Koszul algebras, Veronese subrings and rings with linear resolutions

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Koszul Algebras

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Abstract

A graded k-algebra A is called a Koszul algebra if the minimal graded free A-resolution of k has only linear maps. This article is a survey on results obtained about Koszul algebras since they were introduced by Priddy in 1970. We start with giving several equivalent conditions to being Koszul, and then give lots of examples of Koszul algebras from different fields. We show that the class of Koszul algebras is closed under a number of natural operations. Almost no proofs are given, but ample references to the literature are provided.

1 Introduction

Let k be a field and V a vector space over k with basis $(x_1,\ldots,x_n)$. The tensor algebra (or the noncommutative polynomial ring) $T(V)=\oplus_{i\geq 0}(T(V))_i$ is a graded k-algebra with the monomials $x_{m_1}\cdots x_{m_i}$ as a k-basis for $(T(V))_i$. We will use the notation $k\langle x_1,\ldots,x_n\rangle$ for $T(V)$. With a *graded algebra* we will mean an algebra $A=k\langle x_1,\ldots,x_n\rangle/I$, where I is a two-sided ideal generated by homogeneous elements. The graded algebra A is called *quadratic* if I is generated by elements of degree two. As an example, the usual commutative polynomial ring is quadratic, since $k[x_1,\ldots,x_n]=k\langle x_1,\ldots,x_n\rangle/I$, where I is generated by all commutators $x_ix_j-x_jx_i$. Also algebras $k[x_1,\ldots,x_n]/I$, where I is generated by quadratic forms, are quadratic. The *Hilbert series* $A(z)$ of a graded algebra $A=\oplus A_i$ is the generating function for the k-dimensions of A_i,

$$A(z)=\sum_{i\geq 0}\dim_k A_i\cdot z^i.$$

If $A=k\langle x_1,\ldots,x_n\rangle$ we have $\dim_k A_i=n^i$, so $A(z)=1+nz+n^2z^2+\cdots=1/(1-nz)$, and if $A=k[x_1,\ldots,x_n]$ we have $\dim_k A_i=\binom{i+n-1}{n-1}$, so $A(z)=$

$1/(1-z)^n$. It is well known that if A is a commutative graded algebra, $A = k[x_1, \ldots, x_n]/I$, then $A(z)$ is a rational function, namely $A(z) = p(z)/(1-z)^n$ for some polynomial $p(z) \in \mathbb{Z}[z]$. If A is a graded algebra, then $\oplus_{i>0} A_i$ is a graded maximal ideal which we will denote by A_+.

For any graded algebra A there exists a minimal free graded A-resolution of k

$$\mathbf{F} : \cdots \xrightarrow{\phi_3} A^{b_2} \xrightarrow{\phi_2} A^{b_1} \xrightarrow{\phi_1} A \longrightarrow k.$$

That the resolution is graded means that the nonzero entries of the matrices ϕ_i are homogeneous, that the resolution is minimal means that all nonzero entries have positive degrees. Since $\phi_i \otimes k = 0$ in a minimal resolution we have

$$\mathrm{Tor}_i^A(k,k) \simeq A^{b_i} \otimes k \simeq k^{b_i} \simeq \mathrm{Ext}_A^i(k,k).$$

The *Poincaré series* $P_A(z)$ of A is the generating function for the k-dimensions of $\mathrm{Tor}_i^A(k,k)$,

$$P_A(z) = \sum_{i \geq 0} \dim_k \mathrm{Tor}_i^A(k,k) \cdot z^i = \sum_{i \geq 0} \dim_k \mathrm{Ext}_A^i(k,k) \cdot z^i.$$

If we shift the degrees such that all ϕ_i becomes maps of degree 0, we see that the grading on A induces a grading on $\mathrm{Tor}_i^A(k,k) = \oplus_j (\mathrm{Tor}_i^A(k,k))_j$ and on $\mathrm{Ext}_A^i(k,k) = \oplus_j (\mathrm{Ext}_A^i(k,k))_j$ and we can define a Poincaré series in two variables

$$\mathbf{P}_A(x,y) = \sum_{i,j} \dim_k (\mathrm{Tor}_i^A(k,k))_j \cdot x^i y^j = \sum_{i,j} \dim_k (\mathrm{Ext}_A^i(k,k))_j \cdot x^i y^j.$$

The existence of a minimal resolution gives that $(\mathrm{Tor}_i^A(k,k))_j = (\mathrm{Ext}_A^i(k,k))_j = 0$ if $j < i$.

