation of the fundamental group of a "sphere" around the singular point of $\mathbf{C}^2/G$, and that group, quite clearly, is G itself. Now let $x_1, x_2, \ldots, x_p$ be the generators along a branch of length p towards the branch point x_p. The given relations yield $x_1^2 x_2^{-1} = 1$, $x_1^{-1} x_2^2 x_3^{-1} = 1, \ldots$, whence if $x_1 \equiv x$ then $x_a = x^a$ for $(1 \leq a \leq p)$. Similarly on the other branches $y_q = y^q$, $z_r = z^r$ and $x_p = y_q = z_r$. The relation at the branch point yields $(x^p)^2 = x^{p-1} y^{q-1} z^{r-1}$. Thus $xyz = x^p = y^q = z^r$. As is well known [**C**, 11.7, 7.4] this is a presentation of the Kleinian group of type (p, q, r). Thus G and the graph corresponding to it have the same type, as required.

(10) McKay's correspondence can be extended to yield Dynkin diagrams with multiple bonds in several different ways. One way, used in [**H**], is to start with representations over fields that are not algebraically closed. This yields most Dynkin diagrams, but not all of them. Another way, suggested in [**S**, App.III], which does yield all diagrams, is to start with certain pairs $G \lhd H$ of finite subgroups of SU_2, or, equivalently, with a single subgroup G and an automorphism σ of G which stabilizes the defining representation of G, and then to associate a node to each σ-orbit of irreducible representations of G, or, dually, to each representation of $\langle G, \sigma \rangle$ induced by an irreducible representation of G. One can then carry out large parts of the above development in this new context (with weighted nodes, multiple bonds, etc.) or else notice that the coalescence of irreducible representations into σ-orbits corresponds exactly to the foldings of Dynkin diagrams according to their symmetries.

2. Affine Coexter elements. Our purpose is to develop the principal properties of these elements, including two normal forms and a proof of the property $(*)$ used in the proof of (8) above. Π will be a simple system for an irreducible root system. We write λ_α for the fundamental weight corresponding to α and α^* for the coroot $2\alpha/(\alpha, \alpha)$. We can decompose Π into disjoint parts Π_1 and Π_2 so that each is an orthogonal set of roots. We exclude type A_n mostly. Then μ, the highest root, is orthogonal to all roots of Π but one, so that the notation can be chosen so that μ is orthogonal to Π_2. We write w_1 (resp. w_2) for the product of the simple reflections in Π_1 (resp. Π_2), then $w_i = w_1$ (resp. w_2) when i is odd (resp. even) (and similarly for Π_i), and finally $w^1 = w_1$, $w^2 = w_2 w_1$,

$w^3 = w_3w_2w_1, \ldots$. We observe that w_1, w_2 and each odd w^i is an involution. Since type A_n has been excluded, the Coxeter number, the order of w_1w_2, is even: $h = 2g$. For as can be easily proved or read off from the classification, h is odd only for type A_{2n}.

(1) We have $1 < w^1 < w^2 \cdots < w^{2g}$ and $w^{2g} = w_0$, the element of the Weyl group that makes all positive roots negative.

Here $w < w'$ means that w is the terminal segment of a minimal expression for w' as a product of simple reflections, i.e., that the length of w' is the sum of those of w and $w'w^{-1}$. The more general Bruhat order could also be used in all that follows. The fact that $w^{2g} = w_0$ is proved in **[St]**. In the expression $w_0 = w_{2g} \cdots w_2w_1$ with w_i written as the product of the reflections for the roots in Π_i, the number of roots listed is $g|\Pi| = (h/2)|\Pi|$, which, as is known **[St]**, is equal to the number of positive roots. It follows that the expression is minimal, as in each terminal segment, whence (1).

(2) Let λ be a dominant (integral) weight, and $w < w'$ in the Weyl group. Then $w\lambda \geq w'\lambda$. Hence $\lambda \geq w'\lambda \geq w^2\lambda \geq \cdots \geq w^{2g}\lambda$.

