General linear groups as reflection group "wannabes"

Vic Reiner
Univ. of Minnesota

Aisenstadt Lectures
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Three reflection group counting stories where the general linear group GL_n wants in on the game...

TALK 1: Cycling subsets
(today!)

TALK 2: Catalan numbers

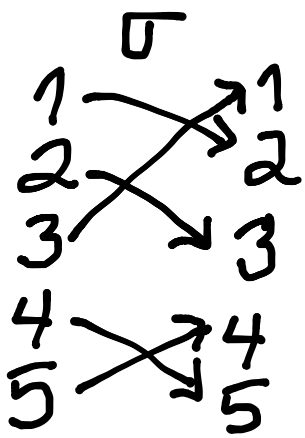
TALK 3: Factorizations into reflections

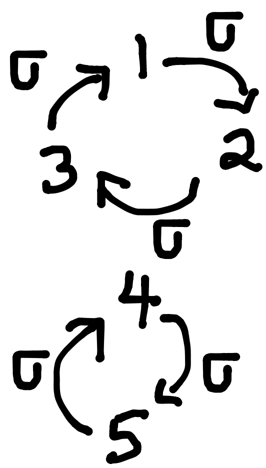
Cycling subsets

Our cycles will live in the symmetric group

$$\mathfrak{S}_n := \{\text{all permutations or bijections } \{1,2,\dots,n\} \to \{1,2,\dots,n\}\}$$

EXAMPLE Here is a σ in $\mathfrak{S}_5$:





directed graph notations

$\sigma = (1,2,3)(4,5)$

cycle notation

An important role will be played by

n-cycles in $\mathfrak{S}_n$, like

$$c_n = \quad n \to 1 \to 2 \to 3 \to \cdots \to n-1 \to n \quad = (1,2,\dots,n-1,n)$$

— and by —

$(n-1)$-cycles in $\mathfrak{S}_n$, like

$$c_{n-1} = \quad n-1 \to 1 \to 2 \to 3 \to \cdots \to n-1 \qquad n \to n \quad = (1,2,\dots,n-1)(n)$$

They also permute the k-element subsets of $\{1,2,\dots,n\}$ in a natural way:

EXAMPLE $k=2$, $n=4$

$c_4=(1,2,3,4)$

$$\{1,2\} \xrightarrow{c_4} \{2,3\} \xrightarrow{c_4} \{3,4\} \xrightarrow{c_4} \{1,4\} \xrightarrow{c_4} \{1,2\}$$

$$\{1,3\} \xrightarrow{c_4} \{2,4\} \xrightarrow{c_4} \{1,3\}$$

$c_3=(1,2,3)(4)$

$$\{1,2\} \xrightarrow{c_3} \{2,3\} \xrightarrow{c_3} \{1,3\} \xrightarrow{c_3} \{1,2\}$$

$$\{1,4\} \xrightarrow{c_3} \{2,4\} \xrightarrow{c_3} \{3,4\} \xrightarrow{c_3} \{1,4\}$$

We know how in total many k-element subsets of $\{1,2,\ldots,n\}$ there are: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

binomial coefficient

where $n! := n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

But do we know their orbit sizes as they are cycled by c_n, or c_{n-1}?

Equivalently, do we know how many k-element subsets are fixed by various powers c_n^d or c_{n-1}^d?

Not 100% obvious, but not hard

THEOREM (R.-Stanton-White 2004)

When $\mathfrak{S}_n$ permutes k-element subsets of $\{1,2,\ldots,n\}$, the number fixed by the d^{th} power c^d of an n-cycle or (n-1)-cycle c is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=\left(e^{\frac{2\pi i}{n}}\right)^d} \quad \text{or} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=\left(e^{\frac{2\pi i}{n-1}}\right)^d}$$

---

where $\begin{bmatrix} n \\ k \end{bmatrix}_q := \dfrac{[n]!_q}{[k]!_q\,[n-k]!_q}$ q-binomial coefficient

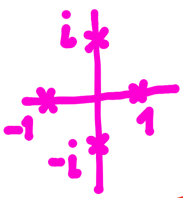
with $[n]!_q := [n]_q [n-1]_q \cdots [2]_q [1]_q$

