THE MCKAY CORRESPONDENCE

YI SUN

Abstract. The McKay correspondence gives a bijection between the finite subgroups of $SU(2)$ and the affine simply laced Dynkin diagrams. In particular, this bijection associates naturally to each finite dimensional representation of $SU(2)$ a vertex of the corresponding diagram. The goal of this talk will be to construct this correspondence and to discuss some proofs and generalizations.

1. Introduction

Let G be a finite group and V a *faithful* representation of G over $\mathbb{C}$, that is, an injective map $G \to GL(V)$. Knowledge of such a representation gives G as a group of matrices. Further, it is a classical fact that all irreducible representations of G are contained in $V^{\otimes N}$ for some N. In some sense, the size of V is a measure of the complexity of G.

One might therefore be led to consider which groups G have faithful representations in rank n, especially when n is small. But recall that every finite dimensional representation of a finite group may be given a unitary structure, meaning that it suffices to consider finite subgroups of $U(n)$. Recall now that central elements of $U(n)$ are given by the diagonal embedding $U(1) \hookrightarrow U(n)$ and that the quotient $U(n)/U(1)$ by this center is $SU(n)$. Therefore, if we restrict our attention to simple groups, it is enough to consider finite subgroups of $SU(n)$.

The purpose of this talk is to present an especially elegant classification of these subgroups for the case $n = 2$. Of course, to obtain the classification, it is possible to simply consider the standard double cover

$$\pi : SU(2) \twoheadrightarrow SO(3)$$

and note that the only element of even order in $SU(2)$ is the generator -1 of the kernel of π. Therefore, any finite subgroup of $SU(2)$ either has even order and is the preimage of a finite subgroup of $SO(3)$ or has odd order and is isomorphic to a finite subgroup of $SO(3)$ of odd order, hence a cyclic group. From this and the classical classification of finite subgroups of $SO(3)$ as symmetry groups of regular polyhedra, we obtain the following.

Proposition 1. Any finite subgroup of $SU(2)$ is one of the following groups:

- The cyclic group $\mathbb{Z}/n\mathbb{Z}$ for $n > 1$.
- The binary dihedral group $\mathbb{B}D_{2n}$ for $n > 1$, the preimage of the dihedral group D_{2n} under π.
- The binary tetrahedral group $\mathbb{BT}$, the preimage of the tetrahedral group $\mathbb{T}$ under π.
- The binary octahedral group $\mathbb{BO}$, the preimage of the octahedral group $\mathbb{O}$ under π.
- The binary dodecahedral group $\mathbb{BD}$, the preimage of the dodecahedral group $\mathbb{D}$ under π.

2. Statement of the correspondence

Let G be a finite subgroup of $SU(2)$, and let V be the faithful representation of G obtained from the embedding $G \hookrightarrow SU(2)$. Let $\{V_i\}$ be the irreducible representations of G. Define the *McKay quiver* of G to be the directed multi-graph with vertices V_i and m_{ij} edges from V_i to V_j if V_j occurs m_{ij} times in the decomposition of $V_i \otimes V$ into irreducibles. In other words, we have

$$V_i \otimes V = \bigoplus_j V_j^{\oplus m_{ij}}.$$

The McKay correspondence classifies the possible groups G via their McKay quivers. More precisely, we have the following.

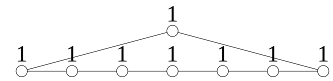
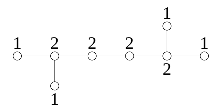
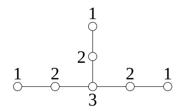
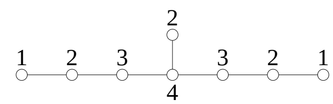
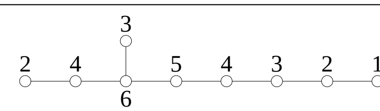
Date: December 1, 2010.

Notes for a talk given on December 1, 2010 as part of the 2010 Part III Seminar Series at the University of Cambridge.

Theorem 1 (McKay 1980)**.** Let G be a non-trivial finite subgroup of $SU(2)$. Then, the McKay quiver of $SU(2)$ is an affine simply laced Dynkin diagram.

Before sketching the proof, we exhibit each case of this correspondence explicitly in Table 1.

TABLE 1. The explicit form of the McKay correspondence.