Koszul algebras were first introduced in [Pr] (under the name homogeneous Koszul algebras). We define a *Koszul algebra* to be a graded algebra such that $(\mathrm{Tor}_i^A(k,k))_j = 0$ if $i \neq j$ or, equivalently, such that $(\mathrm{Ext}_A^i(k,k))_j = 0$ if $i \neq j$. Another way to say this is that the minimal graded A-resolution of k is *linear*, i.e., all nonzero entries of all ϕ_i are of degree one. If $A = k\langle x_1, \ldots, x_n\rangle$ a minimal A-resolution of k looks like

$$0 \longrightarrow A^n \xrightarrow{\phi_1} A \longrightarrow k \longrightarrow 0$$

with $\phi_1 = (x_1 \ \cdots \ x_n)$ and hence it is linear. We get that the Poincaré series of $k\langle x_1, \ldots, x_n\rangle$ equals $1 + nz$, the double Poincaré series equals $1 + nxy$.

If $A = k[x_1, \ldots, x_n]$, the Koszul complex

$$0 \longrightarrow A^{\binom{n}{n}} \xrightarrow{\phi_n} \cdots \longrightarrow A^{\binom{n}{2}} \xrightarrow{\phi_2} A^{\binom{n}{1}} \xrightarrow{\phi_1} A \longrightarrow k \longrightarrow 0$$

is a minimal graded resolution of k with free A-modules. If we denote a basis for $A^{\binom{n}{i}}$ by $\{e_{m_1 \cdots m_i} \ ; \ 1 \leq m_1 < \cdots < m_i \leq n\}$, then $\phi_i(e_{m_1 \cdots m_i}) =$

$\sum_{k=1}^{i}(-1)^{k-1}x_{m_k}e_{m_1\cdots\widehat{m_k}\cdots m_i}$ and hence ϕ_i is linear. The Poincaré series of $k[x_1,\ldots,x_n]$ equals $1+\binom{n}{1}z+\cdots+\binom{n}{n}z^n=(1+z)^n$, the double series $(1+xy)^n$.

Thus both $k\langle x_1,\ldots,x_n\rangle$ and $k[x_1,\ldots,x_n]$ are examples of Koszul algebras since k has a linear resolution over these algebras.

If $A=k\langle x_1,\ldots,x_n\rangle/I$, then $\mathrm{Tor}_2^A(k,k)\simeq I/(A_+I+IA_+)$, so $\mathrm{Tor}_2^A(k,k)=(\mathrm{Tor}_2^A(k,k))_2$ if and only if A is quadratic. Hence a Koszul algebra is necessarily quadratic.

The formula above for $\mathrm{Tor}_2^A(k,k)$ is just a special case of a general result in [Go]. There it is shown that

$$\mathrm{Tor}_{2i}^A(k,k)\simeq(A_+I^{i-1}A_+\cap I^i)/(A_+I^i+I^iA_+)$$

and that

$$\mathrm{Tor}_{2i-1}^A(k,k)\simeq(A_+I^{i-1}\cap I^{i-1}A_+)/(A_+I^{i-1}A_++I^i)$$

if $A=k\langle x_1,\ldots,x_n\rangle/I$. This is used in [B1] to get another equivalent condition for an algebra to be Koszul. Let $L(A)$ be the lattice associated to $A=k\langle x_1,\ldots,x_n\rangle/I$, i.e., the lattice generated by $\{A_+^fI^gA_+^h\ ;\ f,g,h\geq 0\}$ under $+$ and $\cap$. Then A is Koszul if and only if A is quadratic and $L(A)$ is *distributive*. It is also shown in [B1] that this is equivalent to that A is quadratic and all lattices $L_j(A)$ generated by $\{A_+^fI^gA_+^h\ ;\ f,g,h\geq 0, f+g+h=j\}$ are distributive.