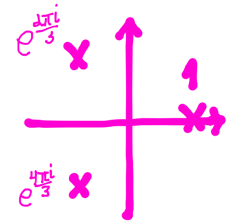
$$[n]_q := \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}$$

EXAMPLE: $\begin{bmatrix}4\\2\end{bmatrix}_q$

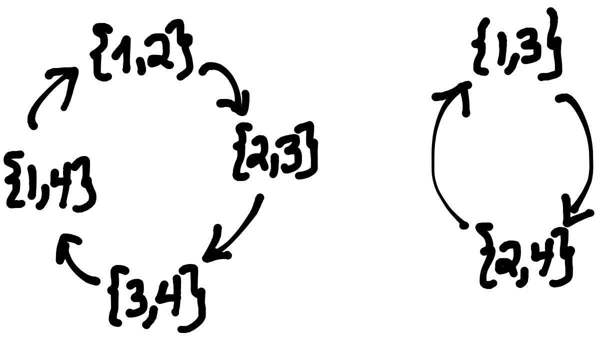
$$= \frac{[4]!_q}{[2]!_q\,[2]!_q} = \frac{[4]_q[3]_q}{[2]_q[1]_q} = (1+q^2)(1+q+q^2)$$

$$= 1+q+2q^2+q^3+q^4$$

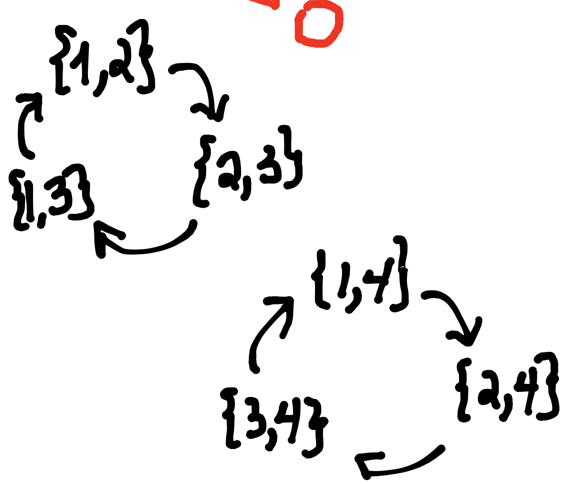




$q=\pm i \rightsquigarrow 0$ $\quad$ $q=-1 \rightsquigarrow 2$ $\quad$ $q=1 \rightsquigarrow 6$ $\quad$ $q=e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}} \rightsquigarrow 0$



$C_4=(1,2,3,4)$



$C_3=(1,2,3)(4)$

We call a situation like this, where

X is a finite set (like k-element subsets of $\{1,2,\ldots,n\}$)

with the action of a cyclic group

$C=\{1,c,c^2,\ldots,c^{m-1}\}$ (like C = powers of n-cycle or $(n-1)$ cycle in $\mathfrak{S}_n$)

and polynomial $X(q)$ in q (like $\begin{bmatrix} n \\ k \end{bmatrix}_q$)

$$\#\{x\in X : c^d(x)=x\} = X(q)\Big|_{q=\left(e^{\frac{2\pi i}{m}}\right)^d}$$

a cyclic sieving phenomenon (CSP).

They show up a lot!

This CSP for $\mathfrak{S}_n$ looked like so much fun that the (finite) general linear group

$$GL_n(\mathbb{F}_q) := \{\text{invertible } n \times n \text{ matrices with entries in } \mathbb{F}_q\}$$

finite field with $q = p^r$ elements

wanted to have one too.

What does $GL_n(\mathbb{F}_q)$ act on?

$$GL_n(\mathbb{F}_q) = \left\{ \mathbb{F}_q^n\text{-linear bijections } \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n \right\}$$

so it acts on

$\{k\text{-dimensional } \mathbb{F}_q\text{-subspaces in } \mathbb{F}_q^n\}$

(the finite Grassmannian)

How many are there in total?

