Rational Catalan Combinatorics: Intro

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Goals of the workshop

1. Reinforce existing connections and forge new connections between two groups:
   - Catalan combinatorialists
   - Representation theorists,
     particularly rational Cherednik algebra experts.
2. Advertise to the RCA people the main combinatorial mysteries/questions.
3. Have the RCA people explain what they perceive as the most relevant theory, directions, and questions.

Goals of this talk

1. Outline Catalan combinatorics and objects, and 4 directions of generalization, mentioning keywords[1] here.
2. Describe my own favorite combinatorial mystery: Why are two seemingly different families of objects,
   - noncrossing and
   - nonnesting,

   equinumerous, both counted by W-Catalan numbers?
3. Describe (with example) a conjecture that would resolve it.

[1] History and real definitions (mostly) omitted.

Catalan numbers

The Catalan number

$$\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$$

surprises us that it counts these "noncrossing" families

1. noncrossing partitions of $[n] := \{1, 2, \ldots, n\}$
2. clusters of finite type A_{n-1}
3. Tamari poset on triangulations of an $(n+2)$-gon

but no longer surprises us counting these "nonnesting" families

1. nonnesting partitions of $[n] := \{1, 2, \ldots, n\}$
2. antichains of positive roots of type A_{n-1}
3. dominant Shi arrangement regions of type A_{n-1}
4. increasing parking functions of length n
5. $(n, n+1)$-core integer partitions

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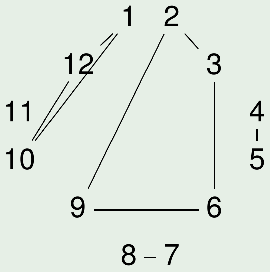
Noncrossing partitions

Definition

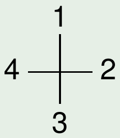
A partition of $[n] := \{1, 2, \ldots, n\}$ is noncrossing if its blocks have disjoint convex hulls when $\{1, 2, \ldots, n\}$ are drawn cyclically.

Example



noncrossing

$1, 10, 12 | 2, 3, 6, 9 | 4, 5 | 7, 8 | 11$



crossing

$1, 3 | 2, 4$

Noncrossing partitions

Example

The $\mathrm{Cat}_4 = 14$ noncrossing partitions of [4]

number of blocks k		tally
1	1234	1
2	123|4, 124|3, 134|2, 1|234, 12|34, 14|23	6
3	12|3|4, 13|2|4, 1|23|4, 1|2|34, 14|2|3, 1|24|3	6
4	1|2|3|4	1

Nonnesting partitions

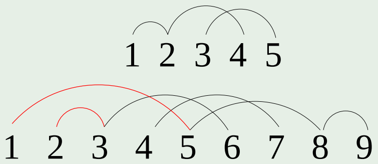
Plot $\{1, 2, \ldots, n\}$ along the x-axis, and depict set partitions by semicircular arcs in the upper half-plane, connecting i, j in the same block if no other k with $i < k < j$ is in that block.

Definition

Say the set partition is nonnesting if no pair of arcs nest.

Example

124|35 is nonnesting,
while 1589|234|67 is nesting as arc 15 nests arc 23.



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4	1|2|3|4	1

More shared numerology: Narayana, Kreweras

There are Cat_n total noncrossing or nonnesting partitions of $[n]$, and in addition, the

1. number with k blocks is the Narayana number,

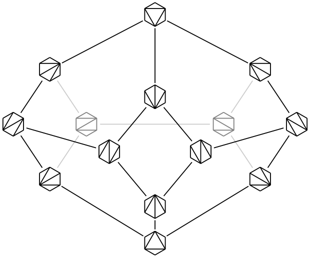
$$\mathrm{Nar}_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

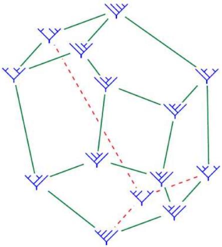
2. number with m_i blocks of size i is the Kreweras number

$$\mathrm{Krew}(1^{m_1}2^{m_2}\cdots) = \frac{n!}{(n-k+1)! \cdot m_1! m_2! \cdots}.$$

Cat_n counts

- triangulations of an $(n+2)$-gon,
- vertices of the $(n-1)$-dimensional associahedron,
- elements of the Tamari poset,
- clusters of type A_{n-1}.