It is enough, by induction, to prove this when $w' = w_\beta w$, with $\beta > 0$ and $w^{-1}\beta > 0$. Here and elsewhere w_β is the reflection relative to β. Now $w'\lambda = w_\beta w\lambda = w\lambda - (w\lambda, \beta^*)\beta$, and $(w\lambda, \beta^*) = (\lambda, w^{-1}\beta^*) \geq 0$ since λ is dominant and $w^{-1}\beta$ is positive, whence (2).

(3) Assume that λ is dominant, $w_0\lambda = -\lambda$, and $\operatorname{Supp}\lambda \subseteq \Pi_1$. Then $w^i\lambda = -w^{2g-1-i}\lambda$ for $0 \leq i < g$.

Here the third condition on λ is that in its expression in terms of the fundamental weights λ_α only those with $\alpha \in \Pi_1$ are needed. We have

$$\begin{aligned} w^i\lambda &= w_i \cdots w_2w_1\lambda \\ &= w_{i+1} \cdots w_{2g} \cdot w_{2g} \cdots w_{i+1}w_i \cdots w_1\lambda = -w_{i+1} \cdots w_{2g}\lambda \quad \text{(by (1))} \\ &= -w^{2g-i-1}\lambda \quad \text{since } w_{2g} = w_2 \text{ fixes } \lambda. \end{aligned}$$

(4) In (3) $w^{g-1}\lambda$ is a nonnegative combination of roots in Π_g.

We have $w^{g-1}\lambda = (w^{g-1}\lambda - w^g\lambda)/2$ by (3). This is ≥ 0 by (2), and, since it equals $(1 - w_g)w^{g-1}\lambda/2$, it involves only the simple roots in Π_g.

(5) $w^{g-1}\mu$ is a simple root, an element of Π_g. It is the unique long simple root b at which there is a branch point or a multiple bond. (Recall that μ is the highest root).

First, by (4), which is applicable since μ as a dominant weight has its support in Π_1, $w^{g-1}\mu = b$ is a nonnegative combination of roots in Π_g. Since b is a root and the elements of Π_g are mutually orthogonal, it easily follows that b is an element of Π_g. And since μ is a long root, so is b.

Since type A_n is being excluded, μ is connected to a unique simple root α. Assume first that α is shorter than μ. To prove (5) in this case we show that there is only one long simple root. In the extended Dynkin diagram $1-\mu$ is joined only to α, by a multiple bond, and in the ordinary diagram α is connected to a nearest long root by a chain C, ending with a multiple bond. It is enough to show that C is the full Dynkin diagram. If it were not, then $C \cup \{1-\mu\}$ would be a proper connected subdiagram of the extended Dynkin diagram, hence a Dynkin diagram in its own right, but one with two multiple bonds, namely those at its two ends, which is impossible. Now assume that α has the same lengths as μ. We have

$$(w^g\mu, w^{g+1}\mu) = (w^{2g+1}\mu, \mu) = -(w_1\mu, \mu) \quad (\text{by } (3)) = -|\mu|^2/2$$

since $w_1\mu = w_\alpha\mu$ and α and μ have equal length and form an angle of $60°$ in the present case. Thus $|w^g\mu - w^{g+1}\mu|^2 = 3|\mu|^2$. However $w^g\mu - w^{g+1}\mu = (1 - w_{g+1})w^g\mu = (1 - w_{g+1})(-b) = \Sigma(b, \gamma^*)\gamma$, summed over the elements of Π_{g+1} that are not orthogonal to b, i.e., over the neighbors of b. Since μ and b have the same length, the last two equations imply that $\Sigma(\gamma, b^*)(b, \gamma^*) = 3$. Thus 3 bonds come together at b, which is therefore a branch point or a point with a multiple bond.

In the development in this section so far we have borrowed ideas from Kostant **[Ko]**, who in turn has borrowed ideas from an earlier version of this paper. In that version, the transition from μ to b was effected differently, namely by alternate applications of (1) the reflection corresponding to b, (2) the product of the other simple reflections ordered so as to move away from b. That method brings up other points of interest, but we shall not pursue them here.

