ECCO 2018 Vic Reiner Lecture 1 exercises

① Let V be a $\mathbb{C}$-vector space and $V \xrightarrow{\pi} V$ a $\mathbb{C}$-linear map which is idempotent : $\pi^2 = \pi$

(a) Show that one has a $\mathbb{C}$-vector space decomposition

$$V = \pi(V) \oplus (1_V - \pi)V$$

where $\pi(V) = \mathrm{im}(\pi)$

and $(1_V - \pi)(V) = \ker(\pi)$

(b) Deduce that $\dim_{\mathbb{C}}(\mathrm{im}(\pi)) = \mathrm{Trace}_V(\pi)$ (assume $\dim_{\mathbb{C}} V$ finite here)

(c) Show that for any representation

$$G \xrightarrow{\rho} GL(V)$$

of a finite group G on a (finite dimensional) $\mathbb{C}$-vector space V, the averageing map

$$V \xrightarrow{\pi_G} V$$
$$v \longmapsto \frac{1}{|G|} \sum_{\sigma \in G} \rho(\sigma)(v)$$

is idempotent, and has image $\mathrm{im}(\pi_G) = V^G$ the G-fixed subspace

(d) Deduce that in the setting of (c), one has

$$\dim_{\mathbb{C}}(V^G) = \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Trace}_V(\rho(\sigma))$$ (used in lecture)

(e) Use (d) to prove Burnside's lemma: When a finite group G permutes a finite set X, the number of G-orbits on X is

$$\frac{1}{|G|} \sum_{\sigma \in G} |\{x \in X : \sigma(x) = x\}|.$$

(HINT: Consider a vector space V with $\mathbb{C}$-basis $\{e_x\}_{x \in X}$ and $\sigma(e_x) = e_{\sigma(x)}$. What is $\dim_{\mathbb{C}}(V^G)$? Can you compute $\mathrm{Trace}_V(\sigma)$ for $\sigma \in G$?)

(2) Let $V = \mathbb{C}^2$ with $\mathbb{C}$-basis $\{b, w\}$

and $V^{\otimes n} = \bigoplus_{k=0}^{n} (V^{\otimes n})_k$ where $(V^{\otimes n})_k$ has $\mathbb{C}$-basis $\{e_S\}_{S \in \binom{[n]}{k}}$

having action of $\mathfrak{S}_n$ positionally via $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$.

(a) Prove that a permutation $\sigma \in \mathfrak{S}_n$ has $\sigma(e_S) = e_S$

$\iff$ the subset S is a union of the cycles of σ

(b) Deduce that $\sum_{k=0}^{n} q^k \, \mathrm{Trace}_{(V^{\otimes n})_k}(\sigma) = \prod_{\substack{\text{cycles } C \\ \text{of } \sigma}} (1 + q^{|C|})$.

(used in lecture)

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(3) Recall the map $(V^{\otimes n})_k \xrightarrow{U_k} (V^{\otimes n})_{k+1}$ defined in lecture.

$$e_S \longmapsto \sum_{\substack{T \in \binom{[n]}{k+1}: \\ S \subset T}} e_T$$

(a) Prove that U_k commutes with the $\mathfrak{S}_n$-action on $V^{\otimes n}$

(used in lecture)

(b) Prove that the map $(V^{\otimes n})_{k+1} \xrightarrow{D_{k+1}} (V^{\otimes n})_k$

$$e_T \longmapsto \sum_{\substack{S \in \binom{[n]}{k}: \\ S \subset T}} e_S$$

is actually the transpose/adjoint map $D_{k+1} = U_k^T$ with respect to our usual bases on $(V^{\otimes n})_k$.

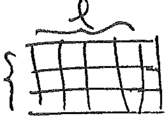
(c) Explain why this implies $D_{k+1} U_k$ and $U_{k-1} D_k$ are both symmetric and nonnegative definite (all eigenvalues ≥ 0)

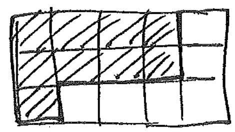
(d) Prove that $(D_{k+1} U_k - U_{k-1} D_k)(e_S) = (n-2k) e_S$ for any $S \in \binom{[n]}{k}$,

and hence $D_{k+1} U_k = U_{k-1} D_k + (n-2k) \mathbb{1}_{(V^{\otimes n})_k}$

(e) Explain why this implies $D_{k+1} U_k$ is positive definite for $k < \frac{n}{2}$.

(f) Explain why this implies U_k is injective for $k < \frac{n}{2}$ (used in lecture)

(4) When $G = \mathfrak{S}_k[\mathfrak{S}_\ell] \subset \mathfrak{S}_{k\ell}$, recall from lecture that the G-orbits $2^{[k\ell]}/G$ biject with Ferrers diagrams inside $k\{$  $\}$ (of width ℓ), that is, to number partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ with $0 \leq \lambda_j \leq \ell$

e.g. $\lambda = (4,4,1) \longleftrightarrow$ 

$k=3$
$\ell=5$

Let $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_k$, so that the rank-generating function for $2^{[k\ell]}/G$ is $r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n = \sum_{\text{such } \lambda} q^{|\lambda|} \underset{\text{DEF'N}}{=:} \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q$, called a q-binomial coefficient.

(a) Prove the q-Pascal recurrences (here $N := k+\ell$, or $\ell := N-k$)

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = q^k \begin{bmatrix} N-1 \\ k \end{bmatrix}_q + \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_q$$

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \begin{bmatrix} N-1 \\ k \end{bmatrix}_q + q^{N-k} \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_q$$

(b) Prove the formula $\begin{bmatrix} N \\ k \end{bmatrix}_q = \dfrac{[N]!_q}{[k]!_q \, [N-k]!_q}$

where $[N]!_q := [N]_q [N-1]_q \cdots [3]_q [2]_q [1]_q$

and $[m]_q := 1 + q + q^2 + \cdots + q^{m-1} = \dfrac{1-q^m}{1-q}$

(c) Prove that when $q = p^d$ is the power of a prime, and hence the cardinality of a finite field $\mathbb{F}_q$,

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \#\{k\text{-dimensional } \mathbb{F}_q\text{-linear subspaces of } \mathbb{F}_q^N\}.$$