SYZYGIES AND WALKS

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ABSTRACT

The major theme of this article is combinatorial applications of the relationship between the Hilbert series of a graded (not necessarily commutative) algebra R, the Hilbert series of a graded R-module M, and the Poincaré biseries of M, notably in the case where M has a linear resolution. We interpret this relation as a combinatorial reciprocity law which for example connects the number of walks in a digraph with that in its complement. Furthermore we establish theorems on the existence of linear resolutions for certain residue class rings of algebras with straightening law and polynomial rings.

Let K be a field, and R a finitely generated homogeneous K-algebra, i.e. $R = \bigoplus_{i=0}^{\infty} R_i$ as a vector space, the multiplication on R satisfies the condition $R_iR_j \subset R_{i+j}$, and as a K-algebra R is generated by R_1. It is essential for this paper that R is not necessarily commutative. Since R is finitely generated, the K-dimension of R_i is finite for each i, and so the *Hilbert series* $H_R(t) = \sum_{n=0}^{\infty}(\dim_K R_n)t^n$ is well defined. Similarly, if $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a finite graded R-module, then it has a well defined Hilbert series $H_M(t) = \sum_{n=0}^{\infty}(\dim_K M_n)t^n$. If not indicated otherwise, a module is always a right module.

Though R need not be Noetherian (in the non-commutative case), the modules that we will consider have a minimal graded resolution by finitely generated free R-modules

$$F_\bullet : \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{ij}} \longrightarrow \bigoplus_j R(-j)^{\beta_{i-1,j}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0.$$

Here $R(-j)$ is the graded R-module $\bigoplus_{i=j}^{\infty} R_{i-j}$. That the resolution is graded means that its differential is a homogeneous homomorphism of degree 0. That it is

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minimal is equivalent to the fact that $F_\bullet \otimes K$ has zero differential. Therefore one may compute the *Poincaré biseries*

$$P_M^R(t,u) = \sum_{i,j} (\dim_K \operatorname{Tor}_i^R(M,K)_j) t^j u^i$$

from $F_\bullet$: since $F_\bullet \otimes K$ has zero differential, $\dim_K \operatorname{Tor}_i^R(M,K)_j = \beta_{ij}$. The *Poincaré series* of M is given by $P_M^R(u) = P_M^R(1,u)$. The numbers β_{ij} are called the *graded Betti numbers* of M, and the coefficients $\beta_i(M) = \sum_j \beta_{ij}$ of $P_M^R(u)$ are its ordinary *Betti numbers*.

The Hilbert series of R and M and the Poincaré biseries of M are related by the formula

$$H_M(t) = H_R(t) P_M^R(t,-1). \tag{1}$$

In fact, since the resolution is minimal, we have $\min_j \beta_{ij} > \min_j \beta_{i-1,j}$. Consequently, for a given j there exist only finitely many i with $\beta_{ij} \neq 0$. So the series $P_M^R(t,-1)$ is well defined, and (1) follows easily if one splits $F_\bullet$ into its graded components and uses that the Euler characteristic of an exact complex of vector spaces vanishes.

The most interesting case is that in which M has a *linear resolution*, i.e. $\beta_{ij} = 0$ for all $i \neq j$. Such a resolution is called linear since the matrices representing the maps between the free modules $\bigoplus R(-j)^{\beta_{ij}}$ have entries which are zero or homogeneous elements of degree 1 in R. If $F_\bullet$ is linear, then we may essentially identify the Poincaré biseries with the Poincaré series, and the formula (1) becomes

$$H_M(t) = H_R(t) P_M^R(-t). \tag{2}$$

Let $\mathfrak{m}_R$ denote the two-sided ideal $\bigoplus_{i>0} R_i$; it is called the *irrelevant* maximal ideal of R. Since K is naturally isomorphic to $R/\mathfrak{m}_R$, we may consider K as an R-module. For all the algebras R below, K has a minimal graded free resolution by finite free R-modules. If it is linear, then one says R is a *Koszul algebra.* For such an algebra, (2) shows that $H_R(t)$ and $P_K^R(-t)$ are truly reciprocal:

$$H_R(t) P_K^R(-t) = 1.$$

The major theme of this article is combinatorial applications of (2). We start with a discussion of algebras with straightening law, which in the discrete case are just the Stanley-Reisner rings of order complexes of posets Π. For them the combinatorial application is a reciprocity formula relating the number of multi-chains in Π and that of the sequences $\pi_1 \not\leq \cdots \not\leq \pi_n$ which we will call *neg-chains.*

If we represent the partial order on Π by a directed graph, then the neg-chains are just the walks in the complementary directed graph, and a generalization from posets to directed graphs suggests itself. However, to obtain it we must definitely use polynomial rings in non-commuting variables.

A further generalization concerns the number of words over a finite alphabet that do not contain any subword belonging to a list of forbidden words. As soon as one of the forbidden subwords contains more than 2 letters, the resolutions to be considered are no longer linear, and therefore the results are not as crisp as in the case of graphs.

In the last section of the paper we use standard methods to show that all our Hilbert series and Poincaré biseries are rational functions.

It is natural in the context of this paper to look for Koszul algebras. Interrupting the combinatorial development after the discussion of algebras with straightening law, we show that a commutative homogeneous K-algebra R is Koszul if its defining ideal has a Gröbner basis of 2-forms.

Our combinatorial terminology follows Stanley[19]. A sequence $x_1, \dots, x_n$ will represent a ring element of degree n; therefore n will be called the *degree* of such a sequence.

Our combinatorial results are not entirely new. For example, Theorem 3.1 was proved by Gessel[12] in his thesis, and its most important case is contained in Carlitz, Scoville, and Vaughan[6]. Furthermore, closely related results were obtained by Jackson and Alelounias[16] and Goulden and Jackson[13].

After the work on this paper had been completed, we learnt that the algebraic approach to a proof of Theorem 3.1 was already used by Kobayashi[18].

1. Homogeneous ASLs

Recall that an *algebra with straightening law*, briefly an ASL, over K on a poset Π (which we always assume to be finite) is a *commutative* ring A containing K and Π and satisfying the following conditions:
(ASL-1) the products $\pi_1 \cdots \pi_n$, $\pi_i \in \Pi$, $n \in \mathbb{N}$, with $\pi_1 \leq \cdots \leq \pi_n$ (including $n = 0$, for which $\pi_1 \cdots \pi_n = 1$) form a K-basis of A; they are called *standard monomials*;
(ASL-2) if π and ρ are incomparable elements of Π, then every standard monomial μ appearing in the *straightening relation*

$$\pi\rho = \sum \mu b_\mu, \qquad \mu \text{ standard monomial}, \quad b_\mu \neq 0,$$

contains a factor σ such that $\sigma \leq \pi$ and $\sigma \leq \rho$.

We say that A is *homogeneous* if $A = \bigoplus_{i \in \mathbb{N}} A_i$ is a graded K-algebra with $A_0 = K$ and if the elements of Π are homogeneous of degree 1. It follows that all the standard monomials in a straightening relation are homogeneous of degree 2. (We refer the reader to Eisenbud[10], Bruns and Vetter[5], or Bruns and Herzog[4] for the theory of ASLs.)

For each poset Π there exists at least one homogeneous ASL over an arbitrary ring K, namely the discrete ASL $K[\Pi]$. It is the residue class ring of the polynomial ring $K[T_\pi \colon \pi \in \Pi]$ with respect to the ideal generated by the products $T_\pi T_\rho$ for which π and ρ are incomparable (the indeterminates T_π have degree 1). In other words, $K[\Pi]$ is the Stanley-Reisner ring of the poset Π, or more precisely, of the simplicial complex formed by the chains of Π. An important class of (in general) non-discrete ASLs is given by the coordinate rings of Grassmannians and their Schubert subvarieties (see Bruns and Vetter[5]).

Note that the standard monomials of degree n in a homogeneous ASL over Π correspond bijectively to the degree n multi-chains in Π. Thus if $\chi_n(\Pi)$ is the number of these multi-chains, the Hilbert series of A is given by

$$H_A(t) = \sum_{n=0}^{\infty} \chi_n(\Pi) t^n.$$

Let $\Omega \subset \Pi$ be an ideal. (This means: if $\pi \in \Omega$ and $\rho \leq \pi$, then $\rho \in \Omega$.) It is easy to see that the residue class ring $A/\Omega A$ is again a homogeneous ASL on $\Pi \setminus \Omega$ (considered as a subset of $A/\Omega A$ in a natural way and with the partial order inherited from Π). We want to prove and to interpret combinatorially that $A/\Omega A$ has a linear resolution over A.