2 The Koszul Dual

Let $k\langle x_1,\ldots,x_n\rangle=\Lambda$ and Λ/I a quadratic algebra. Let $\Lambda^*=\mathrm{Hom}_k(\Lambda,k)=\oplus_{i\geq 0}\mathrm{Hom}_k(\Lambda_i,k)=\oplus_{i\geq 0}\Lambda_i^*$ with multiplication induced by $\mu\nu(ab)=\mu(a)\nu(b)$, where $\mu\in\Lambda_i^*,\nu\in\Lambda_j^*,a\in\Lambda_i,b\in\Lambda_j$. Let $I_2^\perp=\{\mu\in\Lambda_2^*\ ;\ \mu(I_2)=0\}$ and let $I^\perp$ be the ideal generated by $I_2^\perp$. The *Koszul dual* of A is $A^!=\Lambda^*/I^\perp$. The Koszul dual is a quadratic algebra and $(A^!)^!=A$.

To calculate a presentation of $A^!$ from the presentation of A is just some elementary linear algebra. We give some examples. If $A=k\langle x_1,\ldots,x_n\rangle$, then $A^!=k\langle y_1,\ldots,y_n\rangle/(y_iy_j,1\leq i,j\leq n)$. If $A=k[x_1,\ldots,x_n]$, then $A^!=k\langle y_1,\ldots,y_n\rangle/(y_i^2,1\leq i\leq n,y_iy_j+y_jy_i,1\leq i<j\leq n)$. If $A=k\langle x_1,\ldots,x_n\rangle/I$, where I is generated by monomials of degree two, then $A^!=k\langle y_1,\ldots,y_n\rangle/J$, where J is generated by those monomials y_iy_j such that $x_ix_j\notin I$. If $A=k[x_1,\ldots,x_n]/(f_1,\ldots,f_r)$, where $f_i=\sum_{j\leq k}b_{ijk}x_jx_k$, then $A^!=k\langle y_1,\ldots,y_n\rangle/J$, where $J=(g_1,\ldots,g_s)$, $s=\binom{n}{2}-r$, and $g_i=\sum_{j\leq k}c_{ijk}[y_i,y_j]$ (here $[y_i,y_j]=y_iy_j+y_jy_i$ if $i\neq j$ and $[y_i,y_i]=y_i^2$) and $(c_{ijk})_{jk},i=1,\ldots,s$ is a basis of the solutions to the linear system $\sum_{j\leq k}b_{ijk}X_{jk}=0,i=1,\ldots,r$, cf. [L]. As an example, if

$$A=k[x_1,x_2,x_3]/(x_1^2,x_2x_3,x_1x_3-x_3^2),$$

then

$$A^!=k\langle y_1,y_2,y_3\rangle/(y_2^2,y_1y_2+y_2y_1,y_1y_3+y_3y_1+y_3^2).$$

For any quadratic algebra A there is a natural differential $d : A_i \otimes_k (A^!_j)^* \longrightarrow A^!_{i+1} \otimes_k (A^!_{j-1})^*$, where * indicates vector space dual, cf. [Pr], [L], or [Ma], namely if $f \in (A^!)^*_n$, then $df \in A_1 \otimes (A^!)^*_{n-1}$ is defined by identifying $A_1 \otimes (A^!)^*_{n-1}$ with $(A^!_1 \otimes A^!_{n-1})^*$ and letting $(df)(x \otimes m) = f(xm)$, where $x \in A^!_1, m \in A^!_{n-1}$ and extend A-linearly. This makes $A \otimes_k A^!$ into a complex U. (If $A = k[x_1, \ldots, x_n]$, then U is the usual Koszul complex.) This "generalized Koszul complex" is exact if and only if A is Koszul.

