KOSZUL ALGEBRAS, VERONESE SUBRINGS AND RINGS WITH LINEAR RESOLUTIONS

JÖRGEN BACKELIN and RALF FRÖBERG

0. INTRODUCTION

The results in this paper mainly concern graded algebras over a field which are unitary, associative but not necessarily commutative. They have, however, applications to commutative algebra and algebraic geometry.

In section 1 we make definitions and collect, for reference, results that we need in the sequel. In particular we define (homogeneous) *Koszul algebras* (in some articles called Fröberg rings) to be graded algebras R for which $\mathrm{Tor}^R_{i,j}(k, k) = 0$ for $i \neq j$ (see 1.16 for seventeen other equivalent conditions). To each graded algebra R there is a sequence of associated lattices $L_i(R)$ of vector spaces. We use, as one of our main techniques, the fact developped in [2] that the distributivity of the $L_i(R)$'s have homological implications.

The main result in section 2, Theorem 4, shows that Koszul algebras constitute a natural class, in that it is closed under a number of operations such as taking Veronese subrings, Segre products, tensor products, fibre products and coproducts. This generalizes e.g. the result of S. Bărcănescu and N. Manolache that a Segre product of Veronese subrings of (commutative) polynomial rings is a Koszul algebra (see [5]). As preparations for the main theorem we have two results which may have some interest of their own. First we show (Theorem 1) that the distributivity of the associated lattices is preserved when taking Veronese subrings, Segre products, products and coproducts. Secondly (Proposition 3) we give a bound for the degrees of the relations in Veronese subrings and Segre products, given the degrees of the relations in the original rings. This generalizes a result of D. Mumford ([12]). In the above mentioned results we do not in general assume commutativity.

In sections 3 and 4 we restrict to commutative algebras. The main result in section 3 is that the number of non-isomorphic Veronese subrings of an algebra of Krull dimension one is finite. In section 4 we tie the concept of Koszul algebras to some perhaps more well known concepts, in showing that a graded algebra has a 2-linear resolution if and only if it is both a Koszul algebra and a Golod ring. We also give a relative version of this statement.

1. CONVENTIONS AND NOTATIONS

A *graded algebra* will denote a ring of type $R = k\langle T_1, \ldots, T_n\rangle / P$, where $V = k\langle T_1, \ldots, T_n\rangle$ is the free (non-commutative) associative algebra over a commutative field k in variables T_i of *degree one*, and P is a two-sided ideal generated by *finitely many* homogeneous elements of degree $\geqq 2$. All ideals are *two-sided.* $I = (T_1, \ldots, T_n)$ and I/P are called the *augmentation ideals* in V and R respectively. All tensor products, products and coproducts are over k.

1. For a graded algebra $R = \bigoplus_{i\geqq 0} R_i$ we define the *dth Veronese subring* of R as $R^{(d)} = \bigoplus_{i\geqq 0} R_{id}$. An element in R_{id} has degree i in $R^{(d)}$.

2. For two graded algebras $R' = \bigoplus_{i\geqq 0} R'_i$ and $R'' = \bigoplus_{i\geqq 0} R''_i$ we define their *Segre product* as $R' \circ R'' = \bigoplus_{i\geqq 0} R'_i \otimes R''_i$. An element in $R'_i \otimes R''_i$ has degree i in $R' \circ R''$.

3. For two graded algebras R' and R'' we define their *coproduct* (or free product or amalgamated sum or fibre sum) over k, $R' \amalg R''$, as the pushout of $R'' \leftarrow k \rightarrow R'$.

If $R' = k\langle T_1, \ldots, T_n\rangle/(f_1, \ldots, f_r)$ and $R'' = k\langle S_1, \ldots, S_m\rangle/(g_1, \ldots, g_s)$, then $R' \amalg R'' \simeq k\langle T_1, \ldots, T_n, S_1, \ldots, S_m\rangle/(f_1, \ldots, f_r, g_1, \ldots, g_s)$.

4. For two graded algebras R' and R'' we define their (*fibre*) *product* over k, $R' \sqcap R''$, as the pullback of $R' \rightarrow k \leftarrow R''$.

If $R' = k\langle T_1, \ldots, T_n\rangle/(f_1, \ldots, f_r)$ and $R'' = k\langle S_1, \ldots, S_m\rangle/(g_1, \ldots, g_s)$, then $R \sqcap R'' \simeq k\langle T_1, \ldots, T_n, S_1, \ldots, S_m\rangle/(f_1, \ldots, f_r, g_1, \ldots, g_s, T_iS_j, S_jT_i)$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$.

5. If $R' = k\langle T_1, \ldots, T_n\rangle/(f_1, \ldots, f_r)$ and $R'' = k\langle S_1, \ldots, S_m\rangle/(g_1, \ldots, g_s)$, the tensor product over k is $R' \otimes R'' \simeq k\langle T_1, \ldots, T_m, S_1, \ldots, S_m\rangle/(f_1, \ldots, f_r, g_1, \ldots, g_s, T_iS_j - S_jT_i)$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$. We define the *nti-commutative tensorproduct* of R' and R'' as $R' \mathbin{\overline{\otimes}} R'' = k\langle T_1, \ldots, T_n, S_1, \ldots, S_m\rangle/(f_1, \ldots, f_r, g_1, \ldots, g_s, T_iS_j + S_iT_j)$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$.

6. The *Hilbert series* of $R = \bigoplus_{i\geqq 0} R_i$ is $|R|(Z) = \sum_{i\geqq 0} (\dim_k R_i) Z^i$ and correspondingly for graded modules.

7. For a graded algebra R, $\mathrm{Tor}_R(k, k)$ and $\mathrm{Ext}^R(k, k)$ are bigraded and we define the *double Poincaré series* of R as

$$P_R(X, Y) = \sum_{i,j\geqq 0} (\dim_k \mathrm{Tor}^{ij}_R(k, k)) X^i Y^j = \sum_{i,j\geqq 0} (\dim_k \mathrm{Ext}^R_{ij}(k, k)) X^i Y^j$$

(first degree homological, second induced by the grading of R) and the *Poincaré series* of R as $P_R(Z) = P_R(Z, 1)$.