This will follow quite easily from the theory of MSLs developed in Bruns[3]. Let A be an ASL. Then an A-module is called a *module with straightening law*, briefly an MSL, on a finite poset $\mathcal{X} \subset M$ if the following conditions are satisfied:
(MSL-1) for every $x \in \mathcal{X}$ there exists an ideal $\mathcal{I}(x) \subset \Pi$ such that the elements

$$x\xi_1 \cdots \xi_n, \qquad x \in \mathcal{X}, \quad \xi_1 \notin \mathcal{I}(x), \quad \xi_1 \leq \cdots \leq \xi_n, \quad n \geq 0,$$

constitute a K-basis of M; these elements are called *standard elements*;
(MSL-2) for every $x \in \mathcal{X}$ and $\xi \in \mathcal{I}(x)$ one has a *straightening relation*

$$x\xi \in \sum_{y<x} yA.$$

It follows easily that the straightening relations in (MSL-2) can always be chosen of the form

$$x\xi = \sum_{y<x} y(\sum \mu b_{x\xi y\mu}), \qquad b_{x\xi y\mu} \in K,\ b_{x\xi y\mu} \neq 0,$$

in which each $y\mu$ is a standard element. An MSL over a homogeneous ASL A is *homogeneous* if it is a graded A-module in which $\mathcal{X}$ consists of elements of degree 0. In this case, if the straightening relations are chosen as just discussed, then the elements μ appearing on its right hand side have degree 1 and therefore are elements of Π. In particular the straightening relations are homogeneous of degree 1. Furthermore, (MSL-1) immediately yields the Hilbert series

$$H_M(t) = \sum_{x\in\mathcal{X}} H_{A/\mathcal{I}(x)A}(t) = \sum_{n=0}^{\infty} \Big(\sum_{x\in\mathcal{X}} \chi_n(\Pi \setminus \mathcal{I}(x)) \Big) t^n.$$

We will see below that a homogeneous MSL has a linear resolution. For a combinatorial description of its Poincaré series we introduce a special class of sequences in a poset.

Definition. Let Π be a poset. Then a sequence $\pi_1, \ldots, \pi_n$ is called a *neg-chain* if $\pi_1 \not\leq \cdots \not\leq \pi_n$. For a subset Ω of Π we denote the number of degree n neg-chains $\pi_1, \ldots, \pi_n$ with $\pi_1 \in \Omega$ by $\nu_n(\Pi, \Omega)$.

For compatibility with the notation introduced for MSLs, we set $\mathcal{I}(\pi) = \{\rho \in \Pi\colon \pi \not\leq \rho\}$; it is easy to see that $\mathcal{I}(\pi)A$ is the annihilator of π modulo the ideal generated by the elements $\sigma \in \Pi$, $\sigma < \pi$.

Theorem 1.1. *Let A be a homogeneous ASL on a poset Π, and M a homogeneous MSL over A. Then M has a linear resolution, and its j-th Betti number is*

$$\beta_j(M) = \sum_{x \in \mathcal{X}} \nu_j(\Pi, \mathcal{I}(x)).$$

The proof is based on the following proposition which we quote from Bruns[3], (4.5), (4.6):

Proposition. (a) *Let A be an ASL on Π over K, and M an MSL on $\mathcal{X}$ over A. Let e_x, $x \in \mathcal{X}$, denote the elements of the canonical basis of the free module $A^{\mathcal{X}}$. Then the kernel $N_{\mathcal{X}}$ of the natural epimorphism*

$$A^{\mathcal{X}} \longrightarrow M, \qquad e_x \longmapsto x,$$

is generated by the relations required for (MSL-2),

$$\rho_{x\xi} = e_x\xi - \sum_{y<x} e_y a_{x\xi y}, \qquad x \in \mathcal{X},\ \xi \in \mathcal{I}(x);$$

(b) *$N_{\mathcal{X}}$ is an MSL if we let $\mathcal{I}(\rho_{x\xi}) = \{\pi \in \Pi\colon \pi \in \mathcal{I}(\xi)\}$ and*

$$\rho_{x\xi} \leq \rho_{yv} \iff x < y \quad \text{or} \quad x = y,\ \xi \leq v.$$

Proof of 1.1. We must show that M has a free resolution

$$F_{\bullet}\colon \cdots \longrightarrow F_j \xrightarrow{\varphi_j} F_{j-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

in which $\operatorname{rank} F_j = \nu_j(\Pi, \mathcal{I}(x))$ for all j and, in case $j > 0$, φ_j maps the basis elements of F_j to elements which are homogeneous of degree 1 if we assign the degree 0 to the basis elements of F_{j-1}.

Note that, by induction, the proposition yields a free resolution of M in which all syzygy modules are MSLs. We claim that this resolution satisfies our needs, provided in each case the straightening relations are chosen to be homogeneous. To this end we let Γ_j denote the set of sequences

$$x, \pi_1, \ldots, \pi_j \qquad \text{with} \quad x \in \mathcal{X},\ \pi_1 \in \mathcal{I}(x),\ \pi_i \in \mathcal{I}(\pi_{i-1}),\ i = 2, \ldots, j,$$

partially ordered lexicographically.

We use induction on j. For $j = 0$ the assertion amounts to the fact that the elements of $\mathcal{X}$ form a minimal system of generators of M. This holds since the elements of $\mathcal{X}$ are homogeneous of degree 0 and linearly independent over K by (MSL-1).

Let $j > 0$. By the induction hypothesis we may assume that $N = \operatorname{Ker}\varphi_{j-1}$ is an MSL on the poset $\{\rho_\gamma : \gamma \in \Gamma_j\}$ (ordered in the same way as Γ_j) and such that ρ_γ is homogeneous of degree 1 for each $\gamma = (x, \pi_1, \ldots, \pi_j)$ and, furthermore, $\mathcal{I}(\rho_\gamma) = \mathcal{I}(x)$ if $j = 1$ and $\mathcal{I}(\rho_\gamma) = \mathcal{I}(\pi_j)$ otherwise.

After a shift of the graduation of F_{j-1} the submodule $\operatorname{Ker}\varphi_{j-1}$ is therefore a homogeneous MSL. We choose homogeneous straightening relations and use the proposition to find a suitable epimorphism $F_j \to N$. Its kernel has exactly the properties we need. $\square$

Corollary 1.2. *Let Ω be an ideal in Π.*

(a) *Then the residue class ring $A/\Omega A$ has a linear minimal graded free resolution over A, and its j-th Betti number is the number $\nu_j(\Pi, \Omega)$ of neg-chains $\pi_1, \ldots, \pi_j$ in Π with $\pi_1 \in \Omega$.*

(b) *In particular $K = A/\Pi A$ has a linear resolution over A, in other words, a homogeneous ASL is a Koszul algebra.*

(c) *The following conditions are equivalent:*

(i) *$A/\Omega A$ has finite projective dimension;*

(ii) *every element of Ω is comparable to every element of Π;*

(iii) *the elements of Ω form an A-regular sequence.*

Proof. Part (a) follows immediately from the theorem if one observes that $A/\Omega A$ is an MSL on $\{1\}$ with $\mathcal{I}(1) = \Omega$.

The equivalence of (i) and (ii) in (c) is immediate from (a). That (iii) $\Longrightarrow$ (i) is a general fact. For (ii) $\Longrightarrow$ (iii) one can use for instance that the linear resolution of $A/A\Omega$ constructed above is the Koszul complex if (ii) is satisfied. As the elements of Ω are homogeneous, the acyclicity of the Koszul complex implies that they form a regular sequence. $\square$

Let us call a positively graded K-algebra A *strongly Koszul* if its irrelevant maximal ideal has a system of generators $x_1, \ldots, x_m$ such that $A/(x_1, \ldots, x_j)$ has a linear resolution for all j. Then part (a) of 1.2 implies that a homogeneous ASL is strongly Koszul; in fact, the poset Π may be enumerated in such a way that every initial subsequence is a poset ideal.

Part (b) of 1.2 is also covered by a theorem of Kempf[17] which states (b) more generally for homogeneous Hodge algebras (cf. De Concini, Eisenbud, and Procesi[9] or Bruns and Herzog[4] for the notion of a Hodge algebra).