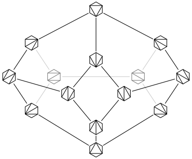
More generally, Kirkman numbers

$$\text{Kirk}_{n,k} := \frac{1}{k+1}\binom{n+k+1}{k}\binom{n-1}{k}$$

count the

- $(n-1-k)$-dim'l faces, or the
- $(n+2)$-gon dissections using k diagonals.

k	$\text{Kirk}_{4,k}$	
3	14	vertices
2	21	edges
1	9	2-faces
0	1	the 3-face



Kirkman is to Narayana as f-vector is to h-vector

The relation between Kirkman and Narayana numbers is the (invertible) relation of the f-vector $(f_0, \ldots, f_n)$ of a simple n-dimensional polytope to its h-vector $(h_0, \ldots, h_n)$:

$$\sum_{i=0}^{n} f_i t^i = \sum_{i=0}^{n} h_i (t+1)^{n-i}.$$

Example

The 3-dimensional associahedron has f-vector $(14, 21, 9, 1)$, and h-vector $(1, 6, 6, 1)$.

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 9 & & \\
 & 1 & & 8 & & 21 & \\
1 & & 7 & & 13 & & 14 \\
\hline
(1, & & 6, & & 6, & & 1)
\end{array}$$

Catalan, Narayana, Kirkman, Kreweras

This is one of the 4 directions of generalization:

$$\mathrm{Cat}_n = \sum_k \mathrm{Nar}_{n,k}$$

$$\mathrm{Nar}_{n,k} = \sum_{\ell(\lambda)=k} \mathrm{Krew}(\lambda)$$

and

$$\begin{array}{rcl} \mathrm{Nar}_{n,k} & \leftrightarrow & \mathrm{Kirk}_{n,k} \\ h\text{-vector} & \leftrightarrow & f\text{-vector} \end{array}$$

4 directions of generalization/refinement for Cat_n

W-Catalan

Narayana, Kirkman, Kreweras ← Catalan → q-Catalan, (q,t)-Catalan

Fuss-Catalan, rational Catalan

Another fascinating direction: $\mathfrak{S}_n$-harmonics $\to$ diagonal harmonics $\to$ tridiagonal harmonics $\to \cdots$?

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The Fuss and rational Catalan direction

- Catalan number

$$\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$$

- Fuss-Catalan number

$$\mathrm{Cat}_n = \frac{1}{mn+1}\binom{(m+1)n}{n} = \frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n}$$

($m = 1$ gives Catalan)

- Rational Catalan number

$$\mathrm{Cat}_n = \frac{1}{a+b}\binom{a+b}{a} \text{ with } \gcd(a,b) = 1$$

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Fuss, rational Catalan and the RCA parameter

This direction is related to the parameter c
in the definition of the RCA H_c (for W of type A_{n-1}).

H_c has an irreducible highest weight module $L(\mathbf{1})$,
and it will be finite-dimensional if and only if $c = \frac{b}{a}$ with
$1 < a < b$ and $\gcd(a, b) = 1$.

The dimension of its W-fixed subspace $L(\mathbf{1})^W$ is

- the Catalan number for $c = \frac{n+1}{n}$,
- the Fuss-Catalan number for $c = \frac{mn+1}{n}$,
- the Rational Catalan number for $c = \frac{b}{a}$

The Kirkman direction from the RCA viewpoint

One can also reinterpret the Kirkman generalization in terms of the RCA in type A_{n-1} as follows:

$$\begin{aligned} \mathrm{Cat}_n &= \#\text{vertices of associahedron} \\ &= \#\text{clusters} \\ &= \#\dim L(\mathbf{1})^W \\ &= \text{ multiplicity of } \wedge^0 V \text{ in } L(\mathbf{1}) \end{aligned}$$

but more generally

$$\begin{aligned} \mathrm{Kirk}_{n,k} &= \#(n-1-k)\text{-dim'l faces of associahedron} \\ &= \#\text{compatible sets of } k \text{ (unfrozen) cluster variables} \\ &= \text{ multiplicity of } \wedge^{n-1-k} V \text{ in } L(\mathbf{1})^W \end{aligned}$$

