Topology of Augmented Bergman complexes

arXiv: 2108.13394

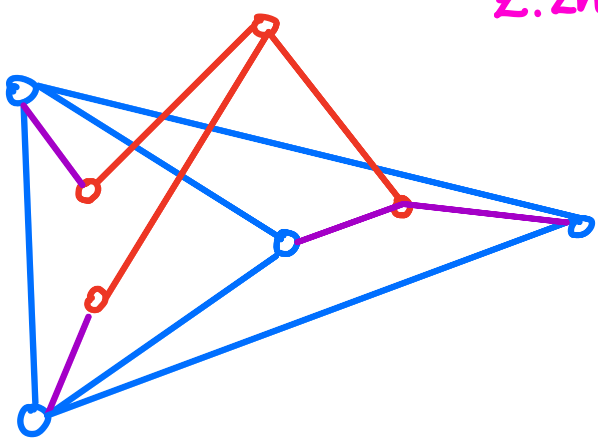
E. Bullock
A. Kelley
K. Ren
G. Sherry
D. Shen
B. Sun
A. Tao
Z. Zhang

UMN Summer 2021 REU students

+ V. Reiner (REU mentor)

+ T. Karn
S. Pevzner

REU TAs



Copenhagen-Jerusalem Combinatorics Seminar

Nov. 18, 2021

1. Review matroids M
   - independent sets $\mathcal{I}(M)$
   - flats $\mathcal{F}(M)$

2. Shellability

3. Augmented Bergman complex Δ_M

4. Two kinds of shellings of Δ_M and corollaries

1. Review matroids M

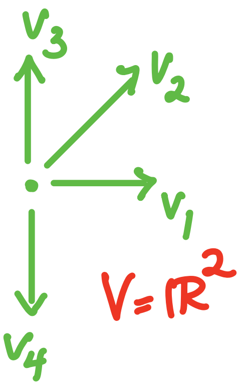
A matroid M of rank r on groundset $E = \{1, 2, \ldots, n\}$ abstracts vectors $v_1, v_2, \ldots, v_n$ spanning an r-dimensional vector space V over some field k

EXAMPLE

$n = 4$

$k = \mathbb{R}$

$r = 2$

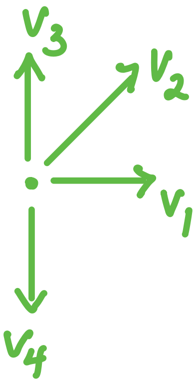


$$\leftrightsquigarrow \quad \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

an $r \times n$ full rank matrix having v_i as its columns

The matroid M associated to $v_1, v_2, \dots, v_n$ forgets their coordinates, but records the subscripts of (linearly) independent sets

$$\mathcal{I}(M) \underset{\text{DEF'N}}{:=} \left\{ I \subseteq \{1,2,\dots,n\} : \begin{array}{l} \{v_i\}_{i\in I} \text{ are} \\ \text{linearly independent} \end{array} \right\}$$



$$\rightsquigarrow \mathcal{I}(M) = \left\{ \emptyset, \begin{array}{ll} 1, & 12, \\ 2, & 13, \\ 3, & 14, \\ 4 & 23, \\ & 24 \end{array} \right\}$$

Note: $34 \notin \mathcal{I}(M)$ since $\{v_3, v_4\}$ are dependent

$ijk \notin \mathcal{I}(M)\ \forall i,j,k$

$\mathcal{I}(M)$ always satisfies these independent set axioms:

(I0) $\emptyset \in \mathcal{I}(M)$

(I1) $I \subseteq J$ and $J \in \mathcal{I}(M) \Rightarrow I \in \mathcal{I}(M)$

(I2) $I, J \in \mathcal{I}(M)$ and $\#I < \#J$

$\Rightarrow \exists\, j \in J \setminus I$ with $I \cup \{j\} \in \mathcal{I}(M)$

and this is our first definition of a matroid M:

a collection $\mathcal{I}(M)$ of subsets of $E = \{1, 2, \ldots, n\}$

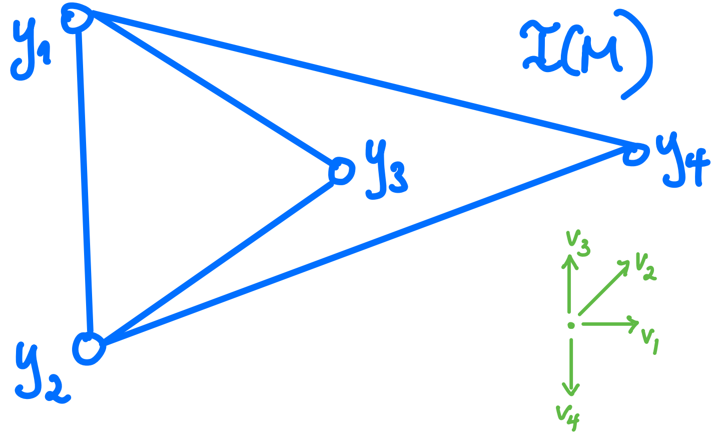
satisfying axioms (I0), (I1), (I2).