The combinatorial interpretation is an identity involving the generating functions

$$H_\Pi(t) = \sum_{n=0}^{\infty} \chi_n(\Pi) t^n, \; H_{\bar\Omega}(t) = \sum_{n=0}^{\infty} \chi_n(\Pi \setminus \Omega) t^n, \text{ and } \bar H_\Omega(t) = \sum_{n=0}^{\infty} \nu_n(\Pi, \Omega) t^n.$$

Corollary 1.3. *Let Π be a poset and Ω be an ideal in Π. Then*

$$H_{\bar{\Omega}}(t) = \bar{H}_{\Omega}(-t)H_{\Pi}(t).$$

In particular, for $\Omega = \Pi$ one has $\bar{H}_{\Pi}(-t)H_{\Pi}(t) = 1$.

Proof. Choose $A = K[\Pi]$ and apply (2) and 1.2. $\square$

Remark 1.4. In the discrete case, which is completely sufficient for the combinatorial interpretation, $A/\Omega A$ is an ASL for arbitrary subsets Ω of A, and $H_{\Pi}(t)$ and $H_{\bar{\Omega}}(t)$ are the Hilbert series of A and $A/\Omega A$. Even more, it is not difficult to see that $A/\Omega A$ has a linear resolution and if we replace $\bar{H}_{\Omega}(t)$ by its Poincaré series, then 1.3 remains valid. However, the combinatorial interpretation of this Poincaré series is easier if one chooses a non-commutative approach. See Remark 3.5 and Theorem 4.1 below.

We conclude this section with a combinatorial application of 1.2.

Example 1.5. Let m, n be positive integers. We consider the set $\Pi = \{(i,j) : 1 \leq i \leq m,\ 1 \leq j \leq n\}$, partially ordered by

$$(i,j) \leq (u,v) \quad \Longleftrightarrow \quad i \leq u,\ j \leq v,$$

and ask for the number of multi-chains and neg-chains in Π of a given degree k.

It is an easy exercise in elementary combinatorics to obtain the number of degree k multi-chains in Π: since Π is just the direct product of the linearly ordered sets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, the number of degree k multi-chains in Π is

$$\binom{m+k-1}{k} \cdot \binom{n+k-1}{k} = \binom{m+k-1}{m-1} \cdot \binom{n+k-1}{n-1}.$$

It is harder to count the degree k neg-chains in Π. Let $\Omega_{(i,j)}$ be the ideal in Π cogenerated by (i,j) (i.e. $\Omega_{(i,j)} = \{(u,v) \colon u < i \text{ or } v < j\}$), and set $\Omega_{(m+1,n+1)} = \Pi$. Then

$$\nu_1(\Pi, \Omega_{(i,j)}) = m(j-1) + (i-1)n - (i-1)(j-1),$$
$$\nu_k(\Pi, \Omega_{(i,j)}) = \sum_{u<i \text{ or } v<j} \nu_{k-1}(\Pi, \Omega_{(u,v)}), \quad k \geq 2.$$

To determine $\nu_k(\Pi, \Pi)$ from this formula is not easy, not even for special m, n. So we try the approach suggested by Corollary 1.3. The poset Π may be identified in an obvious way with the underlying poset of the determinantal ring $R = R_2 = K[X]/I_2(X)$ where $X = (X_{ij})$ is an $m \times n$ matrix of indeterminates over K and $I_2(X)$ denotes the ideal in the polynomial ring $K[X] = K[X_{11}, \ldots, X_{mn}]$, generated by all 2-minors of X. If we let (i,j) correspond to the residue class of X_{ij} in R, then R is a homogeneous ASL on Π.

Let $\delta \in \Pi$ or $\delta = (m+1, n+1)$, and $R(X;\delta) = R/\Omega_\delta R$. Using Corollary 1.2(a) we obtain

$$P^R_{R(X;\delta)}(t) = \sum_{k=0}^{\infty} \nu_k(\Pi, \Omega_\delta)\, t^k.$$

Since

$$P^R_{R(X;\delta)}(t) = H_R(-t)^{-1} \cdot H_{R(X;\delta)}(-t),$$

we will get a recursion formula for $\nu_k(\Pi, \Omega_\delta)$ once we know the Hilbert series of R and $R(X;\delta)$. But $\dim_K(R_k)$ is the number of standard monomials of degree k, that is the number of degree k multi-chains in Π. So

$$H_R(t) = \sum_{k=0}^{\infty} \binom{m+k-1}{m-1}\binom{n+k-1}{n-1} t^k.$$

Since the Krull dimension of R is $m+n-1$, the Hilbert series $H_R(t)$ can be written in the form $H_R(t) = Q_R(t)/(1-t)^{m+n-1}$. To find $Q_R(t)$ we rewrite $H_R(t)$ as

$$\begin{aligned} H_R(t) &= \frac{1}{(m-1)!} \sum_{k=0}^{\infty} (k+m-1)\cdots(k+1)\binom{n+k-1}{n-1} t^k \\ &= \frac{1}{(m-1)!}\, \frac{d}{dt}\left(\frac{t^{m-1}}{(1-t)^m}\right). \end{aligned}$$

An inductive evaluation of this expression yields

$$Q_R(t) = \sum_{k=0}^{\min(m,n)} \binom{m-1}{k}\binom{n-1}{k} t^k.$$

(The numerator polynomials $Q_R(t)$ have been determined by Conca and Herzog[8] for all the determinantal rings $R_{r+1}(X)$.) For $\delta = (u,v) \in \Pi$ the residue class ring $R(X;\delta)$ of R is of the form $R_2(X')$ with an $(m-u+1) \times (n-v+1)$ matrix X' of indeterminates; thus we obtain

$$H_{R(X;\delta)}(t) = (1-t)^{-m+u-n+v+1} \sum_{k=0}^{\min(m-u,n-v)+1} \binom{m-u}{k}\binom{n-v}{k} t^k.$$

Altogether our considerations yield

$$P_{R(X;\delta)}(t) = Q_R(-t)^{-1} \cdot (1+t)^{u+v-2} \sum_{k=0}^{\min(m-u,n-v)+1} (-1)^k \binom{m-u}{k}\binom{n-v}{k} t^k.$$

The computation of $\nu_k = \nu_k(\Pi, \Omega_\delta)$ depends on the 'complexity' of $Q_R(t)$. In case $m = 2$, we have $Q_R(t) = 1 + (n-1)t$, so

$$\nu_k = \sum_{i+j=k} (n-1)^i \sum_{\rho+\sigma=j} (-1)^\rho \binom{2-u}{\rho} \binom{n-v}{\rho} \binom{u+v-2}{\sigma},$$

for example

$$\nu_k(\Pi, \Pi) = \sum_{i+j=k} (n-1)^i \binom{n+1}{j}$$

and

$$\nu_k(\Pi, \Omega_{(1,2)}) = (n-1)^{k-2} n, \quad k \geq 2.$$

For $m = 3$ one has

$$Q_R(t) = 1 + 2(n-1)t + \binom{n-2}{2} t^2,$$

and a routine computation shows that

$$Q_R(-t)^{-1} = \frac{2}{(n-1)(n-2)(a_1 - a_2)} \sum_{k=0}^{\infty} (a_2^{-(k+1)} - a_1^{-(k+1)}) t^k,$$

where

$$a_{1,2} = \frac{1}{n-2} \left(2 \pm \sqrt{\frac{2n}{n-1}} \right)$$

are the zeros of $Q_R(t)$. So in this case

$$\nu_k(\Pi, \Pi) = \frac{2}{(n-1)(n-2)(a_1 - a_2)} \sum_{i+j=k} (a_2^{-(i+1)} - a_1^{-(i+1)}) \binom{n+2}{j}$$

for all $k \geq 0$.

2. Linear resolutions and Gröbner bases

Not all commutative homogeneous algebras defined by equations of degree 2 are Koszul algebras as it is the case for homogeneous ASLs. We will see however that a homogeneous algebra also has this nice property if its defining ideal is generated by a Gröbner basis of elements of degree 2. In particular the following holds.

