MATH. SCAND. 37 (1975), 29–39

DETERMINATION OF A CLASS OF POINCARÉ SERIES

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1. Introduction.

CONVENTION. A ring is a ring with unit element, a local ring is a commutative Noetherian local ring.

In his proof of the fact that a local ring Q is regular if and only if $\mathrm{gl.dim.}\, Q < \infty$ (in [7]), Serre showed that the Poincaré series of a regular local ring is $(1+Z)^n$ where n is the Krull dimension of Q, the Poincaré series of a local ring (Q,m) being defined as the formal power series

$$B^Q(Z) = \sum_{i \geq 0} b_i Z^i$$

with $b_i = \dim_k \mathrm{Tor}_i^Q(k,k)$, $k = Q/m$. This was done by using the Koszul complex for Q. Since then it has been shown for certain other classes of local rings that the Poincaré series is rational. The most important of these classes are:

1. Complete intersections (in [12]).
2. Golod rings (defined by the vanishing of certain homology operators on the Koszul complex of the ring) (see [1]).

(These two classes cover the case, where m is generated by two elements ([6]).)

3. Regular local rings modulo an ideal generated by two elements ([9], [2]).

Other results are given in [13], [10], [11] and [4]. Most of the results mentioned can be found in [3], which we also give as a general reference on the subject.

In all cases we know of, the Poincaré series has the form $(1+Z)^n/p(Z)$ where $p(Z)$ is a polynomial with integer coefficients and n the embedding dimension of Q. In case 1 above $p(Z) = (1-Z^2)^{n-m}$ (m the Krull dimension of Q), in case 2

$$p(Z) = 1 - \sum_{i \geq 1} c_i Z^{i+1}$$

($c_i = \dim_k H_i(K)$, K the Koszul complex of Q) and in case 3 $p(Z) = 1 - 2Z^2 - Z^3$ if Q is not a complete intersection.

Received April 21, 1975.

The purpose of this paper is to show the rationality of the Poincaré series for the quotient of a regular local ring modulo an ideal, generated by any set of monomials of degree two in a minimal system of generators for the maximal ideal. For such rings we obtain a formula for the Poincaré series of the form $(1+Z)^n/p(Z)$ with

$$p(Z) = 1+\sum c_{ij}(-1)^j Z^{i+j},$$

where $c_{ij}=\dim_k H_i^{i+j}(K)$, the upper index indicating a graduation on $H(K)$, induced by a filtration of the Koszul complex K to Q (see Section 5).

Our method consists in constructing a minimal Q-resolution of $k=Q/m$, by means of a generalization of the Koszul complex. This resolution is closely related to a $G(Q)$-resolution of K where $G(Q)$ is the graded ring associated with the m-filtration of Q. $G(Q)$ is a commutative k-algebra, but the proper scope of our method is a certain class of non-commutative k-algebras. So we shall start by studying such rings in Section 2, deriving the "associated" resolutions in Section 3 and determine their "Poincaré series" in Section 4. Finally, in Section 5, we pass to local rings.

2. Certain quotient rings of polynomial rings in non-commuting variables.

Notations. If R is a ring (commutative or not), we let $R\langle X_1,\ldots,X_n\rangle$ denote the polynomial ring in n non-commuting variables over R (the variables commute with the elements of R).

The rings we are going to study in this section are quotients of

$$K\langle X_1,\ldots,X_n\rangle\langle Y_1,\ldots,Y_n\rangle$$

(K a commutative field) modulo certain systems of n^2 relations, the images of X_i and Y_i being denoted x_i and y_i respectivly. The systems of relations considered can be described as follows:

(1) For every i $(1 \leqq i \leqq n)$ one of the relations $X^2{}_i = 0$ and $Y^2{}_i = 0$

For every pair (i,k) where $i \neq k$ one of the following five types of relations, where $c_{ki} \in K-\{0\}$:

$$X_iX_k+c_{ki}X_kX_i = Y_iY_k-c_{ki}Y_kY_i = 0 \tag{2}$$

$$X_iX_k = X_kX_i = 0 \tag{3}$$

(4) $$Y_iY_k = Y_kY_i = 0$$

(5) $$X_iX_k = Y_kY_i = 0$$

(6) $$X_kX_i = Y_iY_k = 0$$

In case (2) we say that the pair (i,k) commutes.