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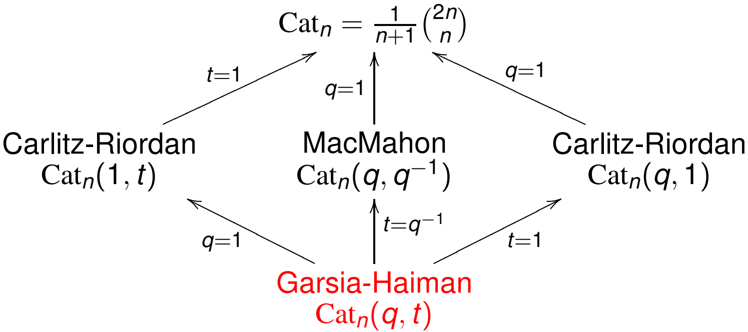
The q- and (q,t)-direction



The Garsia-Haiman (q,t)-Catalan can be thought of as a bigraded (or rather, filtered and graded) dimension for $L(\mathbf{1})^W$.

Parking and increasing parking functions

Before the last direction, a review of more nonnesting families...

Definition

Increasing parking functions of length n are weakly increasing sequences $(a_1 \leq \ldots \leq a_n)$ with a_i in $\{1, 2, \ldots, i\}$.

Definition

A parking function is sequence $(b_1, \ldots, b_n)$ whose weakly increasing rearrangement is an increasing parking function.

- There are $(n+1)^{n-1}$ parking functions of length n, of which
- Cat_n many are increasing parking functions.

Parking functions of length $n = 3$

Example

The $(3+1)^{3-1} = 16$ parking functions of length 3, grouped into the $C_3 = \frac{1}{4}\binom{6}{3} = 5$ different $\mathfrak{S}_3$-orbits, with increasing parking function representative shown leftmost:

111					
112	121	211			
113	131	311			
122	212	221			
123	132	213	231	312	321

Parking functions and $L(\mathbf{1})$

By definition parking functions have an $\mathfrak{S}_n$-action on positions

$$w(b_1, \ldots, b_n) = (b_{w^{-1}(1)}, \ldots, b_{w^{-1}(n)})$$

and increasing parking functions represent the $\mathfrak{S}_n$-orbits. Thus Cat_n is the dimension of the $\mathfrak{S}_n$-fixed space for this $\mathfrak{S}_n$-permutation action.

On the RCA side:
Character computation shows that for parameter $c = \frac{n+1}{n}$, the irreducible H_c-module $L(\mathbf{1})$ carries $\mathfrak{S}_n$-representation isomorphic to

- the $\mathfrak{S}_n$-permutation action on parking functions,
- with $\mathfrak{S}_n$-fixed space $L(\mathbf{1})^{\mathfrak{S}_n}$ of dimension Cat_n.

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Regions, dominant regions in the Shi arrangement

Definition

The Shi arrangement of hyperplanes is

$$\{x_i - x_j = 0, 1\}_{1 \le i < j \le n}$$

inside $\mathbb{R}^n$, or the subspace where $x_1 + \cdots + x_n = 0$.

It dissects these spaces into

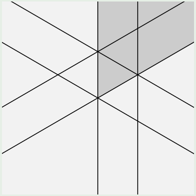
- a total of $(n+1)^{n-1}$ regions, of which
- Cat_n lie in the dominant chamber $x_i \ge x_j$ for $i < j$

Regions, dominant regions in the Shi arrangement

Example

Here for $W = \mathfrak{S}_3$ of type A_2 are shown the

- $(3+1)^{3-1} = 16$ Shi regions, and
- the $\mathrm{Cat}_3 = 5$ dominant Shi regions (shaded)



Simultaneous (n+1,n)-cores

Definition

A partition λ is an *n*-core if it has no hooklengths divisible by *n*.