(I0) $\emptyset \in \mathcal{I}(M)$

(I1) $I \subseteq J$ and $J \in \mathcal{I}(M) \Rightarrow I \in \mathcal{I}(M)$

(I2) $I, J \in \mathcal{I}(M)$ and $\#I < \#J$
$\Rightarrow \exists j \in J \setminus I$ with $I \cup \{j\} \in \mathcal{I}(M)$

Axioms (I0), (I1) say $\mathcal{I}(M)$ is an **abstract simplicial complex** on vertices $\{y_1, y_2, \ldots, y_n\}$



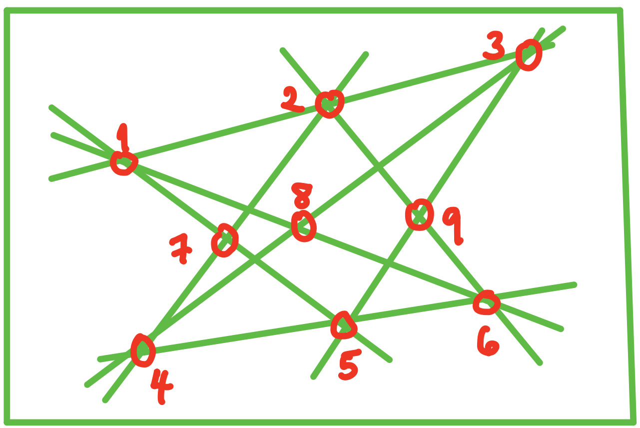
Axiom (I2) implies that all inclusion-maximal independent sets, called the **bases** $\mathcal{B}(M)$, have same cardinality r, called the **rank** $r(M)$.

In other words, $\mathcal{I}(M)$ is a **pure** simplicial complex of dimension $r(M)-1$.

Not all matroids M are representable by vectors $v_1, v_2, \ldots, v_n$

EXAMPLE The non-Pappus matroid M on $E = \{1, 2, \ldots, 9\}$ of rank 3 has

$$\mathcal{I}(M) = \left\{ \begin{array}{l} \text{all } I \subset \{1,2,\ldots,9\} \text{ with } \#I \leq 3, \\ \text{except the collinear triples shown} \end{array} \right\}$$



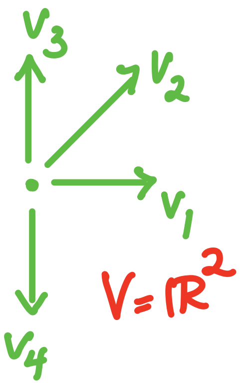
$789 \in \mathcal{I}(M)$ violates Pappus's Theorem

but does not violate axioms (I0), (I1), (I2).

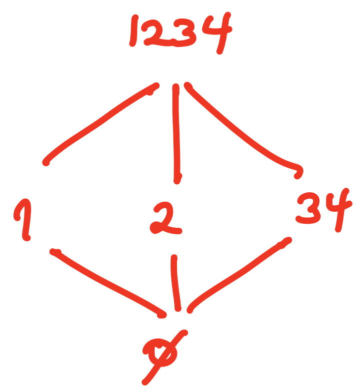
An alternate axiomatization of M uses the flats $\mathcal{F}(M)$ which are (when M is represented by $v_1, v_2, \ldots, v_n$ in V)

$$\mathcal{F}(M) := \left\{ F \subseteq \{1,2,\ldots,n\} \; : \; \{v_i\}_{i \in F} = W \cap \{v_1, v_2, \ldots, v_n\} \text{ for some subspace } W \text{ of } V \right\}$$

EXAMPLE



flats $\mathcal{F}(M) = \{ \varnothing, 1, 2, 34, 1234 \}$



the poset $\mathcal{F}(M)$ ordered via inclusion

We could have defined a matroid M on $E = \{1, 2, \_\_, n\}$ as a collection $\mathcal{F}(M)$ of subsets $F \subseteq E$, satisfying the flat axioms:

(F0) $E = \{1, 2, \_, n\} \in \mathcal{F}(M)$

(F1) $F, G \in \mathcal{F}(M) \Rightarrow F \cap G \in \mathcal{F}(M)$

(F2) $F \in \mathcal{F}(M)$ and $i \in E \setminus F \Rightarrow$
$\exists!\, G \in \mathcal{F}(M)$ covering F with $i \in G$.

(F0), (F1) $\Rightarrow$ the poset $\mathcal{F}(M)$ is a lattice, with $F \wedge G = F \cap G$.