On a ring R of the described type, we will define K-linear maps d_i and D_i $(i=1,\ldots,n)$. We call an element

$$x_{i_1}\cdot\ldots\cdot x_{i_r}\cdot y_{j_1}\cdot\ldots\cdot y_{j_s} = x^{(\mu)}y^{(\nu)} = m \neq 0$$

in R a monomial. It suffieces to define d_i and D_i on monomials in R, and then extend them K-linearly to R. We say that x_i can be factored out to the right in a monomial m (respectively y_i can be factored out to the left in m) if

$$m = c_i x^{(\mu')} x_i y^{(\nu)} \text{ for some } c_i \in K$$

(respectively $m=c_i' x^{(\mu)} y_i y^{(\nu')}$ for some $c_i' \in K$).

Definitions.

$D_i(m) = c_i x^{(\mu')} y_i y^{(\nu)}$ if $m = x^{(\mu)}y^{(\nu)} = c_i x^{(\mu')} x_i y^{(\nu)}$,

$D_i(m) = 0$ if x_i cannot be factored out to the right in m.

$d_i(m) = c_i' x^{(\mu)} x_i y^{(\nu')}$ if $m = x^{(\mu)}y^{(\nu)} = c_i' x^{(\mu)} y_i y^{(\nu')}$,

$d_i(m) = 0$ if y_i cannot be factored out to the left in m.

Lemma 1.

(7) $d_i^2 = 0$.

(8) $d_i d_k + d_k d_i = 0$ if $i \neq k$.

(9) $d_i D_k + D_k d_i = 0$ if $i \neq k$.

(10) *If m is a monomial with $d_i(m) \neq 0$, then $D_i(m) = 0$.*

(11) *If m is a monomial with $d_i(m) \neq 0$, then $D_i d_i(m) = m$.*

(For symmetry reasons the analogues of the above statements, obtained by interchanging d_i and D_i are also true.)

Proof. (7) follows directly from (1). To prove (8), let m be a monomial. If none of y_i and y_k can be factored out to the left in m, then $d_i(m)=$

$d_k(m)=0$, and thus (8) follows in this case. Now suppose that y_i but not y_k can be factored out to the left in m, so $d_k(m)=0$, If now y_k cannot be factored out to the left in $d_i(m)$, we have $d_k d_i(m)=0$. If y_k can be factored out to the left in $d_i(m)$ one of the relations (3) or (5) must hold for (i,k), but then $x_i x_k=0$ which gives $d_k d_i(m)=0$, so (8) follows also in this case. Finally, if both y_i and y_k can be factored out to the left in m, relation (2) must hold for (i,k), and that relation implies $d_i d_k(m)=-d_k d_i(m)$, so (8) is proved in full. (9) is proved similarly, (10) and (11) are fairly obvious from the definitions.

DEFINITION. If m is a monomial in R, we let

$$\operatorname{Index}(m) = \{i\,;\; d_i(m) \neq 0 \text{ or } D_i(m) \neq 0\}\,.$$

NOTE. Index(1) is always empty but Index(m) could be empty also if m is a monomial of positive degree. An example is: Let $n=4$, only $(2,3)$ commutes, $X_1X_3=X_3X_4=Y_1Y_2=Y_2Y_4=0$ (the other relations chosen arbitrarily). Then Index$(x_1x_2y_3y_4)$ is empty. But in the cases of interest for us, Index(m) is always nonempty for monomials of positive degree (see Lemma 4 below).

It is now easy to prove that $(d_iD_i+D_id_i)(m)=n_m m$ for some integer $n_m>0$ if every monomial of positive degree has non-empty Index. This would be sufficient for our needs if $\operatorname{char}(K)=0$ (see the Theorem in next section), but to cover also the case $\operatorname{char}(K)>0$, we must argue further.