(6) Assume as in (3) and (4) except that $\operatorname{Supp} \lambda \subseteq \Pi_2$. Then $w^i\lambda = -w^{2g+1-i}\lambda$ and $w^g\lambda$ is a nonnegative combination of the roots in Π_{g+1}.

This easily follows from (3) and (4) with the roles of Π_1 and Π_2 interchanged.

(7) Let λ be a weight such that $w_0\lambda = -\lambda$, and λ_1 (resp. λ_2) the parts supported by the λ_α with $\alpha \in \Pi_1$ (resp. Π_2). Then $w^g\lambda_1$ (resp. $w^g\lambda_2$) is the part of $w^g\lambda$ supported by the roots of Π_g (resp. Π_{g+1}).

This follows from (4) and (6).

(8) Write $\mu = \Sigma_1 n_\alpha \alpha + \Sigma_2 n_\beta \beta$, the sums over Π_1 and Π_2. Then, with b as in (5),

(a) $w^g \Sigma_1 n_\alpha \alpha = -2\lambda_b$.

(b) $w^g \Sigma_2 n_\beta \beta = 2\lambda_b - b$.

By the orthogonality relations between the simple coroots and the fundamental weights we have $b = 2\lambda_b + \Sigma(b, \gamma^*)\lambda_\gamma$. Here b is in Π_g and the sum, equal to $b - 2\lambda_b$, is over the neighbors of b, all in Π_{g+1}. Using (4) we get $-\mu = (w^g)^{-1}(2\lambda_b) + (w^g)^{-1}\Sigma(b,\gamma^*)\lambda_\gamma$. By (7) with w_g, w_{g+1}, $(w^g)^{-1}$ in the roles of w_1, w_2, w_g, the first term on the right has support in Π_1, the second in Π_2, whence (a) and (b).

(9) With the notation as in (8)

$$\sum_1 n_\alpha \alpha = 2\sum_1 n_\alpha \lambda_\alpha - 2\sum_2 n_\beta \lambda_\beta.$$

If we dot with α^* we get $2n_\alpha$ on both sides. If we dot with β^* instead we get $-2n_\beta$ since the left side equals $\mu - \Sigma_2 n_\beta \beta$ and μ is orthogonal to all elements of Π_2.

(10) With the notation as in (8)

(a) $w^g \Sigma_1 n_\alpha \lambda_\alpha = -\lambda_{b,g}$

(b) $w^g \Sigma_2 n_\beta \lambda_\beta = \lambda_{b,g+1}$

(c) $w^g(\Sigma_1 n_\alpha \lambda_\alpha - \Sigma_2 n_\beta \lambda_\beta) = -\lambda_b$

Here $\lambda_{b,g}$, for example, denotes the part of λ_b supported by Π_g, that one of Π_1 and Π_2 that contains b. By (8) and (9) we have (c). Then (a) and (b) follow from (7).

We turn now to our discussion of affine Coxeter elements. One of these is $c = ww_2w_1$ with w_1 and w_2 as above and $w = w_{1-\mu}$ the reflection corresponding to $1 - \mu$. First we give the proof of (*) of 1(8), thus completing the proof of that result. We have to show that c', the linear part of c, is conjugate to the product of the reflections other than that at the branch point b. By (5), which is all that is needed, we have $w^g w_\mu (w^g)^{-1} = w_b$. This can be written as $w^g(w_\mu w_2 w_1)(w^g)^{-1} = w_b w_g w_{g+1}$. The left side is conjugate to c' and the right side equals the stated product since the w_b in front cancels the w_b that occurs as a factor of w_g.