THEOREM (Schubert 1889)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \#\{k\text{-dimensional } \mathbb{F}_q\text{-subspaces of } \mathbb{F}_q^n\}$$

that is, plug in $q = p^r$

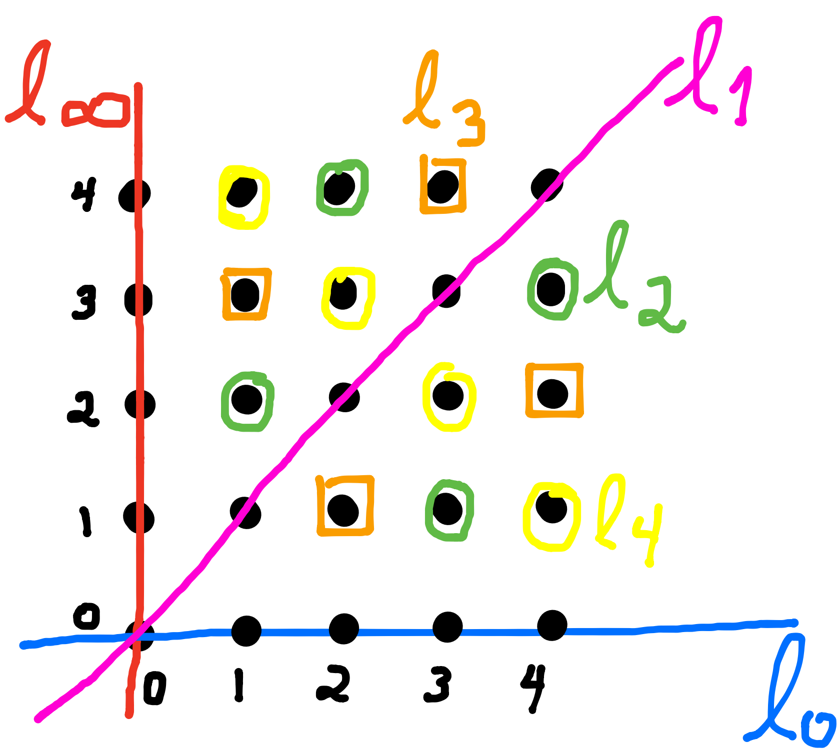
$k=1$:

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{[n]_q}{[1]_q} = 1+q+q^2+\dots+q^{n-1}$$

$$= \#\text{lines through } \bar{0} \text{ in } \mathbb{F}_q^n$$

$n=2$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 1+q \xrightarrow{q=5} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q=5} = 6 \text{ lines in } \mathbb{F}_5^2$$



$\mathbb{F}_5^2$

$\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$
$= \{0,1,2,3,4\}$
modulo 5

Who plays the role of the n-cycles or $(n-1)$-cycles in $\mathfrak{S}_n$ for $GL_n(\mathbb{F}_q)$?

Singer cycles := generators for $\mathbb{F}_{q^n}^\times := \mathbb{F}_{q^n} - \{0\}$ as a cyclic group $= \{1, c, c^2, c^3, \ldots, c^{q^n-2}\}$ of order $q^n - 1$

$\mathbb{F}_{q^n}^\times$ acts invertibly and $\mathbb{F}_q$-linearly on $\mathbb{F}_{q^n} \cong (\mathbb{F}_q)^n$,

i.e. $\mathbb{F}_{q^n}^\times \subset GL_n(\mathbb{F}_q)$

EXAMPLE $\mathbb{F}_{5^2} \cong \mathbb{F}_5[\alpha]/(\alpha^2+\alpha+1)$

$\mathbb{F}_{5^2}^\times$ is cyclic of order $5^2-1=24$.

α has multiplicative order 3 in $\mathbb{F}_{5^2}^\times$

- not Singer

$\alpha+1$ has multiplicative order 6 in $\mathbb{F}_{5^2}^\times$

- not Singer

$\alpha+2$ has multiplicative order 24 in $\mathbb{F}_{5^2}^\times$

- yes, Singer!

$\alpha+2$ acts on $\mathbb{F}_{5^2}$ in the $\mathbb{F}_5$-basis $\{1,\alpha\}$

via $\begin{array}{c|cc} & 1 & \alpha \\ \hline 1 & 2 & -1 \\ \alpha & 1 & 1 \end{array}$, i.e. $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \in GL_2(\mathbb{F}_5)$

$\leftarrow$ a Singer cycle c

Recall...

