Universal parameters for Stanley-Reisner rings and a colorful Hochster formula

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1. Stanley-Reisner ring reminder $\begin{cases} \text{f-vectors} \\ \text{h-vectors} \\ \text{Hilbert series} \end{cases}$

2. Hochster's formula (1977)

3. Colorful Hochster formula (new!)

4. Universal parameters: depth, resolutions (CONJ!)

1. Stanley-Reisner ring reminder

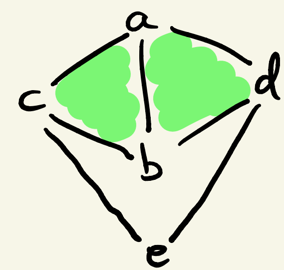
Δ a simplicial complex

has f-vector

$$\underline{f} = (f_{-1}, f_0, f_1, \dots, f_{d-1})$$

where f_i = # faces F of dimension i
(size $\#F = i+1$)

$\Delta =$



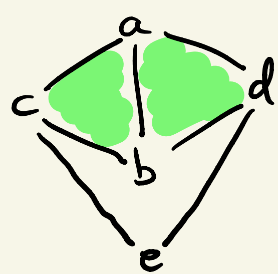
$$= \{ \emptyset, \; \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix}, \; \begin{matrix} ab \\ ac \\ bc \\ ad \\ bd \\ ce \\ de \end{matrix}, \; \begin{matrix} abc \\ abd \end{matrix} \}$$

$$\underline{f} = (\underset{f_{-1}}{1}, \; \underset{f_0}{5}, \; \underset{f_1}{7}, \; \underset{f_2}{2})$$

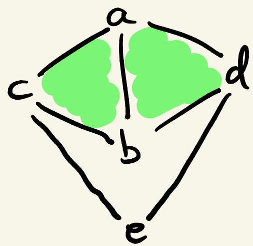
The f-vector of Δ is related to its

Stanley-Reisner ring over a field k

$$k[\Delta] = \underbrace{k[a,b,c,d,e]}_{\substack{S := \\ \text{polynomial ring} \\ \text{in vertex variables}}} \Big/ \underbrace{(cd, ae, be)}_{\substack{I_\Delta := \text{ideal generated} \\ \text{by non-face monomials}}}$$

$\Delta =$ 

$k[\Delta]$ has k-basis given by monomials supported on faces

$\Delta =$ 

$k[\Delta] = k[a,b,c,d,e]/(cd, ae, be)$

has k-basis

$$
\begin{array}{lccccccc}
\text{degree:} & 0 & 1 & 2 & 3 & 4 & \\
& \{\, 1, & a & a^2 & a^3 & a^4 & \ldots \,\} \\
& & b & b^2 & b^3 & b^4 & \\
& & c & c^2 & \vdots & \vdots & \\
& & d & d^2 & & & \\
& & e & e^2 & & & \\
& & & ab & & & \\
& & & ac & & & \\
& & & ad & & & \\
& & & bc & & & \\
& & & bd & & & \\
& & & ce & & & \\
& & & de & & &
\end{array}
$$

Hilbert series

$$\mathrm{Hilb}(k[\Delta],t) := \sum_{d=0}^{\infty} \dim_k k[\Delta]_d \cdot t^d$$

$\underline{f} = (1,5,7,2)$

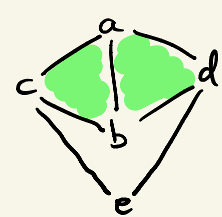
not hard!

$$= \sum_{i=0}^{d} f_{i-1} \left(\frac{t}{1-t}\right)^i = 1 + 5\left(\frac{t}{1-t}\right)^1 + 7\left(\frac{t}{1-t}\right)^2 + 2\left(\frac{t}{1-t}\right)^3$$

$$=: \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^d} = \frac{1+2t+0\cdot t^2 - t^3}{(1-t)^3}$$