Proposition 2.1. *Let $A = K[X_1, \ldots, X_m]/\mathfrak{a}$ be a homogeneous K-algebra where $\mathfrak{a}$ is generated by monomials of degree 2, and let $I \subset A$ be an ideal which is generated* *by the residue classes of a subset of the indeterminates. Then A/I has a linear A-resolution.*

Proof. The case in which I is the irrelevant maximal ideal was shown by Fröberg[11]. In general, A/I is an algebra retract of A, and so

$$P_K^A(t) = P_{A/I}^A(t) P_K^{A/I}(t),$$

according to Herzog[14]. It is easy to see that the initial degree of the formal power series

$$P_M^A(t) H_A(-t) - H_M(-t)$$

gives the first position in the resolution of M where it is not linear. In particular M has a linear resolution if and only if $P_M^A(t)H_A(-t) = H_M(-t)$. By Froeberg's theorem we have $P_K^A(t) = H_A(-t)^{-1}$ and $P_K^{A/I}(t) = H_{A/I}(-t)^{-1}$. This together with the above equation for the Poincaré series yields $P_{A/I}^A(t)H_A(-t) = H_{A/I}(-t)$, as desired. □

There is an alternative, direct way to prove 2.1. Let $A = K[x_1, \ldots, x_m]$ be a homogeneous K-algebra with monomial relations in the x_i, and let M be a finite graded A-module. We call M *monomial* if there is a finite chain of graded submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that each M_i/M_{i-1} is a cyclic A-module whose annihilator is an ideal in A generated by monomials in the x_i.

Choose an epimorphism $\varepsilon\colon \bigoplus_{i=1}^r Ae_i \longrightarrow M$ mapping e_i to a homogeneous element in M which represents a generator of M_i/M_{i-1}. Then it is easy to see that $\operatorname{Ker}\varepsilon$ is again monomial.

We call M *linear* if it has a filtration as above such that $M_i/M_{i-1} \cong A/J_i$ where J_i is generated by a subset of $x_1, \ldots, x_m$. Suppose now that A has monomial relations of degree 2, and that M is linear. Then one checks easily that $\operatorname{Ker}\varepsilon$ is again a linear A-module. From these observations one deduces that a linear module over an algebra with monomial relations of degree 2 has a linear resolution.

We are aiming at a result similar to 2.1 for an ideal $\mathfrak{a}$ whose Gröbner basis consists of forms of degree 2. Let $<$ be a term order on the set of monomials $u = X_1^{a_1} \ldots X_m^{a_m}$. This means that the set of monomials is linearly ordered by $<$ and that $u < v$ implies $uw < vw$; in addition we require that $u < v$ if $\deg u < \deg v$. The largest monomial occurring in a polynomial $f \in R$, $f \neq 0$, is called its *initial term*, and is denoted by $\operatorname{in}(f)$. Let $\mathfrak{a} \subset R$ be an ideal. The ideal $\operatorname{in}(\mathfrak{a})$ is the ideal generated by all monomials $\operatorname{in}(f)$, $f \in \mathfrak{a}$. A set $\{f_1, \ldots, f_s\}$, $f_i \in \mathfrak{a}$, is called a *Gröbner basis* of $\mathfrak{a}$ if $\operatorname{in}(\mathfrak{a}) = (\operatorname{in}(f_1) \ldots, \operatorname{in}(f_s))$. A Gröbner basis is always a system of generators, though not necessarily a minimal one.

We keep the assumptions and notation, and assume in addition that $\mathfrak{a}$ is a homogenous ideal. Further we denote the residue class of X_i modulo $\mathfrak{a}$, $i = 1, \ldots, m$, by x_i.

Theorem 2.2. *Let $\mathfrak{a}$ be an ideal of $R = K[X_1, \dots, X_m]$ which has a Gröbner basis of homogeneous elements of degree 2 (with respect to a given term order). Then*
(a) *$A = R/\mathfrak{a}$ is a Koszul algebra,*
(b) *more generally, if $\operatorname{in}(\mathfrak{a} + (X_{i_1}, \dots, X_{i_l})) = \operatorname{in}(\mathfrak{a}) + (X_{i_1}, \dots, X_{i_l})$, then A/I has a linear A-resolution for the ideal $I = (x_{i_1}, \dots, x_{i_l})$.*

The proof of the theorem could be based on the same ideas as that of the main result in Kempf[17] since one may view A as a deformation of $R/\operatorname{in}(\mathfrak{a})$. We prefer to give a proof of 2.2 not refering to deformations.

It obviously suffices to prove part (b). The following arguments are standard and have appeared in similar form in the literature. Thus we will not prove every detail. We extend the term order in an obvious way to the group G of all monomials in $K[X_1, \dots X_m, X_1^{-1}, \dots, X_m^{-1}]$. Then G is an ordered group and we may define an ascending G-filtration $(\mathbf{F}_u R)_{u \in G}$ on R by setting

$$\mathbf{F}_u R = \sum_{v \in R,\ v \leq u} Kv$$

For all $u \in G$ there is a unique largest element $u' \in G$ with $u' < u$. We consider the associated graded ring $\operatorname{gr}_{\mathbf{F}}(R) = \bigoplus_{u \in G} \mathbf{F}_u R / \mathbf{F}_{u'} R$. It is a G-graded K-algebra which may as well be viewed as a homogeneous K-algebra if one sets

$$\operatorname{gr}_{\mathbf{F}}(R)_a = \bigoplus_{\deg u = a} \operatorname{gr}_{\mathbf{F}}(R)_u$$

for all $a \in \mathbb{Z}$. These notions can be transferred to G-filtered R-modules in a natural way.

As a homogeneous K-algebra $\operatorname{gr}_{\mathbf{F}}(R)$ is isomorphic to R. We denote the induced G-filtration on A also by $\mathbf{F}$. If $\varepsilon \colon R \longrightarrow A$ is the canonical epimorphism, then, by definition, $\mathbf{F}_u A = \varepsilon(\mathbf{F}_u R)$, and ε is a homomorphism of filtered rings inducing a G-graded surjective homomorphism

$$\operatorname{gr}_{\mathbf{F}}(\varepsilon) \colon \operatorname{gr}_{\mathbf{F}}(R) \longrightarrow \operatorname{gr}_{\mathbf{F}}(A).$$

Identifying $\operatorname{gr}_{\mathbf{F}}(R)$ with R we see that $\operatorname{Ker} \operatorname{gr}_{\mathbf{F}}(\varepsilon) = \operatorname{in}(\mathfrak{a})$, so that $\operatorname{gr}_{\mathbf{F}}(A) \cong R/\operatorname{in}(\mathfrak{a})$ as a homogeneous K-algebra.

Set $R' = R/(X_{i_1}, \dots, X_{i_l})$; then we obtain a commutative diagram of K-algebras

$$\begin{array}{ccc} R & \xrightarrow{\ \sigma\ } & R' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\ \tau\ } & A/I \end{array}$$

We equip all algebras in the diagram with the G-filtration induced by the G-filtration $\mathbf{F}$ on R. Note that the induced G-filtration on R' is again defined by the induced order of the monomials in R'. Identifying $\mathrm{gr}_{\mathbf{F}}\, R$ with R and $\mathrm{gr}_{\mathbf{F}}(R')$ with R' we obtain a commutative diagram of G-graded (or $\mathbb{Z}$-graded) R-modules with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{in}(\mathfrak{a}) & \longrightarrow & \mathrm{in}(\mathfrak{a}') & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (X_{i_1}, \dots, X_{i_l}) & \longrightarrow & R & \longrightarrow & R' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\,\mathrm{gr}_{\mathbf{F}}(\tau) & \longrightarrow & \mathrm{gr}_{\mathbf{F}}(A) & \longrightarrow & \mathrm{gr}_{\mathbf{F}}(A/I) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $\mathfrak{a}'$ is the kernel of $R' \longrightarrow A/I$. The snake lemma yields that $(X_{i_1}, \dots, X_{i_l}) \to \mathrm{Ker}\,\mathrm{gr}_{\mathbf{F}}(\tau)$ is surjective provided $\mathrm{in}(\mathfrak{a}) \longrightarrow \mathrm{in}(\mathfrak{a}')$ is surjective. But this follows immediately from the assumption $\mathrm{in}(\mathfrak{a} + (X_{i_1}, \dots, X_{i_l})) = \mathrm{in}(\mathfrak{a}) + (X_{i_1}, \dots, X_{i_l})$. We conclude that

$$\mathrm{gr}_{\mathbf{F}}(A/I) \cong \mathrm{gr}_{\mathbf{F}}(A)/J$$

where J is generated by the images of $X_{i_1}, \dots, X_{i_l}$ in $\mathrm{gr}_{\mathbf{F}}(A)$.