THEOREM (R.-Stanton-White 2004)

When $\mathfrak{S}_n$ permutes k-element subsets of $\{1,2,\ldots,n\}$, the number fixed by the d^{th} power c^d of an n-cycle or (n-1)-cycle c is

$$\left.\begin{bmatrix} n \\ k \end{bmatrix}_q\right|_{q=\left(e^{\frac{2\pi i}{n}}\right)^d} \text{ or } \left.\begin{bmatrix} n \\ k \end{bmatrix}_q\right|_{q=\left(e^{\frac{2\pi i}{n-1}}\right)^d}$$

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where $\begin{bmatrix} n \\ k \end{bmatrix}_q := \dfrac{[n]!_q}{[k]!_q[n-k]!_q}$ q-binomial coefficient

with $[n]!_q := [n]_q[n-1]_q \cdots [2]_q[1]_q$

$$[n]_q := \frac{1-q^n}{1-q} = 1+q+q^2+\ldots+q^{n-1}$$

THEOREM (R.-Stanton-White 2004)

When $GL_n(\mathbb{F}_q)$ permutes k-dimensional subspaces of $\mathbb{F}_q^n$, the number fixed by the d^{th} power c^d of a Singer cycle c is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} \Bigg|_{t=\left(e^{\frac{2\pi i}{q^n-1}}\right)^d}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} := \dfrac{n!_{q,t}}{k!_{q,t}\,(n-k)!_{q,t^{q^k}}}$ (q,t)-binomial coefficient

$$n!_{q,t} := \left(1-t^{q^n-q^0}\right)\left(1-t^{q^n-q^1}\right)\cdots\left(1-t^{q^n-q^{n-1}}\right)$$

Is $\begin{bmatrix} n \\ k \end{bmatrix}_{q,t}$ a polynomial in t, not just a rational function? Yes, even with nonnegative integer coefficients !

EXAMPLE

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q=2,t} = \frac{4!_{2,t}}{2!_{2,t} \cdot 2!_{2,t^{2^2}}}$$

$$= \frac{(1-t^{2^4-2^0})(1-t^{2^4-2^1})}{(1-t^{2^2-2^0})(1-t^{2^2-2^1})} = \frac{(1-t^{15})(1-t^{14})}{(1-t^{3})(1-t^{2})}$$

$$= (1+t^3+t^6+t^9+t^{12})(1+t^2+t^4+t^6+t^8+t^{10}+t^{12})$$

Where do reflection groups play any role in the above?

First let's define them...

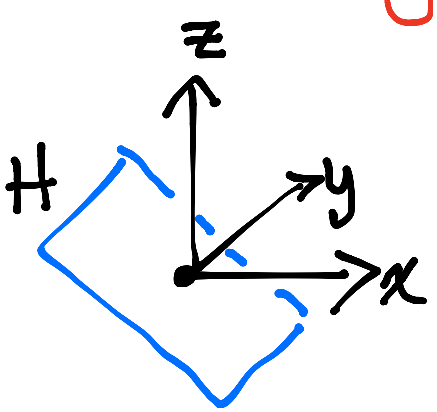
DEFINITION:

A reflection t in $GL_n(\mathbb{F})$ for any field $\mathbb{F}$ $(\mathbb{R}, \mathbb{C}, \mathbb{F}_q, \text{etc})$ is an element whose fixed space

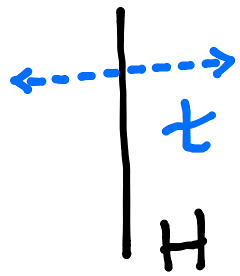
$$(\mathbb{F}^n)^t := \{v \in \mathbb{F}^n : t(v) = v\}$$

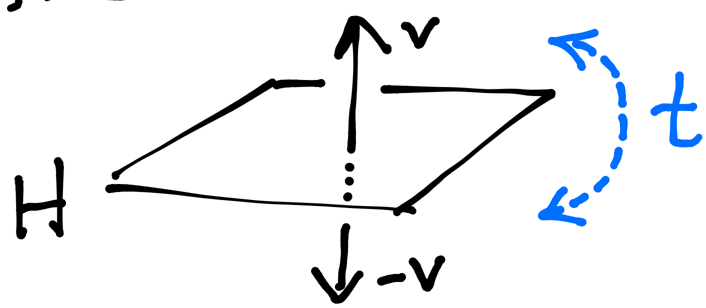
is a hyperplane H

(n-1)-dimensional linear subspace



- Euclidean reflections





- Unitary reflections

$$t = \begin{bmatrix} e^{\frac{2\pi i}{d}} & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix}$$