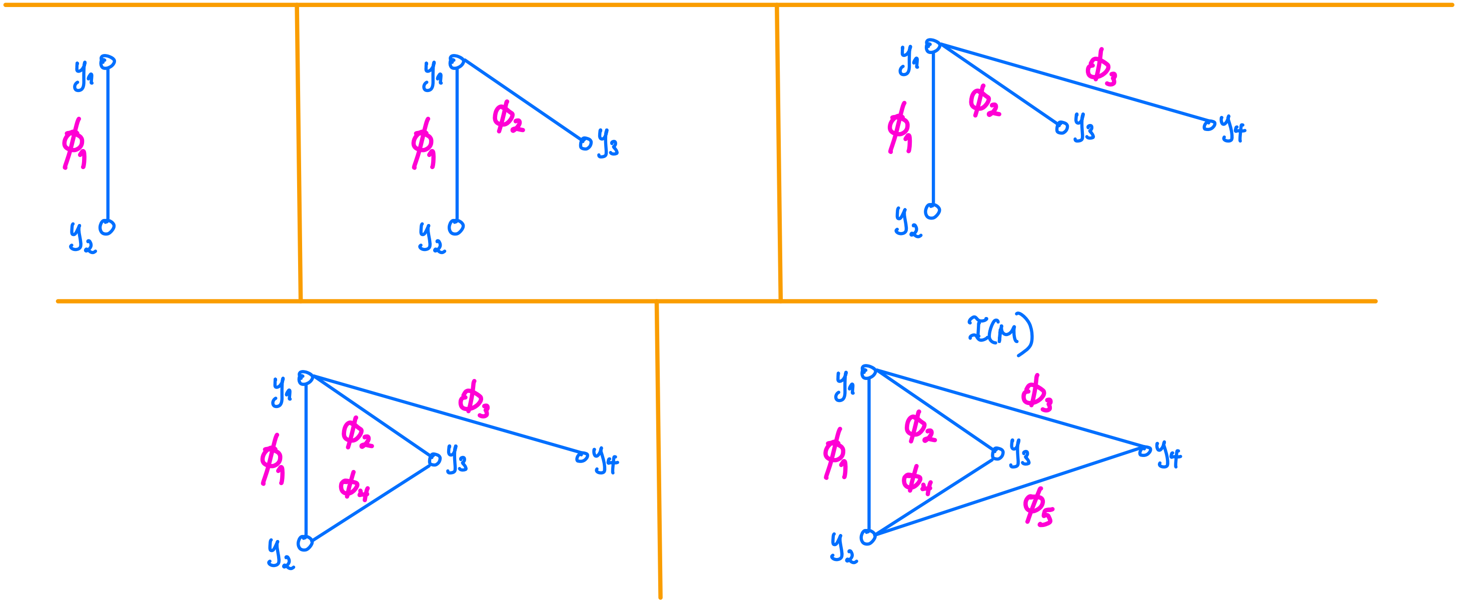
(F2) $\Rightarrow$ $\mathcal{F}(M)$ is actually a geometric lattice.

atomic + upper semimodular

2. Shellability

DEF'N: A pure $(r-1)$-dimensional simplicial complex Δ is **shellable** if we can order its **facets** $\phi_1, \phi_2, \dots, \phi_t$ in **a shelling order**:

$\forall j \geq 2$, ϕ_j intersects the subcomplex generated by $\phi_1, \phi_2, \dots, \phi_{j-1}$ in a pure $(r-2)$-dim'l subcomplex



Shelling determines the homotopy type of Δ

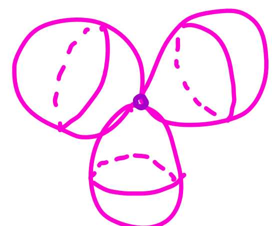
DEF'N: Call ϕ_j a homology facet in the shelling $\phi_1, \phi_2, \ldots, \phi_t$ if ϕ_j intersects the subcomplex gen'd by $\phi_1, \phi_2, \ldots, \phi_{j-1}$ in the entire boundary $\mathrm{Bd}\,\phi_j$

PROPOSITION: When Δ is pure d-dimensional and shellable,

then $\|\Delta\| \approx \underbrace{S^d \vee S^d \vee \cdots \vee S^d}_{\beta\text{-fold 1-point wedge of } d\text{-spheres } S^d}$

geometric realization of Δ

homotopy equivalent

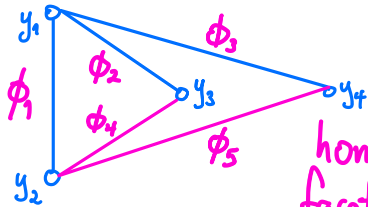


where $\beta :=$ # of homology facets ϕ_j in any shelling order

In fact, whenever Δ is shellable,

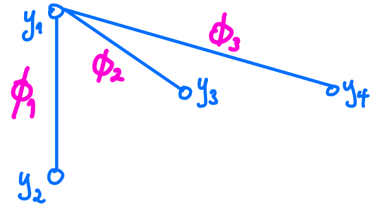
then $\Delta' := \Delta - \left\{ \begin{smallmatrix} \text{all} \\ \text{homology} \\ \text{facets } \phi_j \end{smallmatrix} \right\}$ is contractible:

$\Delta = \mathcal{I}(M)$



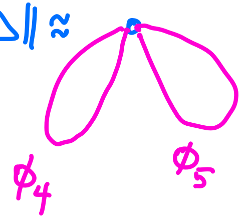
homology facets ϕ_4, ϕ_5

$\Delta' =$

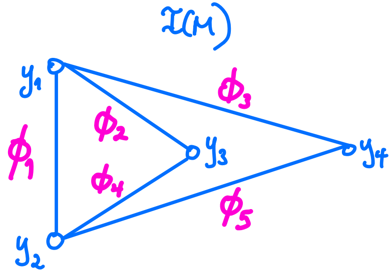


contract Δ' to a point

$\|\Delta\| \approx$



THEOREM (Provan-Billera 1980) For a matroid M, the independent set complex $\mathcal{I}(M)$ is shellable, via lexicographic order on the bases $\mathcal{B}(M)$.



$$\begin{array}{ccccccccc} \phi_1 & & \phi_2 & & \phi_3 & & \phi_4 & & \phi_5 \\ 12 & <_{lex} & 13 & <_{lex} & 14 & <_{lex} & 23 & <_{lex} & 24 \end{array}$$

Furthermore, the number of homology facets is

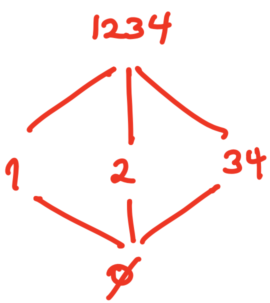
$\beta = T_M(0,1)$ = Tutte polynomial $T_M(x,y)$ evaluated at $x=0, y=1$

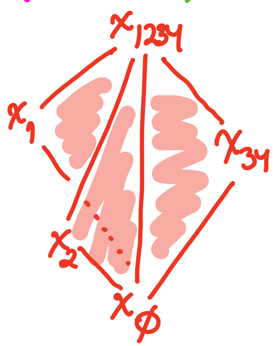
= # bases $B \in \mathcal{B}(M)$ of internal activity zero

COROLLARY: $\|\mathcal{I}(M)\| \simeq \underbrace{\mathbb{S}^{r(M)-1} \vee \cdots \vee \mathbb{S}^{r(M)-1}}_{T_M(0,1)\text{-fold wedge}}$

The flats $\mathcal{F}(M)$ as a poset P gives us another simplicial complex, the order complex $\Delta P :=$ simplicial complex with vertex set $\{x_p\}_{p\in P}$ and simplices/faces the totally ordered subsets $\{x_{p_1}, x_{p_2}, \dots, x_{p_\ell}\}$ if $p_1 < p_2 < \dots < p_\ell$ in P

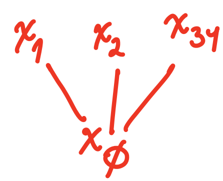
flat poset $\mathcal{F}(M)$ $\overset{\text{order complex}}{\rightsquigarrow}$ $\Delta\mathcal{F}(M)$





contractible

$\mathrm{Cone}(\Delta_M) \overset{\text{DEF}}{:=} \Delta(\mathcal{F}(M) - \{E\})$



contractible

Bergman complex $\Delta_M \overset{\text{DEF}}{:=} \Delta(\mathcal{F}(M) - \{\varnothing, E\})$

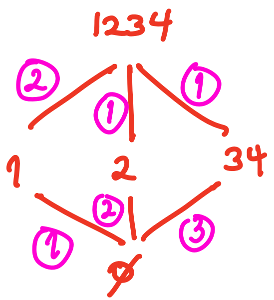
$\mathbb{S}^0 \vee \mathbb{S}^0$

a 2-fold wedge of 0-spheres

THEOREM (Garsia 1980) For a matroid M, all three of
$$\begin{cases} \Delta \mathcal{F}(M) \\ \Delta(\mathcal{F}(M) - \{E\}) \\ \Delta(\mathcal{F}(M) - \{\emptyset, E\}) =: \underline{\Delta}_M \end{cases}$$
are shellable, via lexicographic order on the edge-label sequences on maximal chains $\emptyset \subset F_1 \subset F_2 \subset F_3 \subset \cdots \subset F_{r(M)-1} \subset E$ in $\mathcal{F}(M)$

edge-labels: $(\min F_1, \min(F_2 \setminus F_1), \min(F_3 \setminus F_2), \ldots, \min(E \setminus F_{r(M)-1}))$

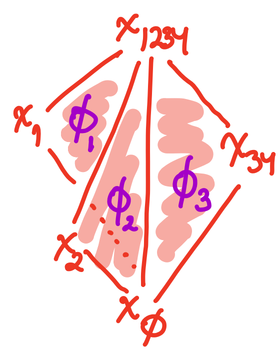
$\mathcal{F}(M)$



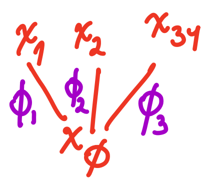
$(1,2) <_{lex} (2,1) <_{lex} (3,1)$

$\phi_1 \quad \phi_2 \quad \phi_3$

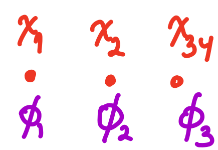
$\Delta\mathcal{F}(M)$



$\Delta(\mathcal{F}(M) - \{E\})$



$\Delta(\mathcal{F}(M) - \{\emptyset, E\})$



Furthermore, the number of homology facets is

$$\beta = T_M(1,0) = \text{Tutte polynomial } T_M(x,y) \text{ evaluated at } x=1, y=0$$

$$= \#\text{ bases } B \in \mathcal{B}(M) \text{ of external activity zero}$$

COROLLARY: $\|\Delta(M)\| \simeq \underbrace{S^{r(M)-2} \vee \dots \vee S^{r(M)-2}}_{T_M(1,0)\text{-fold wedge}}$