There is a well known product $\mathrm{Ext}^i_A(k,k) \times \mathrm{Ext}^j_A(k,k) \longrightarrow \mathrm{Ext}^{i+j}(k,k)$, the Yoneda multiplication, making $\mathrm{Ext}_A(k,k)$ into an associative graded algebra. It is shown in [L] (also cf. [Pr]) that $A^!$ is the subalgebra $[\mathrm{Ext}^1_A(k,k)]$ of $\mathrm{Ext}_A(k,k)$ generated by its one-dimensional elements. It is clear that for any graded algebra A we have $\mathrm{Ext}^1_A(k,k) = (\mathrm{Ext}^1_A(k,k))_1$, so $[\mathrm{Ext}^1_A(k,k)] \subseteq \oplus_i (\mathrm{Ext}^i_A(k,k))_i$, but in fact there is equality, cf. [L]. Hence A is a Koszul algebra if and only if $\mathrm{Ext}_A(k,k) = [\mathrm{Ext}^1_A(k,k)]$. In particular we see that if A is a Koszul algebra, then $\mathrm{Ext}_A(k,k)$ is finitely generated.

For any graded algebra A, for each j, the restriction of the minimal A-resolution of k to degree j is a finite exact complex of finite dimensional vector spaces. Using that the alternating sum of the dimensions in this finite dimensional complex equals the alternating sum of its homologies one gets the formula $A(z)\mathbf{P}_A(-1,z) = 1$ for each graded algebra A. If A is a Koszul algebra we have $\mathbf{P}_A(x,y) = P_A(xy)$, so for a Koszul algebra we have the formula $A(z)P_A(-z) = 1$. This formula is in fact *equivalent* to A being Koszul, cf. [L].

We sum up with a theorem

Definition-Theorem 1 *A graded algebra A is Koszul if and only if the following equivalent conditions are satisfied*
i) $\mathrm{Tor}^A_i(k,k) = (\mathrm{Tor}^A_i(k,k))_i$ *for all i.*
ii) $\mathrm{Ext}^i_A(k,k) = (\mathrm{Ext}^i_A(k,k))_i$ *for all i.*
iii) The minimal graded A-resolution of k is linear.
iv) $\mathrm{Ext}_A(k,k) = [\mathrm{Ext}^1_A(k,k)]$.
v) $A(z)P_A(-z) = 1$.
vi) $A(xy)\mathbf{P}_A(-x,y) = 1$.
vii) $P_A(z) = A^!(z)$.
viii) $\mathbf{P}_A(x,y) = A^!(xy)$.
ix) A is quadratic and $L(A)$ is distributive.
x) A is quadratic and $L_j(A)$ are distributive for all j.
xi)-xx) A is quadratic and $A^!$ satisfies any of the above conditions.
xxi) A is quadratic and the complex U is exact.

It follows that a Koszul algebra has a finitely generated Ext-algebra and rational Poincaré series.

There are some further more technical characterizations of Koszul algebras. The algebra A is Koszul if and only if the map $A \longrightarrow A/A^2_+$ is "small", cf. [BF1]. In [L3] there are conditions (for a commutative algebra A) in terms of a "minimal model" for A and of the "homotopy Lie algebra" of A.

3 Examples of Koszul Algebras

We will now give some examples of classes of Koszul algebras. Since Koszul algebras are necessarily quadratic, we will in this section assume that all algebras are of the form $k\langle x_1, \ldots, x_n \rangle / I$, where I is generated by homogeneous elements of degree two.