Next we present a normal form for the affine Coxeter element $c = ww_2w_1$. Let F denote the standard fundamental domain for the affine Weyl group, defined as the region (a simplex) where $\alpha \geq 0$ for all $\alpha \in \Pi$, and $1 - \mu \geq 0$. Then w_1 is the reflection across the facet F_2 of F where all $\alpha \in \Pi_1$ are 0 (since the elements of Π_1 are mutually orthogonal), and ww_2 is the reflection across the opposite facet F_1 where $1 - \mu$ and all $\beta \in \Pi_2$ are 0. We seek points γ_1 and γ_2 in F_1 and F_2 such that the line L joining them is orthogonal to F_1 and to F_2. Then c will be a screw displacement along L (in 3 dimensions the motion of a screw whose axis is L): translation by the vector $2(\gamma_1 - \gamma_2)$ in the direction of L composed with a rotation around L (i.e. an isometry fixing the points of L), the two factors necessarily commuting and being determined by the stated conditions. Since c', the linear part of c, has 1 as an eigenvalue of multiplicity 1 (with corresponding eigenvector in the direction of L), L, in the present case, may also be described as the set of points moved the least distance by c.

(11) Write $\mu = \Sigma_1 n_\alpha \alpha + \Sigma_2 n_\beta \beta$ as in (8) set $\delta = \Sigma_1 n_\alpha \alpha$, so that $(\delta, \delta) = \Sigma_1 n_\alpha^2(\alpha, \alpha)$. Then the solution to our problem is $\gamma_1 = 2\Sigma_1 n_\alpha \lambda_\alpha/(\delta, \delta)$ and $\gamma_2 = 2\Sigma_2 n_\beta \lambda_\beta/(\delta, \delta)$. Thus c is a rotation around the line joining these points composed with the translation by the vector $2(\gamma_1 - \gamma_2) = 2\delta/(\delta, \delta)$, of length $2/|\delta|$, along the line. Further γ_2 is the point on L closest to the origin.

First δ is orthogonal to F_2 clearly and also to F_1 since $\delta = \mu - \Sigma_2 n_\beta \beta$. Write the equation of (9) as $\delta = \delta_1 - \delta_2$. Then by the definitions $\delta_2 \in F_2$ and $\delta_1 \in F_1$ except for the condition $(\mu, \delta_1) = 1$. Now $(\mu, \delta_1) = 2\Sigma_1 n_\alpha^2(\alpha, \lambda_\alpha) = \Sigma_1 n_\alpha^2(\alpha, \alpha) = (\delta, \delta)$. It follows that $\gamma_1 = \delta_1/(\delta, \delta)$ is in F_1 and γ_2 is in F_2, and that $2(\gamma_1 - \gamma_2) = 2(\delta_1 - \delta_2)/(\delta, \delta) = 2\delta/(\delta, \delta)$ is orthogonal to F_1 and to F_2. Finally γ_2 is the point of L closest to the origin since it is orthogonal to the vector δ along L.

In the normal form for c just given, the axis L and the translational part can be calculated quite explicitly in any given case, but the same can not be said of the rotational part. Here is another normal form which remedies this deficiency.

(12) Let b be the root in (5) above, and write $\lambda_b = \lambda_{b,g} + \lambda_{b,g+1}$ with $\lambda_{b,g}$ the part supported by Π_g (which includes b) and $\lambda_{b,g+1}$ the part supported by Π_{g+1}. Let L be the line through $\varepsilon = \lambda_{b,g+1}/2(\lambda_b, \lambda_b)$ in the direction of λ_b. Then ε is the point of L closest to the origin. Form the rotation around L whose linear part is the product of the simple reflections other than that for b and compose it with the translation by $-\lambda_b/(\lambda_b, \lambda_b)$ in the direction of L. Then the result is an affine Coxeter element c expressed in standard form as a screw displacement.

With γ_1, γ_2, δ as in (11) we have from (8) and (10) that $w^g\delta = -2\lambda_b$, $w^g\gamma_1 = -2\lambda_{b,g}/(\delta, \delta)$ and $w^g\gamma_2 = 2\lambda_{b,g+1}/(\delta, \delta)$. Thus (12) follows from (11).

We observe that since $\Pi - \{b\}$ is a union of systems of type A_n, the eigenvalues and eigenvectors of the linear part of c above, as well as its order, can be easily determined. So can λ_b, hence the other items of (12) also, especially when all roots have the same length, so that b is a branch point:

(13) If $\lambda_b = \Sigma m_\alpha \alpha$ ($\alpha \in \Pi$), then $m_b = (p^{-1} + q^{-1} + r^{-1} - 1)^{-1}$ in terms of the branch lengths, and along the branch of length p, for example, starting at the end point the m_α's are $p^{-1}m_b, 2p^{-1}m_b, \ldots$.

