Topology of Augmented Bergman Complexes

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Topological Methods in Discrete Math
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1. Review matroids $M$
   - independent sets $I(M)$
   - flats $F(M)$

2. Shellability

3. Augmented Bergman complex $\Delta M$

4. Two kinds of shellings of $\Delta M$
   and corollaries
1. **Review matroids \( M \)**

A matroid \( M \) of rank \( r \) on ground set \( E = \{1,2,\ldots,n\} \) abstracts vectors \( v_1, v_2, \ldots, v_n \) spanning an \( r \)-dimensional vector space \( V \) over some field \( k \).

**EXAMPLE**

\( n=4 \)

\( k=\mathbb{R} \)

\( r=2 \)

\( V=\mathbb{R}^2 \)

\( \vdots \)

\( \vdots \)

\( \rightarrow \) a (representable) matroid \( M \), specified either by its independent sets \( \mathcal{I}(M) \) or flats \( \mathcal{F}(M) \).
**independent sets**

\[ I(M) := \{ I \subseteq \{1, 2, \ldots, n\} : \{v_i\}_{i \in I} \text{ are linearly independent} \} \]

\[ \mapsto I(M) = \{ \emptyset, 1, 12, 2, 13, 3, 14, 4, 23, 24 \} \]

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**flats**

\[ F(M) := \{ F \subseteq \{1, 2, \ldots, n\} : \{v_i\}_{i \in F} = W \cap \{v_1, v_2, \ldots, v_n\} \text{ for some subspace } W \text{ of } V \} \]

\[ \mapsto F(M) = \{ \emptyset, 1, 2, 3, 4, 1234 \} \]

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*the poset \( F(M) \) ordered via inclusion*
I(M) satisfies these independent set axioms:

(II) \( \emptyset \in I(M) \)

(I1) \( I \subseteq J \) and \( J \in I(M) \) \( \Rightarrow I \in I(M) \)

(I2) \( I, J \in I(M) \) and \( |I| < |J| \)

\( \Rightarrow \exists j \in J \setminus I \) with \( I \cup \{j\} \in I(M) \)

F(M) satisfies these flat axioms

(F0) \( E = \{1, 2, \ldots, n\} \in F(M) \)

(F1) \( F, G \in F(M) \) \( \Rightarrow F \cap G \in F(M) \)

(F2) \( F \in F(M) \) and \( i \in E \setminus F \)

\( \Rightarrow \exists! G \in F(M) \) covering \( F \) with \( i \in G \).
Not all matroids $M$ are representable by vectors $v_1, v_2, \ldots, v_n$.

**EXAMPLE** The non-Pappus matroid $M$ on $E = \{1, 2, \ldots, 9\}$ of rank 3 has

$I(M) = \{\text{all } I \subseteq \{1, 2, \ldots, 9\} \text{ with } |I| \leq 3, \text{ except the collinear triples shown}\}$

$789 \in I(M)$ violates Pappus's Theorem.
Axioms (I0), (I1) say $\mathcal{I}(M)$ is an abstract simplicial complex on vertices \{y_1, y_2, \ldots, y_n\}.

Axiom (I2) implies that all inclusion-maximal independent sets, called the bases $\mathcal{B}(M)$, have the same cardinality, called the \textbf{rank} $r(M)$.

In other words, $\mathcal{I}(M)$ is a pure simplicial complex of dimension $r(M) - 1$. 

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\( (\text{I0}) \) $\emptyset \in \mathcal{I}(M)$

\( (\text{I1}) \) $I \in \mathcal{J}$ and $J \in \mathcal{I}(M) \Rightarrow I \in \mathcal{I}(M)$

\( (\text{I2}) \) $I, J \in \mathcal{I}(M)$ and $\#I < \#J$

$\Rightarrow \exists j \in J \setminus I$ with $I \cup \{j\} \in \mathcal{I}(M)$
(F0) \[ E = \{1, 2, \ldots, n\} \in F(M) \]

(F1) \[ F, G \in F(M) \Rightarrow F \cap G \in F(M) \]