3.1 Commutative Examples

We start with commutative algebras. If A is a complete intersection, i.e., $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ where $(f_1, \ldots, f_r)$ is a regular sequence (of forms of degree two), then A is Koszul. The exact sequences

$$0 \longrightarrow k[\mathbf{x}]/(f_1, \ldots, f_{i-1}) \xrightarrow{f_i \cdot} k[\mathbf{x}]/(f_1, \ldots, f_{i-1}) \longrightarrow k[\mathbf{x}]/(f_1, \ldots, f_i) \longrightarrow 0$$

easily give that $A(z) = (1 - z^2)^r/(1 - z)^n$. The Poincaré series of a complete intersection was determined in [T], $P_A(z) = (1+z)^n/(1-z^2)^r$. (Tate considered local rings, but the same arguments can be applied.) Since $A(z)P_A(-z) = 1$, A is Koszul.

If $A = k[x_1, \ldots, x_n]/I$ and I is generated by an arbitrary set of monomials (of degree two), then A is Koszul, cf. [F1]. A concrete example: If $A = k[x_1, \ldots, x_n]/(x_i x_j \; ; \; i \neq j)$ then $A(z) = 1 + nz + nz^2 + \cdots = (1+(n-1)z)/(1-z)$, so we can conclude that $P_A(z) = (1 + z)/(1 - (n - 1)z)$. There are interesting classes of quadratic monomial ideals coming from combinatorics. If P is a poset on $\{x_1, \ldots, x_n\}$, the associated Stanley-Reisner ring $k[P] = k[\mathbf{x}]/(x_i x_j \; ; \; x_i \not\leq x_j, x_j \not\leq x_i)$ is Koszul. When studying general Stanley-Reisner rings $k[\Delta]$ one can sometimes reduce problems to the barycentric subdivision Δ' of Δ, and $k[\Delta']$ is Koszul.

The result above on monomial ideals was extended in [Ko] to algebras $k[\mathbf{x}]/I$, where I is generated by monomials and certain binomials.

If the (finite) $k[\mathbf{x}]$-resolution of I is linear, i.e., if $(\mathrm{Tor}_i^{k[\mathbf{x}]}(A, k))_j = 0$ if $j \neq i + 1$ for all $i \geq 1$ $(\mathrm{Tor}_1^{k[\mathbf{x}]}(A, k) = (\mathrm{Tor}_1^{k[\mathbf{x}]}(A, k))_2$ is always true since A is quadratic), then A is Koszul, cf. [BF1]. If A is CM then e.dim$A \leq e(A) + \dim A - 1$, where $e(A)$ is the multiplicity or degree of A. If there is equality, A is said to be of *maximal embedding dimension* or *minimal multiplicity.* CM algebras of maximal embedding dimension have linear resolutions. We give some concrete examples. If (x_{ij}) is a $2 \times n$-matrix of indeterminates and I the ideal in $k[x_{ij}]$ generated by all maximal minors in (x_{ij}), then $k[x_{ij}]/I$ has a linear resolution. If (x_{ij}) is a *symmetric* 3×3-matrix of indeterminates and I the ideal of 2×2-minors then $k[x_{ij}]/I$ is another example, cf. [Sc] and [F2] and the references in them. There are more examples of CM rings with linear resolutions in [FLa].

Extremal Gorenstein rings were introduced in [Sc]. These are Koszul algebras (if they are quadratic), cf. [F2]. A concrete example is $k[x_{ij}]/I$ where I is generated by the 4×4-Pfaffians in a skew-symmetric 5×5-matrix (x_{ij}). If I is the ideal of 2×2-minors in a 3×3-matrix (x_{ij}) we get another example of a quadratic extremal Gorenstein ring.

If $f_1, \ldots, f_r$ are "generic" quadratic forms in $k[x_1, \ldots, x_n]$, then we have that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ is Koszul if and only if $r \leq n$ or $r \geq \binom{n+1}{2} - [n^2/4]$, cf [L2].

