Low degrees in a Gröbner basis may force the Koszul property.

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0. Introduction.

The origin of this paper was a question about precisely why the existence of a quadratical Gröbner basis for an homogeneous ideal in a non-commutative polynomial ring guarantees that the quotient by that ideal is a Koszul algebra. The 'conventional' answer is 'because there is an associated spectral sequence (whose existence is guaranteed by some handwaving)'. The demand for somewhat more robust proofs is not unreasonable; in particular, since the involved spectral sequences arise from somewhat unusual filtrations, which do not always yield conventional spectral sequences.

The koszulness result does hold, and in fact, the present paper contains two proofs thereof (and of a stronger variant, also involving Veronese subrings). The philosophy behind the first proof is to substantiate the following fairly 'wellknown facts', sufficiently well:

- A spectral sequence essentially arises whenever we have a filtered chain complex.
- A Gröbner basis (in a rather generalised meaning) essentially starts a 'filt-good' (free) resolution with respect to some filtration (i. e., one whose graded associated complex also is exact).
- Hence, Gröbner bases give rise to spectral sequences.

The 'essentially' in the first point, however, is not quite obvious; and it also should be modified a little. The 'obvious' and general fact is that the components of the graded associated of the homology of a filtered complex are subquotients of the corresponding components of the homology of the graded associated of that complex. Spectral sequences are a means of passing from the latter to the former; and the existence of a spectral sequence is often employed principally to establish that subquotient property.

However, in my opinion, this is putting the cart before the horse. In order for a spectral sequence to exist, you first must have a filtered complex; which always is enough to guarantee the subquotient property. You then also have whole families of 'intermediate' subquotients, and of morphisms between some of these (for different components). Depending on the properties of the filtration, you may or may not be able to organise some of these intermediates into a sequence of complexes, where the homology of one essentially is its successor, and such that these intermediates allow you to pass from the homologies of the graded associated to the graded associates of the homology in a calculable and controled manner. If you may, the sought spectral sequence does exist; but whether or not it does, the subquoutient property holds.

In particular, the subquotient property is quite sufficient for the aforementioned results about Koszul algebras. Indeed, if an ideal in a non-commutative monomial ring has a Gröbner basis consisting of homogeneous polynomials of degree two, then the corresponding quotient ring A must be Koszul, since then its graded associated $G(A)$ is a quadratically related noncommutative monomial algebra, which indeed is Koszul by [F75], and since moreover a Koszul algebra is characterised by having 'torsion only on the diagonal'. We thus directly get the implications

$$G(A) \text{ is Koszul} \implies \operatorname{Tor}^{G(A)}(k,k)_i = \operatorname{Tor}^{G(A)}(k,k)_{i,i} \implies \operatorname{Tor}^{A}(k,k)_i = \operatorname{Tor}^{A}(k,k)_{i,i} \implies A \text{ is Koszul.}$$

As an afterthought, we also may prove the existence of corresponding spectral sequences for quotients with homogeneous ideals in the noncommutative polynomial ring.

The second proof for the Koszulness covers less of the general theory, but offers greater possibilities for calculations. It mainly consists of employing the known minimal free resolutions of the augmentation field k for non-commutative monomial rings in order to 'construct' a filt-good free resolution of k for any (not necessarily homogeneous) quotient of that polynomial ring, for which a Gröbner basis is 'sufficiently wellknown'. A known finite Gröbner basis always should suffice; but recall that the ideal need not possess one, although it in itself is finitely generated. It should also be enough that we have an algorithm, which for this ideal and any total degree d allows us to calculate all elements of total degree $\leq d$ in a finite time. (Do note, that such algoritms not always exist; if they did, then all Thue systems would have solvable word-problems; which they do not all have.) For our specific situation, we actually find that the filt-good resolution is a minimal one, and generated 'on the diagonal' (i. e., is such that the i'th constituend is generated in total degree i); whence we get the implications

$$G(A) \text{ Koszul} \implies (G(X)_i \text{ is generated in degree } i) \implies (X_i \text{ is generated in degree } i) \implies A \text{ Koszul}.$$

With both approaches, we actually may get a somewhat sharper result, generalising a classical result of Mumford almost 'for free'. If A is a quotient of a twosided non-commutative homogeneous ideal in a polynomial ring, and that ideal has a finite Gröbner basis D, then it also has a finite highest degree h of the elements in D. Hence, and by [B78, théorème 1], the graded associated algebra $G(A)$ has a finite rate of growth of the total degrees of its Tor groups:

$$\operatorname{rate} G(A) \stackrel{def}{=} \sup_{(m,d)} \frac{d-1}{m-1} \leq h-1$$

with the supremum taken over the $(m,d) \in (\mathbf{N} - \{0,1\})^2$ such that $\operatorname{Tor}_m^{G(A)}(k,k)_d \neq 0$.

Now, the same arguments as in either of the proofs of the Koszulness yield that $\operatorname{rate} A \leq \operatorname{rate} G(A)$. Thus, as is shown in [B86], all Veronese subrings $A^{(s)} := \bigoplus_{i\in\mathbf{N}} A_{is}$ with sufficiently high s are Koszul algebras:

Theorem 1. *If k is a field, and $A = k\langle T_1, \ldots, T_n\rangle/\mathfrak{a}$, where $\mathfrak{a}$ is a twosided ideal with a Gröbner basis D, then $A^{(s)}$ is Koszul for all $s \geq \sup\{\deg x : x \in D\} - 1$.*

The original result is the special case where $\sup\{\deg x : x \in D\} \leq 2$.

The disposition of the rest of this paper is as follows. In sections 1, 2, and the beginning of section 3, definitions and some basic properties for augmented algebras, complexes, and monoid orders and filtrations are collected. The rest of section 3 contains proofs of the general subquotient relation between the graded associates of homologies and homologies of graded associates of monoid filtered complexes, and of torsion modules. Section 4 treats 'Gröbner bases', but in an unusual way, and in much more general situations than usual; in particular, it concludes the first proof of theorem 1. Section 5 treats the spectral sequences. Finally, section 6 introduces some concrete resolutions, leading to the the second proof.

In particular, a reader just interested in theorem 1 may choose to forego the entire section 5 and most of section 4, and either most of section 3 and (the entire) sections 4 and 5, or section 6, without loss of consistency. However, on the other hand, the reader will be assumed to be acquaintainced with ordinarily filtered or graded associated modules and complexes. For section 5, but not otherwise, the reader may need some understanding of spectral sequences. All this yields a somewhat uneven treatment; some aspects which may be new are treated in detail, while other areas are glossed over.

Thus, a reader with vague ideas about spectral sequences should have little trouble to follow most of the paper. Likewise, except once, we here are not concerned with any algorithms for calculation of Gröbner bases, but only with consequences of the existence of one Gröbner basis.

The general theory indeed covers much more than what ordinarily is referred to as Gröbner basis theory.

Example. Let $S = k[x_1, \dots, x_4]$ be a (commutative) polynomial ring over a field k, and

$$\mathfrak{a} = (x_2^3 + x_1^2 x_2 - x_3^2, x_2^2 x_4 - x_1 x_4),$$

the S-ideal generated by these two polynomials. Now, S is graded and filtered by (the natural) total degree, with $F_d(S) = \{f \in S : \deg f \leq d\}$; but $\mathfrak{a}$ is not so graded. Its graded associated ideal is

$$G(\mathfrak{a}) = (x_2^3 + x_1^2 x_2, x_2^2 x_4, x_2 x_3^2 x_4 - x_1^3 x_4);$$

indeed, $\{x_2^3 + x_1^2 x_2 - x_3^2, x_2^2 x_4 - x_1 x_4, x_2 x_3^2 x_4 - x_1^3 x_4 - x_1^2 x_4\}$ is a reduced 'Gröbner basis' for $\mathfrak{a}$ (in the general sense employed in this paper). Thus, with $A = S/\mathfrak{a}$ and thus (essentially) $G(A) = S/G(\mathfrak{a})$, for each $i \in \mathbf{N}$, $\mathrm{Tor}^A(k,k)_i$ is a subquotient of $\mathrm{Tor}^{G(A)}(k,k)_i$ (whence in particular the Betti numbers for A are bounded by those for $G(A)$). Moreover, S is (*inter alia*) also graded by the x_4-degree of each monomial, and $\mathfrak{a}$ is homogeneous with respect to this particular grading. Thus, both the $\mathrm{Tor}^A(k,k)_i$ and the $\mathrm{Tor}^{G(A)}(k,k)_i$ are naturally x_4-degree graded, making both Tor bigraded; whence in fact $\mathrm{Tor}^A(k,k)_{i,j}$ is a subquotient of $\mathrm{Tor}^{G(A)}(k,k)_{i,j}$ for each pair (i,j).

1. Augmented algebras.

We start by fixing some notation.

Let k be a field, and A a k-algebra, by which we mean a unitary and associative but not necessarily commutative ring, say with the unit (multiplicative neutral element) 1_A, together with a k (vector) space structure on A, compatible with the arithmetic operations $+_A$ and $\cdot_A$ (in the usual manner). In particular, k may be considered as a subring of A, by the natural identification of any $\kappa \in k$ with $\kappa 1_A \in A$; and we also demand this k copy to be in the centre of A; i. e., that

$$x \cdot_A \kappa 1_A = \kappa x$$

for any $\kappa \in k$ and $x \in A$.

An *augmentation* of A is a ring epimorphism

$$\varepsilon : A \twoheadrightarrow k,$$

such that $\varepsilon(\kappa 1_A) = \kappa$ for all $\kappa \in k$. Two examples are the equicharacteristic (commutative) local rings, and various (ordinarily) graded algebras

$$A = \bigoplus_{n=0}^{\infty} A_n$$

with $A_0 = k$. The *augmentation ideal* is $\mathrm{Ker}\,\varepsilon$; for the two examples, this is the unique maximal ideal or the ideal $\bigoplus_{n\geq 1} A_n$, respectively.

The augmentation forms 'the right end' of some (free) resolution

$$\dots \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 = A \xrightarrow{\varepsilon} k;$$

a long exact sequence where all the X_i are free (say left) A-modules.

2. Monoid orders and filtrations.

Throughout this section, let $(M, \cdot, 1_M)$ be any monoid,[1] i. e., set M equipped with an associative binary operation $\cdot$, and a neutral element (a unit) 1_M with respect to this operation (thus satisfying $1_M \cdot a = a \cdot 1_M = a$ for all $a \in M$). A *strict monoid order* $\leq$ on M is a (total) order of M (as a set), which respects the monoid structure in the following sense: For any $a, b, c \in M$ such that $a < b$ (i. e., that $a \leq b$ but $a \neq b$), we have

$$\begin{aligned} a \cdot c &< b \cdot c, \\ c \cdot a &< c \cdot b, \text{ and} \\ 1_M &\leq c. \end{aligned}$$

$(M, \cdot, 1_M, \leq)$ then is a *strictly ordered monoid*, which we in the sequel by the usual slight abuse of notation just call M.