- Transvections

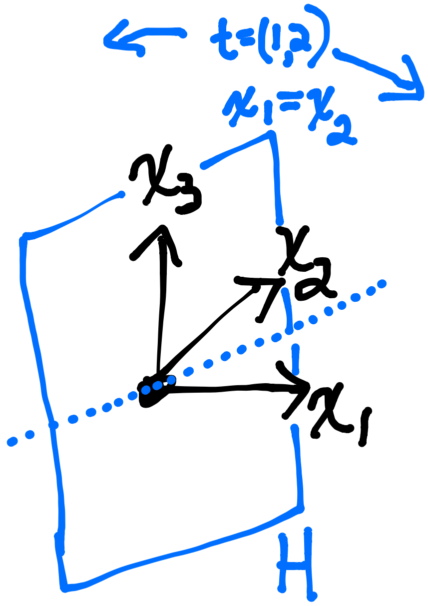
$$t = \begin{bmatrix} \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

- Infinite order is OK!

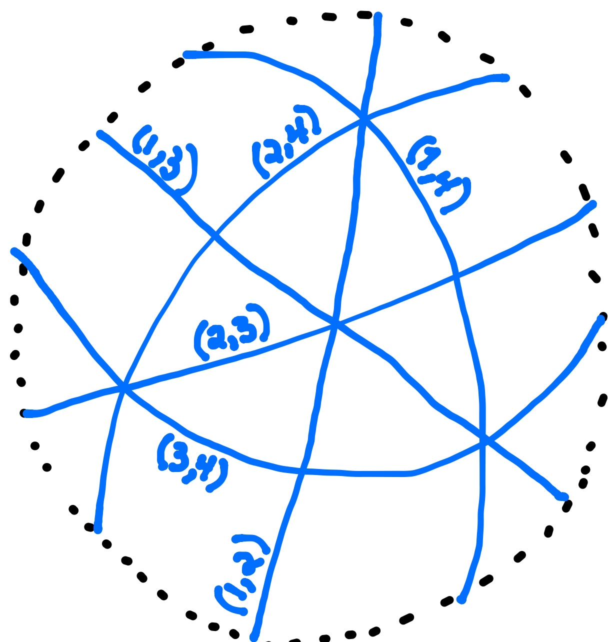
$$t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R})$$

A reflection group is a subgroup G of $GL_n(\mathbb{F})$ generated by reflections.

EXAMPLE $\mathfrak{S}_n \subset GL_n(\mathbb{R})$ is generated by transpositions (i,j)
= reflection through hyperplane $\{x_i = x_j\}$



$\mathfrak{S}_3$



$\mathfrak{S}_4$ (acting inside $x_1+x_2+x_3+x_4=0$)

WARNING!

A (non-standard) DEFINITION:

Call a finite subgroup G of $GL_n(\mathbb{F})$ a *finite reflection group* if, when G acts by linear substitutions $\bar{x} \mapsto g\bar{x}$ on the polynomial ring

$$S := \mathbb{F}[x_1, \dots, x_n],$$

its *G-invariant subalgebra*

$$S^G = \{f(\bar{x}) \in S : f(g\bar{x}) = f(\bar{x}) \text{ for all } g \text{ in } G\}$$

is again a *polynomial* ring

$$S^G = \mathbb{F}[f_1, f_2, \dots, f_n].$$

This is very special!

EXAMPLE (classical, Newton)

$\mathfrak{S}_n$ permutes the variables $x_1, x_2, \ldots, x_n$ in $S = \mathbb{F}[x_1, \ldots, x_n]$

with invariant (symmetric) polynomials

$$S^{\mathfrak{S}_n} = \mathbb{F}[e_1(\bar{x}), e_2(\bar{x}), \ldots, e_n(\bar{x})]$$

where $e_i(\bar{x})$ are elementary symmetric polynomials

$$e_1(\bar{x}) = x_1 + x_2 + \ldots + x_n$$
$$e_2(\bar{x}) = x_1x_2 + x_1x_3 + x_2x_3 + \ldots + x_{n-1}x_n$$
$$\vdots$$
$$e_n(\bar{x}) = x_1x_2 \cdots x_n$$

REMARKS:

- **THEOREM** (Serre 1967): Finite subgroups G of $GL_n(\mathbb{F})$ with S^G polynomial are necessarily generated by reflections.