Example

$\lambda = (5, 3, 1, 1)$ is a 3-core:

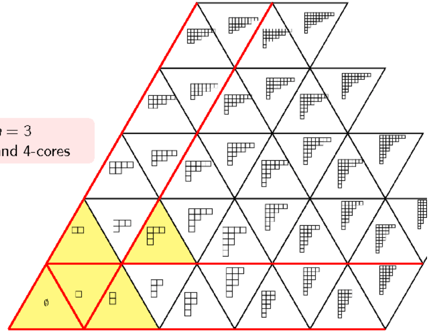
8	5	4	2	1
5	2	1		
2				
1				

Simultaneous (n+1,n)-cores

- Naturally *n*-cores label dominant alcoves for the affine Weyl group $\tilde{\mathfrak{S}}_n$.
- There are Cat_n of the *n*-cores of them which are simultaneously $(n+1)$-cores and *n*-cores. They label minimal alcoves in dominant Shi chambers.

Alcoves $\Longleftrightarrow$ *n*-cores

$m = 1,\ n = 3$
3-cores and 4-cores



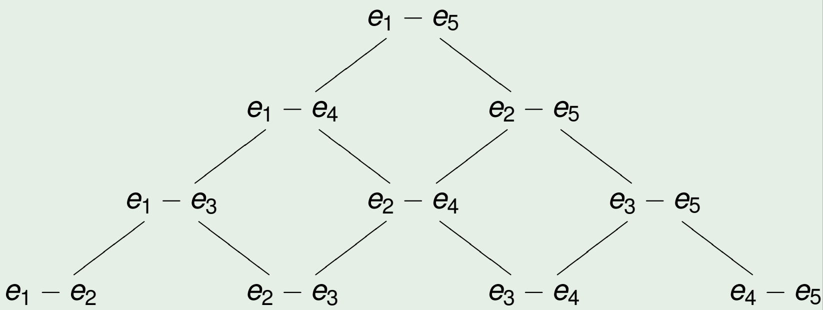
Antichains of positive roots

Definition

The root order on Φ_+ says that $\alpha < \beta$ if $\beta - \alpha$ is a nonnegative combination of roots in Φ_+.

Example

For $W = \mathfrak{S}_5$, the root order on $\Phi_+ = \{e_i - e_j : 1 \leq i < j \leq 5\}$ is

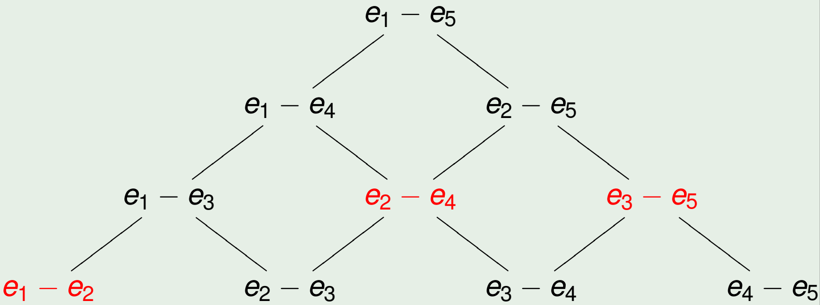


Nonnesting partitions for Weyl groups

Nonnesting partitions of $[n]$ biject with antichains in Φ_+ for $\mathfrak{S}_n$: to each arc $i < j$ associate the root $e_i - e_j$.

Example

124|35 is nonnesting, corresponding to antichain $\{e_1 - e_2, e_2 - e_4, e_3 - e_5\}$:



The reflection group W direction

For W a finite real reflection group[2], acting irreducibly on $V = \mathbb{R}^\ell$, define the W-Catalan number

$$\mathrm{Cat}(W) := \prod_{i=1}^{\ell} \frac{d_i + h}{d_i}$$

where $(d_1, \ldots, d_\ell)$ are the fundamental degrees of homogeneous W-invariant polynomials $f_1, \ldots, f_n$ in

$$S = \mathrm{Sym}(V^*) \cong \mathbb{R}[x_1, \ldots, x_\ell]$$

with $S^W = \mathbb{R}[f_1, \ldots, f_n]$, and Coxeter number

$$h := \max\{d_i\}_{i=1}^n = \frac{\#\{\text{reflections}\} + \#\left\{\begin{array}{c}\text{reflecting}\\ \text{hyperplanes}\end{array}\right\}}{n}.$$

[2] ... or even a complex reflection group, with suitably modified definition.