For, as is easily seen, the scalar product of the proposed vector with b^* is 1, with all other simple coroots is 0.

We conclude our paper with some further remarks about the McKay correspondence that arise from the ideas of this section. G will be a

Kleinian group, as in §1, Π the corresponding simple root system, and the other notations as above.

(14) $2\Sigma_2 n_\alpha^2 = g$, the order of $|G|$.

Regarding the n_α's as the degrees of the irreducible representations of G we have $\Sigma_1 n_\alpha^2 + \Sigma_2 n_\beta^2 + 1$ (trivial representation) $= g$. The same for the group $G/\{\pm 1\}$ yields $\Sigma_2 + 1 = g/2$ (see the proof of 1(4b)), whence (14).

(15) $2\Sigma_1 n_\alpha^2 = 4(p^{-1} + q^{-1} + r^{-1} - 1)^{-1}$ in terms of the branch lengths.

For, $(\delta, \delta) = \Sigma_1 n_\alpha^2\ (\alpha, \alpha)$ in (11) and $(2\lambda_b, 2\lambda_b) = 4(\lambda_b, \Sigma m_\alpha \alpha) = 2m_b(b, b)$. These are equal by (8a), and then (15) follows from (13) and the equality of all root lengths.

These equations show that m_b in (13) is just $g/4$. They also lead to another, nontrigonometric, proof of §1(8), with which we close our paper. We have $g = 4(p_1^{-1} + q_1^{-1} + r_1^{-1} - 1)$ since the decomposition of G into conjugacy classes yields $g = 1 + 1 + (p_1 - 1)g/2p_1 + (q_1 - 1)g/2q_1 + (r_1 - 1)g/2r_1$. Since $r = r_1 = 2$, this, (14) and (15) yield $p^{-1} + q^{-1} = p_1^{-1} + q_1^{-1}$. But also $p + q = p_1 + q_1$ since $p + q + r - 1$ is the number of irreducible representations of G and $p_1 + q_1 + r_1 - 1$ is the number of conjugacy classes. Dividing one equation by the other we get $pq = p_1 q_1$. Thus $(p, q) = (p_1, q_1)$, as required.

REFERENCES

[B] N. Bourbaki, *Groupes et Algebres de Lie*, vol. IV, V, VI, Hermann and Co., Paris (1968).

[C] H. S. M. Coxeter, *Regular Complex Polytopes*, Cambridge University Press, Cambridge (1974).

[G] G. Gonzalez-Sprinberg and J. L. Verdier, *Construction géométrique de la correspondance de McKay*, Ann. Sci. E.N.S., **16** (1983), 409–449.

[H] D. Happel, U. Preiser and C. M. Ringel, *Binary polyhedral groups and Euclidean diagrams*, Manuscripta Math., **31** (1980), 317–329.

[K] H. Knörrer, *Group representations and resolution of rational double points*, Bonn University preprint.

[Ko] B. Kostant, *On finite subgroups of $SU(2)$, simple Lie algebras and the McKay correspondence*, M.S.R.I. preprint.

[M] J. McKay, *Graphs, singularities, and finite groups*, Amer. Math. Soc. Proc. Sym. Pure Math., **37** (1980), 183–186.

[Mu] D. Mumford, *The topology of normal singularities of an algebraic surface and a condition for simplicity*, Publ. Math. I.H.E.S., **9** (1961), 5–22.

[S_1] P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, Lecture Notes in Math. 815, Springer-Verlag, New York (1980).

[S_2] _____, *Platonic Solids, Kleinian Singularities and Lie Groups*, Lecture Notes in Math. 1008, Springer-Verlag, New York (1980), 102–138.
[St] R. Steinberg, *Finite reflection groups*, Trans. Amer. Math. Soc., **91** (1959), 493–504.

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UNIVERSITY OF CALIFORNIA
LOS ANGELES, CA 90024