Bergman complex

$\Delta_M := \Delta\left(\mathcal{F}(M) - \{\bar{\phi}, E\}\right)$

3. Augmented Bergman complex Δ_M

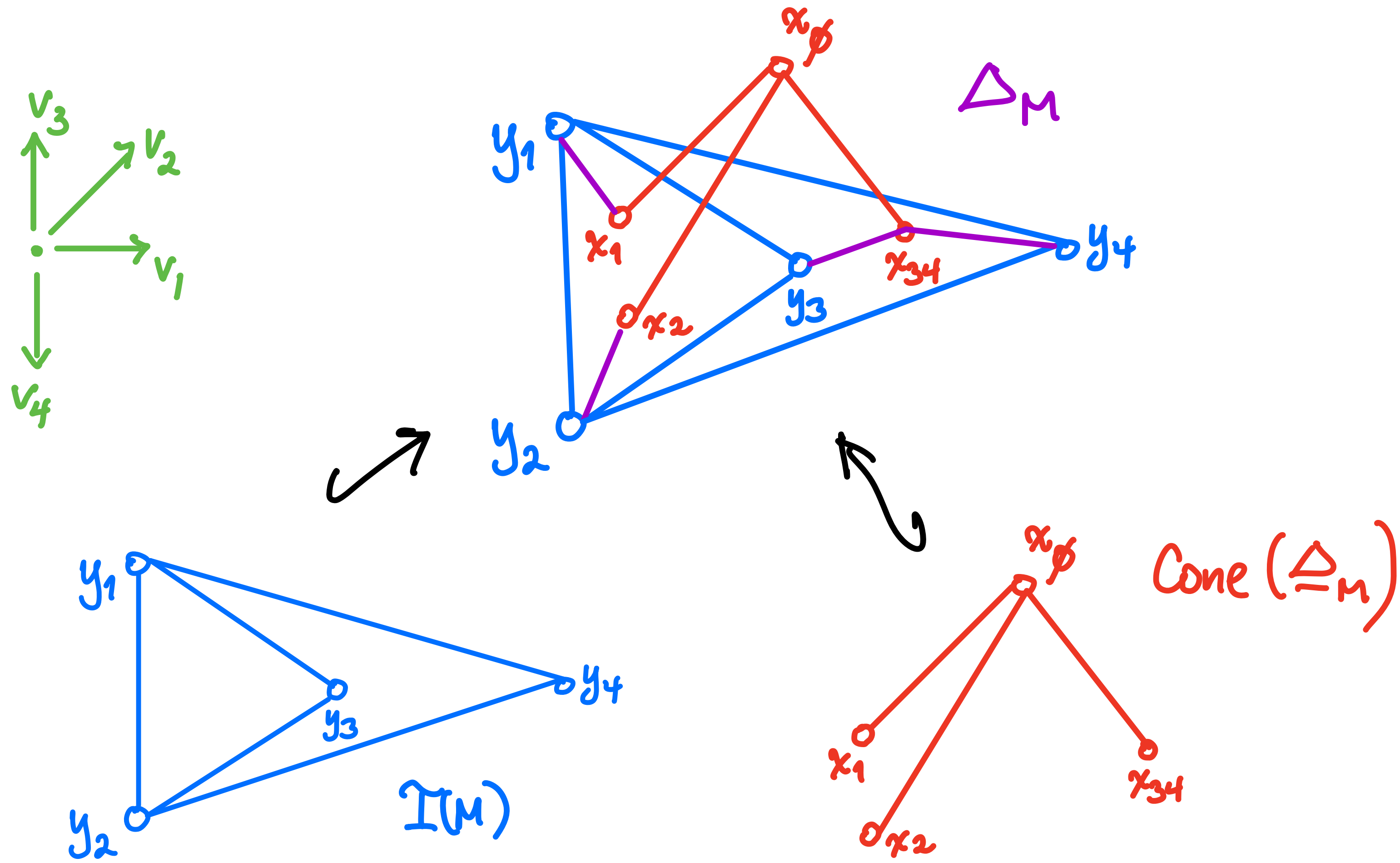
In a monumental pair of 2020 papers, Braden-Huh-Matherne-Proudfoot-Wang introduced a hybrid.