$\underline{h} = (1,2,0,-1)$
$h_0 \; h_1 \; h_2 \; h_3$

Def'n of h-vector
$\underline{h} = (h_0, h_1, \ldots, h_d)$



2. Hochster's formula (1977)

Can also compute $\mathrm{Hilb}(k[\Delta], t)$ from a finite (minimal) free resolution of $k[\Delta]$ as an S-module:

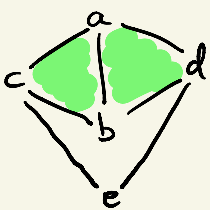
syzygies: 0th, 1st, 2nd, 3rd

$$0 \leftarrow k[\Delta] \leftarrow S \leftarrow S(-2)^3 \leftarrow \begin{array}{c} S(-3)^1 \\ \oplus \\ S(-4)^2 \end{array} \leftarrow S(-5)^1 \leftarrow 0$$

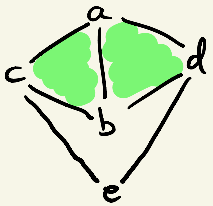
$k[\Delta] = S/I_\Delta$

$S = k[a,b,c,d,e]$

$$\begin{aligned} cd &\leftarrow\!\shortmid\; e_{cd} \\ ae &\leftarrow\!\shortmid\; e_{ae} \\ be &\leftarrow\!\shortmid\; e_{be} \end{aligned}$$



$S(-d)$:= free S-module with 1 basis element in degree d



syzygies: 0th 1st 2nd 3rd

$$0 \leftarrow \underset{\substack{\parallel \\ S/I_\Delta}}{k[\Delta]} \leftarrow \underset{\substack{\parallel \\ k[a,b,c,d,e]}}{S^1} \leftarrow S(-2)^3 \leftarrow \begin{matrix} S(-3)^1 \\ \oplus \\ S(-4)^2 \end{matrix} \leftarrow S(-5)^1 \leftarrow 0$$

---

exactness

$\Longrightarrow$

$$\mathrm{Hilb}(k[\Delta],t) = \mathrm{Hilb}(S,t)\cdot\left(t^0 - 3t^2 + (t^3 + 2t^4) - t^5\right)$$

$$= \frac{1}{(1-t)^5}\left(1 - 3t^2 + t^3 + 2t^4 - t^5\right) \overset{\checkmark}{=} \frac{1+2t-t^3}{(1-t)^3}$$

requires some cancellation!

THEOREM (Hochster 1977) The minimal free S-resolution of $k[\Delta]$ has

for each vertex subset T its vertex-selected subcomplex

$$\Delta_T := \{ F \in \Delta : F \subseteq T \}$$

contributing $\dim_k \tilde{H}^{\#T-1-i}(\Delta_T ; k)$

(reduced) simplicial homology

copies of $S(-\#T)$ to the i^{th} syzygies.

syzygies: 0th 1st 2nd 3rd

$$0 \leftarrow k[\Delta] \leftarrow S^1 \leftarrow S(-2)^3 \leftarrow \begin{matrix} S(-3)^1 \\ \oplus \\ S(-4)^2 \end{matrix} \leftarrow S(-5)^1 \leftarrow 0$$

$\tilde{H}^{0-1-0}(\Delta_\emptyset) = k^1$

$\{\emptyset\}$

$\tilde{H}^{2-1-1}(\Delta_{cd}) = k^1$

c d

$\tilde{H}^{2-1-1}(\Delta_{ae}) = k^1$

a

e

$\tilde{H}^{2-1-1}(\Delta_{be}) = k^1$

b

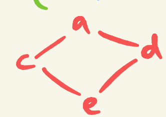
e

$\tilde{H}^{3-1-2}(\Delta_{abe}) = k^1$

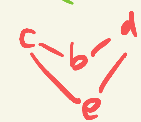
a | b

e

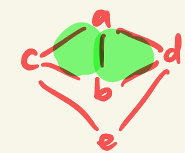
$\tilde{H}^{4-1-2}(\Delta_{acde}) = k^1$



$\tilde{H}^{4-1-2}(\Delta_{bcde})$



$\tilde{H}^{5-1-3}(\Delta_{abcde}) = k^1$



3. Colorful Hochster formula

The S-free resolution of $K[\Delta] = S/I_\Delta$ is unnecessarily long:

THEOREM (Auslander-Buchsbaum 1959)

It will always take

$$\underbrace{\#\text{variables in } S}_{\#\text{vertices in } \Delta} - \underbrace{\operatorname{depth} k[\Delta]} \quad \text{syzygy steps}$$

$\operatorname{depth} k[\Delta]$ = max. length of a regular sequence $\theta_1, \theta_2, \ldots, \theta_s$ in $k[\Delta]_+$

θ_1: nonzero divisor

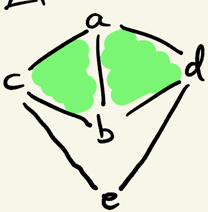
θ_2: nonzero divisor mod θ_1

θ_s: nonzero divisor mod $\theta_1, \ldots, \theta_{s-1}$

$$\geq \#\text{vertices in } \Delta - (\dim \Delta + 1)$$

sometimes huge!

$\Delta =$



has $\operatorname{depth} k[\Delta] = 2$, so S-free res'n had $5-2=3$ syzygy steps

There are shorter resolutions of $k[\Delta]$ ...

Given any proper t-coloring of the graph/1-skeleton of Δ
(= vertices + edges)

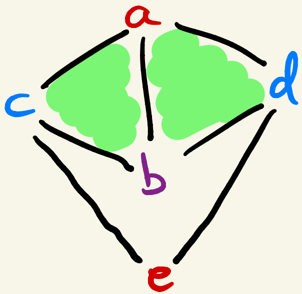
define colorful parameters $\Theta_1, \Theta_2, \ldots, \Theta_t \in k[\Delta]$

$$\Theta_i := \sum_{\substack{\text{vertices } j \\ \text{of color } i}} x_j$$

Then $k[\Delta]$ is a fin. gen'd module over $A = k[z_1, z_2, \ldots, z_t]$,
z_i acting on $k[\Delta]$ as multiplication by Θ_i.

The (minimal) A-free resolution has length (by Auslander - Buchsbaum Thm)

$$t - \operatorname{depth} k[\Delta] \leq t = \#\text{ colors}$$

$\Delta =$ 

has colorful parameters

$\Theta_1 = a + e$

$\Theta_2 = c + d$

$\Theta_3 = b$

minimal A-free resolution over $A = k[z_1, z_2, z_3]$:

$$0 \leftarrow k[\Delta] \leftarrow \underset{0\text{th}}{A^1 \oplus A(-1)^2 \oplus A(-2)^1} \leftarrow \underset{1\text{st}}{A(-2)^1 \oplus A(-3)^1} \leftarrow 0$$

$$\Rightarrow \mathrm{Hilb}(k[\Delta], t) = \mathrm{Hilb}(A, t) \cdot \left(t^0 + 2t^1 + t^2 - (t^2 + t^3)\right)$$

$$\overset{\checkmark}{=} \frac{1}{(1-t)^3}(1 + 2t - t^3)$$

THEOREM (Adams-R. 2020) "Colorful Hochster formula"

For any proper t-coloring of the graph of Δ, minimal free A-resolution of $k[\Delta]$ has for each color subset $T \subseteq \{1,2,\ldots,t\}$ the color-selected subcomplex

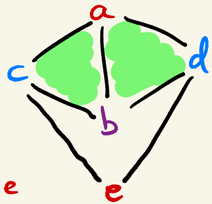
$$\Delta_T := \{ F \in \Delta : \text{colors of } F \subseteq T \}$$

contributing $\dim_k \tilde{H}^{\#T-1-i}(\Delta_T ; k)$

(reduced) simplicial homology

copies of $S(-\#T)$ to the i^{th} syzygies.