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W-Catalan and the RCA

The RCA $H_c(W)$ has its irreducible rep'n $L(\mathbf{1})$ finite-dimensional only for certain parameter values c.

Among these values is $c = \frac{h+1}{h}$, constant on all conjugacy classes of reflections.

- This irreducible $L(\mathbf{1})$ has dimension $(h+1)^n$, and
- W-fixed subspace $L(\mathbf{1})^W$ of dimension $\mathrm{Cat}(W)$, by a standard[3] character computation.

[3] Using a theorem of Solomon

My favorite mystery

We are about to list the names of several

1. *W*-nonnesting families of objects, and
2. *W*-noncrossing families of objects

Although we won't explain it here, we understand well

- via bijections why they are equinumerous within in each family, and
- via character computation why the nonnestings are counted by $\mathrm{Cat}(W)$.

Mystery

Why are the noncrossings also counted by $\mathrm{Cat}(W)$*?*

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The W-nonnesting family for Weyl groups W

Parking functions generalize to ...

1. sign types as defined by Shi
2. Shi arrangement regions
3. the finite torus $Q/(h+1)Q$

where Q is the root lattice for W

Increasing parking functions generalize to

1. $\oplus$-sign types or antichains of positive roots Φ_W^+
2. dominant Shi arrangement regions
3. W-orbits on $Q/(h+1)Q$.

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The W-noncrossing family

Fix a choice of a Coxeter element $c = s_1 s_2 \cdots s_n$ in a finite reflection group W with Coxeter generators $\{s_1, \ldots, s_n\}$. Then

- noncrossing partition lattice,
- clusters of type A_{n-1},
- Tamari poset on triangulations,

will generalize to

1. the lattice $NC(W, c) := [e, c]_{\mathrm{abs}}$, an interval in $<_{\mathrm{abs}}$ on W
2. c-clusters
3. c-Cambrian lattice on c-sortable elements, and
4. reduced subwords for w_0 within the concatenation[4] $\mathbf{cw_0(c)}$.

I'll focus here on NC(W,c).

[4] Here $\mathbf{c} = (s_1, s_2, \cdots, s_n)$ and $\mathbf{w_0}(\mathbf{c})$ is the $\mathbf{c}$-sorting word for w_0.

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Absolute length and absolute order on W

Define the absolute order on W using the absolute length[5]

$$\ell_T(w) := \min\{\ell : w = t_1 t_2 \cdots t_\ell \text{ with } t_i \in T\}$$

where $T := \bigcup_{w \in W, s \in S} wsw^{-1}$.
Then say $u \leq v$ in the absolute order if

$$\ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v)$$

that is, v has a T-reduced expression

$$v = \underbrace{t_1 t_2 \cdots t_m}_{u:=} t_{m+1} \cdots t_\ell$$

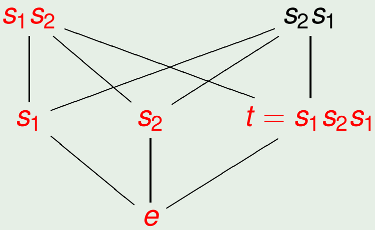
with a prefix that factors u.

[5] Not the usual Coxeter group length!

Define $NC(W, c)$ to be the interval $[e, c]_{\text{abs}}$ from the identity e to the chosen Coxeter element $c = s_1 s_2 \cdots s_n$ in $<_{\text{abs}}$ on W.

Example

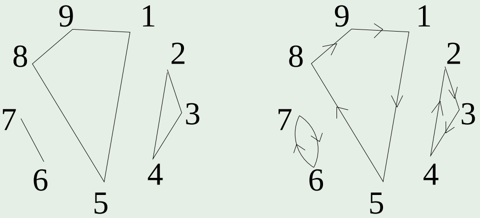
$W = \mathfrak{S}_3$ of type A_2, with $S = \{s_1, s_2\}$ and $c = s_1 s_2$.
Absolute order shown, with $NC(W, c) = [e, c]_{\text{abs}}$ in red.