Finite subgroup of $SU(2)$		Affine simply laced Dynkin diagram	
$\mathbb{Z}/n\mathbb{Z}$	$\langle x \mid x^n = 1\rangle$	$\widetilde{A}_{n-1}$	
$\mathbb{B}D_{2n}$	$\langle x, y, z \mid x^2 = y^2 = y^n = xyz\rangle$	$\widetilde{D}_{n-2}$	
$\mathbb{BT}$	$\langle x, y, z \mid x^2 = y^3 = z^3 = xyz\rangle$	$\widetilde{E}_6$	
$\mathbb{BO}$	$\langle x, y, z \mid x^2 = y^3 = z^4 = xyz\rangle$	$\widetilde{E}_7$	
$\mathbb{BD}$	$\langle x, y, z \mid x^2 = y^3 = z^5 = xyz\rangle$	$\widetilde{E}_8$	

3. PROOF OF THE CORRESPONDENCE

We give here a proof intended to minimize the number of necessary prerequisites, but we note that there are other "deeper" proofs possible. Of course, it is also possible to give a proof based on case-by-case verification (which is how this correspondence was first discovered), but we would like to give a more uniform interpretation.

Proof of Theorem 1. We will slowly obtain more and more combinatorial properties of the McKay quiver G until the affine simply laced Dynkin diagrams pop out as the only graphs satisfying these properties.
Claim 1: The McKay quiver of any G is an undirected graph, that is, $m_{ij} = m_{ji}$. For this, let χ_i be the character of the representation V_i, and notice that

$$m_{ij} = \langle \chi_i, \chi_V \chi_j\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_i(g)\overline{\chi_V(g)\chi_j(g)} = \frac{1}{|G|}\sum_{g\in G} \chi_i(g)\chi_V(g)\overline{\chi_j(g)} = \langle \chi_i\chi_V, \chi_j\rangle = m_{ji},$$

where we note that χ_V is real because each element of $SU(2)$ has real trace.
Claim 2: The McKay quiver is connected. This follows from the fact that every irreducible representation of G is contained in some tensor power of the faithful representation V.
Claim 3: The McKay quiver has no self-loops, that is, $m_{ii} = 0$. For this, observe that

$$m_{ii} = \langle \chi_V \chi_i, \chi_i\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)|\chi_i(g)|^2.$$

If G has even order, then it contains the element $-1 \in SU(2)$, so multiplication by -1 defines an involution on G with $\chi_V(g) = -\chi_V(-g)$; hence we have

$$2m_{ii} = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)(|\chi_i(g)|^2 - |\chi_i(-g)|^2) = 0.$$

Otherwise, if G has odd order, then its image in $SO(3)$ under π has odd order, hence G is a cyclic group $\mathbb{Z}/n\mathbb{Z}$ for some n. Let $A \in SU(2)$ generate G, where $A \neq 1$ satisfies $A^n = 1$, that is, $(1+A+A^2+\cdots+A^{n-1})(1-A) = 0$. But $\det(1-A) \neq 0$ if $A \neq 1$ because $A \in SU(2)$, so we see that

$$m_{ii} = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)|\chi_i(g)|^2 = \frac{1}{|G|}\sum_{g\in G}\chi_V(g) = \frac{1}{|G|}\operatorname{Tr}(1+A+\cdots+A^{n-1}) = 0.$$

Claim 4: The McKay quiver is a simple graph, that is, $m_{ij} \in \{0,1\}$. For this, notice that

$$m_{ij} = \frac{1}{|G|}\sum_{g\in G}\chi_i(g)\chi_V(g)\overline{\chi_j(g)} \le \frac{1}{|G|}\sqrt{\sum_{g\in G}|\chi_i(g)|^2|\chi_V(g)|^2}\sqrt{\sum_{g\in G}|\chi_j(g)|^2}$$
$$< \frac{1}{|G|}\sqrt{4\sum_{g\in G}|\chi_i(g)|^2}\sqrt{\sum_{g\in G}|\chi_j(g)|^2} = 2.$$

Remainder of proof: These four properties along with the definition of the McKay quiver will be enough for us to characterize them as the affine simply laced Dynkin diagrams. Let $d_i = \dim V_i$ be the dimension of V_i and label vertex V_i by d_i. Then, notice that $2d_i = \sum_{j\sim i} d_j$, where $j \sim i$ means i and j are neighbors.

Step 1: If the McKay quiver contains a cycle, then it is of type $\widetilde{A}_{n-1}$.

If $i_1, \ldots, i_k$ are vertices in a cycle, then we have $2d_{i_k} \ge d_{i_{k-1}} + d_{i_{k+1}}$, so summing these shows we must have equality, meaning there are no other vertices in the graph. Further, the trivial representation has dimension 1, meaning that all d_i are equal to 1.