LEMMA 2. *If $i,k \in \operatorname{Index}(m)$, then (i,k) commutes.*

PROOF. If $d_i(m)\neq 0$ and $d_k(m)\neq 0$ (or if $D_i(m)\neq 0$ and $D_k(m)\neq 0$), (i,k) must commute, so we can assume that $d_k(m)\neq 0$ and $D_i(m)\neq 0$. But then

$$m = c\, x^{(\mu')}x_i y_k y^{(\nu')}\,.$$

Now if (i,k) does not commute, either $x_ix_k=0$ or $y_iy_k=0$, which contradicts the assumption.

LEMMA 3. *If $d_k(m)\neq 0$, then* $\operatorname{Index}(d_k(m))=\operatorname{Index}(m)$.
If $D_k(m')\neq 0$, then $\operatorname{Index}(D_k(m'))=\operatorname{Index}(m')$.

PROOF. It suffices to show the first statement. Since $d_k(m)\neq 0$, $D_kd_k(m)=m\neq 0$ so $k\in\operatorname{Index}(d_k(m))$. If $i\neq k$ and $i,k\in\operatorname{Index}(m)$, then $i\in\operatorname{Index}(d_k(m))$ since (i,k) commutes according to Lemma 2. But then

$$\operatorname{Index}(m) \subset \operatorname{Index}(d_k(m)) \subset \operatorname{Index}(D_kd_k(m)) = \operatorname{Index}(m)\,.$$

DEFINITIONS.

$$i(m) = \min(\text{Index}(m)) \quad \text{and} \quad S(m) = D_{i(m)}(m)$$

where we assume Index(m) to be non-empty.

THEOREM.

(12) $$\text{If } d = d_1 + \ldots + d_n \text{ then } d^2 = 0$$

and

(13) $$(Sd + dS)(m) = m$$

if m is a monomial with Index(m) *non-empty.*

PROOF. (12) follows from (7) and (8). On a monomial m, with Index(m) non-empty, we have

$$Sd + dS = D_{i(m)}d + dD_{i(m)} = D_{i(m)}(d_1 + \ldots + d_n) + \\ + (d_1 + \ldots + d_n)D_{i(m)} = D_{i(m)}d_{i(m)} + d_{i(m)}D_{i(m)} = 1$$

according to (9) and (11).

We single out the types of rings of special interest for us in the following lemma.

LEMMA 4. *Every monomial af positive degree has non-empty Index if:*

A′. *No pair is commutative.*

B′. *For some pairs $i \neq k$, (i,k) commutes, and for all other pairs $(i \neq k)$*

$$X_i X_k = X_k X_i = 0 .$$

PROOF. A′. If $m = x^{(\mu')} x_i y_k y^{(\nu')}$, either $d_k(m) \neq 0$ or $D_i(m) \neq 0$.

B′. If $m = y_k y^{(\nu')}$, then $d_k(m) \neq 0$. If m contains x_i and if $D_i(m) = 0$, we must have $y_i^2 = 0$, but then $x_i^2 \neq 0$ so $d_i(m) \neq 0$.

3. Resolutions.

We shall construct resolutions of the residue class field K for rings of the following two types:

A. $K\langle X_1, \ldots, X_n \rangle / I$ where K is a commutative field and I is an ideal generated by any set of monomials of degree 2 in $\{X_i\}$.

B. $K\langle X_1, \ldots, X_n \rangle / I$ where K is a commutative field and I is an ideal generated by any set of monomials of degree 2 in $\{X_i\}$ and furthermore by one element $X_i X_k + c_{ki} X_k X_i$ for each $i \neq k$, where $c_{ki} \in K - \{0\}$.