(F2) \[ F \in F(M) \text{ and } i \in E \setminus F \Rightarrow \exists! \ G \in F(M) \text{ covering } F \text{ with } i \in G. \]

\[(F0), (F1) \Rightarrow \text{ the poset } F(M) \text{ is a lattice, with } F \cap G = F \cap G.\]

\[(F2) \Rightarrow F(M) \text{ is actually a geometric lattice.}\]

atomic + upper semimodular
2. **Shellability**

**Def'n:** A pure \((r-1)\)-dimensional simplicial complex \(\Delta\) is shellable if we can order its facets \(\phi_1, \phi_2, \ldots, \phi_t\) in a shelling order:

\[
\forall j \geq 2, \quad \phi_j \text{ intersects the subcomplex generated by } \phi_1, \phi_2, \ldots, \phi_{j-1} \text{ in a pure } (r-2)\text{-dim'l subcomplex}
\]
Shelling determines the homotopy type of $\Delta$

**DEF'N:** Call $\phi_j$ a homology facet in the shelling $\phi_1, \phi_2, \ldots, \phi_k$ if $\phi_j$ intersects the subcomplex generated by $\phi_1, \phi_2, \ldots, \phi_{j-1}$ in the entire boundary $Bd \phi_j$.

**Proposition:** When $\Delta$ is pure $d$-dimensional and shellable, then

$$\|\Delta\| \cong S^d \vee S^d \vee \ldots \vee S^d$$

is homotopy equivalent to a $\beta$-fold 1-point wedge of $d$-spheres $S^d$,

where $\beta := \#$ of homology facets $\phi_j$ in any shelling order.
In fact, whenever $\Delta$ is shellable,

then $\Delta' := \Delta - \{\text{all homology facets } \phi_j\}$ is contractible.
THEOREM (Provan-Billera 1980) For a matroid $M$, the independent set complex $\mathcal{I}(M)$ is shellable, via lexicographic order on the bases $\mathcal{B}(M)$.

Furthermore, the number of homology facets is

$$\beta = T_M(0, 1) = \text{Tutte polynomial } T_M(x, y) \text{ evaluated at } x=0, y=1 = \# \text{ bases } B \in \mathcal{B}(M) \text{ of internal activity zero}$$

**COROLLARY:** $|\mathcal{I}(M)| \approx S_1^{r(n)-1} \vee \ldots \vee S_1^{r(n)-1}$

$$T_M(0, 1)$$-fold wedge
The flats $F(M)$ as a poset $P$ gives us another simplicial complex, the order complex $\Delta P := \text{simplicial complex with vertex set } \{x_p \mid p \in P\}$ and simplices/faces the totally ordered subsets $\{x_{p_1}, x_{p_2}, \ldots, x_{p_k}\}$ if $p_1 < p_2 < \ldots < p_k$ in $P$.
Theorem (Garsia 1980) For a matroid $M$, all three of
\[
\begin{cases}
\Delta F(M) \\
\Delta (F(M) - \{E\}) \\
\Delta (F(M) - \{\emptyset, E\}) =: \Delta_M
\end{cases}
\]
are shellable, via lexicographic order on the edge-label sequences on maximal chains $\emptyset \subset F_1 \subset F_2 \subset F_3 \subset \ldots \subset F_{r(n)-1} \subset E$ in $F(M)$

\[\text{edge-labels: } (\min_{F_1}, \min_{F_2-F_1}, \min_{F_3-F_2}, \ldots, \min_{E-F_{r(n)-1}})\]
Furthermore, the number of homology facets is

$$\beta = T_M(1,0) = \text{Tutte polynomial } T_M(x,y) \text{ evaluated at } x=1, y=0$$

$$= \# \text{ bases } B \in B(M) \text{ of external activity zero}$$

**Corollary:**

$$|\Delta(M)| \approx S^{r(M)-2} \vee ... \vee S^{r(M)-2}$$

Bergman complex

$$\Delta_m := \Delta(F(M) - \{\bar{\phi}, E\})$$
Augmented Bergman complex $\Delta_M$

In a monumental pair of 2020 papers, Braden-Huh-Matherne-Proudfoot-Wang introduced a hybrid.