(= Hochster's formula for the trivial coloring)

$\Delta =$ 

$\theta_1 = a+e$

$\theta_2 = c+d$

$\theta_3 = b$

$$\begin{array}{ccccccccc} & & & & \text{0th} & & \text{1st} & & \\ 0 & \leftarrow & k[\Delta] & \leftarrow & \begin{array}{c} A^1 \\ \oplus \\ A(-1)^2 \\ \oplus \\ A(-2)^1 \end{array} & \leftarrow & \begin{array}{c} A(-2)^1 \\ \oplus \\ A(-3)^1 \end{array} & \leftarrow & 0 \end{array}$$

$\tilde{H}^{0-1-0}(\Delta_\emptyset) = k^1$

$\{\emptyset\}$

$\tilde{H}^{1-1-0}(\Delta_1) = k^1$

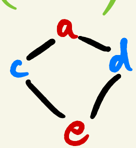
a

e

$\tilde{H}^{1-1-0}(\Delta_2) = k^1$

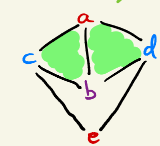
c d

$\tilde{H}^{2-1-0}(\Delta_{12}) = k^1$



$\tilde{H}^{2-1-1}(\Delta_{13}) = k^1$

$\tilde{H}^{3-1-1}(\Delta_{123}) = k^1$



4. Universal parameters

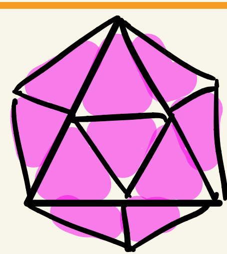
Two complaints:

- What if $d = \dim\Delta + 1$, but we can't properly d-color the graph of Δ?
  Can we still find d parameters $\theta_1, \theta_2, \ldots, \theta_d$ to resolve with?

- What if Δ has some group G of symmetries?
  Can we make a resolution that helps describe $k[\Delta]$
  as a G-representation in each degree $k[\Delta]_d$?

$\Delta =$ boundary of icosahedron

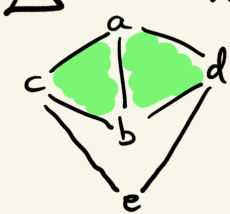
(a polytope, woo-hoo!)



not 3-colorable!

has symmetry group $G = H_3$ with $\#G = 120$

$\Delta =$



has symmetry group

$$G = \{e, (a,b), (c,d), (a,b)(c,d)\}$$

$$= \langle (a,b), (c,d) \rangle$$

$$\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

It helps to have parameters $\theta_1, \theta_2, \ldots, \theta_d \in k[\Delta]$ fixed pointwise by all symmetries of Δ.

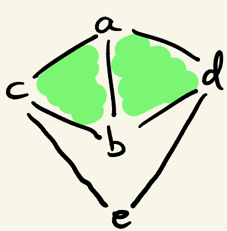
DEFINITION (De Concini–Eisenbud–Procesi 1982, Garsia–Stanton 1984, D.E. Smith 1990, Herzog–Moradi 2020)

The **universal parameters** $\theta_1, \theta_2, \ldots, \theta_d$ in $k[\Delta]$ are

$$\theta_i := \sum_{\substack{\text{faces } F \in \Delta \\ \text{with } \#F = i}} \underline{x}^F \qquad \text{for } i = 1, 2, \ldots, d$$

$\underline{x}^F = \prod_{j \in F} x_j$

where $d = \dim \Delta + 1$

$\Delta =$ 

$$k[\Delta] = k[a,b,c,d,e] / (cd, ae, be)$$

$$\theta_1 = a+b+c+d+e$$

$$\theta_2 = ab+ac+bc+ad+bd+ce+de$$

$$\theta_3 = abc+abd$$

(easy)
PROPOSITION $k[\Delta]$ is fin. gen'd over $A = k[z_1, \ldots, z_d]$, z_i acting by the universal parameter Θ_i

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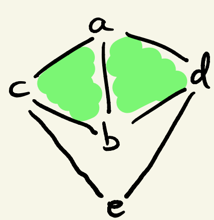
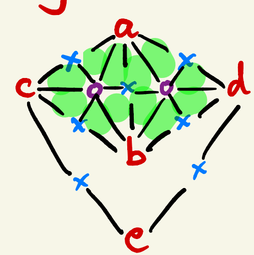
THEOREM (D.E. Smith 1990 for Δ pure, Adams–R. 2020 in general)