Remark. There is a somewhat more natural and slightly weaker concept of monoid order, where we only demand $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. However, what I here call strictness is needed in order to make multiplication well-defined in the graded associated ring $G(R)$ (*vide infra*). It is actually equivalent to M being a cancellative monoid (so that $a \neq b \implies a \cdot c \neq b \cdot c \wedge c \cdot a \neq c \cdot b$).

Lemma 2.1. *If $(M, \leq)$ and $(M', \leq')$ are strictly ordered monoids, and $\phi : M \longrightarrow M'$ is any monoid homomorphism, then the order $\leq''$ defined on M by the rule*

$$a \leq'' b \overset{def}{\iff} \big(\phi(a) <' \phi(b)\big) \vee (\phi(a) = \phi(b) \wedge a \leq b)$$

also is a strict monoid order.

Note, that by the assumptions ϕ maps 1_M to $1_{M'}$ and respects multiplication, but in general does not respect order.

Proof. Say $a, b, c \in M$ with $a <'' b$. There are two distinct cases: Either $\phi(a) <' \phi(b)$, or $\phi(a) = \phi(b)$ but $a < b$. In the first case, by the strictness of $\leq'$,

$$\phi(ac) = \phi(a)\phi(c) <' \phi(b)\phi(c) = \phi(ac);$$

and in the second case we (similarly and by the strictness of $<$) have $\phi(ac) = \phi(bc) \wedge ac < bc$. Thus, and analogously, indeed $ac <'' bc$ and $ca <'' cb$. □

An *M-filtration* of a (unitary but not necessarily commutative) ring R is a family $\big(F_a(R)\big)_{a \in M}$ of additive subgroups of R, satisfying

$$\begin{aligned} 1_R &\in F_{1_M}(R), \\ x \in F_a(R) \wedge y \in F_b(R) &\implies xy \in F_{a \cdot b}(R), \text{ and} \\ a \leq b &\implies F_a(R) \subseteq F_b(R) \end{aligned}$$

for all $a, b \in M$.

[1] For a reader acquaintained with the usual Gröbner basis theory for ideals in commutative or non-commutative polynomial rings, the first application would be the monoid of all monomials; i. e., they could consider $M = [x_1, \ldots, x_n]$ or $M = \langle T_1, \ldots, T_n \rangle$, the free commutative or noncommutative monoid on n generators, respectively. They then may note that indeed the strict monoidal wellorders on either of these two M are precisely the usual 'admissible term-orders'.

It is an easy consequence of the definitions, that then $F_{1_M}(R)$ is a subring (in the unitary sense) of R, and that each $F_a(R)$ is an $F_{1_M}(R)$-module. Furthermore, in most of our present applications, R will be a k-algebra for a fixed field k; and it will be natural also to demand that

$$k \subseteq F_{1_M}(R).$$

As a consequence, each $F_a(R)$ then also is a k (vector) space.

For each $a \in M$, let

$$F_{<a}(R) \stackrel{def}{=} \{0_R\} \cup \bigcup_{\substack{b \in M \\ b<a}} F_b(R).$$

Then clearly $F_{<a}(R)$ is an additive subgroup of $(F_a(R), +)$ for each $a \in M$; and actually is a $F_{1_M}(R)$-submodule (and in the k-algebra case a k-subspace), too. Moreover, by the strictness of $<$, for any $a, b \in M$, we get

$$x \in F_{<a}(R) \wedge y \in F_b(R) \implies xy \in F_{<a \cdot b}(R) \wedge yx \in F_{<b \cdot a}(R). \tag{2.1}$$

Thus, and putting

$$G(R)_a \stackrel{def}{=} F_a(R)/F_{<a}(R),$$

this is a $F_{1_M}(R)$ bimodule (and, in our case, a k-space); and moreover (2.1) ensures that the R multiplication restriction $F_a(R) \times F_b(R) \longrightarrow F_{a \cdot b}$ induces a multiplication $G(R)_a \times G(R)_b \longrightarrow G(R)_{a \cdot b}$. This makes

$$G(R) \stackrel{def}{=} \bigoplus_{a \in M} G(R)_a$$

to a ring, and actually to an $F_{1_M}(R) \simeq G(R)_{1_M}$ algebra (and here in particular often a k-algebra).

M-filtered R modules, and their graded associated, are defined analogously. An M-filtered (left or right) module homomorphism $f : L \longrightarrow N$ between two such modules should 'respect filt-degrees', i. e., have $f\big(F_a(L)\big) \subseteq F_a(N)$ for all $a \in M$. This gives rise to a graded associated morphism $G(f) : G(L) \longrightarrow G(N)$ (of M-graded $G(R)$-modules). In particular, G is a functor from the category of M-filtered (left, say) R-modules to the one of M-graded (left) $G(R)$-modules.

If N is an M-filtered R-module, and L a submodule of N, then L inherits a structure of M-filtered R-module by the prescription $F_a(L) \stackrel{def}{=} L \cap F_a(N)$ (for any $a \in M$). Likewise, then the quotient module N/L is M-filtered by $F_a(N/L) \stackrel{def}{=} \frac{F_a(N)+L}{L}$. These filtrations are *induced* by the one of N.

In the sequel, also assume that $<$ is a wellordering (so that any nonempty subset of M has a minimal element), and that the considered M-filtrations are *exhaustive*, so that $\bigcup_a F_a(R) = R$, and correspondingly. Then, each $x \in R$ has a well-defined *filt-deg* $\operatorname{fdeg} x \stackrel{def}{=} \min\{a \in M : x \in F_a(R)\}$; and there is a well-defined function $\operatorname{lt} : R \longrightarrow G(R)$ defined by

$$\operatorname{lt}(x) \stackrel{def}{=} x + F_{<\operatorname{fdeg} x}(R) \in G(R)_{\operatorname{fdeg} x} \subseteq G(R).$$

In this case, if $L = L_R$ and $N = {}_RN$ are an M-filtered right respectively left R-module, then these filtrations induce one of the tensor product $L \otimes_R N$ by prescribing that

$$F_a(L \otimes N) \stackrel{def}{=} \{x \in L \otimes N : x \text{ may be written as } \sum_{i=1}^{s} l_i \otimes n_i \text{ with } \operatorname{fdeg}(l_i) \cdot \operatorname{fdeg}(n_i) \leq a \text{ for each } i\}.$$

Definition. An *admissible* order on a monoid is a strict monoid order, which also is a wellorder. From now on, for an admissibly ordered monoid, we only consider exhaustive filtrations with respect to that monoid.

Remarks. Note, that lt is not an homomorphism. In fact, the restriction of lt to $F_{1_M}(R)$ indeed is the natural isomorphism between $F_{1_M}(R)$ and $G(R)_{1_M}$; but if $|M| \geq 2$, then (the entire) f does not even respect addition.

Moreover, readers familiar with Gröbner bases may recognise $\operatorname{lt} x$ as a notation for 'the leading term' of x (if $x \neq 0$). However, they also may feel a bit surprised by the fact that these 'leading terms' do not belong to R. In fact, most common Gröbner basis theories concern algebras $R = \oplus_M R_a$ which already are M-graded, where the filtration is derived from the grading:

$$F_a(R) := \bigoplus_{\substack{b \in M \\ b \leq a}} R_b\,,$$

and where thus R and $G(R)$ are naturally isomorphic, making it somewhat unnecessary to separate them. (The main exception I can recall is the theory for Gröbner bases in the Weyl algebra A_n on n variables and n derivations, where instead $G(A_n)$ is a commutative polynomial ring in $2n$ variables.) However, even in the simplest case, where $R = k[x_1, \ldots, x_n]$ is an ordinary polynomial polynomial ring and M is the set of (monic) monomials in $x_1, \ldots, x_n$, the Gröbner basis technique is applied for an ideal $\mathfrak{a} \subset R$, which in general is not M-homogeneous. Thus, 'the associated monomial ideal' $G(\mathfrak{a})$ is very far from naturally isomorphic to $\mathfrak{a}$.

Finally, such readers may note that for M being a free commutative or noncommutative monoid there is a naturally defined monoid homomorphism $\deg : M \longrightarrow \mathbf{N}$, taking the total degree of any monomial; and that thus lemma 2.1 generalises the formation of a new admissible order out of any given one on M by the prescription 'first consider the total order'; as when DEGLEX is formed from PURELEX on $M = [x_1, \ldots, x_n]$.

3. Filt-good exact sequences and resolutions.

Throughout this section, let M be a fixed strictly wellordered monoid. Then any *M-filtered complex*, i.e., any complex of (increasingly) M-filtered abelian groups

$$\ldots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \ldots$$

with 'differentials' d_i respecting the M-filtration, give rise to an *M-graded associated complex*

$$\ldots \xrightarrow{G(d_{i+2})} G(C_{i+1}) \xrightarrow{G(d_{i+1})} G(C_i) \xrightarrow{G(d_i)} G(C_{i-1}) \xrightarrow{G(d_{i-1})} \ldots$$

(since $G(d_{i+1})G(d_i) = G(d_{i+1}d_i) = 0$).

The complexes may or may not have extra structure, as being graded with respect to some monoid (different from or equal to M), and/or being complexes of modules over some M-filtered ring R. If so, both the filtrations and the differentials are assumed to respect that extra structure. For gradings on filtered complexes, this means that also all $F_a(C_i)$ are graded, and that the d_i take homogeneous elements to elements of the same degree. In this case, the graded associalted complex inherits the extra structure, in the form of an extra (compatible) grading, and/or a $G(R)$-module structure, respectively; all structures with full compatibility.