**DEF'N:** The augmented Bergman complex $\Delta_M$ has vertex set $\{y_1, y_2, \ldots, y_n\} \cup \{x_F\}_{\emptyset \subseteq F \subsetneq E}$ (proper flats $F \in \mathcal{F}(M)$).

with simplices/faces $\{y_i\}_{i \in I} \cup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\}$

when

- $I \subseteq \mathcal{I}(M)$ is independent
- $F_1, F_2, \ldots, F_\ell$ are proper flats
- $I \subseteq F_1 \subset F_2 \subset \ldots \subset F_\ell$ ($\neq E$)
$\Delta_m$ is pure of dimension $r(M)-1$, containing both $I(M)$ and $\text{Cone}(\Delta_m)$ as subcomplexes:
SPECIAL CASE: Boolean matroid $M$ of rank $n$

$I(M) = (n-1)$-simplex

$2^{\{1,2,\ldots,n\}}$

$\text{Cone } (\Delta_n) = \text{barycentric subdivision of } (n-1)$-simplex

$\Delta_M = \text{boundary of stellohedron}$

$n=2$

$\begin{align*}
\text{y}_1 & \quad \text{y}_2 \\
\end{align*}$

$n=3$

$\begin{align*}
\text{y}_1 & \quad \text{y}_2 & \quad \text{y}_3 \\
\end{align*}$
Why did BHMPW introduce $\Delta n$?

Its Stanley-Reisner ring has an amazing Artinian quotient by certain linear forms

\[ CH(M) = \prod \{ y_1, \ldots, y_n, x_F \} \text{ flats} \text{F \& F} \]

\[ \chi_F y_i, \text{ F \& F} \]

\[ y_i x_F, \text{ i \& F} \]

\[ y_i - \sum \frac{x_F}{i \& F}, i = 1, 2, \ldots, n \]

in which the $y_1, \ldots, y_n$ generate a subalgebra $H(M)$ = graded M"obius algebra of $M$.

with a crucial $H(M)$-submodule $IH(M)$ = intersection cohomology of $M$ and remarkable properties...
• $H(M) \leftrightarrow IH(M) \leftrightarrow CH(M)$  
  these satisfy Kähler package

• Hilbert series for $H(M)$ interprets rank sizes $W_k$ of $F(M)$
  and Kähler package for $IH(M)$ $\Rightarrow$ Dowling-Wilson's
  Top Heavy Conj. (1974)

• Hilbert series for $IH(M)$ interprets $Z$-polynomial for $M$
  and Kähler package for $IH(M)$ $\Rightarrow$ unimodality for $Z$

• Hilbert series for $IH(M)/(y_1, ..., y_n) IH(M)$
  interprets Kazhdan-Lustzig polynomial for $M$
  $\Rightarrow$ nonnegativity of K-L polynomials.
They used this weaker property of $\Delta M$ than shellability:

**Proposition:** For any matroid $M$, $\Delta M$ is gallery-connected, that is, any two facets $\Phi, \Phi'$ are connected by a gallery of facets $\Phi = \Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_{t-1}, \Phi_t = \Phi'$ with each $\Phi_i \cap \Phi_{i+1}$ of dimension $r(M) - 2$ ($= \text{codimension } 1$).
4. Two kinds of shellings of $\Delta M$ and corollaries

**THEOREM (UMN REU 2021)** For any matroid $M$, the augmented Bergman complex has two families of shellings:

(i) some that shell the facets of $\text{Cone}(\Delta M)$ first, and facets of $\mathcal{I}(M)$ last.

(ii) some that shell the facets of $\mathcal{I}(M)$ first, and facets of $\text{Cone}(\Delta M)$ last.
Type (i) shellings

Type (ii) shellings

\[ \text{Cone}(\Delta_M) \]
COROLLARY: The augmented Bergman complex $\Delta_M$ has $\|\Delta_M\| \approx \underbrace{S^{r(M)-1} \vee \ldots \vee S^{r(M)-1}}_{\beta\text{-fold wedge}}$

where $\beta$ has two expressions:

(i) $\beta = T_M(1,1) = \#B(M)$

because the homology facets in type (i) shellings are $\{y_i\}_{i \in B}$ indexed by bases $B$ of $M$.