- **THEOREM** (Shephard-Todd, Chevalley 1955): When $\mathbb{F}$ has characteristic zero $(\mathbb{R}, \mathbb{C}, \dots)$ a finite subgroup G of $GL_n(\mathbb{F})$ has S^G polynomial if and only if G is generated by reflections.

THEOREM (R.-Stanton-White 2004 $\underset{=0}{\mathrm{char}(\mathbb{F})}$
Broer-R.-Smith-Webb 2011)

For W a finite reflection group in $GL_n(\mathbb{F})$ transitively permuting a set $X = W/W'$, and $w \in W$ of order m which is regular in the sense of Springer 1974 (w has an eigenvector in $\mathbb{F}^n$ fixed by no reflections of W), then w fixes exactly $X(t)\big|_{t=e^{\frac{2\pi i}{m}}}$ elements of X

where $$X(t) = \frac{\mathrm{Hilb}(S^{W'}, t)}{\mathrm{Hilb}(S^{W}, t)}$$

DEFN: A graded $\mathbb{F}$-vectorspace
$R = \bigoplus_{d=0}^{\infty} R_d$ has **Hilbert series**

$$\mathrm{Hilb}(R,t) := \sum_{d=0}^{\infty} \dim_{\mathbb{F}} R_d \cdot t^d$$

EXAMPLE

$\mathfrak{S}_n$ acting on $S = \mathbb{F}[x_1, \ldots, x_n]$
has $S^{\mathfrak{S}_n} = \mathbb{F}[e_1, e_2, \ldots, e_n]$
and degree of e_i is i, so that

$$\mathrm{Hilb}(S^{\mathfrak{S}_n}, t)$$
$$= (1+t+t^2+t^3+\ldots)(1+t^2+t^4+t^6+\ldots)\cdots(1+t^n+t^{2n}+t^{3n}+\ldots)$$

$$= \frac{1}{1-t^1}\frac{1}{1-t^2}\cdots\frac{1}{1-t^n} = \prod_{i=1}^{n}\frac{1}{1-t^i}$$

Meanwhile, $W = \mathfrak{S}_n$ permutes

$$X = \{k\text{-subsets of } \{1,2,\ldots,n\}\}$$

$$= \mathfrak{S}_n / \mathfrak{S}_k \times \mathfrak{S}_{n-k} = W/W'$$

with $S^{W'} = \mathbb{F}[e_1(x_1,\ldots,x_k),\ldots,e_k(x_1,\ldots,x_k),$
$e_1(x_{k+1},\ldots,x_n),\ldots,e_{n-k}(x_{k+1},\ldots,x_n)]$

$\Rightarrow$

$$X(t) = \frac{\mathrm{Hilb}(S^{W'},t)}{\mathrm{Hilb}(S^{W},t)} = \frac{(1-t^1)(1-t^2)\cdots(1-t^n)}{(1-t^1)\cdots(1-t^k)\cdot(1-t^1)\cdots(1-t^{n-k})}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix}_t$$

Furthermore, the Springer regular elements w inside $W = \mathfrak{S}_n$ are exactly the powers of n-cycles and $(n-1)$-cycles:

— $c = (1, 2, \ldots, n)$ has eigenvector $(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$ if $\zeta = e^{\frac{2\pi i}{n}}$ avoiding all hyperplanes $x_i = x_j$

— $c = (1, 2, \ldots, n-1)(n)$ has eigenvector $(1, \zeta, \zeta^2, \ldots, \zeta^{n-2}, 0)$ if $\zeta = e^{\frac{2\pi i}{n-1}}$ similarly avoiding all $x_i = x_j$

Certainly $GL_n(\mathbb{F}_q)$ is finite, but is it a finite reflection group? Yes.