Now, putting

$$B_i := \operatorname{Im} d_{i+1} \subseteq Z_i := \operatorname{Ker} d_i \subseteq C_i,$$

the subgroups B_i and Z_i receive induced M-filtrations; and so do the homology groups $H_i = H_i(C.) := Z_i/B_i$. For each one of these there is a graded associated object; e. g., $(G(Z_i)_a)_{a \in M}$, with $G(Z_i)_a := F_a(Z_i)/F_{<a}(Z_i)$. (Also any extra structure is inherited by these objects.) Thus, we naturally have two doubly indexed families of graded objects, depending on whether we first take graded associates or first take homology:

$$\big(H_i(G(C.)_a)\big)_{i,a} \text{ and } \big(G(H_i)_a\big)_{i,a}\,, \text{ respectively.}$$

Here, clearly, in general, the former objects should be expected to be larger that the latter, index pair by index pair. The reason is that (up to isomorphisms)[1]

$$G(H_i)_a = G(Z_i/B_i)_a = \frac{G(Z_i)_a}{G(B_i)_a} = \frac{F_{<a}(C_i) + F_a(Z_i)}{F_{<a}(C_i) + F_a(B_i)} = \frac{F_a(Z_i)}{F_{<a}(Z_i) + F_a(B_i)} \,,$$

while

$$\big(H_i(G(C.)_a)\big)_{i,a} = \frac{\operatorname{Ker} d_{i,a}}{\operatorname{Im} d_{i+1,a}} = \frac{F_{<a}(C_i) + F_a(Z_i^{<a})}{F_{<a}(C_i) + F_a(B_i^a)} = \frac{F_a(Z_i^{<a})}{F_{<a}(Z_i^{<a}) + F_a(B_i^a)} \,,$$

where the filtered subgroups $Z_i^{<a}$ and B_i^a of C_i are defined by

$$Z_i^{<a} = \{x \in C_i : d_i(x) \in F_{<a}(C_{i-1})\}, \quad \text{and} \quad B_i^a = d_{i+1}\big(F_a(C_{i+1})\big).$$

Thus, we have the (often strict) inclusions

$$0 \subseteq B_i^a \subseteq F_a(B_i) \subseteq F_a(Z_i) \subseteq F_a(Z_i^{<a}) \subseteq F_a(C_i), \tag{3.1}$$

yielding that indeed

$$G(H_i)_a \text{ (canonically) is a subquotient of } H_i(G(C.)_a), \tag{3.2}$$

which itself is a subquotient of $G(C_i)_a$.

So far, we have employed no spectral sequence theory at all; but we already have derived (3.2), one common application of that theory. In other words, we already have proved

Theorem 2. *Let M be a strictly wellordered monoid, and (C_*, d_*) a complex, which is exhaustively M-filtered. Then, for every $a \in M$ and every integer i such that the i-homology of the complex is defined, the a-component of the graded associated to the i-homology in a canonical manner is a subquotient of the a-component of the i-homology of the graded associated to the complex. In other words, $H_i(G(C.)_a)$ has a subgroup (or submodule) $\overline{Z}_{i.a}$, which has a subgroup $\overline{B}_{i.a}$, such that*

$$G\big(H_i(C.)\big)_a \simeq \frac{\overline{Z}_{i.a}}{\overline{B}_{i.a}} \,; \tag{3.3}$$

and this isomorphism behaves well with respect to morphisms of M-filtered complexes, and to any reasonable extra structure on the original complex.

In particular, if $H_i(G(C.)_a) = 0$ for any such (a, i), then also $G\big(H_i(C.)\big)_a = 0$. □

The next object is to draw the usual homological algebra kinds of conclusions from theorem 2 applied for (co)homology of M-filtered modules. Mostly, this goes through with no trouble; but we should ensure that there are 'enough sufficiently free/projective/flat' modules to create the appropriate complexes for which the theorem is to be applied. Let us focus our attention on the $\operatorname{Tor}_*^R(L_R, {}_RN)$, and on what here 'sufficiency' should mean for free resolutions.

Thus, and for convenience, from now on, let R be a fixed exhaustively M-filtered ring, only consider M-filtered abelian groups where the groups are R-modules and the filtrations are exhaustive R-module filtrations, and make the corresponding demands for M-filtered complexes.

Call an M-filtered complex an *(M)-filt-good exact sequence*, if both the complex and its graded associated complex are exact sequences.

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[1] We freely employ Noether's canonical isomorphisms. In general, if $L_1 \subset L_2 \subset L_3 \subset L_4 \subseteq L$ is an inclusion chain of modules, then $L_3/L_2 \simeq \frac{L_3/L_1}{L_2/L_1}$, which very concretely is a subquotient of L_4/L_1. Likewise, if moreover $L' \subseteq L$, then L/L' has a subquotient $\frac{L_3+L'}{L_2+L'} = \frac{L_3+(L_2+L')}{L_2+L'} \simeq \frac{L_3}{(L_2+L')\cap L_3} = \frac{L_3}{L_2+(L'\cap L_3)}$.

Lemma 3.1. *Let (C_*, d_*) be an M-filtered complex. Then, the following properties are equivalent.*

(a) *(C_*, d_*) is a filt-good exact sequence.*

(b) *$\big(G(C_*), G(d_*)\big)$ is exact.*

(c) *Any cycle z in the complex is a boundary of an element of the same filt-degree as z.*

Proof. $(a) \implies (b)$ is immediate; and $(c) \implies (a)$ is elementary and almost immediate. Also $(b) \implies (c)$ follows by elementary means; but these include a transfinite induction (with respect to the wellordered set M). Indeed, if we assume (b) to hold (in full generality), and, for some C_i (in non-extremal position) and any $a \in M$, that (c) holds for all cycles in $F_{<a}(C_i)$, and consider any cycle $z \in C_i$ with $\operatorname{fdeg} z = a$, then its class $\overline{z} := z + F_{<a}(C_i) \in G(C_i)_a$ is a cycle therein, and thus a boundary by(b). Thus, there is some $y \in F_a(C_{i+1})$, such that $\overline{y} := y + F_{<a}(C_{i+1}) \mapsto \overline{z}$; or, in other words, such that $z' := z - d_{i+1}(y) \in F_{<a}(C_i)$. Say $\operatorname{fdeg} z' = b < a$. Since z' is a cycle, and by the inductive assumption, $z' = d_{i+1}(y')$ for some $y' \in F_b(C_{i+1})$; whence indeed $z = d_{i+1}(y + y')$ and $\operatorname{fdeg}(y + y') = a$. Thus, here, $\big((c)$ holds in $F_{<a}(C_i)\big) \implies \big((c)$ holds in $F_a(C_i)\big)$, which is what is needed for the transfinite induction to go through. □

We shall in particular be interested in the case where C_* also is a free resolution of some M-filtered module N; i. e., where in addition all C_i are R-free, and the complex is equipped with an *augmentation* $\eta : C_0 \longrightarrow N$ (respecting the M-filtrations), such that

$$C_1 \longrightarrow C_0 \longrightarrow N \longrightarrow 0$$

is exact. We of course also want the augmented sequence to be filt-good; i. e., we want

$$G(C_1) \longrightarrow G(C_0) \longrightarrow G(N) \longrightarrow 0$$

to be exact. However, seemingly, even all this does not in itself suffice for the functorial properties we need.

Hence, define any free and M-filtered left R-module X (or, technically, any pair (X, P)) to be *filt-free*, if X has a *specified basis* $P = \{p_j\}_{j\in J}$, such that indeed

$$X = \bigoplus_{j\in J} Rp_j\ ,$$

and that for any $a \in M$ and any $x \in X$ we have the equivalence

$$\big(x \in F_a(X)\big) \iff (x = \sum_{j\in J'} x_j p_j \text{ with } J' \subseteq J, \text{ and } \operatorname{fdeg}(x_j)\cdot\operatorname{fdeg}(p_j) \leq a \text{ for all } j \in J'). \tag{3.4}$$

In other words, after we also have specified the filt-degrees of the X generators in the specified basis P, the filt-degrees of the elements in X are determined in the way one would expect. It is an elementary exercise to verify that any such specification also yields an R-module M-filtration to X; i. e., that (3.4) also works as a recipee for constructing M-filt-free modules.

We reserve the term (M)-*filt-good free resolution* of ${}_RN$ for an M-filtered resolution

$$\ldots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\eta} N \longrightarrow 0$$

which forms an filt-good exact sequence, and where in addition all the C_i are filt-free. Now, this seems to be ‘the right concept’, in the sense that, on the one hand, by the usual kinds of arguments, any (M-filtered left) module $N = {}_RN$ indeed has a filt-good free resolution, while, on the other, such resolution allows the usual kind of comparison homomorphisms, chain homotopies, *et cetera*, as we soon shall see.

For the first property, it is enough to be able to construct a filt-good surjection $f : X \twoheadrightarrow L$ onto any given filtered module L, from some appropriate filt-free X. Do this by considering any basis (R-module generating system) $\{y_j\}_{j\in J}$ for L, say with $\mathrm{fdeg}\, y_j = g_j$, and by constructing a filtration of $X = \bigoplus_J Rp_j$ by means of (3.4) and the prescriptions $\mathrm{fdeg}\, p_j = g_j$, and, finally, by prescribing $f(p_j) = y_j$. (Then, indeed, for any $j \in J$ and any $r \in R \setminus \{0\}$,

$$rp_j \in F_a(X) \implies \mathrm{fdeg}(r)\cdot g_j \le a \implies \mathrm{fdeg}(ry_j) \le a \implies f(rp_j) = ry_j \in F_a(L);$$

and this easily extends to all $x \in X$.)

Similarly, the salient point for the second property is that given any diagramme

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow f & & \\ K & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

of M-filtered modules and homomorphism, where X is filt-free with specified basis $P = \{p_j\}_J$ and the bottom line is filt-good exact, there exists a filtered homomomorphism $h : X \longrightarrow K$, such that $f = g \circ h$. For each p_j, say with $\mathrm{fdeg}\, p_j = g_j$, we have $\mathrm{fdeg}\big(f(p_j)\big) \le g_j$, and thus by the goodness some $y_j \in F_a(K) \cap g{-}1(f(p_j))$; choose $h(p_j) := y_j$.

Since M-filtrations also are compatible with e.g. forming quotient modules and taking tensor products, and employing filt-free resolutions whenever projective or flat ones ordinarily are prescribed, we indeed may employ theorem 2 in order to deduce the following theorem, *except* its statements about spectral sequences:

THEOREM 3. *Let M be a monoid with an admissible order, R an M-filtered ring, and $L = L_R$ and $N = {}_RN$ be an (exhaustively) M-filtered right respectively left R-module. Then (canonically) the torsion modules $\mathrm{Tor}_i^R(L,N)$ are M-filtered, and the $\mathrm{Tor}^{G(R)}\big(G(L),G(N)\big)$ are M-graded, and for every $i \in \mathbf{N}$ and every $a \in M$ $\mathrm{Tor}^{G(R)}\big(G(L),G(N)\big)_a$ is a subquotient of $G\big(\mathrm{Tor}_i^R(L,N)\big)_a$.*
Moreover, if in addition these M-filtrations are exhaustive, and the ordinal number of $(M,\le)$ is at most ω, then there is a convergent spectral sequence (in the usual sense)

$$\mathrm{Tor}_*^{G(R)}\big(G(L)_*, G(N)_*\big) = E^1_{*,*} \;\Rightarrow\; E^\infty_{*,*} = G\big(\mathrm{Tor}_*^R(L,N)\big)_*.$$

Finally, any further gradation respected by the involved ring and modules and their filtration subgroups also are respected by these torsions (and, if existing, by this spectral sequence).

(Recall that the ordinality of M is at most ω if and only if there exists some order preserving injection $\iota : M \hookrightarrow \mathbf{N}$, which otherwise need not respect the algebraic structures.) The remaining part of theorem 3 is proved in section 5.

In the following detailed example, much of the terminology from the preceding constructions is concreticised. However, the multiplicative notation in the general (not necessarily commutative) monoid M is replaced by the additive one in $\mathbf{N}$.