8. If f is a homogeneous element of positive degree in a graded algebra R we have $|R/(f)|(Z) \geqq |R|(Z)/(1 + Z^{\deg f}|R|(Z))$ (coefficientwise). f is called *strongly free* if we have equality (cf [1]).

9. If f is a homogeneous element of positive degree in a *commutative* graded algebra R we have $|R/(f)|(Z) \geqq (1 - Z^{\deg f})|R|(Z)$ (coefficientwise), with equality if and only if f is a non-zerodivisor.

10. The *lattice associated* to a graded algebra $R = V/P$ is the lattice $L(R)$ of subspaces to the graded k-vectorspace V, generated by $\{I^f P^g I^h;\ f, g, h \geqq 0\}$ under $+$ and $\cap$. Here $I^0 = P^0 = V$.

11. The *ith local lattice associated* to a graded algebra $R = V/P$ is the lattice of subspaces to V_i, generated by $\{I_f P_g I_h;\ f, g, h \geqq 0, f + g + h = i\}$ under $+$ and $\cap$. This lattice is denoted $L_i(R)$.

12. A lattice $(L, +, \cap)$ is *distributive* if $p \cap (q + r) = (p \cap q) + (p \cap r)$ for all p, q, r in L. If $(L, +, \cap)$ is a finitely generated lattice

of subspaces to a vectorspace V, then L is distributive if and only if there is a basis B for V such that $B \cap p$ is a basis for p, for each p in L. Such a basis is said to *distribute* L (cf [2, lemma 1.2]).

13. A graded algebra $R = V/P$ is called *r-related* if P is generated by elements of degree $\leqq r$.

14. Let $R = V/P$ be 2-related and let $V^* = \mathrm{Hom}_k(V, k) = \bigoplus_{i \geqq 0} \mathrm{Hom}_k(V_i, k) = \bigoplus_{i \geqq 0} V_i^*$ with multiplication induced by $\mu\nu(ab) = \mu(a)\nu(b)$ ($\mu \in V_i^*$, $\nu \in V_j^*$, $a \in V_i$, $b \in V_j$). Let $P_2^0 = \{\mu \in V_2^* ; \mu(P_2) = 0\}$ and let $P^\circ = (P_2^0) \subset V^*$. The *dual ring* to R is defined to be $R^\circ = V^*/P^\circ$ (cf [2, ch. 3]).

15. A homomorphism $\Phi : R' \to R''$ of graded algebras is called *small* if and only if the induced homomorphism $\mathrm{Tor}^{R'}_{*,*}(k, k) \to \mathrm{Tor}^{R''}_{*,*}(k, k)$ is injective, or equivalently, if and only if the induced homomorphism $\mathrm{Ext}^{*,*}_{R''}(k, k) \to \mathrm{Ext}^{*,*}_{R'}(k, k)$ is surjective.

16. A graded algebra R is called a *Koszul algebra* if and only if the following equivalent conditions are satisfied (cf. e.g. [14], [11] and [2]).

(1) $\mathrm{Ext}^1_R(k, k)$ generates $\mathrm{Ext}_R(k, k)$ as an algebra with Yoneda multiplication.

(2) $\mathrm{Ext}^{p,q}_R(k, k) = 0$ for $p \neq q$.

(3) $\mathrm{Tor}^R_{p,q}(k. k) = 0$ for $p \neq q$.

(4) $P_R(X, Y)\,|R|\,(-XY) = 1$.

(5) $P_R(Z)\,|R|\,(-Z) = 1$.

(6) $(I/P)^2$ is small (i.e., $R \to R/(I/P)^2$ is small).

(7) R is 2-related and $L(R)$ is distributive.

(8) R is 2-related and $L_i(R)$ is distributive for all $i \geqq 2$.

(9) R is 2-related and V_i has a $L_i(R)$-distributing basis for all $i \geqq 2$.

(10) R is 2-related and R° satisfies (1)—(9).

(The equivalence of (1)—(4) could be found in [11, thm. 1.2], and that these conditions are equivalent to (7)—(10) is shown in [2, thm. 3.3]. For (1)$\Leftrightarrow$(6), cf e.g. [3, bottom of page 2]. (4) obviously implies (5). Assuming (5) and by [11,1.11] we have $P_R(-X, 1)\,|R|\,(X) = P_R(-1, Y)\,|R|\,(Y) = 1$. Using $\mathrm{Tor}^R_{i,j}(k, k) = 0$ for $j < i$ this easily gives (3).)

17. The following classes of graded algebras are examples of Koszul algebras.

(a) $k\langle T_1, \ldots, T_n\rangle/I$ where I is generated by an arbitrary set of monomials of degree two, [6, cor. 1 in sec. 4].

(b) $k[X_1, \ldots, X_n]/I$ where I is generated by an arbitrary set of monomials of degree two, [6, cor. 2 in sec. 4].

(c) $k[X_1, \ldots, X_n]/I$ where I is generated by some special classes of monomials and binomials of degree two, [9].

(d) $k[X_1, \ldots, X_{n_1}]^{(d_1)} \circ \ldots \circ k[X_1, \ldots, X_{n_i}]^{(d_i)}$, [5, thm 2.1].

(e) "Most" *commutative* 2-related algebras in embedding dimension $\leqq 3$, [4, thm 1].

(f) "Most" 2-related algebras with at most two relations, [2, thm 4.6].

(g) All 2-related *commutative* graded algebras R with $\dim_k R_2 \leqq 2$, [2, thm 4.8].

(h) $k[X_1, \ldots, X_n]/I$ where I is generated by a regular sequence of elements of degree two, [4, lemma 2].

2. KOSZUL ALGEBRAS

The main result in this section is that the class of Koszul algebras is closed under a number of operations. In some cases we get slightly more general results.

