(1)

A q-analogue of Ehrhart theory

(joint with B. Rhoades arXiv: 2407.06511)

1. Ehrhart theory for P
   $\Uparrow$
2. Semigroup ring $\Bbbk[\Lambda_P]$ $\left\{ \begin{array}{l} \text{Noeth,} \\ \text{C-M. with} \\ \omega_{\Bbbk[\Lambda_P]} \text{ known} \end{array} \right.$
3. q-Ehrhart theory CONJECTURES
   $\Uparrow$
4. Harmonic algebra $\mathcal{H}_P$ CONJECTURES

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1. Ehrhart theory for P a lattice polytope in $\mathbb{R}^n$
   so vertices $(P) \subset \mathbb{Z}^n$

DEF'N: $i_P(m) := \# \mathbb{Z}^n \cap mP$ Ehrhart function/polynomial
for $m=0,1,2,\ldots$

$$E_P(m) := \sum_{m=0}^{\infty} t^m \underbrace{(\# \mathbb{Z}^n \cap mP)}_{i_P(m)}$$ Ehrhart series

THEOREM(s)

(1) (Ehrhart 1962)
$i_P(m)$ is a polynomial in m of degree $d := \dim(P)$
or equivalently

$$E_P(t) = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}} \in \mathbb{Q}(t) \text{ with } h_i^* \in \mathbb{Z}$$

(2) (Stanley 1980) $(h_0^*, h_1^*, \ldots, h_d^*)$ all lie in $\mathbb{N} = \{0,1,2,\ldots\}$

(3) (Ehrhart-Macdonald reciprocity)
1959 1971 (relative)
If one defines interior count versions

for $m=1,2,3,\ldots$
$$\bar{i}_P(m) = \# \mathbb{Z}^n \cap \mathrm{int}(mP)$$
$$\bar{E}_P(t) = \sum_{m=0}^{\infty} \bar{i}_P(m) t^m$$

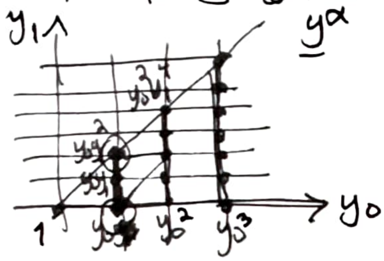
then $\bar{i}_P(m) = (-1)^d i_P(-m)$
or equivalently

$$E_P(t^{-1}) = (-1)^{d+1} \bar{E}_P(t)$$

(2)

2. Affine semigroup ring for $\mathrm{Cone}(\{0\}\times P) \subset \mathbb{R}^{d+1}$

$$\Bbbk[\Lambda_P] := \mathrm{span}_{\Bbbk}\{ y_0^m \underbrace{y_1^{\alpha_1}\cdots y_n^{\alpha_n}}_{y^\alpha} : \alpha \in \mathbb{Z}^n \cap mP \} \subset \Bbbk[y_0, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$$

$P =$  $\rightsquigarrow$ 

with $\mathbb{N}$-grading via $\deg(y_0^m y^\alpha) = m$

has $\underset{\text{Hilbert series}}{\mathrm{Hilb}(\Bbbk[\Lambda_P], t)} = \sum_{m=0}^{\infty} t^m \cdot i_P(m) = E_P(t)$.

THEOREM(S)

(1) (a) (Gordan 1873) $\Bbbk[\Lambda_P]$ is Noetherian, even ~~finitely~~ gen'd as a module over its subalgebra gen'd in degree one.

(b) (Noether 1926) That subalgebra contains a linear system of parameters $\theta_1, \theta_2, \ldots, \theta_{d+1}$ $\Big\} \Rightarrow$ Ehrhart Thm(1)

(2) (Hochster 1972) $\Bbbk[\Lambda_P]$ is Cohen-Macaulay, so a free $\Bbbk[\theta_1, \theta_2, \ldots, \theta_{d+1}]$-module. $\Big\} \Rightarrow$ Stanley Thm(2)

(3) (Danilov 1978) The canonical module

$$\omega_{\Bbbk[\Lambda_P]} \underset{\substack{\text{up to}\\ \text{a grading}\\ \text{shift}}}{\cong} \Bbbk[\Lambda_{\mathrm{int}(P)}] = \mathrm{span}_{\Bbbk}\{ y_0^m y^\alpha : \alpha \in \mathbb{Z}^n \cap \mathrm{int}(mP) \}.$$