If R is a ring of type A (respectively B), there is exactly one ring R' of type A′ (respectively B′) (definition in Lemma 4 in Section 2), with the same number of X-variables and the same relations between these variables. This is so, because the ralations between the X-variables determine the relations between the Y-variables in the rings examined in Section 2. We shall say that R' belongs to R. Now R' is in a natural way a graded algebra over R,

$$R' = \oplus_{i \geqq 0} R'_i$$

(R'_i consists of the homogenous elements of degree i in $\{y_i\}$). Clearly d becomes a homogenous R-linear map of degree -1.

THEOREM. *If R is a ring of type* A (*respectively* B) *and R' $\oplus_{i \geqq 0} R'_i$ the ring of type* A′ (*respectively* B′) *belonging to R, then*

$$\ldots \xrightarrow{d} R'_i \xrightarrow{d} \ldots \xrightarrow{d} R'_1 \xrightarrow{d} R \xrightarrow{n} K \to 0$$

is a resolution of K (n is the natural map.)

PROOF. Lemma 4 shows that every monomial in R'_i $(i > 0)$ has non-empty Index. The Theorem preceeding Lemma 4 then gives that we have a complex with a chain homotopy, i.e. an exact sequence (n is obviously an augmentation map).

We illustrate the connection between R and R' with some simple examples, which in particular shed light on the obvious duality between the X- and Y-variables.

1a. If $R = K\langle X_1, \ldots, X_n\rangle$, then $R' = R\langle Y_1, \ldots, Y_n\rangle / M^2$ where $M = (Y_1, \ldots, Y_n)$.

1b. If $R = K\langle X_1, \ldots, X_n\rangle / m^2$ where $m = (X_1, \ldots, X_n)$, then $R' = R\langle Y_1, \ldots, Y_n\rangle$.

2a. If

$$R = K[X_1 \ldots, X_n] = K\langle X_1, \ldots, X_n\rangle / (\{X_i X_j - X_j X_i \,;\, i \neq j\}),$$

then

$$R' = R\langle Y_1, \ldots, Y_n\rangle / (\{Y_i^2\}, \{Y_i Y_j + Y_j Y_i \,;\, i \neq j\})$$

(the Koszul complex of R).

2b. If

$$R = K\langle X_1, \ldots, X_n\rangle / (\{X_i^2\}, \{X_i X_j + X_j X_i \,;\, i \neq j\})$$

(R "Koszul") then $R' = R[Y_1, \ldots, Y_n]$.

NOTE. If $R=\oplus_{i\geqq 0}R_i$ is a graded ring with R_0 a field, then gl.dim.$R=$ $=$h.d.R_0 (see [5]). So a ring R of type A will have finite global dimension if R' has finite rank as an algebra over R, i.e. if $(y_1,\ldots,y_n)$ is nilpotent in R'.

4. "Poincaré series".

NOTATIONS. If $R'=\oplus_{i\geqq 0}R'_i$ is a graded algebra over a ring R, with R'_i free R-modules, we let

$$H_R{}^{R'} = \textstyle\sum_{i\geqq 0}h_iZ^i, \quad \text{where } h_i = \operatorname{rank}_R R'_i\,.$$

If $R=K\langle X_1,\ldots,X_n\rangle/I$ is a ring of type A, we study $\bar{R}=K\langle X_1,\ldots,X_n\rangle/\bar{I}$, where $\bar{I}$ is generated by monomials of degree two, such that $X_iX_j\in\bar{I}$ if and only if $X_iX_j\notin I$. We call $\bar{R}$ the complement K-algebra to R.

THEOREM. *If R is a ring of type* A, *$\bar{R}$ the complement K-algebra to R, then*

$$H_K{}^R(-Z)\cdot H_K{}^{\bar{R}}(Z) = 1\,.$$

PROOF. Let

$$H_K{}^R(Z) = \textstyle\sum_{i\geqq 0}h_iZ^i \quad \text{and} \quad H_K{}^{\bar{R}}(Z) = \sum_{i\geqq 0}\bar{h}_iZ^i\,.$$

We shall show, that $\bar{h}_0\cdot h_0=1$ (clear) and that

$$\bar{h}_n\cdot h_0-\bar{h}_{n-1}\cdot h_1+\ldots+(-1)^n\bar{h}_0\cdot h_n = 0 \quad \text{if } n>0\,.$$