THEOREM 1. (a) *Let R be a graded algebra with $L_i(R)$ distributive for all $i \geq 2$. Then $L_i(R^{(d)})$ is distributive for all $i \geq 2$ and all $d \geq 2$.*

(b) *Let R' and R'' be graded algebras with $L_i(R')$ and $L_i(R'')$ distributive for all $i \geq 2$. Then $L_i(R' \circ R'')$ is distributive for all $i \geq 2$.*

(c) *Let R' and R'' be graded algebras. Then $L_i(R' \sqcap R'')$ is distributive for all $i \geq 2$ if and ony if both $L_i(R')$ and $L_i(R'')$ are distributive for all $i \geq 2$.*

(d) *Let R' and R'' be graded algebras. Then $L_i(R' \sqcup R'')$ is distributive for all $i \geq 2$ if and only if both $L_i(R')$ and $L_i(R'')$ are distributive for all $i \geq 2$.*

Proof. (a) Fix $d \geq 2$ and $i \geq 2$. Let $Q_i = \sum_{t=0}^{i-1} V_{td} P_d V_{(i-t-1)d}$. Then $V_i^{(d)} = V_{id}/Q_i$. Let $\pi_i : V_{id} \to V_i^{(d)}$ be the projection. For f, g, h such that $f + g + h = i$ we have

$$V_f^{(d)} P_g^{(d)} V_h^{(d)} = (Q_i + V_{fd} P_{gd} V_{hd})/Q_i.$$

Thus the restriction of π_i^{-1} induces a monomorphism of lattices $L_i(R^{(d)}) \hookrightarrow L_{id}(R)$. Now use the fact that a sublattice of a distributive lattice is distributive.

(b) Fix an $i \geq 2$. Let $R = R' \circ R''$. Then $V_i = V_i' \oplus V_i''$, and if $f + g + h = i$ then

$$V_f P_g V_h = (V_f' P_g' V_h' \otimes V_i'') + (V_i' \otimes V_f'' P_g'' V_h'').$$

Now apply [2, lemma 1.3].

(c) Let $R' = V'/P'$, $R'' = V''/P''$, and $R = R' \sqcap R'' = V/P$ in the natural manner. Fix an $i \geq 2$. Then V_i is the direct sum of all kinds of mixed products of length i of copies of V_1' and V_1''. More precisely, let us write

$$V_i = \bigoplus_{\mathfrak{a} \in A(i)} V_i^{\mathfrak{a}},$$

where $A(i) = \{\mathfrak{a} = (a_1, \ldots, a_i)\,;\, a_1, \ldots, a_i \in \{', ''\}\}$, and where $V_i^{\mathfrak{a}} = V_1^{a_1} V_1^{a_2} \ldots \ldots V_1^{a_i}$ for $\mathfrak{a} \in A(i)$.

Thus if we put $'(i) = (', ', \ldots, ') \in A(i)$ and $''(i) = ('', '', \ldots, '') \in A(i)$, then $V_i' = V_i^{'(i)}$ and $V_i'' = V_i^{''(i)}$, and for $f + g + h = i$ we get

$$\begin{aligned} V_f P_g V_h &= \Big(\sum_{\mathfrak{a} \in A(f)} \sum_{\mathfrak{b} \in A(h)} V_f^{\mathfrak{a}} P_g' V_h^{\mathfrak{b}} + V_f^{\mathfrak{a}} P_g'' V_h^{\mathfrak{b}} \Big) + \sum_{\mathfrak{a} \in A(f)} \sum_{\mathfrak{b} \in A(h)} \sum_{\mathfrak{c}} V_f^{\mathfrak{a}} V_g^{\mathfrak{b}} V_h^{\mathfrak{c}} \subseteq \\ &\subseteq \bigoplus_{\mathfrak{a} \in A(i)} (V_i^{\mathfrak{a}} \cap V_f P_g V_h), \end{aligned} \tag{1}$$

where the last sum in the first row is over $\mathfrak{c} \in A(g) \setminus \{'(g), ''(g)\}$. Now assume that $L_j(R')$ and $L_j(R'')$ are distributive for $j \leq i$. For $j = 2, \ldots, i$, let B_j'

(resp. B''_j) be a $L_j(R')$-distributing basis (resp. a $L_j(R'')$-distributing basis). Let

$$B_i = \{b_1 b_2 \ldots b_s;\ b_t \in B'_{j_t} \text{ for } t \text{ odd and } b_t \in B''_{j_t} \text{ for } t \text{ even } or\ b_t \in B'_{j_t} \text{ for } t \text{ even and } b_t \in B''_{j_t} \text{ for } t \text{ odd, where } \sum_{t=1}^{s} j_t = i\}.$$

(1) yields that B_i distributes $L_i(R)$. Conversely, assume that $L_i(R)$ is distributive. By (1) there are surjective homomorphisms of lattices $\pi'_i : L_i(R) \to L_i(R')$ and $\pi''_i : L_i(R) \to L_i(R'')$, defined by $\pi_i(p) = p \cap V'_i$ and $\pi''_i(p) = p \cap V''_i$. Now use the fact that a homomorphic image of a distributive lattice is distributive.

(d) With the same notations (mutatis mutandis) as in the proof of (c),

$$V_f P_g V_h = \sum_{c \geqq f} \sum_{d \geqq h} \sum_{\mathfrak{a} \in A(c)} \sum_{\mathfrak{b} \in A(d)} V_c^{\mathfrak{a}} P_{i-c-d} V_d^{\mathfrak{b}} \subseteq \bigoplus_{\mathfrak{a} \in A(i)} (V_i^{\mathfrak{a}} \cap V_f P_g V_h). \text{ Now}$$

proceed as before.

Let R' and R'' be 2-related algebras. Using the representation with generators and relations in 1.3—1.5 and the definition of the dual ring in 1.14 yields:

LEMMA 2. (a) $(R' \sqcup R'')^\circ \simeq (R')^\circ \sqcap (R'')^\circ$
(b) $(R' \sqcap R'')^\circ \simeq (R')^\circ \sqcup (R'')^\circ$
(c) $(R' \otimes R'')^\circ \simeq (R')^\circ \overline{\otimes} (R'')^\circ$
(d) $(R' \overline{\otimes} R'')^\circ \simeq (R')^\circ \otimes (R'')^\circ$

If R' and R'' are r'-related and r''-related, respectively, then directly from 1.3—1.5 it follows that $R' \sqcup R''$, $R' \sqcap R''$, $R' \otimes R''$ and $R' \overline{\otimes} R''$ are max $(r', r'', 2)$-related. In order to obtain similar estimates for Veronese subrings and Segre products, we may use the interpretation of the i'th degree relations in $R = V/P$ as

$$P_i/(P_{i-1}V_1 + V_1 P_{i-1}).$$

Note also that any *commutative* algebra $R = V/P$ has as relations all commutators ab-ba $(a, b \in V_1)$ of degree two. We denote the ideal in V generated by these commutators C.