The picture in type A_{n-1}

It's not hard to see that for $W = \mathfrak{S}_n$ of type A_{n-1} with $c = (123 \cdots n) = s_1 s_2 \cdots s_{n-1}$, a permutation w lies $NC(W, c) = [e, c]_{\text{abs}}$ if and only if the cycles of w are noncrossing and oriented clockwise.

Example



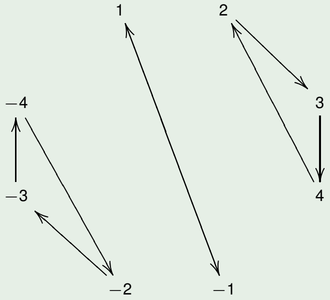
The picture in type B_n

Similarly, for W the hyperoctahedral group of type B_n of $n \times n$ signed permutation matrices, with c sending

$$e_1 \mapsto e_2 \mapsto \cdots e_{n-1} \mapsto e_n \mapsto -e_1$$

one has the same description of $NC(W, c)$, imposing the extra condition that the cycles of w are centrally symmetric.

Example



A conjecture

We wish to phrase a conjecture[6] that would explain why

$$|NC(W,c)| = \mathrm{Cat}(W)$$

along with some other remarkable numerology,
at least for real reflection groups W.

There is a good deal of evidence in its favor,
and evidence that RCA theory can play a role in proving it.

[6] ... from Armstrong-R.-Rhoades "Parking spaces"

A conjecture

One needs the existence of a magical set of polynomials

$$\Theta = (\theta_1, \ldots, \theta_n)$$

inside

$$S := \mathbb{C}[x_1, \ldots, x_n] = \mathrm{Sym}(V^*)$$

having these properties:

1. each θ_i is homogeneous of degree $h + 1$,
2. Θ is a system of parameters, meaning that the quotient $S/(\Theta) = S/(\theta_1, \ldots, \theta_n)$ is finite-dimensional over $\mathbb{C}$,
3. the subspace $\mathbb{C}\theta_1 + \cdots + \mathbb{C}\theta_n$ is a W-stable copy of V^*, so that one can make the map

$$\begin{array}{ccc} V^* & \cong & \mathbb{C}\theta_1 + \cdots + \mathbb{C}\theta_n \\ x_i & \longmapsto & \theta_i \end{array}$$

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Do such magical Θ exist?

Yes, they exist, but it's subtle.
For classical types A_{n-1}, B_n, D_n, there are ad hoc constructions.

Example

For types B_n, D_n, one could take $\Theta = (x_1^{h+1}, \ldots, x_n^{h+1})$.

Example

For type A_{n-1}, Mark Haiman gives an interesting construction in his 1994 paper[a], that works via the prime factorization of n.

[a] "Conjectures on the quotient ring by diagonal invariants"

Why do they exist in general?

For general real reflection groups, RCA theory gives such a Θ: the image of V^* under a map in the BGG-like resolution of $L(\mathbf{1})$:

$$\begin{array}{ccccccc} \cdots \to M(\wedge^2 V^*) \to & M(V^*) & \to & M(\mathbf{1}) & \to L(\mathbf{1}) \to 0 \\ & \| & & \| & \\ & S \otimes_{\mathbb{C}} V^* & \to & S & \end{array}$$

In fact, the quotient $S/(\Theta)$ will again carry a W-representation isomorphic to the W-representation on $Q/(h+1)Q$, or on $L(\mathbf{1})$.

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Resolve the singularity

So $S/(\Theta)$ will have W-fixed space of dimension $\mathrm{Cat}(W)$.
We don't understand geometry of $S/(\Theta)$: it's the coordinate ring for a fat point of multiplicity $(h+1)^n$ at 0 in $V = \mathbb{C}^n$.