The universal parameters detect depth:

$$\text{depth } k[\Delta] = \max\{\delta : \Theta_1, \Theta_2, \ldots, \Theta_\delta \text{ are a regular sequence in } k[\Delta]\}$$

The universal parameters Θ_i are closely related to the colorful parameters for the natural d-coloring of $\mathrm{Sd}\,\Delta :=$ barycentric subdivision of Δ

$\Delta =$ 

$\mathrm{Sd}\,\Delta =$ 

CONJECTURE (Adams–R. 2020) The shape of the resolution of $k[\Delta]$ over the universal parameters Θ_i

is the same as those predicted by the colorful Hochster formula for the resolution of $k[\mathrm{Sd}\Delta]$ over its colorful parameters.

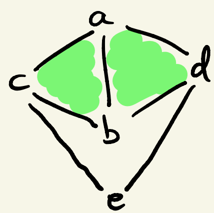
For each subset $T \subseteq \{1,2,\ldots,d\}$, the subcomplex $(\mathrm{Sd}\,\Delta)_T$ contributes $\dim_k \tilde{H}^{\#T-1-i}((\mathrm{Sd}\Delta)_T\,;k)$ copies of

- proven: $A(-\#T)$ to the i^{th} syzygies of $k[\mathrm{Sd}\Delta]$ over colorful parameters,
- conjecture: $A(-\sum_{j\in T} j)$ to the i^{th} syzygies of $k[\Delta]$ over universal parameters.

$$0 \leftarrow k[\Delta] \leftarrow \overset{0\text{th}}{\begin{matrix} A^1 \\ \oplus \\ A(-1)^4 \\ \oplus \\ A(-2)^6 \\ \oplus \\ A(-3)^4 \\ \oplus \\ A(-4)^1 \end{matrix}} \longleftarrow \overset{1\text{st}}{\begin{matrix} A(-4)^1 \\ \oplus \\ A(-5)^2 \\ \oplus \\ A(-6)^1 \end{matrix}} \leftarrow 0$$

resolved over universal $\theta_1, \theta_2, \theta_3$

$\Delta =$



degree	1	2	3

$A = k[z_1, z_2, z_3]$

$\theta_1 = a+b+c+d+e$

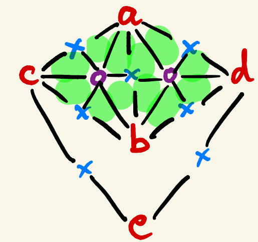
$\theta_2 = ab+ac+bc+ad+bd+ce+de$

$\theta_3 = abc+abd$

$$\Longrightarrow \mathrm{Hilb}(k[\Delta], t) = \mathrm{Hilb}(A, t) \cdot \left(t^0 + 4t^1 + 6t^2 + 4t^3 + t^4 - (t^4 + 2t^5 + t^6)\right)$$

$$= \frac{1}{(1-t)(1-t^2)(1-t^3)}\left(1 + 4t + 6t^2 + 4t^3 - 2t^5 + t^6\right) \overset{\checkmark}{=} \frac{1+2t-t^3}{(1-t)^3}$$

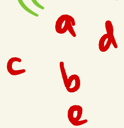
$$0 \leftarrow k[\Delta] \leftarrow \overset{0^{\text{th}}}{A^1} \oplus A(-1)^4 \oplus A(-2)^6 \oplus A(-3)^4 \oplus A(-4)^1 \longleftarrow \overset{1^{\text{st}}}{A(-4)^1} \oplus A(-5)^2 \oplus A(-6)^1 \leftarrow 0$$

$\mathrm{Sd}\,\Delta =$ 

$\tilde H^{0-1-0}\big((\mathrm{Sd}\Delta)_\emptyset\big) = k^1$

$(\mathrm{Sd}\Delta)_\emptyset = \{\emptyset\}$

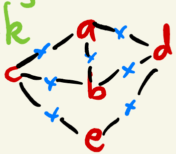
$\tilde H^{1-1-0}\big((\mathrm{Sd}\Delta)_1\big) = k^4$