The first half of the following proposition generalizes a result of D. Mumford ([12, thm 1]).

PROPOSITION 3. (a) *If R is an r-related graded algebra, then $R^{(d)}$ is $[2 + (r-2)/d]$-related; and if furthermore R is commutative, then $R^{(d)}$ is* max $([1 + (r-1)/d], 2)$*-related.*

In particular, if $d \geqq r - 1$ *or if R is commutative and $d \geqq r/2$, then $R^{(d)}$ is 2-related.*

(b) *If R' and R'' are r'-related and r''-related, respectively, then $R' \circ R''$ is max (r', r'')-related.*

(Here $[c]$ *denotes the integer part of* c.)

Proof. (a) Let $d \geqq 2$ and let $i > [2 + (r-2)/d]$, whence $i \geqq 2 + (r-1)/d$, i.e.

(1) $$id \geqq r + 2d - 1.$$

Let $Q_i = \sum_{t=0}^{i-1} V_{td} P_d V_{(i-t-1)d}$. Then $P_i^{(d)} = (P_{id} + Q_i)/Q_i$ and $P_{i-1}^{(d)} V_1^{(d)} + V_1^{(d)} P_{i-1}^{(d)} = (P_{(i-1)d} V_d + V_d P_{(i-1)d} + Q_i)/Q_i = (P_{(i-1)d} V_d + V_d P_{(i-1)d})/Q_i$. If $0 \leqq f \leqq d-1$ then by (1) $V_f P_r V_{id-f-r} \subset P_{(i-1)d} V_d$, while if $d \leqq f \leqq 2d-1$ then $V_f P_r V_{id-f-r} \subset V_d P_{(i-1)d}$. Thus and since R is r-related,

$$P_{id} = \sum_{f=0}^{id-r} V_f P_r V_{id-f-r} \subset P_{(i-1)d} V_d + V_d P_{(i-1)d} \subset P_{id}.$$

Thus indeed $P_i^{(d)} = P_{i-1}^{(d)} V_1^{(d)} + V_1^{(d)} P_{i-1}^{(d)}$.

If R is commutative these arguments may be slightly improved. For arbitrary integers $d \geqq 2$ and $i \geqq 3$ we then have

$$P_{(i-1)d} V_d + V_d P_{(i-1)d} = \sum_{f=0}^{d} V_f P_{(i-1)d} V_{d-f}. \tag{2}$$

This is so because both sides in (2) contain the graded component C_{id} of the commutator ideal C, where (2) follows from the trivial equality in commutative algebra.

$$(C_{id} + P_{(i-1)d} V_d)/C_{id} = (C_{id} + \sum_{f=0}^{d} V_f P_{(i-1)d} V_{d-f})/C_{id}.$$

Hence, if $i > \max\,[(1 + (r-1)/d],\ 2)$, i.e. if $(i-1)d \geqq r$ and $i \geqq 3$, then by (2)

$$P_{(i-1)d} V_d + V_d P_{(i-1)d} \supset \sum_{f=0}^{id-r} V_f P_r V_{id-f-r} = P_{id},$$

whence, as above, $P_i^{(d)} = P_{i-1}^{(d)} V_1^{(d)} + V_1^{(d)} P_{i-1}^{(d)}$.

Finally, if $d \geqq r-1$ (or if R is commutative and $d \geqq r/2$), then $[2 + (r-2)/d] = 2$ ($[1 + (r-1)/d] = 2$, respectively), whence R is 2-related.

b) Assume that $R' = V'/P'$ is r'-related, that $R'' = V''/P''$ is r''-related, that $R' \circ R'' = V/P$ and that $i > \max\,(r', r'')$. Then

$$\begin{aligned} P_{i-1} V_1 + V_1 P_{i-1} &= (P'_{i-1} V'_i \otimes V''_i) + (V'_i \otimes P''_{i-1} \otimes V''_1) + (V'_1 P'_{i-1} \otimes V''_i) + \\ &+ (V'_i \otimes V''_1 P''_{i-1}) = ((P'_{i-1} V_1 + V_1 P'_{i-1}) \otimes V''_i) + (V'_i \otimes (P''_{i-1} V''_1 + \\ &+ V''_1 P''_{i-1})) = (P'_i \otimes V''_i) + (V'_i \otimes P''_i) = P_i. \end{aligned}$$ ∎

Examples. If $R = k[X_1, \ldots, X_d]/(X_1^r, \ldots, X_d^r)$, then $R^{(d)}$ *has* relations of degree $\max\,[1 + (r-1)/d], 2)$. Thus, this bound in the commutative case cannot be improved. Furthermore, $R^{(d)}$ is *not* 2-related unless $d \geqq \frac{1}{2} r$.

If S is the 4-related algebra $k\langle T_1, T_2\rangle/(T_1 T_2, T_1^2 - T_2^2) I^2$, then $S^{(2)}$ is 3-related but not 2-related. Thus, there is a real difference for the bounds in the commutative and the non-commutative case.

Remark. If we drop the assumption that P is finitely generated, for a graded algebra V/P some strange phenomena may occur. Let $R = k\langle T_1, T_2, T_3\rangle/P$, where $P = (T_1 T_2, T_2^2, T_3 T_1, T_3 T_2, T_3^2, T_2 T_1^i T_3;\ i = 0, 1, \ldots)$ is not finitely generated. Then $R^{(d)} \simeq R$ for all $d \geqq 1$.

In the following theorem we collect results on the preservation of being Koszul algebra. Some special cases of the theorem are already known. E. g. (a) and (b) generalizes the result in 1.17 (d) of S. Barcanescu and N. Manolache. The equivalence (i)$\Leftrightarrow$(iv) in (c) is essentially due to S. Priddy (i)$\Rightarrow$(ii) in (c) follows easily from results of J. M. Lemaire (see the alternative proof on this point). Case (iii) in (e) has a simple homological proof in the commutative case.