Since $\mathrm{gr}_{\mathbf{F}}(A) \cong R/\mathrm{in}(\mathfrak{a})$ and since, by hypothesis, $\mathrm{in}(\mathfrak{a})$ is generated by monomials of degree 2, Proposition 2.1 implies that $\mathrm{gr}_{\mathbf{F}}(A/I)$ has a linear $\mathrm{gr}_{\mathbf{F}}(A)$-resolution. Thus the theorem follows from the next proposition (or its Corollary 2.4).

Proposition 2.3. *Let $R = K[X_1, \dots, X_m]$, $\mathfrak{a}$ a graded ideal of R, and $A = R/\mathfrak{a}$. Suppose I is a graded ideal of A. Then there exists a (possibly non-minimal) graded free A-resolution $(H_\bullet, d_\bullet)$ of A/I which is a complex of G-filtered modules with the property that the associated G-graded complex*

$$\mathrm{gr}_{\mathbf{F}}(H_\bullet) : \cdots \longrightarrow \mathrm{gr}_{\mathbf{F}}(H_i) \xrightarrow{\mathrm{gr}_{\mathbf{F}}(d_i)} \mathrm{gr}_{\mathbf{F}}(H_{i-1}) \longrightarrow \dots$$

is a minimal free G-graded $\mathrm{gr}_{\mathbf{F}}(A)$-resolution of $\mathrm{gr}_{\mathbf{F}}(A/I)$. Furthermore the filtration $\mathbf{F}$ on $H_\bullet$ is such that

$$\dim_K (H_i)_a = \sum_{\deg u = a} \dim_K \mathrm{gr}_{\mathbf{F}}(H_i)_u \tag{3}$$

for all $i, a \in \mathbb{Z}$.

Proof. We will construct the desired complex $(H_\bullet, d_\bullet)$ with the additional property that H_i has a *standard G-filtration* for all i. This means, if $H_i = \bigoplus_j A(-j)$, then

$\mathbf{F}_u H_i = \bigoplus_j \mathbf{F}_{u-u_j} A$ for some $u_j \in G$ with $\deg u_j = j$. For this filtration, $\operatorname{gr}_{\mathbf{F}}(H_i)$ is a free G-graded $\operatorname{gr}_{\mathbf{F}}(A)$-module and satisfies condition (3).

We construct H_i and d_i by induction on i. We let $H_0 = A$ (with the given G-filtration), and let d_0 be the canonical epimorphism $A \to A/I$. Suppose H_i and d_i have been constructed. If $z \in H_i$, then there exists a unique element $u = \nu(z) \in G$ such that $z \in \mathbf{F}_u H_i$ and $z \notin \mathbf{F}_v H_i$ for all $v < u$. We set $\operatorname{in}(z) = z + \mathbf{F}_{u'} H_i$.

Let $y_1, \dots, y_s$ be a set of G-homogeneous generators of $\operatorname{Ker} \operatorname{gr}_{\mathbf{F}}(d_i)$. Since $\operatorname{Ker} \operatorname{gr}_{\mathbf{F}}(d_i)$ is generated by the elements $\operatorname{in}(z)$, $z \in \operatorname{Ker} d_i$, z homogeneous, we can find homogeneous elements $z_1, \dots, z_s \in \operatorname{Ker} d_i$ with $\operatorname{in}(z_j) = y_j$ for $j = 1, \dots, s$. Now let $a_j = \deg \nu(z_j)$ for $j = 1, \dots, s$, and set $H_{i+1} = \bigoplus_{j=1}^{s} A(-a_j)$ with standard filtration $\mathbf{F}_u H_{i+1} = \bigoplus_{j=1}^{s} \mathbf{F}_{u-\nu(z_j)} A$.

Finally we let $d_{i+1} \colon H_{i+1} \longrightarrow H_i$ be the homomorphism mapping the standard basis of H_{i+1} to $z_1, \dots, z_s$. $\square$

Proposition 2.3 yields the following inequality for the Poincaré biseries of A/I and $\operatorname{gr}_{\mathbf{F}}(A/I)$.

Corollary 2.4. *$P^{A}_{A/I}(t,u) \le P^{\operatorname{gr}_{\mathbf{F}}(A)}_{\operatorname{gr}_{\mathbf{F}}(A/I)}(t,u)$, where the inequality is to be understood coefficientwise. In particular, if $\operatorname{gr}_{\mathbf{F}}(A/I)$ has a linear $\operatorname{gr}_{\mathbf{F}}(A)$-resolution, then A/I has a linear A-resolution.*

As an example we consider the degrevlex term order: given two monomials $u = X_1^{a_1} \dots X_m^{a_m}$ and $v = X_1^{b_1} \dots X_m^{b_m}$ in $R = K[X_1, \dots, X_m]$, then $u < v$ if and only if the first non-vanishing component of the vector

$$(\deg u - \deg v, b_n - a_n, \dots, b_1 - a_1)$$

is negative. If u and v are monomials of the same degree such that v is a product of powers of $X_1, \dots, X_k$ and X_{k+1} is a factor of u, then $u < v$ with respect to the degrevlex term order.

Corollary 2.5. *Let $\mathfrak{a}$ be an ideal of R which has a Gröbner basis of degree 2 homogeneous elements with respect to the degrevlex term order. Then $A/(x_l, \dots, x_m)$ has a linear A-resolution for all l, $1 \le l \le m$.*

Proof. With the notation of the proof of 2.2 it is enough to show that the map $(X_l, \dots, X_m) \to \operatorname{Ker} \operatorname{gr}_{\mathbf{F}}(\tau)$ is surjective. In fact, let $u \in \operatorname{in}(\mathfrak{a}')$ and pick a homogeneous element $g \in \mathfrak{a}'$ with $\operatorname{in}(g) = u$. There exists a homogeneous element $f \in \mathfrak{a}$ such that $f = g + h$ with $h \in (X_l, \dots, X_m)$. (We consider R' as a subring of R in a natural way.) Note that g contains no factor X_i, $i = l, \dots, m$. Thus in the degrevlex order all monomials of g are larger then those of h. This implies that $\operatorname{in}(f) = u$. Hence $\operatorname{in}(f)$ is mapped identically onto $u \in \operatorname{in}(\mathfrak{a}')$. $\square$

Theorem 2.2 cannot be applied to prove 1.2 since the product of the two incomparable elements in the straighening relation do not necessarily form the initial term. Examples satisfying the condition of 2.2 are given by the ladder determinantal rings defined by 2-minors (see Herzog and Trung[15]) and hence are Koszul; however, they are also ASLs. Non-ASL examples to which one can apply 2.2 are (ladder) determinantal ideals of symmetric matrices generated by 2-minors (see Conca[7]).

Corollary 2.4 shows that $A = R/\mathfrak{a}$ is strongly Cohen-Macaulay if $\mathfrak{a}$ has a Gröbner basis of degree 2 elements with respect to the degrevlex term order. It would be interesting to know whether this holds true for different term orders as well.

3. Walks in directed graphs

Each binary relation on a set V may be represented by a subset of $V \times V$. If we consider V as a set of vertices and draw a directed edge from v to w exactly when the given relation holds for (v, w) (in this order), then we obtain a directed graph G on the vertex set V. Conversely, every such graph G determines a binary relation on V.

In particular, if $V = \Pi$ is a poset, then the partial order $\leq$ on V may be considered as a directed graph G, and the multi-chains of Π are exactly the walks in G. Let $\bar{G} = (V \times V) \setminus G$ be the *complementary graph* of G. If Ω is a poset ideal, then the multi-chains in $\Pi \setminus \Omega$ are the directed walks in G starting outside Ω, and the neg-chains counted by $\nu(\Pi, \Omega)$ correspond to the walks in $\bar{G}$ starting from a vertex in Ω. At this point a generalization of Corollary 1.3 to arbitrary directed graphs suggests itself.