DEF'N: The augmented Bergman complex Δ_M has vertex set $\{y_1, y_2, \ldots, y_n\} \cup \{x_F\}_{\emptyset \subseteq F \subsetneq E}$ (proper flats $F \in \mathcal{F}(M)$)

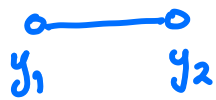
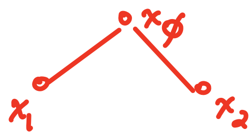
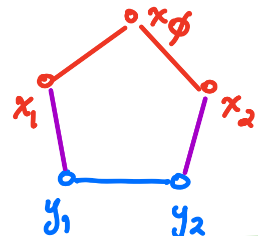
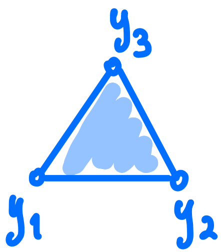
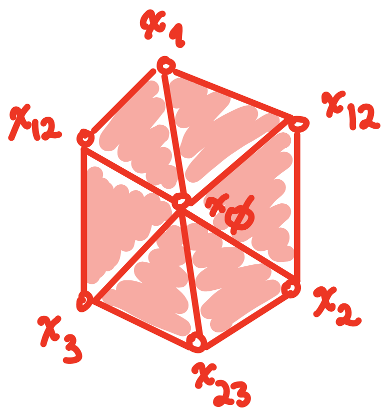
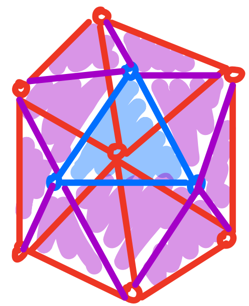
with simplices/faces $\{y_i\}_{i \in I} \cup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\}$

when
- $I \in \mathcal{I}(M)$ is independent
- $F_1, F_2, \ldots, F_\ell$ are proper flats
- $I \subseteq F_1 \subset F_2 \subset \cdots \subset F_\ell$ $(\neq E)$

Δ_M is pure of dimension $r(M)-1$, containing both $\mathcal{I}(M)$ and $\mathrm{Cone}(\Delta_M)$ as subcomplexes:



SPECIAL CASE : Boolean matroid M of rank n

<table>
<tr>
<td></td>
<td>$\mathcal{I}(M)$
$=$ $(n-1)$-simplex
$2^{\{1,2,\ldots,n\}}$</td>
<td>$\mathrm{Cone}(\Delta_M)$
$=$ barycentric subdivision of $(n-1)$-simplex</td>
<td>Δ_M
$=$ boundary of stellohedron</td>
</tr>
<tr>
<td>$n=2$</td>
<td>
</td>
<td>
</td>
<td>
</td>
</tr>
<tr>
<td>$n=3$</td>
<td>
</td>
<td>
</td>
<td></td>
</tr>
</table>

Why did BHMPW introduce Δ_M?

Its Stanley-Reisner ring has an amazing Artinian quotient by certain linear forms

$$CH(M) = \mathbb{R}[y_1, \ldots, y_n, x_F]_{\substack{\text{flats} \\ F \subsetneq E}} \Big/ \begin{pmatrix} x_F x_G, \ F \not\subseteq G, G \not\subseteq F \\ y_i x_F, \quad i \notin F \\ y_i - \sum_{i \notin F} x_F, \ i = 1, 2, \ldots, n \end{pmatrix}$$

= augmented Chow ring of M

in which the $y_1, \ldots, y_n$ generate a subalgebra $H(M)$

= graded Möbius algebra of M

with a crucial $H(M)$-submodule $IH(M)$

= intersection cohomology of M

and remarkable properties ...

- $H(M) \hookrightarrow IH(M) \hookrightarrow CH(M)$ these satisfy Kähler package
- Hilbert series for $H(M)$ interprets rank sizes W_k of $\mathcal{F}(M)$
  and Kähler package for $IH(M)$ $\Rightarrow$ Dowling-Wilson's Top Heavy Conj. (1974)
- Hilbert series for $IH(M)$ interprets Z-polynomial for M
  and Kähler package for $IH(M)$ $\Rightarrow$ unimodality for Z
- Hilbert series for $IH(M)/(y_1,\dots,y_n)IH(M)$
  interprets Kazhdan-Lusztig polynomial for M
  $\Rightarrow$ nonnegativity of K-L polynomial!

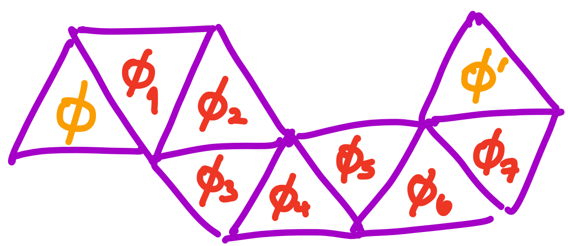
They used this weaker property of Δ_M than shellabity:

PROPOSITION (BHMPW 2020): For any matroid M,

Δ_M is gallery-connected,

that is, any two facets ϕ, ϕ' are connected by a gallery of facets $\phi = \phi_0, \phi_1, \phi_2, \ldots, \phi_{t-1}, \phi_t = \phi'$

with each $\phi_i \cap \phi_{i+1}$ of dimension $r(M)-2$ (= codimension 1)