THEOREM 4. (a) *If R is a Koszul algebra then $R^{(d)}$ is a Koszul algebra for all d.*

(b) *If both R' and R'' are Koszul algebras, then $R' \circ R''$ is a Koszul algebra.*

(c) *The following five conditions are equivalent.*

(i) *Both R' and R'' are Koszul algebras;*

(ii) *$R' \sqcup R''$ is a Koszul algebra;*

(iii) *$R' \sqcap R''$ is a Koszul algebra;*

(iv) *$R' \otimes R''$ is a Koszul algebra;*

(v) *$R' \underline{\otimes} R''$ is a Koszul algebra.*

(d) *If $R' \to R''$ is small and R'' is a Koszul algebra, then R' is a Koszul algebra.*

(e) *Assume that $R'' = R'/(f)$ where f is a homogeneous element in the graded algebra R', and that one of the following four conditions is satisfied.*

(i) *f is strongly free and of degree one or two;*

(ii) *f is a socle element of degree one;*

(iii) *f is a socle element of degree two and (f) is small;*

(iv) *R' is commutative and f is a non-zerodivisor of degree one or two.*

Then R' is a Koszul algebra if and only if R'' is a Koszul algebra.

Proof. (a) follows from Theorem 1 (a), Proposition 3 (a) and 1.16 (8). (b) follows from Theorem 1 (b), Proposition 3 (b) and 1.16 (8).

(c) The equivalence (i)$\Leftrightarrow$(ii) follows from Theorem 1 (c) and 1.16 (8), noting that $R' \sqcup R''$ is 2-related if and only if both R' and R'' are 2-related, but we will also give an alternative proof without using Theorem 1. For any graded algebras R and S we have $P_{R \sqcup S}(Z) = P_R(Z) + P_S(Z) - 1$ and $|R \sqcup S|(Z)^{-1} = |R|(Z)^{-1} + |S|(Z)^{-1} - 1$ ([10, Lemma 5.1.9 and Lemma 5.1.10]). Thus, if R' and R'' are Koszul algebras it easily follows that $R' \sqcup R''$ is a Koszul algebra using 1.16 (5). On the other hand, for any 2-related graded algebra R it is true that R° is the subalgebra of $\mathrm{Ext}_R(k,k)$ generated by $\mathrm{Ext}^1_R(k, k)$ ([11, thm 1.1]), hence $|R^\circ|(Z) \leq P_R(Z)$ with equality if and only if R is a Koszul algebra according to 1.16 (1). Thus, supposing $R' \sqcup R''$ to be a Koszul algebra, we have

$$P_{R'}(Z) + P_{R''}(Z) - 1 = P_{R' \sqcup R''}(Z) = |(R' \sqcup R'')^\circ|(Z) = |(R')^\circ \sqcap (R'')^\circ|(Z) =$$
$$= |(R')^\circ|(Z) + |(R'')^\circ|(Z) - 1 \leq P_{R'}(Z) + P_{R''}(Z) - 1.$$

Thus we have that $|(R')^\circ|(Z) = P_{R'}(Z)$ and $|(R'')^\circ|(Z) = P_{R''}(Z)$ whence R' and R'' are Koszul algebras.

(i)$\Leftrightarrow$(iv) is [14, Prop. 2.1] (in fact only (i)$\Rightarrow$(iv) is stated, but the argument works equally well in the other direction).

(i)$\Leftrightarrow$(iii) (and (iv)$\Leftrightarrow$(v)) follows from Lemma 2 (b) and 1.16 (10) (and Lemma 2 (d) and 1.16 (10), respectively).

(d) follows from 1.15 and 1.16 (3).

Suppose R' and R'' to be as in (e).

If f is strongly free it follows from [1, thm 2.10] that $\mathrm{Tor}_i^{R'}(k, k) \simeq \simeq \mathrm{Tor}_i^{R''}(k, k)$ for $i \geqq 3$. Thus, using that for a graded algebra $R = V/P$ we have $\mathrm{Tor}_1^R(k, k) \simeq I/I^2$ and $\mathrm{Tor}_2^R(k, k) \simeq P/(IP + PI)$, it follows that $P_{R''}(Z) = P_{R'}(Z) - Z$ if $\deg(f) = 1$ and $P_{R''}(Z) = P_{R'}(Z) + Z^2$ if $\deg(f) = 2$, repectively. Thus, using 1.8, it follows easily that $|R''|(-Z) P_{R''}(Z) = 1$ if and only if $|R'|(-Z) P_{R'}(Z) = 1$, whence the equivalence follows from 1.16 (5).

If f is a socle element of degree one, we have $R'' \simeq R' \sqcap k[X]/(X^2)$, and $k[X]/(X^2)$ is a Koszul algebra, whence the equivalence follows from (c) above.

Now assume that f is a socle element of degree two. Since (f) is small, by (d), R' is a Koszul algebra if R'' is. We will use 1.16 (8) to show the other direction. By the assumptions

(3) $P'_2 \subset P''_2$ and $P'_{i-1}V_1 + V_1P'_{i-1} = P'_i = P''_i = P'_{i-1}V_1 + V_1P''_{i-1}$ for $i \geqq 3$.

For $i \geqq 2$, we know that $L_i(R')$ (which is generated by $(V_{f-1}P'_2V_{i-1-f};\ f = 1, \ldots, i-1)$) is distributive, and we want to show that $L_i(R'')$ (which is generated by $(V_{f-1}P''_2V_{i-1-f};\ f = 1, \ldots, i-1)$) is distributive. Now the idea is to use (3) in order to show that a necessary and sufficient small family of conditions for distributivity remains valid when successively the generators $V_{f-1}P'_2V_{i-1-f}$ are replaced by generators $V_{f-1}P''_2V_{i-1-f}$. Let us adopt the notation $A(i) = \{a = (a_1, \ldots, a_i;\ a_1, \ldots, a_i \in \{', ''\}\}$, etc., from the proof of Theorem 1 (c). For $i \geqq 2$ and $a \in A(i-1)$, let

$$F(i, a) = (V_{f-1}P_2^{a_f}V_{i-1-f}; \quad f = 1, \ldots, i-1).$$

Then the families $F(i, '(i-1))$ and $F(i, ''(i-1))$ generate $L_i(R')$ and $L_i(R'')$, respectively, as subspaces of V_i.