(ii) $\beta = \sum_{\text{flats } F \in F(M)} T_{M/F}(0,1) T_{M/F}(1,0)$

counting type (ii) shelling homology facets.
REMARK: The equality

\[ T_m(1,1) = \sum_{\text{flats } F} T_{m|F}(0,1) T_{m/F}(1,0) \]

appeared in work of Étienne-Las Vergnas 1998, rediscovered in Kook-R.-Stanton 2000, and is a specialization of a convolution formula

\[ T_m(x,y) = \sum_{\text{flats } F} T_{m|F}(0,y) T_{m/F}(x,0) \]

for Tutte polynomials.
The type (i) shellings show contractibility of
\[ \Delta' = \Delta_m - \{ \text{facets } \{ y_i \mid i \in B : \text{bases } B \in B(M) \} \}\]

Since matroid automorphisms set-wise stabilize the collection of basis facets, one can conclude:

**COROLLARY:** The group \( \text{Aut}(M) \) acts on
\[ H_{r(m)-1}(\Delta_m, \mathbb{Z}) \]
as a signed permutation representation, same as on
\[ C_{r(m)-1}(X(M), \mathbb{Z}) : \]
\[ \sigma(\begin{bmatrix} b_1, b_2, \ldots, b_r \end{bmatrix}) = \begin{bmatrix} b_{\sigma(1)}, \ldots, b_{\sigma(r)} \end{bmatrix} \]
for bases \( B = \{ b_1, \ldots, b_r \} \in B(M) \)

**oriented simplex**
\[ \Delta' = \Delta_M - \{ \text{bases} \} \]

is contractible

\[ \text{Aut}(M) = \{ e, (12), (34), (12)(34) \} \]

\[ H_1(\Delta_M) = \mathbb{Z}^5 \]

\[ (12) = -1 \]

\[ [y_1, y_3] \xrightarrow{(34)} [y_1, y_4] \]

\[ (34) = +1 \]

\[ [y_2, y_3] \xrightarrow{(12)} [y_2, y_4] \]
REMARK: Neither $\text{Im}_M$ nor $\Delta_M$ have simple descriptions for their homology representations in general. Notable special cases:

<table>
<thead>
<tr>
<th></th>
<th>$H_{r(m)-1}(\text{Im})$</th>
<th>$H_{r(m)-2}(\Delta_M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boolean</td>
<td>trivial rep of $S_n$</td>
<td>sign rep of $S_n$</td>
</tr>
<tr>
<td>$q$-Boolean = $F_q$-vector space</td>
<td>known virtually, not so explicit</td>
<td>Steinberg rep of $G_n(F_q)$</td>
</tr>
<tr>
<td>braid arrangement = complete graphic</td>
<td>an $S_n$-rep that restricts nicely to $S_{n-1}$ (Kook 1996)</td>
<td>Lie rep of $S_n$</td>
</tr>
</tbody>
</table>
Thanks for your attention!
(Extra pages)
REMARK: Can generalize $\text{Aut}(M)$-rep description to arbitrary closure operators $2^E \xrightarrow{f} 2^E$.

defining indep. sets $I : f(I \setminus \{i\}) \supseteq f(I)$ $\forall i \in I$.

bases $B : B$ indep. and $f(B) = E$.

defined $F : f(F) = F$.

and augmented Bergman complex $\Delta_f$.

with vertices $\{y_1, \ldots, y_n\} \cup \{x_F\}$ proper flats $F \subseteq E$.

simplices $\{y_i\} \cup \{x_F, \ldots, x_{Fl}\}$

with

- $I$ indep.
- $F_1, \ldots, F_l$ flats
- $I \subseteq F_1 \subset \ldots \subset F_l$.
If $\Delta f$, $\Delta f$, $\Delta f$ are not shellable in general.

Nevertheless:

**Theorem** (UMN REU 2021) \( \|\Delta f\| \cong \bigvee_{\text{bases } B} \# B - 1 \)

and Aut(M) acts on $H_c(\Delta f)$ as a signed permutation rep on oriented chains \([b_1, b_2, \ldots, b_r]\) indexed by bases $B = \{b_1, \ldots, b_r\}$.

Again \( \Delta' = \Delta f - \{ \{y_i\}_{i \in B} : B \text{ a basis} \} \) is contractible.