So, let V be a finite set whose elements we consider as vertices, G a directed graph on V, $\bar{G} = (V \times V) \setminus G$ the complementary graph, and W a subset of V. Let $\chi_n(G)$ denote the number of degree n walks in G, $\chi_n(G, \bar{W})$ the number of degree n walks in G starting in $\bar{W} = V \setminus W$, and $\chi_n(\bar{G}, W)$ the number of degree n walks in $\bar{G}$ starting in W. (By convention, the number of degree 0 walks starting in a subset of V is 1.) As above, we define the generating functions

$$H_G(t) = \sum_{n=0}^{\infty} \chi_n(G)t^n, \;\; H_{\bar{W}}(t) = \sum_{n=0}^{\infty} \chi_n(G, \bar{W})t^n, \;\text{ and }\; \bar{H}_W(t) = \sum_{n=0}^{\infty} \chi_n(\bar{G}, W)t^n.$$

Theorem 3.1. *With the notation just introduced, $H_{\bar{W}}(t) = \bar{H}_W(-t)H_G(t)$, equivalently*

$$\chi_n(G, \bar{W}) = \sum_{i=0}^{n} (-1)^i \chi_i(\bar{G}, W)\chi_{n-i}(G) \qquad \textit{for all} \quad n \in \mathbb{N}.$$

In particular one has $H_{\bar{G}}(-t)H_G(t) = 1$.

Proof. Let $K\langle G \rangle$ be the residue class algebra of the free K-algebra $K\langle V \rangle$ on V modulo the two-sided ideal $\mathfrak{a}$ generated by the products vv' for which $(v, v') \notin G$ (for simplicity we identify a vertex and its corresponding variable), and set $A = K\langle G \rangle$. It is clear that $H_G(t)$ is the Hilbert series of A: the monomials (in non-commuting variables) which form a K-basis of A are presented by the walks in G. Now we choose I as the right ideal generated by the elements $w \in W$. The monomials whose leftmost factor belongs to W form a K-basis of I, and so the residue classes of those monomials whose leftmost factor is outside W form a K-basis of the right A-module A/I. Thus $H_{\bar{W}}(t)$ is the Hilbert series of A/I.

We start the free resolution of A/I with the natural choice $F_0 = A$. Next let $F_1 = A^{(W)}$ be a free right A-module with basis e_w, $w \in W$. Then the assignment $e_w \mapsto w$ induces a homomorphism $\varphi_1 \colon F_1 \to F_0$ with $\operatorname{Im} \varphi_1 = I$. Note that $I = \bigoplus_{w \in W} wA$. Thus $\operatorname{Ker} \varphi_1 = \bigoplus_{w \in W} e_w \operatorname{Ann} w$. Obviously $\operatorname{Ann} w$ is the right ideal generated by those $v \in V$ for which $wv \in \mathfrak{a}$, equivalently, for which $(w, v) \in \bar{G}$.

Applying the same argument to each of $\operatorname{Ann} w$ in place of I and iterating the procedure, we obtain a linear free resolution of A/I in which the basis of F_j corresponds bijectively to the walks $v_1, \dots, v_j$ in $\bar{G}$ that start from a vertex $v_1 \in W$.

The asserted equation follows now as above. For the last statement one chooses $W = V$. □

Corollary 3.2. *The following are equivalent:*
(a) *K has finite projective dimension over $K\langle G \rangle$;*
(b) *$\bar{G}$ contains no cycles;*
(c) *$H_G(t)^{-1}$ is a polynomial.*

We would like to present another proof of 3.1 which uses the *transfer matrix* T of the graph G (over the real numbers $\mathbb{R}$). In order to define T we enumerate the vertices $v_1, \dots, v_m \in V$. Then $T_{ij} = 1$ if $(v_i, v_j) \in G$, and $T_{ij} = 0$ otherwise. Let $\bar{T}$ be the transfer matrix of $\bar{G}$; then $E = T + \bar{T}$ is the matrix with all entries equal to 1. For a subset $W \subset V$ we define its *indicator* e_W as the *row* vector whose i-th component is 1 if $v_i \in W$, and 0 otherwise. It follows immediately by induction that for $n \geq 1$ the number of degree n walks starting from a vertex in a subset $X \subset V$ and ending in a vertex belonging to $Y \subset V$ is

$$\langle e_X T^{n-1}, e_Y \rangle$$

where $\langle \_, \_ \rangle$ denotes the standard scalar product in $\mathbb{R}^m$. In particular, the j-th component of $e_X T^{n-1}$ is the number of degree n walks starting in a vertex $v \in X$ and ending in v_j. The generating function $H_G(t)$ above can be written

$$H_G(t) = 1 + \sum_{n=1}^{\infty} \langle e_V T^{n-1}, e_V \rangle t^n.$$

Furthermore, if we set $\lambda(y) = \langle y, e_V \rangle$, $\tau(y) = yT$, $\varepsilon(y) = yE$, and $\bar{\tau}(y) = (\varepsilon - \tau)(y)$, then the equation for $\chi_n(G, \bar{W})$ in 3.1 reads

$$\lambda(\tau^{n-1}(e_{\bar{W}})) = \lambda(\tau^{n-1}(e_V)) + \sum_{i=1}^{n-1} (-1)^i \lambda(\bar{\tau}^{i-1}(e_W)) \lambda(\tau^{n-i-1}(e_V)) + (-1)^n \lambda(\bar{\tau}^{n-1}(e_W)).$$

The following lemma will show that one has an even stronger equation.

Lemma 3.3. *Let M be a left module over some ring R, $\tau\colon M \to M$ an endomorphism, $e \in M$, and $\lambda\colon M \to R$ an arbitrary map. We define $\varepsilon\colon M \to M$ by $\varepsilon(x) = \lambda(x)e$. Then*

$$(\tau-\varepsilon)^n(y) = \tau^n(y) - \sum_{i=1}^{n} \lambda\big((\tau-\varepsilon)^{n-i}(y)\big)\tau^{i-1}(e)$$

for all $x \in M$ and $n \in \mathbb{N}$.

Proof. One goes by induction on n. For the induction step one writes $(\tau-\varepsilon)^{n+1}(y) = (\tau-\varepsilon)((\tau-\varepsilon)^n(y))$, applies the induction hypothesis, and uses the definition of ε. □

We apply the lemma to the maps introduced above. Note that indeed $\varepsilon(x) = \lambda(x)e_V$. Since λ is now linear, we obtain from the lemma with $y = e_W = e_V - e_{\bar{W}}$ that

$$(-1)^n\bar{\tau}^n(e_W) = \tau^n(e_V - e_{\bar{W}}) - \sum_{i=1}^{n}(-1)^{n-i}\lambda\big(\bar{\tau}^{n-i}(e_W)\big)\tau^{i-1}(e_V).$$

Solving for $\tau^n(e_{\bar{W}})$ yields

$$\tau^n(e_{\bar{W}}) = (-1)^{n+1}\bar{\tau}^n(e_W) + \sum_{i=1}^{n}(-1)^{n+1-i}\lambda\big(\bar{\tau}^{n-i}(e_W)\big)\tau^{i-1}(e_V) + \tau^n(e_V).$$

The j-th component of $\tau^n(e_{\bar{W}})$ is the number $\chi^{(j)}_{n+1}(G,\bar{W})$ of degree $n+1$ walks in G which start in $\bar{W}$ and end in the vertex v_j. If we modify the remaining notation accordingly, then we get a vectorial refinement of the second equation in 3.1 (we have replaced n by $n-1$ and i by $n-i$):

Theorem 3.4. *With the notation introduced,*

$$\chi^{(j)}_n(G,\bar{W}) = \sum_{i=0}^{n-1}(-1)^i\chi_i(\bar{G},W)\chi^{(j)}_{n-i}(G) \;+\; (-1)^n\chi^{(j)}_n(\bar{G},W) \qquad \textit{for} \quad n \geq 1.$$

To obtain 3.1, simply sum the equations in 3.4 over j. The question arises whether one can prove 3.4 homologically. This is indeed possible, and the homological approach explains the structure of the formula very well.

Let $A = K\langle G\rangle$. We observed in the proof of 3.1 that the maps in the free resolution $F_{\bullet}$ of A/I are composed of homomorphisms $A \to A$, $1 \mapsto w$, of right A-modules. But such a homomorphism is left multiplication by w, and left multiplication maps a left ideal into itself. This observation is the starting point for a decomposition of $F_{\bullet}$ that yields the formula in 3.4.