Hence it is sufficient to prove that for all $i \geqq 2$ and all $a \in A(i-1)$,

(4) $F(i, a)$ generates a distributive lattice.

In order to do this we use induction, in the first place with respect to i, and in the second place with respect to the number $n(a) = |\{f;\ a_f = ''\}|$ of times '' appear in a. Obviously (4) is true for $i = 2, 3$ and for any i if $n(a) = 0$ (i.e. if $a = '(i-1)$). Thus, let $j \geqq 4$, $b \in A(i-1)$, $n(b) > 0$ and assume (4) to hold for any (i, a) such that $i < j$ or that $n(a) < n(b)$. There are integers $g, h \in \{1, \ldots, i-1\}$ such that $b_g = ''$ and that $|h - g| = 1$. Let σ be a permutation of the integers $1, \ldots, j-1$ such that $\sigma(j-2) = g$ and that $\sigma(j-1) = h$. By [13] $F(j, b)$ generates a distributive lattice if and only if the following two conditions are satisfied:

(5) Any $(j-2)$-subfamily of $F(j, b)$ generates a distributive lattice, and

(6)
$$\left(\bigcap_{s=1}^{k} V_{(\sigma s)-1}P_2^{b_{\sigma s}}V_{j-1-\sigma s}\right) \sum_{t=k+1}^{j-1} V_{(\sigma t)-1}P_2^{b_{\sigma t}}V_{j-1-\sigma t} =$$
$$= \sum_{t=k+1}^{j-1} \left(V_{(\sigma t)-1}P_2^{b_{\sigma t}}V_{j-1-\sigma t} \cap \bigcap_{s=1}^{k} V_{(\sigma s)-1}P_2^{b_{\sigma s}}V_{j-1-\sigma s}\right) \text{ for } k = 1, \ldots, j-3.$$

Let $F = (P_2^{b_1}V_{j-2}, \ldots, V_{i-1}P_2^{b_i}V_{j-1-i}, \ldots, V_{j-2}P_2^{b_{j-1}};\ i = 1, \ldots, j-1)$ be, an arbitrary $(j-2)$-subfamily of $F(j, b)$. First assume that $1 < i < j-1$. By the induction hypothesis $F(i, (b_1, \ldots, b_{i-1}))$ and $F(j-i, (b_{i+1}, \ldots, b_{j-1}))$ generate distributive lattices of subspaces of V_i and of V_{j-i}, respectively, whence by [2, lemma 1.3] F generates a distributive lattice, indeed. If $i = 1$ or $i = j-1$, proceed correspondingly. Thus (5) holds.

Now fix a k such that $1 \leq k \leq j-3$. Define $c \in A(j)$ by $c_f = 1$ if $f = g$ and $c_f = b_f$ otherwise. Then $n(c) = n(b) - 1$, whence by the induction hypothesis $F(i, c)$ generates a distributive lattice. Furthermore, by (3)

$$\bigcap_{s=1}^{k} V_{(\sigma s)-1}P_2^{b\sigma s}V_{j-1-\sigma s} \cap \sum_{t=k+1}^{j-1} V_{(\sigma t)-1}P_2^{b\sigma t}V_{j-1-\sigma t} =$$

$$= \bigcap_{s} V_{(\sigma s)-1}P_2^{c\sigma s}V_{j-1-\sigma s} \sum_{t} V_{(\sigma t)-1}P_2^{c\sigma t}V_{j-1-\sigma t} =$$

$$= \sum_{t} (V_{(\sigma t)-1}P_2^{c\sigma t}V_{j-1-\sigma t} \cap \bigcap_{s} V_{(\sigma s)-1}P_2^{c\sigma s}V_{j-1-\sigma s} \subset$$

$$\subset \sum_{t} (V_{(\sigma t)-1}P_2^{b\sigma t}V_{j-1-\sigma t} \cap \bigcap_{s} V_{(\sigma s)-1}P_2^{b\sigma t}V_{j-1-\sigma s} \subset$$

$$\subset \bigcap_{s} V_{(\sigma s)-1}P_2^{b\sigma s}V_{j-1-\sigma s} \cap \sum_{t} V_{(\sigma t)-1}P_2^{b\sigma t}V_{j-1-\sigma t},$$

whence (6) holds and we have proved the equivalence for (iii).

Finaly, if R' is commutative and f is a non-zerodivisor of degree one (two) we have $P_{R''}(Z) = P_{R'}(Z)/(1+Z)$ $(P_{R''}(Z) = P_{R'}(Z)/(1-Z^2))$ according to [8, cor 3.4.2 (ii)] ([8, cor 3.4.2 (i)], respectively). Thus, since $|R''|(Z) = (1-Z)|R'|(Z)$ $(|R''|(Z) = (1-Z^2)|R'|(Z)$, respectively) we easily get $|R''|(-Z)P_{R''}(Z) = 1$ if and only if $|R'|(-Z)P_{R'}(Z) = 1$, hence (e) follows in case (iv) by 1.16 (5).

Remark. We believe that the concepts "small socle element" on one side and "strongly free" on the other are dual in the following sense.

Conjecture. Assume that R' is 2-related and the $R'' = R'/(f)$ for some f of degree two in R', whence $(R')^\circ = (R'')^\circ/(g)$ for some g of degree two in $(R'')^\circ$. Then f is strongly free if and only if g is a small socle element. If this is true in general, then clearly case (iii) of (*e*) in the theorem follows from case (i). The conjecture *is* true in the special case when at least one of R' and R'' is a Koszul algebra according to the theorem. If R' is a Koszul algebra and f a socle element of degree two in R', it follows from the theorem that R'' is a Koszul algebra. It is not true that, if f is a socle element of degree two, then R'' a Koszul algebra implies R' a Koszul algebra. A counterexample is

$$R' = k[X_1, X_2, X_3]/(X_1^2, X_2X_3, X_1X_3 + X_2^2),\ f = x_1x_3$$

(the image of X_1X_3).