$\Longrightarrow$ Ehrhart-Macdonald reciprocity Thm(3)

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3. q-Ehrhart theory

Uses a canonical q-analogue for $\#Z$ of any finite point set $Z \subset \mathbb{R}^n$

e.g. $Z = \mathbb{Z}^2 \cap P$ 

$$\mathbb{R}[Z] = \underset{\text{polynomial fns } Z \to \mathbb{R}}{\mathbb{R}[x_1, \ldots, x_n]/I(Z)} \quad \text{of } \dim_{\mathbb{R}} = \#Z = 4$$

$$= \mathbb{R}[x_1, x_2] / \underbrace{\left((x_1, x_2) \cap (x_1 - 1, x_2 - 1) \cap (x_1 - 2, x_2 - 1) \cap (x_1 - 1, x_2 - 2) \right)}$$

$$= (\underline{x_2^3} - 3x_2^2 + 2x_2, \; \underline{x_1^2 - x_2^2} - 3x_1 + x_2, \; \underline{2x_1x_2 - x_2^2} - 2x_1 + x_2)$$

deform

$$\mathbb{R}\left[\begin{smallmatrix}\text{a certain}\\ \text{fat point}\\ \text{scheme at } 0\\ \text{of mult } \#Z\end{smallmatrix}\right] = \mathrm{gr}\,\mathbb{R}[Z] = \mathbb{R}[\underline{x}] / \underbrace{I_{\mathrm{gr}}(Z)}_{= (\mathrm{topdeg}(f)\, :\, f \in I(Z))}$$

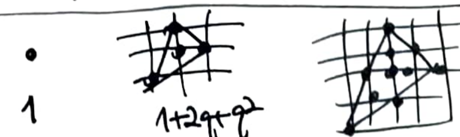
$$= \mathbb{R}[x_1, x_2] / (x_2^3, x_1^2 - x_2^2, 2x_1x_2 - x_2^2) = \mathrm{span}_{\mathbb{R}}\{1, \bar{x}_1, \bar{x}_2, \bar{x}_2^2\}$$

(3)

DEFN: For a lattice polytope $P \subset \mathbb{R}^n$,

$$i_P(m;q) := \mathrm{Hilb}\left(\mathbb{R}[x_1,\dots,x_n]/I_{gr}(\mathbb{Z}^n \cap mP), q\right)$$

e.g.



1 $1+2q+q^2$ $1+2q+3q^2+3q^3+q^4$

$i_P(0;q)$ $i_P(1;q)$ $i_P(2;q)$

and $E_P(t,q) := \sum_{m=0}^{\infty} t^m \cdot i_P(m,q) \in \mathbb{Z}[[t,q]]$

q-Ehrhart (some) CONJECTURE(S): (R.-Rhoades '24)

(1) $E_P(t,q) \in \mathbb{Q}(t,q)$ has form $\dfrac{N_P(t,q)}{\prod_{(i,j)}(1-t^iq^j)}$ ← in $\mathbb{Z}[t,q]$

(2) $E_P(t^{-1},q^{-1}) = (-1)^{d+1} q^d \, \overline{E}_P(t,q)$

4. Harmonic algebra $\mathcal{H}_P$

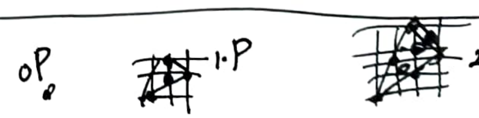
This is a $\mathbb{Z}^2$-graded $\mathbb{R}$-algebra having $\mathrm{Hilb}(\mathcal{H}_P, t, q) = E_P(t,q)$:

$$\mathcal{H}_P \subset \mathbb{R}[\underbrace{y_0}_{\mathbb{Z}^2\text{-deg}\begin{bmatrix}1\\0\end{bmatrix} \text{ i.e. } t^1},\ \underbrace{y_1,\dots,y_n}_{\mathbb{Z}^2\text{-deg}\begin{bmatrix}0\\1\end{bmatrix} \text{ i.e. } q^1}] \quad \text{with } \mathcal{H}_P = \bigoplus_{m=0}^{\infty} y_0^m \cdot V_{\mathbb{Z}^n \cap mP}$$

where $V_Z :=$ Macaulay inverse system $\subset \mathbb{R}[y_1,\dots,y_n]$ to $I_{gr}(Z) \subset \mathbb{R}[x_1,\dots,x_n]$

$$= \left\{ g(\underline{y}) \in \mathbb{R}[\underline{y}] : f\left(\tfrac{\partial}{\partial y_1},\dots,\tfrac{\partial}{\partial y_n}\right) g(\underline{y}) = 0 \ \forall f(\underline{x}) \in I_{gr}(Z) \right\}$$