Let $A^{(j)} = Av_j$ be the *left* ideal of A generated by v_j. Then one has a decomposition $A = K \oplus \bigoplus_{j=1}^{m} A^{(j)}$ of K-vector spaces. Writing the free A-modules F_i in $F_\bullet$ as a direct sum of copies of A, namely $F_i = A^{\beta_i}$ with $\beta_i = \chi_i(\bar{G}, W)$, one may similarly decompose F_i as

$$F_i = K^{\beta_i} \oplus \bigoplus_{j=1}^{m} (A^{(j)})^{\beta_i}.$$

Furthermore, for $i \geq 1$ we split the direct summand K^{β_i} into the direct sum

$$\bigoplus_{j=1}^{m} K^{\chi_i^{(j)}(\bar{G},W)}$$

where for each j we have collected the subspaces eK with base elements e of F_i corresponding to those direct summands A on which the map to a component of F_{i-1} is left multiplication by v_j. Finally we set $F_i^{(j)} = K^{\chi_i^{(j)}(\bar{G},W)} \oplus (A^{(j)})^{\beta_i}$ for $i \geq 1$, and $F_0^{(j)} = A^{(j)}$.

These decompositions are compatible with the grading of $F_\bullet$ and furthermore they even split $F_\bullet$ into a direct sum of complexes, since $F_i^{(j)}$ is mapped into $F_{i-1}^{(j)}$: the maps $A \to A$ which occur in $F_\bullet$ are left multiplications by an element $w \in V$ or 0. Taking both decompositions simultaneously we obtain an acyclic complex of K-vector spaces

$$0 \to (F_n^j)_n \to (F_{n-1}^j)_n \to \ldots \to (F_1^j)_n \to (F_0^j)_n$$

for each $n \geq 1$. Its Euler characteristic is the right hand side of the formula in 3.4 and the degree n piece of its homology is the vector space generated by all degree n monomials in A/I which end in v_j.

One should note that 3.4 improves the special case 1.3 of 3.1. It may be possible to verify 3.4 for posets by commutative methods. We leave this as a problem for the reader. (Certainly the decompositions above must be replaced by filtrations.)

Remark 3.5. (a) Fröberg[11] showed that the residue class algebras of a free algebra with respect to certain classes of homogeneous relations of degree 2 are Koszul algebras. In the case $W = V$ the resolution in the proof of 3.1 is a (very simple) special case of Fröberg's construction, which gives the base elements in a free resolution as monomials in 'complementary' variables modulo 'complementary' relations.

(b) Let $K[X_1, \ldots, X_m]$ be the polynomial ring in m commuting variables over K, and $\mathfrak{a}$ an ideal generated by monomials of degree 2. With $\mathfrak{a}$ we may associate the graph G on $\{v_1, \ldots, v_m\}$ in which there is a directed edge from v_i to v_j if and only if $i \leq j$ and $X_i X_j \notin \mathfrak{a}$. Let $A = K[X_1, \ldots, X_m]/\mathfrak{a}$. Fröberg's result covers the free A-resolution of K and shows that every base element in it may be written in the form $cY_{i_1} \cdots Y_{i_r}$ with $c \in K$ and $v_{i_1}, \ldots, v_{i_r}$ representing a walk in $\bar{G}$ (Y_i is the variable complementary to X_i).

However, if one tries to compute the Hilbert series of A or the Poincaré series $P_K(t)$ of K over A from G or $\bar{G}$, then one gets the correct result almost only in the case in which A is the Stanley-Reisner ring of a partial order on $\{v_1, \ldots, v_m\}$. In fact, the following assertions are equivalent:
(i) the relation on $\{v_1, \ldots, v_m\}$ given by G is anti-symmetric and transitive;
(ii) $H_A(t) = H_G(t)$;
(iii) $P_K(t) = H_{\bar{G}}(t)$.
The equivalence of (ii) and (iii) follows from the equations $H_A(t)P_K(-t) = 1$ and $H_G(t)H_{\bar{G}}(-t) = 1$. Furthermore, (ii) holds exactly when the following condition is fulfilled: a monomial $X_{i_1} \cdots X_{i_r}$ is non-zero modulo $\mathfrak{a}$ if and only if so is $X_{i_j}X_{i_{j+1}}$ for $j = 1, \ldots, r-1$. This is easily seen to be equivalent to the transitivity of the relation; the anti-symmetry is automatically satisfied.

4. Walks avoiding vertices

The previous section contains formulas for the number of walks in a graph G that start in a given subset W of the set of vertices V. These walks correspond to a monomial basis of the right ideal I of $K\langle G\rangle$ generated by the elements of W. We may also consider the two-sided ideal $\mathfrak{b}$ generated by these elements. Then $K\langle G\rangle/\mathfrak{b} = K\langle G_{\bar{W}}\rangle$ where $G_{\bar{W}}$ is the restriction of G to $\bar{W} = V \setminus W$, i.e. $G_{\bar{W}} = G \cap (\bar{W} \times \bar{W})$. The number of degree n walks in G that avoid all the vertices in W is just the K-dimension of $K\langle G_{\bar{W}}\rangle_n$ so that its generating function is the Hilbert series of $K\langle G_{\bar{W}}\rangle$. In this section we want to give a combinatorial description of the quotient $H_G(t)/H_{G_{\bar{W}}}(t)$. The method is the same as that which proved 3.1.

Set $A = K\langle G\rangle$. As a right A-module, $\mathfrak{b}$ has a minimal system of generators given by the monomials $v_1 \cdots v_k$, $k \in \mathbb{N}$, that represent a walk in G and have $v_k \in W$, but $v_1, \ldots, v_{k-1} \notin W$. In fact, $\mathfrak{b}$ is the direct sum of the right ideals generated by each of these monomials, and the right ideal generated by $v_1 \cdots v_k$ is isomorphic to that generated by v_k. Since we know the resolution of v_kA from the proof of 3.1, we know that of $v_1 \cdots v_k$: it is linear in the sense that the non-zero entries of the matrices in it are homogeneous of degree 1. So we can easily build a graded resolution of $A/\mathfrak{b}$. Though this is not a resolution by finite free A-modules in general, the K-dimension of each graded piece is finite. The reader may verify that one obtains the Poincaré biseries

$$P(t,u) = 1 + \sum_{w \in W} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \chi_k(\bar{G}, w)\omega_j(\bar{W}, w)t^{j+k}u^k$$

where $\omega_j(\bar{W}, w)$ is the number of walks $v_1, \ldots, v_j, v_{j+1} = w$ with $v_1, \ldots, v_j \in \bar{W}$. We set

$$\bar{H}'_{\{w\}}(s) = \bar{H}_{\{w\}}(s) - 1 = \sum_{k=1}^{\infty} \chi_k(\bar{G}, w)s^k \qquad \text{and} \qquad J_{\bar{W},w}(t) = \sum_{j=0}^{\infty} \omega_j(\bar{W}, w)t^j.$$

Since $H_{G_{\bar{W}}}(t) = P(t,-1)H_G(t)$, the formula above yields

Theorem 4.1. *With the notation introduced,*

$$H_{G_{\bar{W}}}(t) = \Big(1 + \sum_{w\in W} \bar{H}'_{\{w\}}(-t)J_{\bar{W},w}(t)\Big)H_G(t).$$

5. Words and worms

The walks in a directed graph G on the vertex set V may be considered as those words over the alphabet V which avoid the 'forbidden' subwords $(v,w) \in \bar{G}$ (a subword is a contiguous subsequence of a word). The association of words avoiding a set of forbidden subwords with walks in a graph works only if the forbidden subwords are of degree 2. However, in the algebraic setting we can cover the general situation by allowing the algebra under consideration to be defined by monomial relations of arbitrary degree.

Let $\bar{\mathcal{F}}$ be an (not necessarily finite) set of words over a finite alphabet V, which we consider as forbidden subwords. We assume that $\bar{\mathcal{F}}$ is minimal, that is, no element of $\bar{\mathcal{F}}$ contains another one as a subword. A *worm* over $\bar{\mathcal{F}}$ is a sequence of words $f_1,\dots,f_j$ over V such that f_1 is a single letter and for each $i = 2,\dots,j$ the concatenation $f_{i-1}f_i$ contains a subword belonging to $\bar{\mathcal{F}}$, but no initial subword of $f_{i-1}f_i$ contains a subword in $\bar{\mathcal{F}}$. We say that $f_1,\dots,f_j$ is of type (j,l) if l is the total number of letters in the concatenation $f_1,\dots,f_j$. (Clearly, in the graph situation the worms are the walks in $\bar{G}$, and each worm is of type (j,j) for some j). The empty worm is of type $(0,0)$.