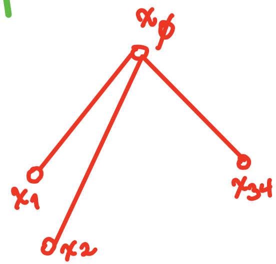
4. Two kinds of shellings of Δ_M and corollaries

THEOREM (UMN REU 2021) For any matroid M, the augmented Bergman complex has two families of shellings:

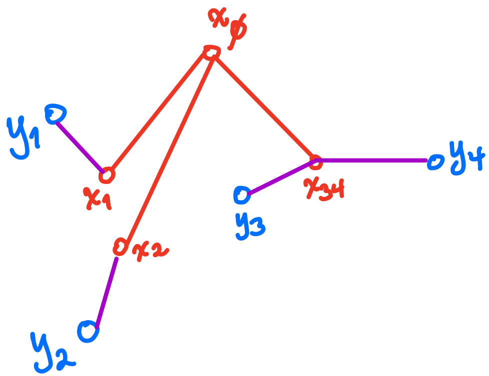
(i) some that shell the facets of $\mathrm{Cone}(\Delta_M)$ first, and facets of $\mathcal{I}(M)$ last.

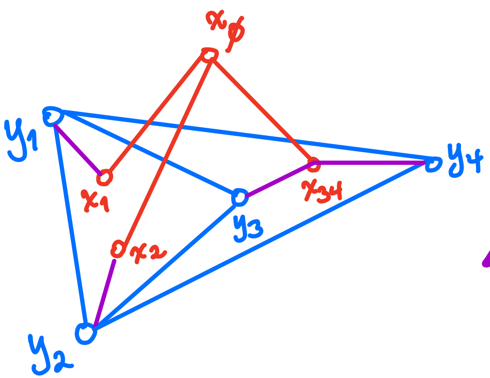
(ii) some that shell the facets of $\mathcal{I}(M)$ first, and facets of $\mathrm{Cone}(\Delta_M)$ last.

Type (i) shellings



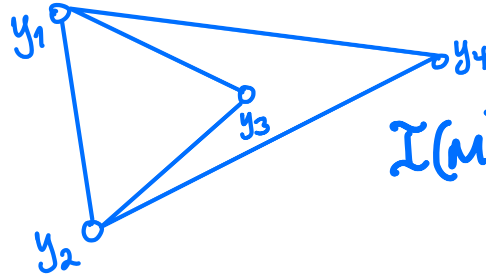
Cone(Δ_M)



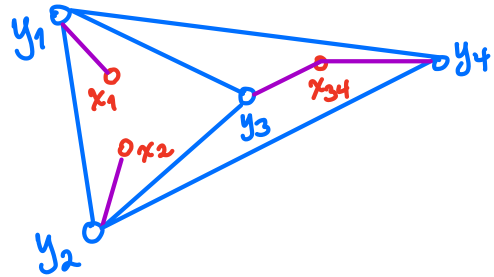


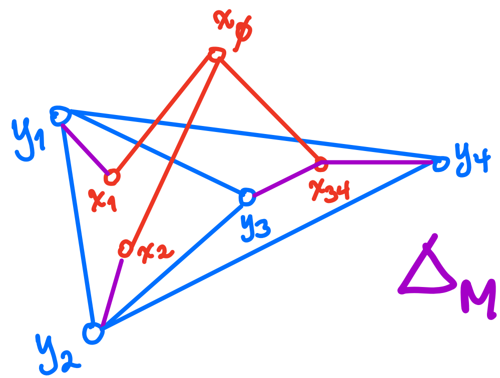
Δ_M

Type (ii) shellings



$\mathcal{I}(M)$





Δ_M

COROLLARY: (UMN REU 2021) The augmented Bergman complex Δ_M has $\|\Delta_M\| \approx \underbrace{S^{r(M)-1} \vee \cdots \vee S^{r(M)-1}}_{\beta\text{-fold wedge}}$

where β has two expressions:

(i) $\beta = T_M(1,1) = \#\mathcal{B}(M)$

because the homology facets in type (i) shellings are $\{y_i\}_{i \in B}$ indexed by bases B of M.

(ii) $\beta = \sum_{\text{flats } F \in \mathcal{F}(M)} T_{M|_F}(0,1)\, T_{M/F}(1,0)$

counting type (ii) shelling homology facets.

REMARK: The equality

$$T_M(1,1) = \sum_{\text{flats } F} T_{M|_F}(0,1)\, T_{M/F}(1,0)$$

appeared in work of Étienne-Las Vergnas 1998,

rediscovered in Kook-R.-Stanton 2000,

and is a specialization of a convolution formula

$$T_M(x,y) = \sum_{\text{flats } F} T_{M|_F}(0,y)\, T_{M/F}(x,0)$$

for Tutte polynomials.