3. VERONESE SUBRINGS

All graded algebras in this section are assumed to be *commutative.*

If R is a graded (commutative) algebra of Krull dimension t, then $\dim_k R_j$ is a polynomial $h(j, R)$, the *Hilbert-Samuel polynomial,* of degree $t-1$ for $j \gg 0$. (If $t = 0$ then $h(j, R) = 0$.) The *regularity index* of R is $i(R) = \max\{j; \dim_k R_j \neq h(j, R)\} + 1$. If $|R|(Z) = p(Z)/(1-Z)^t$, where $t = \dim R$, then $p(1)$ is the *multiplicity* of R.

If R is 0-dimensional we have $R^{(d)} \simeq k$ for $d \geqq i(R)$. We will generalize this to 1-dimensional algebras.

THEOREM 5. *If R is a (commutative) graded algebra (over an infinite field) of Krull dimension 1, there exists a graded algebra R^∞ and an integer d_0 such that $R^{(d)} \simeq R^\infty$ if $d \geqq d_0$. (If R is Cohen-Macaulay we can choose $d_0 = i(R)$.) Moreover R^∞ contains a non-zerodivisor x of degree one such that $R^\infty/(x) \simeq k[X_1, \ldots, X_{m-1}]/(X_1, \ldots, X_{m-1})^2$, where m is the multiplicity of R. In particular R^∞ is a Koszul algebra.*

Proof. If we factor out a socle element s, we have for $\bar{R} = R/(s)$ that $\bar{R}^{(d)} \simeq R^{(d)}$ if $d \gg 0$ and that $\dim \bar{R} = 1$. Thus we can continue until $R/(s_1, \ldots, s_t)$ has no socle, i.e. we can assume R to be Cohen-Macaulay. Suppose $R = k[X_1, \ldots, X_n]/(F_1, \ldots, F_s)$ and that $\bar{Y} = y$ is a non-zerodivisor in R of degree one. It is easy to see that $i(R/(y)) = i(R) + 1$, and hence, if R (and thus $T/(y)$) is r-related but not $(r-1)$-related, $i(R) \geqq \geqq r - 1$ since $i(R/(y)) \geqq r$. $R^{(d)}$ is 2-related for $d \geqq i(R)$ according to proposition 3. Let $\dim_k R_j = m$ (= the multiplicity of R) if $j \geqq i(R)$. Then $|R^{(d)}|(Z) = 1 + mZ + mZ^2 + mZ^3 + \ldots = (1 + (m-1)Z)/(1-Z)$ for $d \geqq i(R)$. Let $R^{(i(R))} = k[g_1, \ldots, g_m]$, g_i in R, and let $\bar{G}_i = g_i$, G_i in $k[X_1, \ldots, X_n]$. Then $R^{(i(R))} \simeq k[Y_1, \ldots, Y_m]/J$, where J is generated by those forms $\sum_{j,k} c_{jk} Y_j Y_k$ for which $\sum_{j,k} c_{jk} G_j G_k$ belongs to $(F_1, \ldots, F_s)$. Since y is a non-zerodivisor in R we have $R^{(i(R)+i)} = k[y^i g_1, \ldots, y^i g_m] \simeq k[Y_1, \ldots, Y_m]/J'$, where J' is generated by those forms $\sum_{j,k} c_{jk} Y_j Y_k$ for which $\sum_{j,k} c_{jk} Y^i G_j Y^i G_k = = Y^{2i} \sum_{j,k} c_{jk} G_j G_k$ belongs to $(f_1, \ldots, F_s)$. Thus $J \subset J'$, but $|R^{(i(R))}|(Z) = = |R^{(i(R)+i)}|(Z)$, whence $R^{(d)} \simeq R^{(i(R))}$ for $d \geqq i(R)$. $R^{(i(R))}$ contains a non-zerodivisor of degree one (e.g. $x = y^{i(R)}$) and $|R/(x)|(Z) = 1 + + (m-1)Z$, whence $R/(x) \simeq k[X_1, \ldots, X_{m-1}]/(X_1, \ldots, X_{m-1})^2$. That R^∞ is a Koszul algebra follows from 1.17 (b) and theorem 4 (e) (iv).

We call a graded algebra S a *limit algebra* if $S \simeq R^\infty$ for some one-dimensional garded algebra R. It follows from the proof of theorem 5 that an algebra R with $|R|(Z) = (1 + (m-1)Z)/(1-Z)$ is a limit algebra if and only if it is Cohen-Macaulay.

Example. We list all limit algebras of embedding dimension $\leqq 3$ (i.e. isomorphic to R^∞ for some R of multiplicity $\leqq 3$). Such an algebra S has $|S|(Z) = (1 + (m-1)Z)/(1-Z)$, $m = 1, 2$ or 3. Any algebra with such a series is isomorphic to one of the following (cf [4] for the case $m=3$):

(1) $k[X]$

(2) $k[X, Y]/(X^2)$ or $k[X, Y]/(XY)$

(3) $k[X, Y, Z]/I$ where $I = (X^2, XY, Y^2)$ or (X^2, XY, YZ) or (XY, XZ, YZ) or $(X^2, XY, XZ + Y^2)$.

It is easy to check that $S^{(d)} \simeq S$ for all d in all these cases, i.e. they are limit algebras.

We can also give examples of limit algebras of higher multiplicity Namely, let $R(j) = k[X_1, \ldots, X_n]/J(j)$ where

$$J(j) = X_1(X_2, \ldots, X_n) + X_2(X_3, \ldots, X_n) + \ldots + X_j(X_{j+1}, \ldots, X_n) + \\ + (X_{j+1}, \ldots, X_{n-1})^2.$$

The $R(j)$ is a limit algebra of multiplicity $n-1$ for each $j=0, 1, \ldots, n-1$. If $R = k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$ is a graded algebra we have

$$|R|(Z)^l \leqq (1-Z)^{-g} \max \left(\prod_{i=1}^{r} (1 - Z^{d_i})/(1-Z)^{n-g}, \quad 1/(1-Z)^{d-g} \right),$$

where $g = \operatorname{depth} R$, $d = \dim R$ and $d_i = \deg f_i$, $i = 1, \ldots, r$. R is called *extremal* of numerical character $(n, d, g, (d_1, \ldots, d_r))$ if there is equality, cf [7].