Step 2: Each vertex of the McKay quiver has degree at most 4, and if such a vertex exists, it is of type $\widetilde{D}_4$.

Let $j_1, \ldots, j_5$ be neighbors of i. Then, we have that

$$\frac{5}{4}d_i \le \frac{1}{2}[d_{j_1} + \cdots + d_{j_5}] \le d_i,$$

a contradiction, where the first inequality holds because $d_{j_k} \ge \frac{1}{2}d_i$. So vertices have degree at most 4. In this case, for neighbors $j_1, \ldots, j_4$ of i, we have

$$d_i \le \frac{1}{2}[d_{j_1} + \cdots + d_{j_4}] \le d_i,$$

so equality holds in all cases, which shows that we are in type $\widetilde{D}_4$.

Step 3: The McKay quiver has a vertex of degree at least 3 connected to a leaf with label 1.

If all vertices have degree 2 or less, then the quiver is a path, which is impossible. So there is at least one vertex of degree 3. It remains to show that there is a leaf with label 1; but if not then all vertices in the quiver will have label greater than 1, which is impossible.

Step 4: If this vertex of degree 3 has label 2, then we are in type $\widetilde{D_n}$ or it is the only vertex of degree 3.

Let i and j be two vertices with degree 3 connected by a path, let x, y be the two neighbors of i not on this path, and let z, w be the two neighbors of j not on this path. Then, we have by the conditions on the path and the fact that $2d_x, 2d_y \ge d_i$ and $2d_z, 2d_z \ge d_j$ that

$$d_i + d_j \ge d_x + d_y + d_z + d_w \ge d_i + d_j,$$

so we have equality. Therefore, there are no other vertices abutting the path and x, y, z, w are the only neighbors of i and j. This is $\widetilde{D}_n$.

Step 5: If there is only one vertex of degree 3, then we are in type $\widetilde{E_6}$, $\widetilde{E_7}$, or $\widetilde{E_8}$.

This is clear by easy computation. □

4. Other interpretations and generalizations

We have presented the McKay correspondence as only between finite subgroups of $SU(2)$ and ADE-type affine Dynkin diagrams. However, it extends also to a different context, which we will only sketch here.

Finite subgroups G of $SU(2)$	$\leftrightarrow$	simply laced affine Dynkin diagrams	$\leftrightarrow$	Resolution graphs of $\mathbb{C}^2/G$

Because $G \subset SU(2)$, it acts on $\mathbb{C}^2$ and we may consider the quotient variety $\mathbb{C}^2/G$, which has a singularity at 0. In this case, there is a minimal resolution of $\mathbb{C}^2/G$, i.e. smooth variety $\pi : X \twoheadrightarrow \mathbb{C}^2/G$ where the preimage $\pi^{-1}(0)$ of $0 \in \mathbb{C}^2/G$ is the union of irreducible components isomorphic to $\mathbb{CP}^1$. Then, the graph with these components as vertices and an edge between two vertices if their components intersect is the *resolution graph* of $\mathbb{C}^2/G$. It turns out that this graph is just the simply laced affine Dynkin diagram of G! Further, the K-theory $K_0(X)$ of X is simply the representation ring of G! And even further, this isomorphism extends to an equivalence between the derived categories $D^b(\mathrm{Coh}(\mathbb{C}^2)^G)$ and $D^b(\mathrm{Coh}(X))$ of G-equivariant coherent sheaves on $\mathbb{C}^2$ and coherent sheaves on X!

Sources

The correspondence was first noticed by J. McKay in [4]. The proof we give here is based loosely on one suggested in [1] together with an exercise assigned by A. Postnikov (Problem 5 in [5]). The connection to the K-theory of Kleinian singularities was given in [2], and its extension to a derived equivalence in [3].

References

[1] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, E. Udovina, and D. Vaintrob, Introduction to Representation Theory, available at: `http://arxiv.org/abs/0901.0827`.

[2] G. Gonzalez-Sprinberg and J. L. Verdier, Construction géométrique de la correspondance de McKay, Ann. ENS **16** (1983), 409-449.

[3] M. Kapranov, E. Vasserot, Kleinian singularities, derived categories and Hall algebras, Math. Ann. **316** (2000), 565-576.

[4] J. McKay, Graphs, singularities, and finite groups, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, California, 1979), Proc. Sympos. Pure Math **37**, American Mathematical Society, Providence, RI 1980, 183-186.

[5] A. Postnikov, Problem Set #2, 18.319 Spring 2008, available at: `http://math.mit.edu/~apost/courses/18.319/pset2.pdf`.