Now $\bar{h}_r$ is the number of different monomials of degree r in $\{X_i\}$, such that their images in $\bar{R}$ is $\neq 0$. We say that a monomial $X_{i_1}\ldots X_{i_n}$ can be divided after X_{i_r}, if $X_{i_1}\cdot\ldots\cdot X_{i_r}$ has an image $\neq 0$ in $\bar{R}$, at the same time as $X_{i_{r+1}}\cdot\ldots\cdot X_{i_n}$ has an image $\neq 0$ in R. If a monomial can be divided in this sense at some place, then it can be divided at exactly two places (besides after X_{i_r} also after $X_{i_{r-1}}$ or $X_{i_{r+1}}$, depending on whether $X_{i_r}X_{i_{r+1}}$ has an image $\neq 0$ in R or in $\bar{R}$). This shows, that there are just as many monomials, that can be divided after an even number of steps, as after an odd number of steps. As the total number of monomials of degree n, that can be divided after r steps is $\bar{h}_r\cdot h_{n-r}$, this shows that

$$\bar{h}_n\cdot h_0+\bar{h}_{n-2}\cdot h_2+\ldots = \bar{h}_{n-1}\cdot h_1+\bar{h}_{n-3}\cdot h_3+\ldots$$

Corollary 1. *If R is a ring of type* A, *R' the ring of type* A′ *belonging to R, then*

$$H_K^R(-Z)\cdot H_R^{R'}(Z) = 1 .$$

Proof. $H_R^{R'}(Z)=H_K^{\bar{R}}(Z)$, where $\bar{R}$ is the complement K-algebra to R.

Corollary 2. *If R is a ring of type* B, *R' the ring of type* B′ *belonging to R, then*

$$H_K^R(-Z)\cdot H_R^{R'}(Z) = 1 .$$

Proof. Let

$$R = K\langle X_1,\ldots,X_n\rangle/I \quad \text{and} \quad R' = R\langle Y_1,\ldots,Y_n\rangle/I' .$$

Now $h_i=\operatorname{rank}_R R'_i$ is the maximum number of linearly independent elements in R'_i. But since R'_i can be generated by monomials in $\{y_i\}, h_i$ is the maximum number of linearly independent monomials of degree i in R'_i, that is the number of different monomials in R'_i, when represented in lowest possible lexicografical way. So $H_R^{R'}(Z)=H_R^S(Z)$, where $S=R\langle Y_1,\ldots,Y_n\rangle/J$, and J is generated by the following monomials of degree two:

$$\{Y_iY_j\,;\ i>j \text{ and } (i,j) \text{ commutes}\} \quad \text{and} \quad \{Y^2{}_i\,;\ Y^2{}_i \in I'\} .$$

It is clear that $H_R^S(Z)=H_K^T(Z)$, where $T=K\langle Y_1,\ldots,Y_n\rangle/L$, and L is generated by "the same" monomials as J. Representing the monomials in R in the highest possible lexicografical way, one sees that $H_K^R(Z)=H_K^U(Z)$, where $U=K\langle X_1,\ldots,X_n\rangle/N$, N generated by the following monomials of degree two:

$$\{X_iX_j\,;\ i<j\} \quad \text{and} \quad \{X_iX_j\,;\ i\geqq j \text{ and } X_iX_j\in I\} .$$

But then $y_i^2=0$ in T if and only if $x_i^2\neq 0$ in U, and $y_iy_j=0$ in $T(i\neq j)$ if and only if $x_ix_j\neq 0$ in U. Hence U is the complement K-algebra to T, and the result follows from the Theorem.