Let's try to resolve it, keeping the same W-representation, but hopefully better geometry, namely

$$S/(\Theta - \mathbf{x}) := S/(\theta_1 - x_1, \ldots, \theta_n - x_n)$$

which is the coordinate ring for the zero locus of $(\Theta - \mathbf{x})$, or equivalently, the fixed points V^Θ of this map Θ:

$$\begin{array}{ccc} V & \xrightarrow{\Theta} & V \\ \mathbf{x} = (x_1, \ldots, x_n) & \longmapsto & (\theta_1(\mathbf{x}), \ldots, \theta_n(\mathbf{x})) \end{array}$$

Example: Type B_2

Example

W of type B_2, the 2×2 signed permutation matrices, with

$$\begin{array}{rcl} S & = & \mathbb{C}[\quad x_1, \qquad\qquad x_2 \qquad] \\ S^W & = & \mathbb{C}[\quad x_1^2 + x_2^2, \qquad x_1^2 x_2^2 \qquad] \\ & & \qquad d_1 = 2 \qquad d_2 = 4 = h \end{array}$$

So W-parking spaces have dimension $(h+1)^n = 5^2 = 25$, and their W-fixed spaces have dimension

$$\mathrm{Cat}(W) = \frac{(d_1 + h)(d_2 + h)}{d_1 d_2} = \frac{(2+4)(4+4)}{2 \cdot 4} = 6$$

Example: Type B_2

Example

For W of type B_2, as $h+1=5$, the ad hoc choice of Θ is

$$\begin{aligned} \Theta &= (x_1^5, x_2^5) \\ \Theta - \mathbf{x} &= (x_1^5 - x_1, x_2^5 - x_2) \\ &= (x_1(x_1^4 - 1), x_2(x_2^4 - 1)) \end{aligned}$$

Here V^Θ consists of $(h+1)^n = 5^2$ distinct points in $\mathbb{C}^2$':

$$V^\Theta = \{(x_1, x_2) \text{ with } x_i \text{ in } \{0, +1, +i, -1, -i\}\}.$$

The conjecture

We conjecture this always happens, and
$NC(W,c)$ describes the W-action on these $(h+1)^n$ points.

Conjecture

For any magical Θ, the set V^Θ has these properties:

1. *V^Θ consists of $(h+1)^n$ distinct points, and*
2. *the W-permutation action on V^Θ has its W-orbits $\mathcal{O}_w$ in bijection with elements w of $NC(W,c)$, and*
3. *the W-stabilizers within $\mathcal{O}_w$ are conjugate to the parabolic that pointwise stabilizes V^w.*

Example

Continuing the example of type B_2, where

$$V^{\Theta} = \{(x_1, x_2) : x_i \in \{0, +1, +i, -1, -i\}\}$$

the W-orbits on V^{Θ} are these 6:

		$(0,0)$		
$(\pm 1, 0)$, $(0, \pm 1)$	$(\pm i, 0)$, $(0, \pm i)$		$\pm(1,1)$, $\pm(1,-1)$	$\pm(i,i)$, $\pm(i,-i)$
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The conjecture

Compare the 6 W-orbits on V^{Θ} ...

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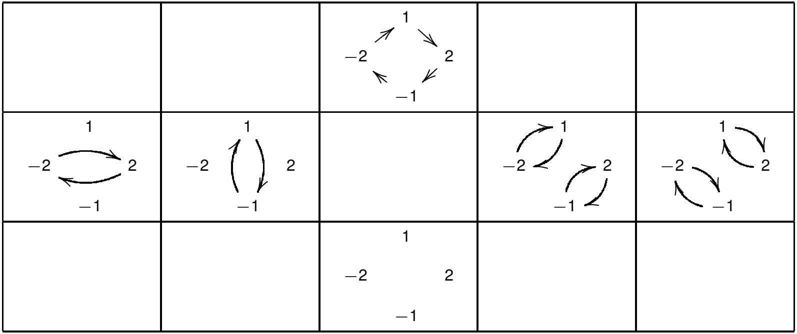

Two remarks on the conjecture

- When Θ comes from RCA theory, Etingof proved for us that, indeed V^{Θ} has $(h+1)^n$ distinct points.

- V^{Θ} actually carries a $W \times C$-permutation action, where $C \cong \mathbb{Z}/h\mathbb{Z}$ acts via scalings $v \mapsto e^{\frac{2\pi i}{h}} v$.

  The full $W \times C$-orbit structure is predicted precisely by the elements of $NC(W) = [e, c]_{\mathrm{abs}}$, where the C-action corresponds to conjugation by c.

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Thanks for listening!