EXAMPLE. Let k be a field, and let $R = k[x]$ (the polynomial ring with one variable x), graded and filtered over $\mathbf{N}$ by the usual polynomial degree. The augmentation $\varepsilon : R \twoheadrightarrow k$ (with $\varepsilon(x) = 0$) makes k to an R-module, with $F_m(k) = F_0(k) = k$ for all $m \in \mathbf{N}$. Now, consider a copy ${}_RN \simeq {}_RR$ of R, but given the trivial filtration with $F_m(N) = N$ for all m. Then N indeed is an $\mathbf{N}$-filtered R-module; and it is free as an R-module; but it is not filt-free.

By the usual considerations, we have $\mathrm{Tor}_*^R(k,N) = \mathrm{Tor}_0^R(k,N) \simeq k$, while

$$\mathrm{Tor}_*^R(k,k) = \mathrm{Tor}_1^R(k,k) \oplus \mathrm{Tor}_0^R(k,k) \simeq k \oplus k.$$

In the latter case, since k is an $\mathbf{N}$-graded R-module, the $\mathrm{Tor}_i^R(k,k)$ inherit this grading; and we get

$$\mathrm{Tor}_i^R(k,k) = \mathrm{Tor}_i^R(k,k)_i$$

for both i; exhibiting R as a Koszul algebra.

If we take graded associated, we find that $G(R) \simeq R$ (as graded rings), but $G(N) = G(N)_0 \simeq \bigoplus_{j\in\mathbf{N}} k$; whence

$$\mathrm{Tor}_i^{G(R)}\big(k, G(N)\big) \simeq \mathrm{Tor}_i^{G(R)}\big(k, \bigoplus_{\mathbf{N}} k\big) \simeq \begin{cases} \bigoplus_{\mathbf{N}} k & \text{if } i \le 1 \\ 0 & \text{else} \end{cases}.$$

We now also consider the filtrations. The minimal R-free resolution of N is

$$0 \longrightarrow RU \overset{\rho}{\longrightarrow} N \longrightarrow 0,$$

where $U \mapsto 1_R$; but this cannot be made to a filt-good free resolution by any filtration on RU. Indeed, if we put all $F_m(RU) = RU$, the resolution is a filt-good exact sequence, but RU then is not filt-free for any specified basis $\{p\}$. On the other hand, putting $F_m(RU) = F_m(R)U$ makes RU filt-free, but then the sequence is not filt-good, since e.g. $\mathrm{fdeg}_N(x) = 0$, but $\rho^{-1}(x) = \{xU\} \subseteq F_1(RU) \setminus F_0(RU)$.

On the other hand, a minimal filt-free resolution of ${}_RN$ is given by

$$0 \longrightarrow C_1 \overset{d_1}{\longrightarrow} C_0 \overset{\eta}{\longrightarrow} N \longrightarrow 0,$$

where

$$C_1 = \bigoplus_{j=1}^{\infty} RS^j, \quad C_0 = \bigoplus_{j=0}^{\infty} RT^j, \quad \mathrm{fdeg}\, S^j = 1,\ \mathrm{fdeg}\, T^j = 0, \quad d_1(S^j) = xT^{j-1} - T^j, \ \text{ and } \ \eta(T^j) = x^j$$

(and where the specified bases for C_1 and C_0 are $\{S^j : j \ge 1\}$ and $\{T^j : j \ge 0\}$, respectively).

Tensoring this resolution with k yields

$$0 \longrightarrow k \otimes C_1 = \bigoplus_{j=1}^{\infty} kS^j \overset{k\otimes d_1}{\longrightarrow} k \otimes C_0 = \bigoplus_{j=0}^{\infty} kT^j, \quad \text{with} \ \ (k \otimes d_1)(S^j) = -T^j \ \ (j \ge 1).$$

This is not filt-good, since $\mathrm{fdeg}(S^j) = 1 > 0 = \mathrm{fdeg}(-T^j)$ (and $k \otimes d_1$ is injective). The graded associated sequence has (essentially) the same modules; but $G(k \otimes d_1) = 0$. Hence, indeed (up to canonical isomorphisms),

$$\mathrm{Tor}_1^{G(R)}\big(k, G(N)\big)_m = H\big(G(k \otimes C_1)\big)_m = G(k \otimes C_1)_m = G(k \otimes C_1)_1 = \bigoplus_{j=1}^{\infty} kS^j, \ \text{and}$$

$$\mathrm{Tor}_0^{G(R)}\big(k, G(N)\big)_m = H\big(G(k \otimes C_0)\big)_m = G(k \otimes C_0)_m = G(k \otimes C_0)_0 = \bigoplus_{j=0}^{\infty} kT^j.$$

On the other hand, $k \otimes d_1 \ne 0$, whence indeed

$$\mathrm{Tor}_1^R(k,N) = H_1(k \otimes C_.) = 0 \ \text{ and } \ \mathrm{Tor}_0^R(k,N) = H_0(k \otimes C_.) \simeq kT_0 \simeq k.$$

Concretely, for the complex $(k \otimes C_*, k \otimes d_*)$, we have

$$0 = B_1^1 = F_1(B_1) = F_1(Z_1) \subset \bigoplus_{j=1}^{\infty} kS^j = F_1(Z_1^{<1}) = F_1(k \otimes C_1),$$

and indeed (up to canonical homomorphisms)

$$G\big(\mathrm{Tor}_1^R(k,N)\big)_1 = \frac{F_1(Z_1)}{F_1(B_1)} \text{ is a subquotient of } \frac{F_1(Z^{<1})}{B_1^1} = \mathrm{Tor}_1^{G(R)}\big(k,G(N)\big)_1.$$

Similarly,

$$0 = B_0^0 \subset F_0(B_0) = \bigoplus_{j=1}^{\infty} kT^j \subset \bigoplus_{j=0}^{\infty} kT^j = F_0(Z_0) = F_0(Z_0^{<0}) = F_0(k\otimes C_0),$$

yielding that $\mathrm{Tor}_0^R(k,N) = \frac{F_0(Z_0)}{F_0(B_0)} = \frac{\bigoplus_{j=0}^{\infty} kT^j}{\bigoplus_{j=1}^{\infty} kT^j} = kT^0$ is a subquotient of $\mathrm{Tor}_0^{G(R)}\big(k,G(N)\big) = H_0\big(G(k\otimes C_.)\big)_0$.

Also note, that here $k\otimes d_1$ induces an injective map from $\mathrm{Tor}_1^{G(R)}\big(k,G(N)\big)$ to $\mathrm{Tor}_0^{G(R)}\big(k,G(N)\big)$.

Alternatively, we may first consider a minimal filt-goodfree resolution

$$0 \longrightarrow C_1' \xrightarrow{d_1'} C_0' \xrightarrow{\eta'} k_R \longrightarrow 0.$$

This time, an ordinary minimal free resolution does work: Put $C_1' = VR$ with a specified basis $\{V\}$ of filtdegree 1, $C_0' = R$ with a specified basis $\{1\}$ of filtdegree 0, and define the homomorphisms by $d_1'(V) = x$ and $\eta' = \varepsilon$. Tensoring with N yields the complex $(C_*'\otimes N, d_*'\otimes N)$ with $C_1'\otimes N = VN$, $C_0'\otimes N = N$, and (still) $(d_1'\otimes N)(V) = x$, but now as an element in N (of filtdeg 0) instead of R (where we had $\mathrm{fdeg}\, x = 1$). For this complex, we get

$$\begin{aligned} 0 = B_1^1 = F_1(B_1) = F_1(Z_1) \subset VN = F_1(Z_1^{<1}) = F_1(C_1'\otimes N), \quad \text{and} \\ 0 = B_0^0 \subset xN = F_0(B_0) \subset N = F_0(Z_0) = F_0(Z_0^{<0}) = F_0(C_0'\otimes N). \end{aligned}$$

and thus $G\big(\mathrm{Tor}_1^R(k,N)\big)_1 = G\big(H_1(C_.'\otimes N)\big)_1 = 0/0 = 0$ as a subquotient of $\mathrm{Tor}\, G(R)_1\big(k,G(N)\big)_1 = VN/0 = VN$, and $G\big(\mathrm{Tor}_0^R(k,N)\big)_0 = G\big(H_0(C_.'\otimes N)\big)_0 = N/xN \simeq k$ as a subquotient of $\mathrm{Tor}\, G(R)_1\big(k,G(N)\big)_1 = N/0 = N$. Again, $d_1'\otimes N$ induces a degree-decreasing map from $\mathrm{Tor}\, G(R)_1\big(k,G(N)\big)$ to $\mathrm{Tor}\, G(R)_0\big(k,G(N)\big)$.

4. Gröbner bases.

This section is devoted to a more detailed discussion of what all this has to do with Gröbner bases.

In this section, let M be a fixed strictly wellordered monoid, R an exhaustively M-filtered ring, and $\mathfrak{a}$ a (two-sided) ideal $\mathfrak{a}$ in R. It is M-filtered in the usual manner (with $F_a(\mathfrak{a}) = \mathfrak{a}\cap F_a(R)$). For the purpose of this *opus*, a subset D of $\mathfrak{a}$ is an *(M-filtration) Gröbner basis*, if D generates $\mathfrak{a}$ and $\mathrm{lt}(D) := \{\mathrm{lt}\, x : x \in D\}$ generates $G(\mathfrak{a})$, as two-sided ideals in R and $G(R)$, respectively. This rather broad usage of the term is a bit unusual; perhaps, *filt-good basis* would be more appropriate, in analogy with the resolution terminology, and in view of the close relations between Gröbner bases and filt-good resolutions illustrated in this section, and later in theorem 4.

Put $A = R/\mathfrak{a}$. Since we anyhow have $G(R/\mathfrak{a}) = G(R)/G(\mathfrak{a})$ (up to canonical homomorphisms, and by applying theorem 3 for $R/\mathfrak{a}$ in lieu of R, and get that, for any $L = L_A$, $N = {}_AN$ and each $(i,a) \in \mathbf{N}\times M$, indeed

$$\text{(4.1)}\quad G\big(\mathrm{Tor}_i^A(L,N)\big)_a \text{ is a subquotient of } \mathrm{Tor}_i^{G(A)}\big(G(L),G(N)\big)_a = \mathrm{Tor}_i^{G(R)/G(R)\,\mathrm{lt}(D)G(R)}\big(G(L),G(N)\big)_a;$$

also respecting any extra grading on all of R, $\mathfrak{a}$, L, and N.