We aim at an analogue of 3.1. Thus, given a subset W of V, we set $\chi_n(\mathcal{F},\bar{W})$ equal to the number of n letter words which have their first letter in $\bar{W} = V \setminus W$ and do not contain any of the forbidden subwords. Furthermore, we let $\psi_{(j,l)}(\bar{\mathcal{F}},W)$ denote the number of worms of type (j,l) over $\bar{\mathcal{F}}$ which have their first letter in W. (Thus $\psi_{(1,1)}(\bar{\mathcal{F}},W) = |W|$.)

Let $A = K\langle\mathcal{F}\rangle$ denote the residue class ring of $K\langle V\rangle$ modulo the two-sided ideal generated by the monomials which are represented by forbidden subwords. Then the Hilbert series $H_A(t) = H_{\mathcal{F}}(t)$ is the generating function for the number $\chi_n(\mathcal{F})$ of n letter words avoiding the forbidden subwords. Finally, set

$$H_{\bar{W}}(t) = \sum_{n=0}^{\infty} \chi_n(\mathcal{F},\bar{W})t^n \qquad \text{and} \qquad P_W(t,u) = \sum_{j,l=0}^{\infty} \psi_{(j,l)}(\bar{\mathcal{F}},W)t^l u^j.$$

Theorem 5.1. *Let V be a finite alphabet, $W \subset V$, and $\bar{\mathcal{F}}$ be a set of forbidden words. Then*

$$H_{\bar{W}}(t) = P_W(t,-1)H_{\mathcal{F}}(t).$$

Proof. As in the proof of 3.1 let I denote the right ideal generated by the elements of W. Again we have $I = \bigoplus_{w\in W} wA$. The kernel of the homomorphism $A^{(W)} \to A$,

$e_w \mapsto w$, is the direct sum $\bigoplus_{w\in W} e_w(\operatorname{Ann} w)$. Therefore we must find the right annihilator of each $w \in W$ in order to determine the first syzygy module of I. Clearly, a minimal system of generators of $\operatorname{Ann} w$ is given by the words f such that $wf \in \bar{\mathcal{F}}$, but no initial subword of wf belongs to $\bar{\mathcal{F}}$. Thus $\operatorname{Ann} w = \bigoplus_f fA$. To continue the resolution we have to find the annihilator of each of the elements f. The annihilator of f is minimally generated by those g for which w, f, g forms a worm etc.

These arguments show that the base elements of F_j in a free resolution $F_\bullet$ of A/I correspond bijectively to the worms of type (j, l) that have their leftmost letter in W. In order to make the maps in the free resolution homogeneous we must give such a base element the degree l. Consequently $P_W(t, u)$ is the Poincaré biseries of A/I. □

Remark 5.2. The conceivably most general case that one might be able to handle with our methods is the following: V is a finite alphabet, $\bar{\mathcal{F}}$ and $\bar{\mathcal{G}}$ are sets of forbidden subwords with $\bar{\mathcal{F}} \subset \bar{\mathcal{G}}$, and $\mathcal{W}$ is a set of forbidden 'initial subwords'. Then one could try to relate the generating function for the number of n letter words over V avoiding subwords in $\bar{\mathcal{F}}$ with the generating function for the number of n letter words which (i) avoid subwords in $\bar{\mathcal{G}}$ and also (ii) all the initial subwords in $\mathcal{W}$.

6. Rationality

In the first part of this section we want to prove that all the generating functions that appeared in 1.3, 3.1, 4.1, and 5.1 are rational. (In 5.1 we must assume that $\bar{\mathcal{F}}$ is finite.) It is enough to do this for the function $H_{\bar{W}}(t)$ in 5.1 since for each equation at least two of the functions in it are special instances of $H_{\bar{W}}(t)$. This is well known (cf. Stanley[19]) and only included for the reader's convenience.

Thus we start from the hypothesis of Section 5 and, in addition, assume that $\bar{\mathcal{F}}$ is finite. Let m be the maximum degree of an element of $\bar{\mathcal{F}}$, and L be the set of all $m-1$ letter words over V that do not contain a forbidden subword. We assume L to be enumerated as $l_1, \ldots, l_p$. Then we define a $p \times p$ matrix T as follows: if the last $m-2$ letters of l_i coincide with the first $m-2$ letters of l_j and furthermore the word which consists of l_i concatenated with the last letter of l_j does not contain a forbidden subword, then $T_{ij} = 1$; otherwise $T_{ij} = 0$. As the indicator e_W of a subset W of V we now choose the row vector which has the entry 1 at exactly those indices i for which l_i starts with a letter $w \in W$. Then the number of 'allowed' degree $m-1+n$ letter words with leftmost letter in $X \subset V$ is

$$\sum_{i=1}^{p} (e_X T^n)_i.$$

(The index i denotes the i-th component.)

Let a_n denote the number of 'allowed' n letter words with first letter in $\bar{W}$. Then

$$H_{\bar{W}}(t) = 1 + a_1 t + \cdots + a_{m-2}t^{m-2} + t^{m-1} \sum_{i=1}^{p} \sum_{n=0}^{\infty} (e_{\bar{W}} T^n)_i t^n.$$

Let $h_i(t) = \sum_{n=0}^{\infty} (e_{\bar{W}} T^n)_i t^n$ and $\mathbf{h}(t) = (h_1(t), \ldots, h_p(t))$. Then $\mathbf{h}(t) = \mathbf{h}(t)tT + e_{\bar{W}}$, and therefore

$$\mathbf{h}(t) = e_{\bar{W}}(1 - tT)^{-1}.$$

Since the entries of $(1 - tT)^{-1}$ are rational functions in t, all the components $h_i(t)$ of $\mathbf{h}(t)$ are rational functions, and the rationality of $H_{\bar{W}}(t)$ follows. It is clear that $H_{\bar{W}}(t)$ has poles at most at the reciprocals of the eigenvalues of T.

Finally we want to show that the Poincaré biseries $P_W(t, u)$ in Section 5 is a rational function. For $I = (X_1, \ldots, X_m)$ this result is due to Backelin[1]. We formulate it in algebraic terms.

Theorem 6.1. *Let K be a field, $X_1, \ldots, X_m$ non-commuting variables over K, and $\mathfrak{a}$ the two-sided ideal in $R = K\langle X_1, \ldots, X_m\rangle$ generated by a set M of finitely many monomials. Let I be the right ideal of $A = R/\mathfrak{a}$ generated by the residue classes of finitely many monomials. Then the A-module A/I has a rational Poincaré biseries.*

Proof. The right ideal I is a direct sum of finitely many right principal ideals generated by a monomial. Therefore it is enough to consider a single monomial s. Its annihilator is again a right ideal generated by finitely many monomials which appear as final subwords of elements of M etc. (Cf. the description of the worms in Section 5. Instead with a single letter we start a worm with the monomial s here.) So it suffices to choose s as a final subword of an element in M. We let $S = \{s_1, \ldots, s_q\}$ be the set of final subwords of the elements of M. Denote the length of s_i by l_i. Then we define a $q \times q$ transfer matrix T by setting $T_{ij} = t^{l_j}$ if $s_i s_j = 0$ in A, but no initial subword of $s_i s_j$ is zero. (As usual, t is a variable.) Otherwise $T_{ij} = 0$. Let $p_i(t, u) = \sum_{j,k} \beta_{jk}^{(i)} t^k u^j$ be the generating function for the number of base elements of degree k in the j-th free module in the free resolution of sA which are mapped to es_i where e is a basis element of the $(j-1)$-th free module. Then for $\mathbf{p}(t, u) = (p_1(t, u), \ldots, p_q(t, u))$ we obtain the recursion

$$\mathbf{p}(t, u) = \mathbf{p}(t, u)uT + \mathbf{p}_0(t)$$

where $\mathbf{p}_0(t) = (\sum_k \beta_{0k}^{(1)} t^k, \ldots, \sum_k \beta_{0k}^{(q)} t^k)$ with $\beta_{0k}^{(i)} = 0$ except that $\beta_{0l_r}^{(r)} = 1$ for the index r with $s = s_r$. As above it follows, that $\mathbf{p}(t, u)$ is a vector of rational functions. The sum of its components is the Poincaré biseries of sA. $\square$

For $I = (X_1, \ldots, X_m)$ Backelin[2] also proved the analogous result in the commutative case.

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