PROPOSITION 6. *If R is a limit algebra of multiplicity n, then R is extremal of numerical character $(n, 1, 1, (2, \ldots, 2))$. $\left(\text{The number of 2's is } \binom{n}{2}.\right)$*

Proof. Since $|R|(Z) = 1 + nZ + nZ^2 + nZ^3 + \ldots =$ $=(1+(n-1)Z)/(1-Z)$ and since R is 2-related, R can be represented as

$$k[X_1, \ldots, X_n]/(f_1, f_2, \ldots, f_{\binom{n}{2}}), \deg f_i = 2 \text{ for } i = 1, \ldots, \binom{n}{2}.$$

It is easy to check that $(1-Z)^{-1} \max ((1 - Z^2)^{\binom{n}{2}}/(1 - Z)^{n-1}, 1) =$ $=(1 - (n - 1)Z)/(1 - Z)$.

4. RINGS WITH 2-LINEAR RESOLUTIONS

All graded algebras in this section are assumed to be *commutative*.

The results in this section are, at least in the absolute case, fairly well-known. They are however, as far as we know, not published (cf [15] where half of our corollary is proved). We first define Golod maps (resp. Golod algebras) and d-linear maps (which is the relativization of a ring with d-linear resolution). We restrict to graded (commutative) algebras.

Let $\Phi : R' \to R$ be a surjective map of graded k-algebras, let X be a graded minimal R'-algebra resolution and let $Y = X \otimes_{R'} R$.

Φ is called a *Golod map* if the following equivalent conditions are satisfied :

(a) For each sequence $v_1, \ldots, v_s$ of elements in $H_+(Y)$ there is an element $\gamma(v_1, \ldots, v_s)$ in $\mathbf{m}Y$ ($\mathbf{m}$ the graded maximal ideal in R) such that

(g_1) $[\gamma(v)] = v$ for each v in $H_+(Y)$ and

(g_2) $d(\gamma(v_1, \ldots, v_s)) = y(v_1, \ldots, v_s) = \sum_{k=1}^{s-1} \overline{\gamma}(v_1, \ldots, v_k)\gamma(v_{k+1}, \ldots, v_s)$,

where $\overline{a} = (-1)^{\deg(a)+1} a$

(b) $P_R(Z)=P_{R'}(Z)/(1-Z(P^R_{R'}(Z)-1))$, where $P^R_{R'}(Z)=\sum_{i=0}^{\infty}\dim_k H_i(Y)Z^i=$ $=\sum_{i=0}^{\infty}\dim_k \operatorname{Tor}_i^{R'}(k, R)Z^i$.

R is called a *Golod algebra* of the natural map $k[X_1,\ldots,X_n]\to R$ is a Golod map.

Φ is called a *d-linear map* if $H_{i,j}(Y)=\operatorname{Tor}^{R'}_{i,j}(k,R)=0$ for $j\neq i+d-1$ $(i>0)$. R has a *d-linear resolution* if the natural map $k[X_1,\ldots,X_n]\to R$ is d-linear.

THEOREM 7. *Let $\Phi : R'\to R$ be a surjective map of graded algebras.*

(a) *If Φ is a d-linear map then Φ is a Golod map. If $d=2$ and R' is a Koszul algebra then R is a Koszul algebra.*

(b) *If Φ is a Golod map and R' and R are Koszul algebras, then Φ is a 2-linear map.*

In the absolute case we immediately get the following corolary.

COROLLARY. *R has a 2-linear resolution if and only if R is both a Koszul algebra and a Golod algebra.*

Proof. (a) Suppose Φ is a d-linear map, whence $H_i(Y)=H_{i,\,i+d-1}(Y)$ for $i>0$. Choose a basis B for $H_+(Y)$ and pick for each v in B a representative z of bidegree $(g, g+d-1)$ in Y. Let $\gamma(v)=z$. For each pair v_1, v_2 in B $y(v_1,v_2)=\bar{\gamma}(v)\gamma(v_2)$ lies in $Z(Y)\cap m^{2d-2}Y\subset B(Y)$, so we could define a $\gamma(v_1,v_2)$ of bidegree $(g, g+2d-3)$ for some g. By induction $y(v_1,\ldots,v_s)$ lies in $Z(Y)\cap m^{sd-2s+2}Y\subset B(Y)$ and we could continue as above. Then γ is extended k-linearly to $H_+(Y)$. Thus Φ is a Golod map.

Without assumption on Φ we have (cf [11, 1 11])

$$|R|(Z)=|H(Y)|(-1,Z)/|X|(-1,Z)=P^R_{R'}(-1,Z)/P_{R'}(-1,Z).$$

If R' is a Koszul algebra this equals $P^R_{R'}(-1,Z)|R'|(Z)$ according to 1.16 (4), and if Φ is 2-linear this equals $(1+Z(P^R_{R'}(Z)-1))|R'|(Z)$. Since Φ is a Golod map we have $P_R(Z)=P_{R'}(Z)/(1-Z(P^R_{R'}(Z)-1))$, so $|R|(Z)P_R(Z)=1$, hence R is a Koszul algebra according to 1.16 (5).

Now suppose Φ is a Golod map and that R' and R are Koszul algebras. Then

$$P_R(Z)=P_{R'}(Z)/(1-Z(P^R_{R'}(Z)-1))=P_{R'}(Z)/(1-\sum_{i>0}c_iZ^{i+1}),$$

where $c_i=\dim_k H_i(Y)$. On the other hand

$$P_R(Z)=1/|R|(-Z)=|X|(-1,-Z)/|H(Y)|(-1,Z)=$$
$$=P_{R'}(Z)/\sum_{i,j}(-1)^{i+j}c_{i,j}Z^j,$$

where $c_{i,j}=\dim_k H_{i,j}(Y)$. This gives $c_1=c_{1,2}$, $c_2=c_{2,3}-c_{1,3}$, $c_3=c_{3,4}-c_{2,4}+c_{1,4}$ etc since $c_{i,j}=0$ for $i>j$. Induction gives $c_i=c_{i,i+1}$ for $i>0$, hence Φ is 2-linear.

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Matematiska Institutionen
Stockholms Universitet
Box 6701
S-11385 Sweden

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