THEOREM (L.E. Dickson 1911):

$S=\mathbb{F}_q[x_1,\ldots,x_n]$ has

$S^{GL_n(\mathbb{F}_q)}=\mathbb{F}_q[f_1,f_2,\ldots,f_n]$ where

$$\prod_{\substack{(c_1,\ldots,c_n)\\ \in\mathbb{F}_q^n}}\left(t+(c_1x_1+\cdots+c_nx_n)\right)=t^{q^n}+f_1(\underline{x})t^{q^{n-1}}+f_2(\underline{x})t^{q^{n-2}}+\cdots+f_{n-1}(\underline{x})t^{q^1}+f_n(\underline{x})t^{q^0}$$

Degree of f_i is q^n-q^{n-i} so that

$$\mathrm{Hilb}\left(S^{GL_n(\mathbb{F}_q)},t\right)=\frac{1}{1-t^{q^n-q^0}}\,\frac{1}{1-t^{q^n-q^1}}\cdots\frac{1}{1-t^{q^n-q^{n-1}}}=\frac{1}{n!_{q,t}}$$

Meanwhile, $GL_n(\mathbb{F}_q)$ permutes

X = k-dimensional subspaces of $\mathbb{F}_q^n$

$= GL_n(\mathbb{F}_q) / P_k$

where $P_k = \left\{ \begin{array}{cc} \overbrace{\hphantom{*}}^{k} & \overbrace{\hphantom{*}}^{n-k} \\ \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] & \end{array} \right\}$

THEOREM (Kuhn and Mitchell 1984)

$$X(t) := \frac{\mathrm{Hilb}(S^{P_k}, t)}{\mathrm{Hilb}(S^{GL_n(\mathbb{F}_q)}, t)}$$

$$= \frac{n!_{q,t}}{k!_{q,t}\,(n-k)!_{q,t^{q^k}}} = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t}$$

Who are Springer regular elements w inside $W = GL_n(\mathbb{F}_q)$?

That is, which w have an $\mathbb{F}_q^n$ eigenvector avoiding all $\mathbb{F}_q$-hyperplanes?

PROPOSITION: (R.-Stanton-Webb 2006)

They are exactly the powers of Singer cycles, that is, elements of $\mathbb{F}_{q^n}^\times$ embedded inside $GL_n(\mathbb{F}_q)$.

COROLLARY:

- $W = \mathfrak{S}_n$ permuting $\{k\text{-subsets}\}$, $w = c^d$ for c an m-cycle with $m = n$ or $n-1$

$$\Rightarrow w \text{ fixes } \underbrace{\begin{bmatrix} n \\ k \end{bmatrix}_q \Big|_{q = \left(e^{\frac{2\pi i}{m}}\right)^d}}_{}$$
k-subsets

- $W = GL_n(\mathbb{F}_q)$ permuting $\{k\text{-subspaces}\}$, $w = c^d$ for c a Singer cycle,

$$\Rightarrow w \text{ fixes } \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} \Big|_{t = \left(e^{\frac{2\pi i}{q^n - 1}}\right)^d}$$
k-subspaces

REMARK:

$GL_n(\mathbb{F}_q)$ is already behaving here more like the real reflection groups

- $W = W(B_n) = \mathfrak{S}_n^{\pm}$
  = hyperoctahedral group of all $n \times n$ signed permutations

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

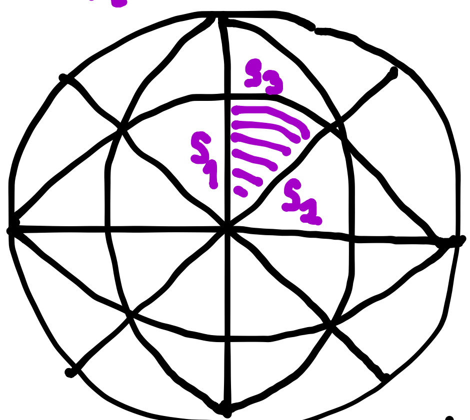
- $W = W(D_n)$
  = its index 2 subgroup with evenly many -1's

Both also have

$$\left\{ \begin{matrix} \text{Springer's} \\ \text{regular elements} \end{matrix} \right\} = \left\{ \begin{matrix} \text{powers of} \\ \text{Coxeter elements} \end{matrix} \right\}$$

What's a Coxeter element c in a finite reflection group $W \le GL_n(\mathbb{R})$?