In the rest of this section, in addition, assume, that k is a field, R is an M-filtered connected k-algebra, and $\mathfrak{a}$ is contained in its augmentation ideal $\mathfrak{c}$. (In other words, we assume, that $F_{1_M}(R) = k$, contained in the centre of R, that we have a short exact sequence

$$0 \longrightarrow \mathfrak{c} \xrightarrow{\iota} R \xrightarrow{\varepsilon} k \longrightarrow 0,$$

where ι is the inclusion and ε is an R-bimodule epimorphism, and where $\varepsilon \circ \iota = 1_k$, and thus $R = k \oplus \mathfrak{c}$ as k-vector spaces for $\mathfrak{c} := \operatorname{Ker} \varepsilon$; and finally we assume that $\mathfrak{a} \subseteq \mathfrak{c}$.) Applying (4.1) for $L = N = k$, we do find that the $G\big(\operatorname{Tor}_i^A(k,k)\big)_a$ are subquotients of the respective $\operatorname{Tor}_i^{G(R)/G(R)\operatorname{lt}(D)G(R)}\big(k,k\big)_a$. Now, in the special case where moreover R is the non-commutative polynomial ring $k\langle T_1, \dots, T_n\rangle$, $M = \langle T_1, \dots, T_n\rangle$, and $\mathfrak{a}$ is homogeneous with respect to the total degree of these polynomials, it follows that $G\big(\operatorname{Tor}_i^A(k,k)\big)_d$ can be non-zero only for $(i,d) \in N \times \mathbf{N}$ such that $\operatorname{Tor}_i^{R/R\operatorname{lt}(D)R}\big(k,k\big)_d$ is non-zero; which together with the known restrictions of the latter indeed proves theorem 1, as we saw in the introduction.

The rest of the section is not technically necessary for this *opus*, but perhaps may clarify 'why' we should expect to have the filt-good resolution yielding (4.1). Assume given an R-free resolution

$$\dots \longrightarrow Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 = R \xrightarrow{\varepsilon} k \longrightarrow 0. \tag{4.2}$$

The projection $\pi : R \twoheadrightarrow A$ extends naturally to Q_* (with $\pi(Q_m) = A \otimes_R Q_i$), yielding a free *complex*

$$\dots \pi(Q_2) \longrightarrow \pi(Q_1) \longrightarrow \pi(Q_0) = A \longrightarrow k \longrightarrow 0;$$

but this in general is not exact at $\pi(Q_m)$ for any $m \geq 1$. However, it may be extended to one; and we now shall investigate how such an extension starts (for the moment forgetting all about M-filtrations).

We immediately see that (w. l. o. g.) then we may assume

$$Q_1 = \bigoplus_{C \in \mathcal{C}} RC,$$

where the set $\{\partial_1(C)\}_{\mathcal{C}}$ is a left ideal generating set of $\mathfrak{c}$ (and thus *a fortiori* a generating set for $\mathfrak{c}$ as a two-sided ideal). For $C \in \mathcal{C}$, put $f_C = \partial_1(C) \in R$, and $d_C = \pi(f_C) \in A$. We thus get a start of an A-free resolution

$$\dots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \longrightarrow k \longrightarrow 0 \tag{4.3}$$

by taking $X_0 = \pi(Q_0) = A$, $X_1 = \pi(Q_1) = A\mathcal{C} = \bigoplus_{\mathcal{C}} AC$, and $d_1(C) = \pi \cdot \partial_1(C) = d_C$. On the other hand, we must take

$$X_2 := \pi(Q_2) \oplus A\mathcal{G} = \pi(Q_2) \oplus \bigoplus_{G \in \mathcal{G}} AG \tag{4.4}$$

for some suitable (in general non-empty) set $\mathcal{G}$, and with suitable $d_2(G) = \sum_{\mathcal{C}} y_{G,C} C \in X_1$. (Here and elsewhere, the (direct or not) sums of sets may be infinite, but for sums of elements in rings or modules only finitely many non-zero summands are allowed.)

The first 'suitability' condition is that $d_2(G)$ should be a cycle; i. e., that $\sum_{\mathcal{C}} y_{G,C} d_C = 0$ (in A). Choosing $Y_{G,C} \in \pi^{-1}(y_{G,C})$, this is equivalent to

$$g_G := \sum_{\mathcal{C}} Y_{G,C} f_C \in \mathfrak{a}, \tag{4.5}$$

for any $G \in \mathcal{G}$. In other words, and putting $D := \{g_G\}_{\mathcal{G}}$, we have the equivalence

$$\operatorname{Im} d_2 \subseteq \operatorname{Ker} d_1 \iff D \subseteq \mathfrak{a}.$$

For the converse inclusion, it is sufficient that D generates $\mathfrak{a}$ as a two-sided ideal (i.e., that $\mathfrak{a} \subseteq ADA$), as we soon shall see; and in 'favourable' situations this also should be necessary. **We thus get a direct parallel between generating sets for the ideal $\mathfrak{a}$ on the one hand, and beginnings of extensions of $\pi(Q_*)$ to A-free resolutions of the other.** Since such parallels form a basic idea underlying this *opus*, we shall investigate this with some care.

Note, that this d_2 may be lifted to an R-module homomorphism $\tilde{d}_2 : \widetilde{X}_2 := Q \oplus R\mathcal{G} \longrightarrow X_1$, by prescribing $\tilde{d}_2(C) = \partial_2(C)$ for $C \in \mathcal{C}$, and $\tilde{d}_2(G) = \sum_{\mathcal{C}} Y_{G,C}C$ for $G \in \mathcal{G}$.

Now, assume that indeed $\mathfrak{a} \subseteq ADA$, and consider any $z \in \operatorname{Ker} d_1$. Say $z = \sum_{\mathcal{C}} z_C C$, where $z_C = \pi(Z_C)$; whence $z = \pi(w)$ for $w := \sum Z_C C \in Q_1$. Then

$$d_1(z) = 0 \implies \sum_{\mathcal{C}} Z_C f_C = \partial_1(w) \in \mathfrak{a} \subseteq ADA = AD + AD\mathfrak{c},$$

whence we may assume

$$\partial_1(w) = \sum_{\mathcal{G}} s_G g_G + \sum_{\mathcal{G}} \sum_{\mathcal{C}} t_{G,C} g_G u_{G,C} f_C \,.$$

Now, putting

$$w' := \tilde{d}_2(\sum_{\mathcal{G}} t_{G,C} G) = \sum_{calC} (\sum_{\mathcal{G}} t_{G,C} Y_{G,C}) C$$

and

$$w'' := \sum_{\mathcal{G}} \sum_{\mathcal{C}} t_{G,C} g_G u_{G,C} C,$$

we find that $\partial_1(w' + w'') = \partial_1(w)$. Thus, $w''' := w - w' - w'' \in Q_1$ is a cycle, and thus a boundary; i. e., $w''' = \tilde{d}_2(x)$ for some $x \in Q_2$. Hence,

$$z = \pi(w'' + w''' + w') = 0 + d_2(\pi(x) + \sum_{\mathcal{G}} \pi(t_{G,C}) G) \in \operatorname{Im} d_2 \,,$$

indeed.

Thus, we have proven, that if D is a generating set or *basis* of $\mathfrak{a}$ (as a two-sided ideal), then indeed X_* is exact at X_1; and, conversely, that this exactness, and indeed just the 'complexity', forces $D \subseteq \mathfrak{a}$.

On the other hand, if we just assume exactness at X_1, and let $\mathfrak{a}' := ADA \subseteq \mathfrak{a}$, then we may lift any $x \in \mathfrak{a}$ to a $y \in Q_1$. Thus, then $\pi(y) \in \operatorname{Ker} d_1 = \operatorname{Im} d_2$; which eventually forces $x \in \mathfrak{a}' + \mathfrak{a}\mathfrak{c}$. In other words, this exactness yields

$$\mathfrak{a} = \mathfrak{a}' + \mathfrak{a}\mathfrak{c}.$$

In 'favourable situations', this should force $\mathfrak{a}' = \mathfrak{a}$. Those situations include those where in addition

$$\bigcap_{i=1}^{\infty} \mathfrak{c}^i = 0, \tag{4.6}$$

holds, together with enough finiteness to make Nakayama's lemma applicable. One situation ensuring both (4.6) and the Nakayama applicability is where in addition R possess an ordinary $\mathbf{N}$ grading, and is connected (so that $k = R_0$ and $\mathfrak{c} = \bigoplus_{i \geq 1} R_i$). In other words, we have proven

Lemma 4.1. *If in addition R also is an $\mathbf{N}$-graded connected k-algebra, then the construction in (4.4) and (4.5) makes (X_*) exact at X_1 if and only if D generates $\mathfrak{a}$.* □

After this general analysis, we return to our specific situation. Thus, now, also assume, that the resolution (4.2) is M-filtered, and indeed is filt-good. Then, given a Gröbner basis D for $\mathfrak{a}$, there is an X_2 and a d_2 as given in (4.4), and suitable choices, such that indeed $D = \{g_G\}_{\mathcal{G}}$, and that $X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0$ is filt-good. Indeed, any $g \in D \subseteq \mathfrak{a} \subseteq \mathfrak{c} = \operatorname{Ker} \varepsilon$ by the filt-goodness and lemma 3.1 may be lifted to a $\tilde{g} \in Q_1$ with $\operatorname{fdeg} \tilde{g} = \operatorname{fdeg} g$; now let $\mathcal{G} = \{G_g\}_{g \in D}$, with $d_2(G_g) := \pi(\tilde{g}) \in X_1$; and, for any finite subset $D' \subseteq D$ and family $(x_g)_{g \in D'}$ of nonzero A elements and $a \in M$, the prescription

$$\sum_{D'} x_g G_g \in F_a(X_2) \iff a \geq \sup\{\operatorname{lt}(x_g) \cdot \operatorname{lt}(g) : g \in D'\}$$

(where the supremum is the maximum, if $D' \neq \emptyset$ and is 1_M else). Elementary verifications (again including transfinite induction) then show the M-filteredness and filt-goodness.

Conversely, e. g., if at least $G(R)$ (also) is a connected k-algebra, and the construction does yield filt-goodness at X_1, then $\operatorname{lt}(D)$ generates $G(\mathfrak{a})$; i. e., then D is a Gröbner basis. For, then, $\big(G(Q_*), G(\partial_*)\big)$ is a free resolution of ${}_{G(R)}k$; and the construction yields a free presentation of k as a left module over $G(A) = G(R)/G(\mathfrak{a})$. Now, apply lemma 4.1 for $G(R)$ and $G(\mathfrak{a})$ in lieu of R and $\mathfrak{a}$. Thus, we get

Lemma 4.2. *If R is M-filtered and also an* **N***-graded connected k-algebra, $\mathfrak{a}$ is a twosided R-ideal, (4.2) is a filt-good M-filtered free resolution of ${}_Rk$, and $X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow {}_Ak$ and D are constructed as described in (4.4) and (4.5), then this construction is filt-good at X_1 if and only if D is a Gröbner basis of $\mathfrak{a}$.* □

5. Subquotient homologies and spectral sequences with respect to general monoid orders.

This section concerns to what extent the 'general' theory of spectral sequences is applicable also for such filtered complexes as we have encountered in earlier sections. It is not strictly necessary for the main conclusions of the article, but more intended for the expert who may wonder about the relation of this approach on the one hand, and e. g. the theory behind Anick's spectral sequence on the other. (However, *en passant*, we shall indeed complete the proof of theorem 3.)

