(1)

ECCO 2018 Vic Reiner Lecture 4 Exercises

① We want to understand the coinvariant algebra for the dihedral group $G = I_2(m) \xrightarrow{\rho_{\mathrm{ref}}} GL_2(\mathbb{C})$

Recall from the Lecture 2 Exercise 2 that $\rho_{\mathrm{ref}} \cong \rho^{(1)}$

where $G = I_2(m) \xrightarrow{\rho^{(1)}} GL_2(\mathbb{C}) = GL(V)$ where V has basis $\{x, y\}$

$\quad\; = \langle s, r \mid s^2 = r^m = e, srs = r^{-1} \rangle$

sends $s \longmapsto \begin{array}{c|cc} & x & y \\ \hline x & 0 & 1 \\ y & 1 & 0 \end{array}$ i.e. $s(x) = y$, $s(y) = x$

$r \longmapsto \begin{array}{c|cc} & x & y \\ \hline x & \zeta & 0 \\ y & 0 & \zeta^{-1} \end{array}$ $\quad r(x) = \zeta x$, $r(y) = \zeta^{-1} y$

$\zeta := e^{2\pi i/m}$

(a) Check that $\mathbb{C}[x,y]^G \supset \mathbb{C}[\underbrace{xy}_{f_1}, \underbrace{x^m + y^m}_{f_2}]$

degrees: $d_1 = 2$ $\quad d_2 = m$

It can be shown that the inclusion above is actually an equality, but let's just assume this.

(b) Explain why the coinvariant algebra

$$\mathbb{C}[x,y]/(f_1, f_2) = \mathbb{C}[x,y]/(xy, x^m + y^m)$$

has the following $\mathbb{C}$-basis in various degrees:

degree	0	1	2	...	$m-1$	m
$\mathbb{C}$-basis	1	x, y	x^2, y^2		x^{m-1}, y^{m-1}	x^m $(= -y^m)$

(c) Prove these fake degree formulas $f^\psi(q)$:

$$f^{\mathbb{1}}(q) = 1, \quad f^{\det}(q) = q^m, \quad f^{\rho_s}(q) = q^{\frac{m}{2}} = f^{\rho_t}(q) \text{ for } m \text{ even}$$

$$f^{\rho^{(j)}}(q) = q^j + q^{m-j} \text{ for } j = 1, 2, \ldots, \left\lfloor \tfrac{m-1}{2} \right\rfloor$$

(d) Check that the answers in (c) are consistent for $m = 3$ with our previous calculations of $f^\psi(q)$ for $\mathfrak{S}_3 = I_2(3)$.

2. Let ζ be a primitive d^{th} root-of-unity, such as $\zeta = e^{\frac{2\pi i}{d}}$

(a) Show that for positive integers a, b having $a \equiv b \bmod d$, one has
$$\lim_{q \to \zeta} \frac{[a]_q}{[b]_q} = \begin{cases} a/b & \text{if } a \equiv b \equiv 0 \bmod d \\ 1 & \text{if } a \equiv b \not\equiv 0 \bmod d. \end{cases}$$

(b) We want to understand how a general q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ behaves when one sets $q = \zeta$.

Write $n = n'd + n''$ uniquely with $n', n'' \in \mathbb{Z}$ and $0 \le n'' \le d-1$

$k = k'd + k''$ uniquely with $k', k'' \in \mathbb{Z}$ and $0 \le k'' \le d-1$

that is, let n', k' be the quotients and n'', k'' be the remainders when dividing n, k by d.

Prove that
$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\zeta} = \binom{n'}{k'} \cdot \begin{bmatrix} n'' \\ k'' \end{bmatrix}_{q=\zeta}$$

(and hence one only needs to understand how $\begin{bmatrix} n' \\ k' \end{bmatrix}_{q=\zeta}$ behave when $0 \le k'', n'' \le d-1$)

(c) Use part (b) to prove the CSP result for $X = \binom{[n]}{k}$ ⟲ $C = \langle (1, 2, \ldots, n) \rangle$ and $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ via brute force evaluation of $[X(q)]_{q=\zeta^k}$, and brute force enumeration of $|\{x \in X : c^k(x) = x\}|$!

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3. Prove that the two statements in Springer's Theorem are equivalent: the isomorphism of $G \times C$-representations versus $\chi_\rho(c) = [f^\rho(q)]_{q=\zeta}$.