EXAMPLE $0P$ $1\cdot P$ $2\cdot P$



$$\mathcal{H}_P = \mathrm{span}_{\mathbb{R}}\{1,\ y_0\cdot 1,\ y_0 y_1,\ y_0 y_2,\ y_0(y_1^2+y_1y_2+y_2^2),\ y_0^2\cdot 1,\ y_0^2 y_1,\ y_0^2 y_2,\ y_0^2 y_1^2,\ y_0^2 y_1 y_2,\ y_0^2 y_2^2,\ y_0^2 y_1^3,\ y_0^2(y_1^2 y_2 + y_1 y_2^2),\ y_0^2 y_2^3,\ y_0^2(y_1^2+y_1y_2+y_2^2)^2,\ \dots\}$$

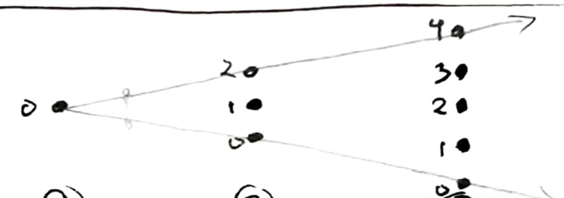
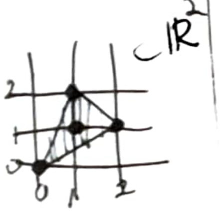
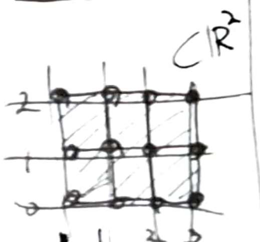
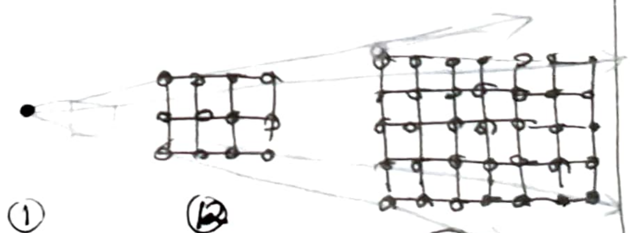
CONJECTURE(S)

(1) $\mathcal{H}_P$ is Noetherian.

(2) $\mathcal{H}_P$ is Cohen-Macaulay

(3) $\omega_{\mathcal{H}_P} = \bigoplus_{m=0}^{\infty} y_0^m \cdot V_{\mathbb{Z}^n \cap \mathrm{int}(mP)}$

These would imply the q-Ehrhart CONJECTURES.

P	$i_P(m)$	$i_P(m)$ vs. $i_P(m;q)$	$m=0$	$m=1$	$m=2$	$E_P(t)$	vs. $E_P(t,q)$
	$1+2m$		1 1	3 $1+q+q^2$	5 $1+q+q^2+q^3+q^4$	$\frac{1+t}{(1-t)^2}$ $=1+3t+5t^2+\dots$	$\frac{1+qt}{(1-t)(1-q^2t)}$
	$1+\frac{3}{2}m+\frac{3}{2}m^2$		1 1	4 $1+2q+q^2$	10 $1+2q+3q^2+3q^3+q^4$	$\frac{1+t+t^2}{(1-t)^3}$ $=1+4t+10t^2+\dots$	$\frac{(1+qt+q^2t^2)(1+qt)}{(1-t)(1-q^2t)(1-q^3t^2)}$
	$(1+2m)(1+3m)$ $=1+5m+6m^2$		1 1	12 $(1+q+q^2)(1+q+q^2+q^3)$	35 $(1+q+q^2+q^3+q^4)\cdot(1+q+q^2+q^3+q^4+q^5+q^6)$	$\frac{1+9t+2t^2}{(1-t)^3}$ $=1+12t+35t^2+\dots$	$\frac{1+2(q+q^2+q^3+q^4)t-(q^3+2q^4+q^5+q^6+q^2)t^2-(q^7+q^8)t^3}{(1-t)(1-q^2t)(1-q^3t)(1-q^5t)}$