So, what do spectral sequences contribute, above an alternative proof of the main subquotient result (3.2)? For a starter, they provide intermediate steps between the larger groups and their final subquotents; and in favourable cases a calculable procedure for passing from the former to the latter, by means of a decreasing sequence of groups, where each is a subquotient of its predecessor.

Classically, the theory of spectral sequences, and its notation, often is developed for for double complexes in the first place. Now, that approach often is rather useful; but in my opinion the accompanying notation is not equally well suited for spectral sequences derived from other (Z-filtered) complexes; and almost impossible to apply for filtrations over 'weird' monoids. Moreover, while 'all' spectral sequences depend on the existence of underlying filtered complexs, typically these complexes are not well defined. Only when the first or second homologies have been calculated, we may derive e.g. certain Tor's or Ext's which are uniquely defined (up to canonical isomorphisms). Hence, the corresponding spectral sequences often are presented as starting with the E^1 or the E^2 terms; which further may obscure the essentially fairly natural underlying mechanisms.

In the presentation here, I employ a modified indexing of the spectral sequence terms; and I derive them as subquotients of somewhat more basic groups. The approach in some respects is close to that in [W94, section 5.4]. We also shall lean heavily on the already developed subquotient exibition in section 3.

Thus, as in section 3, again assume that M is an admissibly ordered monoid; and only consider exhaustive M-filtrations. Recall that for any M-filtered complex

$$\dots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} C_{i-2} \dots ,$$

and any appropriate i and any $a \in M$ we could consider the sequence (3.1) of subgroups of $F_a(C_i)$, ordered under set inclusion, and yielding the main exhibition (3.2) of $G(H_i(C_.))_a$ as a subquotient of $H_i(G(C_.)_a)$, which is a subquotient of $G(C_i)_a$. Now, if in addition $M = \mathbf{N}$ (with its usual order), and thus $a \in \mathbf{N}$, then the conventional way of presenting this would be to make the identifications

$$E^0_{pq} = G(C_i)_a, \quad E^1_{pq} = H_i(G(C.)_a), \text{ and } E^\infty_{pq} = G(H_i(C.))_a,$$

for $p := a$ and $q := i - a$; introduce numerous intermediate other subquotients E^2_{pq}, E^3_{pq}, …, and corresponding differentials d^*_{pq} on these, in such a way that (canonically) E^{r+1}_{pq} is the homology of E^r_{pq}, and then discuss the 'convergence' of the sequence of these intermediates to the smallest and final subquotient E^∞_{pq}. We shall briefly investigate this; but we avoid the "pq index" conventions better apt for spectral sequences arising from double complexes than for our more general one. Thus, we write $E^r_{i.a}$ rather than E^r_{pq}; or, in other words, make the definition $E^r_{p+q.p} = E^r_{pq}$.

Let us give an explanation of this in terms as close to the terminology from section 3 as possible. First, we may extend the $\mathbf{N}$-filtrations to $\mathbf{Z}$-filtration, by putting all $F_m(*) = 0$ for any $m < 0$. Second, since in $\mathbf{Z}$ $m < a \iff m \le a-1$, we may rewrite $Z^{<a}_i$ as Z^{a-1}_i, where we in general set

$$Z^b_i := \{x \in C_i : d_i(x) \in F^b(C_{i-1}).$$

Third, the sequence of inclusions (3.1) may be extended to a longer one

$$(5.3) \quad 0 \subseteq F_a(B^a_i) \subseteq F_a(B^{a+1}_i) \subseteq \ldots \subseteq F_a(B_i) \subseteq F_a(Z_i) = F_a(Z^{-1}_i) \subseteq F_a(Z^0_i) \subseteq F_a(Z^1_i) \subseteq \ldots \subseteq F_a(C_i).$$

Now, for each $a = p$ and $i = p + q$, these subgroups, with $F_{m-1}(C_i)$ added to each, give rise to a multitude of subquotients of $E^0_{i.a} := G(C_i)_a$. For the moment, we are just interested of the

$$E^r_{i.a} := \frac{F_a(Z^{a-r}_i) + F_{a-1}(C_i)}{F_a(B^{a+r_i}) + F_{a-1}(C_i)};$$

in other words, up to also factoring out all of lower filtdegrees, the set of 'quasicycles' consisting of the $F_a(C_i)$ elements 'dropping down' more than r steps when 'differentiated', quoted with the 'restricted' boundaries in $F_a(C_i)$, which 'dropped down' at most r steps.

The 'convergence' of the spectral sequence now amounts to noting that, on the one hand, a true cycle in $F_a(C_i)$ amounts to one which 'drops down' all the way to the negatively indexed $F_m(C_{i-1})$; i.e., by noting that for any $x \in F_a(C_i)$ we have the equivalences

$$x \in F_a(Z_i) \iff d_i(x) = 0 \iff (d_i(x) \in F_m(C_{i-1}) \text{ for all } m \in Z) \iff x \in \bigcap_{r=0}^{\infty} F_a(Z^{a-r}_i);$$

and that, on the other hand, since the filtration of C_{i+1} is exhaustive, that any boundary is a 'restricted' one for a sufficiently high r, so that

$$F_a(B_i) = F_a(C_i) \cap d_{i+1}\big(\bigcup_{r=0}^{\infty} F_{r+a}(C_{i+1})\big) = \bigcup_r F_a(B^r_i).$$

Finally, the differentials crucially depend on the fact that $F_{<a-r}(C_i) = F_{a-r-1}(C_i)$. Thus, any element $x \in F_a(C_i)$ with the property that fdeg $d_i(x) < a-r$ in fact has $d_i(x) \in F_{a-r-1}(C_{i-1})$; and moreover this $d_i(x) \in F_{a-r-1}(Z_{i-1}) \subseteq F_{a-r-1}(Z^{a-2r-2}_{i-1})$. Hence, any element $\overline{x} \in E^r_{i.a}$ may be lifted to some $x \in F_a(Z^{a-r-1}_i)$, whose image $d_i(x) \in F_{a-r-1}(Z^{(a-r-1)-r-1}_{i-1})$ represents some element $\overline{d_i(x)}$ in $E^r_{i-1,a-r-1}$. Moreover, if we pick another representative $x' \in F_a(C_i)$ (thus with $\overline{x'} = \overline{x}$), then $d_i(x') - d_i(x) = d_i(x' - x) \in B^{r-1}_{i-1}$, whence then $\overline{d_i(x')} = \overline{d_i(x)}$. This diagramme chasing thus yields a well-defined function $d^r_{i.a} : E^r_{i.a} \longrightarrow E^r_{i-1,a-r-1}$; by a general diagramme chasing result, an homomorphism. Since moreover clearly $d^r_{i.a} \circ d^r_{i+1.a+r+1} = 0$, this makes $(E^r_{*.*}, d^r_{*.*})$ to a complex, whose cycles correspond to elements 'dropping down' one step more, and correspondingly for its boundaries; indeed essentially making $E^{r+1}_{*.*}$ its homology.

Note, however, that we only used a small part of all the available subquotients resulting from the chain(5.3), and that there also are other choices where induced differentials yield other of these subquotients as homology.

To what extent do spectral sequences exist when we instead consider a M-filtration (for an arbitrary admissibly ordered monoid M)? The short answer is

- The plethora of intermediate subquotients always exists.
- The conventional way to organise some of these as a spectral sequence works, if the ordinal number for $(M, \le)$ is at most ω.
- In the special case where the complex is used for calculating $\mathrm{Tor}^{R/\mathfrak{a}}(k,k)$ for a non-commutative polynomial ring $R = k\langle T_1, \ldots, T_n\rangle$, modulo an ideal which is homogeneous with respect to the totaldegree in R, the order $\textsc{deg} \le$ yields the same Gröbner basis as does $\le$, and $(M, \textsc{deg} \le) \simeq_{\text{poset}} \mathbf{N}$; whence by the preceeding point 'the usual' spectral sequence $\mathrm{Tor}^{R/\mathrm{lt}(\mathfrak{a})}(k,k) \implies \mathrm{Tor}^{R/\mathfrak{a}}(k,k)$ exists.

The first of these points is obvious from (5.3).

For the second point, and thus the completion of the proof of theorem 3, assume, that indeed M is finite, or $M = \{1_M = m_0, m_1, \ldots\}$ with $m_0 < m_1 < m_2 < \ldots$. Then the same arguments work as well for M-filtrations as they did for the case $M = \mathbf{N}$.

The third point may deserve a few more words. In fact, this is the exceptional place in this work where any algorithm for *deriving* a Gröbner basis matters. Indeed, given $R = k\langle T_1, \ldots, T_n\rangle$ and $\mathfrak{a}$ as in the statement, and any admissible order $\le$ on the free monoid $M := \langle T_1, \ldots, T_n\rangle$, to begin with, we note that $\mathfrak{a}$ (as a twosided R-ideal) has a homogeneous basis, which we may organise as a finite or infinite sequence $f_1, f_2, \ldots$ of polynomials of non-decreasing total degrees and with only finitely many generators of each total-degree; and we may assume them monic, so that $\mathrm{lt}\, f_i = \mathrm{fdeg}\, f_i$ for each i. We now may apply the Buchberger algoritm, not for the pairs (f_i, f_j) as such, but for the triplets $\big(\mathrm{lt}(f_i), w, \mathrm{lt}(f_j)\big)$ of M elements, such that $\mathrm{fdeg}(f_i)$ is a left factor and $\mathrm{lt}(f_j)$ a right factor of w, that the total degrees satisfy $\deg w < \deg(f_i) + \deg(f_j)$, and that there is no proper 'in-factor' (neither left nor right factor) of w which is the filt-degree of any already booked polynomial f_l. In particular, for any such triple, we have $w = \mathrm{lt}(f_i)w' = w''\,\mathrm{lt}(f_j)$, say, with w' a proper right factor of $\mathrm{lt}\, f_j$, and w'' a proper right factor of $\mathrm{lt}\, f_i$.

We now process the f_i in order, trying to reduce either one untreated f_j, or an 'S-polynomial' $f_i w' - w'' f_j$ derived from one of the triplets; and always considering a new polynomial of lowest possible degree. The polynomials to be reduced thus always are homogeneous; and the result of one reduction step always preserves this property. Hence, the resulting Gröbner basis consists entirely of homogeneous polynomials. Moreover, wherever the order $\le$ of M is applied in order to determine a leading term or to perform one reduction step, only monomials of the same length (total degree) are compared.