- c conjugate to $s_1 s_2 \cdots s_n$ where $W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} \rangle$



$s_1 \overset{4}{—} s_2 \overset{3}{—} s_3$

- a regular element c with eigenvalue $e^{2\pi i/h}$ where

$h :=$ Coxeter number $=$ max degree $d_i = \deg(f_i)$

if $S^W = \mathbb{F}[f_1, f_2, \ldots, f_n]$

EXTRA for EXPERTS: the proof idea

Springer 1974: (char($\mathbb{F}$)=0)

$$\mathbb{F}[W] \underset{\substack{\uparrow \\ \text{as rep'ns} \\ \text{of } W\times C \\ \| \\ \langle w\rangle}}{\cong} S/(f_1,\ldots,f_n)$$

$$= S/(S_+^W)$$

{ take W'-fixed spaces

$$\mathbb{F}[W/W'] \underset{\substack{\uparrow \\ \text{as rep'ns of } C}}{\cong} S^{W'}/(f_1,\ldots,f_n)$$

Compare trace of w on left, right:

$$\#\{x\in W/W' : w(x)=x\} = X(t)\Big|_{t=e^{\frac{2\pi i}{m}}}$$

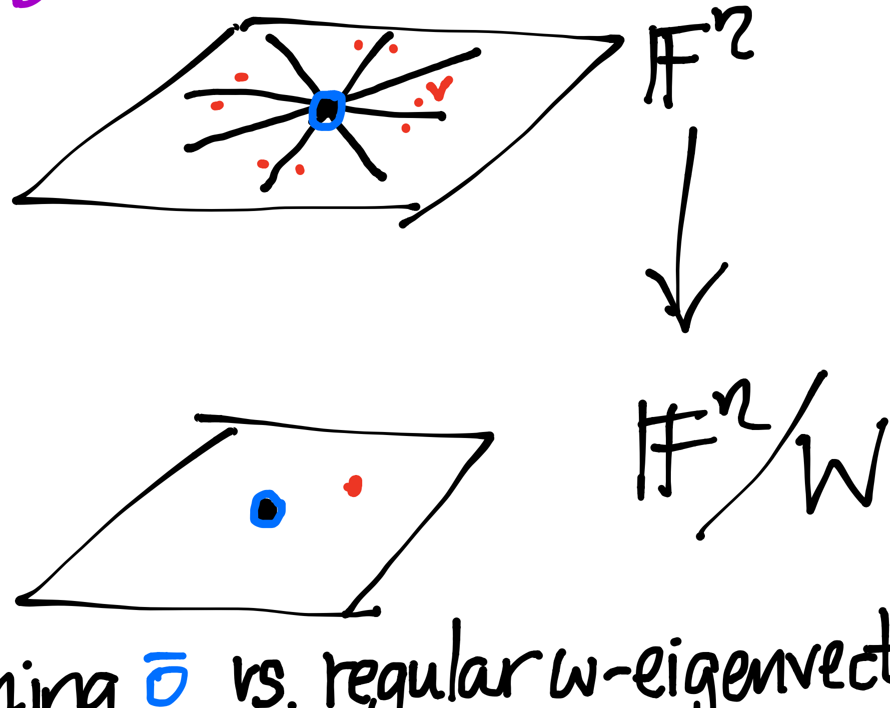
When char$(\mathbb{F})=0$,

$$X(t) := \frac{\mathrm{Hilb}(S^W, t)}{\mathrm{Hilb}(S^{W'}, t)} = \mathrm{Hilb}\left(S^W / (S^{W'}_+), t\right)$$

When char$(\mathbb{F})=p>0$, reinterpret

$$X(t) = \sum_{i \geq 0} (-1)^i \,\mathrm{Hilb}\left(\mathrm{Tor}_i^{S^{W'}}(S^W, \mathbb{F}), t\right)$$

and compare scheme-theoretic fibers of
(Broer)



containing $\bar{0}$ vs. regular w-eigenvector

THESIS (to be continued...)

$GL_n(\mathbb{F}_q)$ likes to pretend it is a real reflection group with

- Coxeter number $h = q^n - 1$.
- Coxeter elements = Singer cycles

e.g.

$$S^{GL_n(\mathbb{F}_q)} = \mathbb{F}_q[f_1, \ldots, f_{n-1}, f_n]$$

with degrees $q^n - q^{n-1}, \ldots, q^n - q, \underset{\underset{\underset{\text{order of all Singer cycles}}{\|}}{\underset{h}{\|}}}{q^n - 1}$

Thanks

for your attention,

and again,

thanks to C.R.M.!