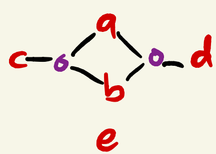
$\tilde H^{1-1-0}\big((\mathrm{Sd}\Delta)_2\big) = k^6$

$\tilde H^{1-1-0}\big((\mathrm{Sd}\Delta)_3\big) = k^1$

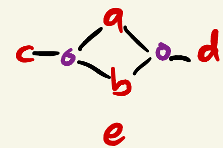
$\tilde H^{2-1-0}\big((\mathrm{Sd}\Delta)_{12}\big) = k^3$



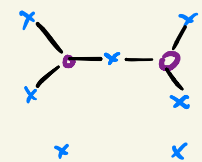
$\tilde H^{2-1-0}\big((\mathrm{Sd}\Delta)_{13}\big) = k^1$



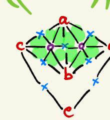
$\tilde H^{2-1-1}\big((\mathrm{Sd}\Delta)_{12}\big) = k^1$



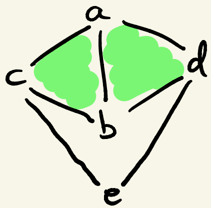
$\tilde H^{2-1-1}\big((\mathrm{Sd}\Delta)_{23}\big) = k^2$



$\tilde H^{3-1-1}\big((\mathrm{Sd}\Delta)_{123}\big) = k^1$



$\Delta =$



$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$G = \langle (a,b), (c,d) \rangle$

$\alpha \mapsto -1 \qquad \gamma \mapsto -1$

$$0 \leftarrow k[\Delta] \leftarrow \begin{array}{rl} 1 & A^1 \\ & \oplus \\ 2+\alpha+\gamma & A(-1)^4 \\ & \oplus \\ 2+\alpha+2\gamma+\alpha\gamma & A(-2)^6 \\ & \oplus \\ 1+\alpha+\gamma+\alpha\gamma & A(-3)^4 \\ & \oplus \\ \alpha\gamma & A(-4)^1 \end{array} \longleftarrow \begin{array}{rl} 1 & A(-4)^1 \\ & \oplus \\ 1+\gamma & A(-5)^2 \\ & \oplus \\ \gamma & A(-6)^1 \end{array} \leftarrow 0$$

G-equivariant resolution of $k[\Delta]$ over universal parameters

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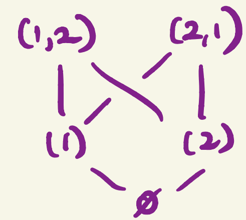
$$\Rightarrow \mathrm{Hilb}_G(k[\Delta], t) = \frac{1}{(1-t)(1-t^2)(1-t^3)} \Big(1 + (2+\alpha+\gamma)t + (2+\alpha+2\gamma+\alpha\gamma)t^2 + (1+\alpha+\gamma+\alpha\gamma)t^3 + \alpha\gamma\, t^4 - \big(t^4 + (1+\gamma)t^5 + \gamma \cdot t^6\big)\Big)$$

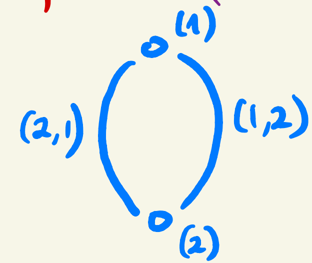
($\Rightarrow$ quasi-polynomial expression for G-representation on $k[\Delta]_d$)

REMARKS :

- Works not just for Stanley-Reisner rings $k[\Delta]$ of simplicial complexes but also for Stanley's face rings of simplicial posets (1991)

EXAMPLE complex of injective words studied by Athanasiadis (2018)





- (S. Murai)

Is $k[\Delta]$ isomorphic as A-module over universal parameters to $k[\mathrm{Sd}\,\Delta]$ as A-module over colorful parameters ?

(i.e. do maps also coincide in the two resolutions ?)

Thanks for
your attention!