The result of this process is the reduced Gröbner basis D with respect to the given admissible order $\le$. This may be infinite; but the process guarantees that we never go back to a lower total order than the one we last treated, whence for any given $d \in \mathbf{N}$ all $f \in D$ with $\deg f \le d$ are calculated within a finite time.

Now, if we instead for $\le$ had applied the 'first total degree; $\le$ for tiebreaks' order $\textsc{deg}{\le}$ (which indeed is admissible by lemma 2.1 and the remarks at the end of section 2), then every calculation and all results of these had been exactly the same as it was for $\le$, since these two orders coincide in outcome for monomials of the same total-degree, and only such ones are compared in this algorithm. Thus, D also is the reduced Gröbner basis with respect to $\textsc{deg}{\le}$. Moreover, for any totaldegree d, $|\{w \in M : \deg w = d\}| = n^d < \infty$. Hence, indeed, $(M, \textsc{deg}{\le}) \simeq (N, \le)$ as ordered sets; whence also the spectral sequence does exist.

6. A concrete filt-good field resolution for a quotient of a non-commutative polynomial ring.

We now specialise further. Let $1 \le n \in \mathbf{N}$, $\Lambda = k\langle T_1, \dots, T_m\rangle$, $I = \bigoplus_{i=1}^{\infty} \Lambda_i$, $P \subseteq I^2$ a (twosided) Λ ideal (respecting the $\mathbf{N}$ grading of Λ, or not), let M be the free monoid on $T_1, \dots, T_n$, $<$ any admissible order (i. e., monoid strict well-order) of M, and let D be a minimal Gröbner basis for P, where we without loss of generality may assume that $W_2 := \mathrm{lt}(D) = \{\mathrm{lt}\, x : x \in D\}$ consists of monic monomials; i. e., that $W_2 \subset M$.

Let $A = \Lambda/P$, and note that A is naturally M-filtered but not necessarily M-graded, and is generated by $t_1, \dots, t_n$, say, with $t_i = \pi(T_i)$, considering the short exact sequence

$$0 \to P \longrightarrow \Lambda \overset{\pi}{\twoheadrightarrow} A \to 0.$$

Every element $x \in A$ is representable as a (non-commutative) polynomial in the t_*, in many different ways; however, there is just one representation, the *normal form* of x, where each appearing monomial $t_{i_1} \cdots t_{i_r}$ is *reduced*, i. e., is such that the corresponding Λ monomial $T_{i_1} \cdots T_{i_r} \notin MW_2M$. If $x \ne 0$, then $\mathrm{lt}\, x$ is on the form $\kappa t_{i_1} \cdots t_{i_r}$, with $\kappa \in k^*$, the monomial reduced, and $\mathrm{fdeg}\, x = T_{i_1} \cdots T_{i_r}$.

In our applications, indeed $P = \sum_i (P \cap \Lambda_i)$; in which case it may be convenient to replace the original order of M with the TOTAL DEG FIRST one, and to assume the Gröbner basis elements to be (tdeg) homogeneous, and to construct the resolution tdeg-graded. However, the first results hold in greater generality, without any such extra assumption of P.

Recall that Λ is a *free ideal ring* (*sensu* P. M. Cohn). In particular,

$$I = \bigoplus_{i=1}^{n} \Lambda T_i = \bigoplus_{i=1}^{n} T_i \Lambda. \tag{6.1}$$

Moreover, since every element in P is a sum of elements yxz with $y, z \in \Lambda$ and $x \in D$, and any such $z = \kappa + z'$ for some $\kappa \in k$ and some $z' \in I$,

$$P = \Lambda D + PI. \tag{6.2}$$

Recall that $\widetilde{A} := G(A) = \Lambda/G(P)$ is a (non-commutative) monomial ring, and an M-graded augmented k-algebra, and that (by similar means as for (6.1) and (6.2)) there is a minimal $\widetilde{A}$-free resolution of ${}_{\widetilde{A}}k$

$$\dots \xrightarrow{\tilde{d}_3} \widetilde{X}_2 \xrightarrow{\tilde{d}_2} \widetilde{X}_1 \xrightarrow{\tilde{d}_1} \widetilde{X}_0 = \widetilde{A} \overset{\tilde{\eta}}{\twoheadrightarrow} k \tag{6.3}$$

with $\widetilde{X}_m = \oplus_{W_m} \widetilde{A} w$, where $W_0 = \{1\}$, $W_1 = \{T_1, \dots, T_n\}$, and W_2 is set before; whence each $w \in W_2$ may be factorised as $w = w_2 w_1$ for some $w_1 = T_i \in W_1$; and, for $m \ge 3$,

$$W_m = \{w_m w_{m-1} \cdots w_1 : w_{m-1} \cdots w_1 \in W_{m-1} \wedge w_m \notin P \ni w_m w_{m-1} \\ \wedge (\text{no proper left factor of } w_m w_{m-1} \text{ belongs to } P)\}.$$

Moreover, for any such $w = w_m w_{m-1} \cdots w_1 \in W_m$, $\tilde{d}_m(w) = \tilde{\pi}(w_m) w_{m-1} \cdots w_1$ (with $\tilde{\pi} : \Lambda \twoheadrightarrow \widetilde{A}$). (See e. g. the proof of [B78, théorème 1] for details.)

By a slight abuse of notation, we also let $t_i = \tilde{\pi}(T_i)$; bearing in mind that thus a monomial $t_{i_1} \cdots t_{i_r}$ is reducible in A if and only if it vanishes in $\widetilde{A}$.

Our next aim is to construct an (in general non-minimal) filt-good free ${}_A k$ resolution

$$\dots \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 = A \overset{\eta}{\twoheadrightarrow} k \tag{6.4}$$

such that $G(6.4) = (6.3)$; in other words, such that $G(X_m) = \widetilde{X}_m$, $G(d_m) = \tilde{d}_m$, *et cetera*. Thus, for each m we should set and verify

$$X_m := \oplus_{W_m} Aw;$$

$$w \in W_m \implies \operatorname{fdeg} w = \operatorname{fdeg} d_m(w) = w; \text{ and} \tag{6.5}$$

$$x \in X_m \implies \operatorname{fdeg}\big(d_m(x)\big) < \operatorname{fdeg} x \vee \operatorname{lt}\big(d_m(x)\big) = \tilde{d}_m(\operatorname{lt} x); \tag{6.6}$$

$$z \in X_{m-1} \wedge d_{m-1}(z) = 0 \wedge \operatorname{fdeg}(z) = a \implies \exists\, u \in F_a(X_m) : d_m(u) = z \tag{6.7}$$

(*mutatis mutandis* for m small). We shall do this by (an ordinary) induction with respect to m in the first place, and by (a transfinite) induction (for a fixed m) with respect to filt-degree in the second.

For any $w \in W_m$, we indeed consider the factorisation $w = w_m \cdots w_1$ employed in the definition of the W_m.

Define d_1 by $d_1(T_i) = t_i$. For d_2, employ the first equality in (6.1), which may be abbreviated $I = \Lambda W_1$, in order to create a combined map $IeraAW_1$; apply this for the defining Gröbner basis element for each $w \in W_2$. In other words, for any $w \in W_2$ there is an $x \in D \subset I$ with $\operatorname{lt} x = w$, and there are unique $x_1, \ldots, x_n \in \Lambda$ such that $x = \sum_i x_i T_i$; put $d_2(w) = \sum_i \pi(x_i) T_i \in X_1$. It is fairly easy to see that the constuctions and the properties (6.5) and (6.6) are satisfied for $m \leq 2$, and that so is the respective modification of (6.7) for $m \leq 1$.

Proof of (6.7) for $m = 2$. Inductively, assume this true for all $a < b$ for a $b \in M$, and consider a cycle $z \in X_1$ with $\operatorname{fdeg} z = b$. To avoid trivialities, also assume $z \neq 0$. Thus, there are a $\kappa \in k^*$, a $v \in M$, and a $T_i \in W_1$, such that $\operatorname{lt} z = \kappa\pi(v)T_i$. Hence, $\pi(v)$ is reduced, but $\pi(v)t_i$ is not. In other words, $vT_i \in \Lambda P$; whence there is an $x \in D$ with (say) $w := \operatorname{lt} x \in W_2$, and an $y \in M$, such that $vT_i = yw$. Moreover, $a := \operatorname{fdeg} z' < b$ for the cycle $z' := z - d_2(\kappa\pi(y)w)$, whence by the inductive assumption there is a $u \in F_a(X_2)$ with $d_2(u) = z'$. Thus, indeed, z is the image of $d_2(\kappa\pi(y)w + u \in F_b(X_2)$. □

Now, for any $m \geq 3$, assume that d_{m-1} (and lower d_*) be constructed, with the claimed properties. For any $w = w_m w' = w_m w_{m-1} w'' = w_m w_{m-1} \cdots w_1 \in W_m$ we then may choose a boundary $d_m(w) \in X_{m-1}$ with $\operatorname{lt} d_m(w) = \pi(w_m)w'$, in the following manner. Consider

$$z := d_{m-1}(\pi(w_m)w') = \pi(w_m)d_{m-1}(w') \in X_{m-2}.$$

Since z is a boundary, it is a cycle. Moreover, since $\pi(w_m)$ is a reduced monomial (in A),

$$a := \operatorname{fdeg} z \leq \operatorname{fdeg}(\pi(w_m)w') = w. \tag{6.8}$$

I claim that the inequality in (6.8) is strict. Indeed, by (6.6) (applied inductively), and since w_m being reduced induces $\operatorname{lt}\pi(w_m) = \tilde{\pi}(w_m)$, and by construction, else we should have

$$\operatorname{lt} z = \tilde{d}_{m-1}\big(\operatorname{lt}(\pi(w_m)w')\big) = \tilde{\pi}(w_m)\tilde{d}_{m-1}(w') = \tilde{\pi}(w_m)\tilde{\pi}(w_{m-1})w'' = \tilde{\pi}(w_m w_{m-1})w'' = 0,$$

since, on the other hand, $\pi(w_m w_{m-1})$ is *not* reduced.

Thus, instead, indeed, $a < w$. Moreover, by (6.7), $z = d_{m-1}(u)$ for some $u \in F_a(X_m) \subseteq F_{<w}(X_m)$. Now, put

$$d_m(w) := \pi(w_1)w' - u,$$

which indeed is a cycle in X_{m-1}, and has a 'leading term' $\pi(w_1)w' = \tilde{d}_m(w)$. (6.5) follows directly.