The type (i) shellings show contractibility of

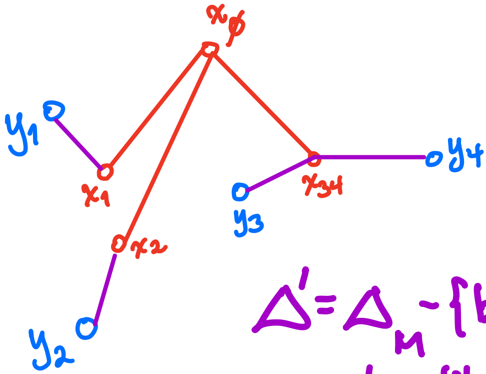
$$\Delta' = \Delta_M - \{\text{facets } \{y_i\}_{i\in B} : \text{bases } B \in \mathcal{B}(M)\}$$

Since matroid automorphisms set-wise stabilize the collection of basis facets, one can conclude:

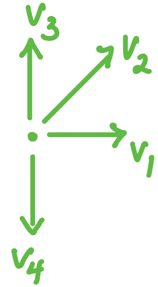
COROLLARY: The group $\mathrm{Aut}(M)$ acts on $H_{r(M)-1}(\Delta_M, \mathbb{Z})$ as a signed permutation representation, same as on $C_{r(M)-1}(\mathcal{I}(M), \mathbb{Z})$:

$$\sigma([b_1, b_2, \ldots, b_r]) = [b_{\sigma(1)}, \ldots, b_{\sigma(r)}] \text{ for bases } B = \{b_1, \ldots, b_r\} \in \mathcal{B}(M)$$

oriented simplex

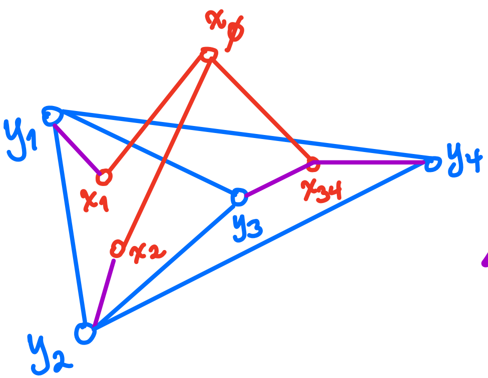


$\Delta' = \Delta_M - \{\text{bases}\}$ is contractible

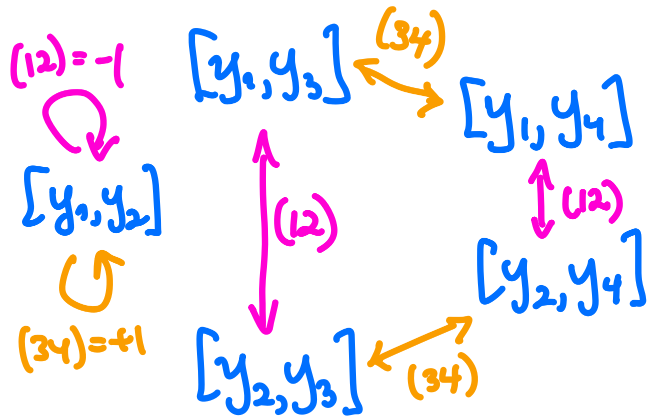


$\mathrm{Aut}(M) = \{e, (12), (34), (12)(34)\}$

$H_1(\Delta_M) = \mathbb{Z}^5$



Δ_M



REMARK: Neither I_M nor Δ_M have simple descriptions for their homology representations in general.

Notable special cases:

matroid M	$H_{r(M)-1}(I_M)$	$H_{r(M)-2}(\Delta_M)$
Boolean	trivial rep of S_n	sign rep of S_n
q-Boolean = $\mathbb{F}_q$-vector space	known virtually, not so explicit	Steinberg rep of $GL_n(\mathbb{F}_q)$
braid arrangement = complete graphic	an S_n-rep that restricts nicely to S_{n-1} (Kook 1996)	Lie rep of S_n

REMARK: Can generalize Aut(M)-rep description to arbitrary closure operators $2^E \xrightarrow{f} 2^E$

defining indep. sets I : $f(I \setminus \{i\}) \subsetneq f(I) \ \forall i \in I$

bases B : B indep. and $f(B) = E$

flats F : $f(F) = F$

and augmented Bergman complex Δ_f

with vertices $\{y_1, \ldots, y_n\} \cup \{x_F\}_{\text{proper flats } F \subsetneq E}$

simplices $\{y_i\}_{i \in I} \cup \{x_{F_1}, \ldots, x_{F_\ell}\}$

with
- I indep.
- $F_1, \ldots, F_\ell$ flats
- $I \subseteq F_1 \subset \ldots \subset F_\ell$

$\mathcal{I}(f)$, $\underline{\Delta}_f$, Δ_f are not shellable in general.

Nevertheless:

THEOREM (UMN REU 2021) $\|\Delta_f\| \simeq \bigvee_{\text{bases } B} \mathbb{S}^{\#B-1}$

and $\mathrm{Aut}(M)$ acts on $H_\bullet(\Delta_f)$ as a signed permutation rep on oriented chains $[b_1, b_2, \ldots, b_r]$ indexed by bases $B = \{b_1, \ldots, b_r\}$.

Again $\Delta' := \Delta_f - \left\{ \{y_i\}_{i \in B} : B \text{ a basis} \right\}$

is contractible.

Thanks
for
your
attention!