5. Local rings.

The Poincaré series for a local ring Q with residue class field k is defined as

$$B^Q(Z) = \textstyle\sum_{i\geqq 0} b_i Z^i ,$$

where $b_i=\dim_k \operatorname{Tor}_i^Q(k,k)$. To determine b_i, one can construct a minimal resolution of k. A resolution

$$\ldots Q^{c_2} \xrightarrow{d_2} Q^{c_1} \xrightarrow{d_1} Q \xrightarrow{d_0} k \to 0$$

is minimal, if c_i has been chosen minimally, which means that

$$d_i \otimes \mathrm{id}_k \colon Q^{c_i} \otimes k \to \mathrm{Im}\, d_i \otimes k$$

is an isomorphism for every i (Nakayamas lemma). Then $c_i = b_i$. See e.g. [8].

We shall determine the Poincaré series for a ring Q of the following type: $Q = \tilde{Q}/\tilde{I}$, where $\tilde{Q}$ is a regular local ring and $\tilde{I}$ is generated by a set of monomials of degree two in a minimal set of generators for the maximal ideal $\tilde{m}$ in $\tilde{Q}$. The associated graded ring

$$R = G(Q) = \oplus_{i \geq 0} m^i/m^{i+1}$$

to Q (m the maximal ideal of Q) is then a ring of type B. Let

$$R' = R\langle Y_1, \ldots, Y_n\rangle/J'$$

be the ring of type B′ belonging to R, and let

$$Q' = Q\langle Y_1, \ldots, Y_n\rangle/I' ,$$

where I' is generated by "the same" elements as J'. On $Q' = \oplus_{i \geq 0} Q'_i$ (graded by the degree in $\{y_i\}$), we define a map ∂, in the same way as the differential d is defined at R'.

THEOREM.

$$\ldots \xrightarrow{\partial} Q'_i \xrightarrow{\partial} \ldots \xrightarrow{\partial} Q'_1 \xrightarrow{\partial} Q \xrightarrow{n} k \to 0$$

is a minimal resolution of $k = Q/m$ (n is the natural quotient map).

PROOF. That $\partial^2 = 0$ is clear. Now let $\partial(\sum_i q_i m_{i,r}) = 0$, where $q_i \in Q$ and $m_{i,r}$ are monomials in $\{y_i\}$ of some degree r. Let $\bar{q}_i$ be the images of q_i in Q/m. Then $d(\sum_i \bar{q}_i m_{i,r}) = 0$ in R' (we identify the monomials in $\{y_i\}$ in Q and $G(Q)$). Since (R', d) is a resolution of k, we have $\sum_i \bar{q}_i m_{i,r} = d(r')$ for some $r' \in R'_{r+1}$. Let q' be an inverse image of r' in Q'_{r+1}. Then $\sum_i \bar{q}_i m_{i,r} - \partial(r') \in Q_r$, so

$$\mathrm{Ker}\, \partial \subset \mathrm{Im}\, \partial + mQ' .$$

In the same way we see that $\mathrm{Ker}\,\partial \subset \mathrm{Im}\,\partial + m^n Q'$ for all n. But then

$$\mathrm{Ker}\, \partial \subset \bigcap_{n=1}^{\infty} (\mathrm{Im}\, \partial + m^n Q') = \mathrm{Im}\, \partial ,$$

the last equality by Artin-Rees' lemma applied to the various Q'_r. The resolutions are clearly minimal.

COROLLARY 1. *Let $\tilde{Q}$ be a commutative regular Noetherian local ring, $\tilde{I}$ an ideal in $\tilde{Q}$ generated by monomials of degree two in a minimal system of generators for the maximal ideal. Let the maximal ideal and residue class field of $Q=\tilde{Q}/\tilde{I}$ be m and k. Let*

$$H^Q(Z) = \sum_{i\geq 0} h_i Z^i, \quad \text{where } h_i = \dim_k(m^i/m^{i+1})$$

(we call $H^Q(Z)$ the Hilbert series for Q).
Then $H^Q(-Z)\cdot B^Q(Z)=1$.

PROOF. If $R=G(Q)$ is the associated graded ring to Q, R' the ring of type B' belonging to R, then $B^Q(Z)=H_R{}^{R'}(Z)$ since the resolution is minimal. Since $H^Q(Z)=H_k{}^R(Z)$ the result follows from Corollary 2 of Section 4.