For proving (6.6) for an $x \in X_m \setminus \{0\}$, let $\operatorname{lt} x = \kappa\pi(v)w$, with $\kappa \in k^*$, $v \in M$, and $w = w_m w' \in W_m$, put $x' := x - \operatorname{lt} x \in F_{<vw}(X_m)$, and still assume $d_m(w) = \pi(w_1)w' - u$. Then $\operatorname{fdeg} x = vw$, and

$$d_m(x) = \kappa\pi(vw_1)w' + d_m(x') - \kappa\pi(v)u.$$

Now, the last two terms belong to $F_{<vw}(X_{m-1}$ (technically, also by applying an inductive assumption with respect to $\operatorname{fdeg} x$); while

$$\operatorname{fdeg}(\kappa\pi(vw_1)w') = \begin{cases} vw & \text{if } \pi(vw) \text{ is reduced,} \\ \text{less} & \text{else.} \end{cases}$$

Thus, indeed, in the former case, $\operatorname{lt} d_m(x) = \kappa\pi(vw_1)w' = \tilde{d}_m(\operatorname{lt} x)$ and $\operatorname{fdeg} d_m(x) = vw$, while in the latter case $\operatorname{fdeg} d_m(x) < vw$.

Proof of (6.7) (given the others, for a fixed m). Follow the proof for $m = 2$, *mutatis mutandis.* Thus, consider a cycle $z \in X_{m-1} \setminus \{0\}$ (under the inductive assumption for $b := \operatorname{fdeg} z$), with $\operatorname{lt} z = \kappa\pi(v)w$ and $z' = z - \operatorname{lt} z$ (say), and thus $\pi(v)$ reduced and $b = vw = vw_{m-1}w'$ (say). As before, $d_{m-1}(\kappa\pi(v)w) = d_{m-1}(z') \in F_{<b}(X_{m-2})$, yielding that $\pi(vw_{m-1})$ is reduced, although $\pi(w_{m-1})$ is not. Thus, and (w. l. o. g.) writing

$$v = T_{i_r}T_{i_{r-1}} \cdots T_{i_1}$$

for some $r \geq 1$ there exists a minimal $s \in \{1, \ldots, r\}$, such that $T_{i_s} \cdots T_{i_1} w_{m-1} \in W_2 M$. Put $w_m := T_{i_s} \cdots T_{i_1}$ and $y := T_{i_r} \cdots T_{i_{s+1}}$ (so that $v = yw_m$), and note that by construction $w_m w \in W_m$. Hence, we may put $z' := z - d_m(\kappa\pi(y)w_m w)$, and proceed as for $m = 2$. □

Summing up, in Gröbner basis terms, we have proved

Theorem 4. *Let k be a field, $\Lambda = k\langle T_1, \ldots, T_n\rangle$ the non-commutative polynomial ring in n variables, $I = (T_1, \ldots, T_n)$ the augmentation ideal of Λ, $P \subseteq I^2$ a (two-sided) Λ-ideal, and let D be the (possibly infinite) reduced Gröbner basis of P with respect to some admissible monomial order. Put $A = \Lambda/P$. Then there is a free resolution*

$$\ldots \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\eta} k$$

of k considered as a left A-module, with an increasing complex filtration (with respect to the monoid of monomials in the T_*, and their admissible order), such that its graded associated complex

$$\ldots \xrightarrow{G(d_3)} G(X_2) \xrightarrow{G(d_2)} G(X_1) \xrightarrow{G(d_1)} G(X_0) \xrightarrow{G(\eta)} k$$

is a minimal free resolution of k as a left module for the non-commutative monomial ring $\Lambda/G(P) = \Lambda/(G(D))$. □

Thus, we directly get the presumed Koszulness result:

Corollary 1. *If in addition P is generated by homogeneous elements of degree 2, and also D consists of quadratic monomials, then $A = \Lambda/P$ is a Koszul algebra.* □

Moreover, indeed, this corollary is a special case of a more general one, concerning the rate of growth of the total-degrees of the non-vanishing homologies for an ($\mathbf{N}$-graded) connected k-algebra $A = \Lambda/P \neq \Lambda$; see [B86] for details. Note, that such an algebra is Koszul if an only if its rate of growth is 1. Now, the rate of growth always is finite for non-commutative monomial rings with finite numbers of generators. (In fact, it then equals one less than the maximal degree for an element in a minimal set of homogeneous generators.) Hence, and again by [B86], in the situation of the theorem, we get theorem 1, in the form of

Corollary 2. *If in addition P is generated by homogeneous elements, then $\operatorname{rate} A \leq \operatorname{rate} \widetilde{A}$; and if moreover $|D| < \infty$, then so is $\operatorname{rate} A$; in fact, then $\operatorname{rate} A \leq \max_{x \in D}(\deg x) - 1$; and then also any Veronese subring*

$$A^{(d)} := \bigoplus_{j=0}^{\infty} A_{jd}$$

of A with $d \geq \operatorname{rate} A$ is a Koszul algebra. □

The existence proof of theorem 4 implicitly also provides a recursive process for constructing the differentials of the resolution; at least, if the Gröbner basis is finite. The following example is done in some detail, in order to clarify how that recursion works. Therein, the M order always is a strict lexicographical one, reading from left to right. It thus is completely determined by the order on $W_1 = \{T_1, \ldots, T_n\}$.

Example. If $n = 3$ and $P = (T^3 - T_1T_2)$, then the order $T_1 > T_2 > T_3$ makes $D = \{T_1T_2 - T_3^2\}$ and $W_2 = \{T_1T_2\}$, whence by the corollary A is Koszul; and it has global homological dimension 2. Thus, the resolution is

$$0 \longrightarrow AT_1T_2 \xrightarrow{d_2} AT_1 \oplus AT_2 \oplus AT_3 \xrightarrow{d_1} A \xrightarrow{\varepsilon} k,$$

with $d_2(T_1T_2) = t_1T_2 - t_3T_3$ (and of course $d_1(T_i) = t_i$, $i = 1, \ldots, 3$).

However, if we instead consider the order $T_1 < T_2 < T_3$, we get $D = \{T_3^2 - T_1T_2, T_3T_1T_2 - T_1T_2T_3\}$, and for $m \geq 2$ $W_m = \{T_3^{m-1}T_1T_2, T_3^m\}$. Thus, $X_2 = AT_3^2 \oplus AT_3T_1T_2$, where $d_2(T_3^2) = t_3T_3 - t_1T_2$ and $d_2(T_3T_1T_2) = t_3t_1T_2 - t_1t_2T_3$.

Likewise, $X_3 = AT_3^3 \oplus AT_3^2T_1T_2$. For constructing (and choosing) $d_3(T_3^2T_1T_2)$, first note that

$$d_2(t_3T_3T_1T_2) = t_3^2t_1T_2 - t_3t_1t_2T_3 = t_1t_2t_1T_2 - t_1t_2t_3T_3 = -t_1t_2t_3T_3 + t_1t_2t_1T_2$$

(in reduced form, with terms in descending filt-degree); whence $\operatorname{lt} d_2(t_3T_3T_1T_2) = -t_1t_2t_3T_3$. Hence and since $d_1 \cdot d_2 = 0$, as seen in the proof, the corresponding Λ monomial has a right factor in W_2; indeed, it is $T_1T_2 \cdot T_3^2$; and so also $\operatorname{lt} d_2(-t_1t_2T_3^2) = -t_1t_2t_3T_3$. Thus, with $z := t_3T_3T_1T_2 + t_1t_2T_3^2$, we must have $\operatorname{fdeg} d_2(z) < T_1T_2T_3^2$; actually, it is 1, since $d_2(z)$ turns out to be 0. We thus may choose to set $d_3(T_3^2T_1T_2) = z = t_3T_3T_1T_2 + t_1t_2T_3^2$.

Similarly, $d_2(t_3T_3^2) = t_3^2T_3 - t_3t_1T_2 = -t_3t_1T_2 + t_1t_2T_3$, and also $d_2(-T_3T_1T_2) = -t_3t_1T_2 + t_1t_2T_3$, whence we may choose $d_3(T_3^3) = t_3T_3^2 + T_3T_1T_2 \in X_2$.

For higher differentials, we need to care a bit more about which 'subword to substitute'. When we construct the differential of $T3^3T_1T_2 \in X_4$, we start by considering the cycle

$$d_3(t_3T_3^2T_1T_2) = t_3^2T_3T_1T_2 + t_3t_1t_2T_3^2 = t_1t_2t_3T_3^2 + t_1t_2T_3T_1T_2$$

in X_2, with filt-degree $T_1T_2T_3^3$. The differential of its leading term has a strictly lower filt-degree; which means that $T_1T_2T_3^3$ must have one right factor in W_3. Indeed, $T_3^3 \in W_3$. We therefore should 'lift' the leading term $t_1t_2t_3T_3^2$ to $d_3(t_1t_2T_3^3) = t_1t_2(t_3T_3^2 + T_3T_1T_2)$; or, rather, 'correct' the cycle by that boundary. We get

$$d_3(t_3T_3^2T_1T_2) - d_3(t_1t_2T_3^3) = (t_1t_2t_3T_3^2 + t_1t_2T_3T_1T_2) - t_1t_2(t_3T_3^2 + T_3T_1T_2),$$

which indeed already is zero. Thus, we may choose $d_4(T_3^3T_1T_2) = t_3T_3^2T_1T_2 - t_1t_2T_3^3$. Similarly, we may let $d_4(T_3^4) = t_3T_3^3 - T_3^2T_1T_2$; and indeed the same pattern works for the higher homological degree differentials, whence in fact for any $m \geq 3$ we may put

$$\begin{cases} d_m(T_3^{m-1}T_1T_2) & = & t_3T_3^{m-2}T_1T_2 & -(-1)^m\, t_1t_2T_3^{m-1} \\ d_m(T_3^m) & = & t_3T_3^{m-1} & -(-1)^m\, T_3^{m-2}T_1T_2 \end{cases}.$$

In this example, the graded associated complex $G(X_*)$ has identical module generators, but different differentials; indeed, here, $G(X_m) = X_m$, but $G(d_m)(w) = \operatorname{lt} d_m(w)$ for $w \in W_m$. Thus, indeed, as complexes, $G(X_*) = \widetilde{X}_*$, and is a minimal $\widetilde{A}$-free resolution of ${}_{\widetilde{A}}k$.

For calculating the $\operatorname{Tor}_*^A(k,k)$, we take the homology of the M-filtered complex $C_* := k \times_A X_*$. Now, the differential in C_* does not vanish; indeed, it maps $T_3^m \in C_m$ to $(-1)m + 1T_3^{m-2}T_1T_2 \in C_{m-1}$ for $m \geq 3$. (This indeed yields the small Tor we should expect from the facts that A is Koszul, as we saw from considering the $T_1 > T_2 > T_3$ order, and that its Koszul dual[1] is artinian.) On the other hand, since the differential in

[1] Koszul duals form one of the well-known concepts I completely gloss over in this paper; but that concept is not employed outside this sentence.

C_* lowers the filt-degrees, its graded associated complex $G(C_*) = k \times_{\widetilde{A}} \widetilde{X}_*$ indeed has trivial differentials, reflecting the fact that the ${}_{\widetilde{A}}k$-resolution $\widetilde{X}_* \twoheadrightarrow k$ indeed is minimal.

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