Let Q be a local ring with maximal ideal $m=(t_1,\ldots,t_n)$ (minimal set of generators), K its Koszul complex in the variables $T_1,\ldots,T_n$. Let $c_{ij}=\dim_k H_i^{i+j}(K)$, where the upper index indicates the succesive quotient in the filtration of $H_i(K)$, induced by the filtration

$$K^p = (t_1\ldots,t_n,T_1,\ldots,T_n)^p K$$

of the Koszul complex. Then we can give the Hilbert series for Q as

$$(1+\sum c_{ij}(-1)^i Z^{i+j})/(1-Z)^n$$

(see [8] chapitre IV. 3). Thus we get

COROLLARY 2. *If Q is as in Corollary 1, then with the notions above*

$$B^Q(Z) = (1+Z)^n/(1+\sum c_{ij}(-1)^j Z^{i+j}) .$$

We conclude with a concrete example. If

$$Q = k[[X_1,X_2,X_3]]/(X_1{}^2,X_2{}^2,X_3{}^2,X_1X_2,X_1X_3) ,$$

then $H^Q(Z)=1+3Z+Z^2$, so

$$B^Q(Z) = 1/(1-3Z+Z^2) = (1+Z)^3/(1-5Z^2-5Z^3+Z^5)$$

(cf. the Gorenstein ring

$$k[[X_1,X_2,X_3]]/(X_1{}^2,X_2{}^2,X_3{}^2,X_1X_2-X_2X_3,X_1X_3-X_2X_3)$$

with the same Poincaré series, see [13]).

NOTE. An example by Gunnar Sjödin shows that our formula of Corollary 1 is false in general if $\tilde{I}$ is generated by forms of degree two. Namely, let

$$Q = k[[X_1, X_2, X_3, X_4]]/(X_1^2 - X_2^2, X_2^2 - X_3^2, X_3^2 - X_4^2, X_1X_2, X_1X_3 - X_2X_4) .$$

Then $H^Q(Z) = 1 + 4Z + 5Z^2$, so $1/H^Q(-Z)$ will have negative coefficients.

Acknowledgement.

The author wants to thank Christer Lech for many fruitful discussions and improvements of the arguments during the work on this paper.

REFERENCES

1. E. S. Golod, *On the homology of some local rings*, Soviet Math. Dokl. 3 (1962), 745–748.
2. T. H. Gulliksen, *A note on the Poincaré series of a local ring*, J. Algebra 13, (1969), 242–245.
3. T. H. Gulliksen and G. Levin, *Homology of local rings*, Queen's Paper no. 20, Queen's University, Kingston 1969.
4. T. H. Gulliksen, *Massey operations and the Poincaré series of certain local rings*, J. Algebra 22 (1972), 225–232.
5. A. Roy, *A note on filtered rings*, Arch. Math. (Basel) 16 (1965), 421–427.
6. G. Scheja, *Über die Bettizahlen lokaler Ringe*, Math. Ann. 179 (1969), 257–274.
7. J.-P. Serre, *Sur la dimension homologique des anneaux et des modules noetheriens*, Procedings of the International Symposium on Algebraic Number theory, Tokyo-Nikko 1955, pp. 175–189, Kasai Publishing co., Tokyo, 1956.
8. J.-P. Serre, *Algébre locale Multiplicités*, (Lecture Notes in Mathematics 11), Springer-Verlag, Berlin, Heidelberg, New York 1965.
9. J. Shamash, *The Poincaré series of a local ring*, J. Algebra 12 (1969), 453–470.
10. J. Shamash, *The Poincaré series of a local ring*, II, J. Algebra 17 (1971), 1–18.
11. J. Shamash, *The Poincare series of a local ring*, III, J. Algebra 17 (1971) 153–170.
12. J. Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. 1, (1957), 14–25.
13. H. Wiebe, *Über homologische Invarianten lokaler Ringe*, Math. Ann. 179 (1969), 257–274.

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