If $A = \oplus_{i \geq 0} A_i$ is a graded algebra, the dth *Veronese subalgebra* is $A^{(d)} = \oplus_{i \geq 0} A_{id}$. An element in A_{id} is considered to have degree i. If $\oplus_{i \geq 0} A_i$ and $\oplus_{i \geq 0} B_i$ are graded algebras their *Segre product* is $A \circ B = \oplus_{i \geq 0} A_i \otimes_k B_i$. An element in $A_i \otimes B_i$ has degree i. It is shown in [BM] that, if one starts with a polynomial ring and performs a finite number of Segre products and Veronese subalgebras in any order, the result will be a Koszul algebra. There is a partial generalization to "weighted" Segre products in [C].

A classical theorem in geometry states that any projective variety can be cut out by quadrics (cf. [Mu], but the result is certainly older). This was generalized in [BF1] where it is shown that for any graded algebra A (commutative or not) we have $A^{(d)}$ quadratic if $d >> 0$ (an actual bound is given). In [B3] it is shown that if A is commutative, then $A^{(d)}$ is even Koszul if $d >> 0$. This was sharpened in [ERT]and [BGT], see below.

3.2 Examples from Geometry

There are several examples of coordinate rings of projective varieties that are Koszul. In [Ke1] it is shown that any algebra A with straightening law whose discrete algebra is defined by quadratic monomials is Koszul (called wonderful in [Ke1]). In [VF] it is shown that the coordinate ring of a general curve of genus ≥ 5 is Koszul. There are more examples in [Ke2], [Pa], [Pol], [PP], and [Ra].

3.3 Noncommutative Examples

If I is generated by an arbitrary set of monomials (of degree two) in $k\langle \mathbf{x} \rangle$, then $k\langle \mathbf{x} \rangle / I$ is Koszul, cf. [F1]. A slight generalization of this result was used in [BHV] to calculate the number of walks of certain kinds in a directed graph.

If $(f_1, \ldots, f_r)$ is a sequence of homogeneous elements of degree two in $k[\mathbf{x}]$ then $k[\mathbf{x}]/(f_1, \ldots, f_r)(z) \geq (1-z^2)^r/(1-z)^n$ with equality if and only if the sequence is regular. There is a corresponding property in $k\langle x_1, \ldots, x_n \rangle$. If $(f_1, \ldots, f_r)$ is a sequence of homogeneous elements of degree two in $k\langle x_1, \ldots, x_n \rangle$ then $k\langle \mathbf{x} \rangle/(f_1, \ldots, f_r)(z) \geq 1/(1 - nz + rz^2)$, cf. [An2]. If there is equality the sequence is called *strongly free*. That $(f_1, \ldots, f_r)$, $r \geq 1$, is strongly free is equivalent to $\mathrm{gl.dim}(k\langle \mathbf{x} \rangle/(f_1, \ldots, f_r)) = 2$. This is shown in [An2], and then it follows easily that these rings are Koszul.

The results above can also be formulated in a relative situation. If f is a quadratic form in a commutative algebra A, then $A/(f)(z) \geq (1-z^2)A(z)$ with equality if and only if f is a nonzerodivisor in A. Similarly if f is a quadratic form in any graded algebra A, then $A/(f)(z) \geq A(z)/(1 + z^2 A(z))$, and f is called strongly free in A if there is equality. We will use this below.

4 Gröbner Bases and Koszul Algebras

If $A = k\langle \mathbf{x} \rangle / I$ and I has a (noncommutative) Gröbner basis consisting of elements of degree two, then A is Koszul. This follows from the spectral sequence $\mathrm{Tor}^{k\langle \mathbf{x} \rangle/\mathrm{in}(I)}(k,k) \Longrightarrow \mathrm{Tor}^{k\langle \mathbf{x} \rangle/I}(k,k)$, where in$(I)$ denotes the ideal generated by the initial monomials of the elements in the Gröbner basis, cf. [An3]. There is a similar result in the commutative case, cf. [An3], so if I has a quadratic (commutative) Gröbner basis then $k[\mathbf{x}]/I$ is Koszul. Another proof of this fact could be found in [BHV]. As an example, the ideal I of 2×2-minors in a matrix (x_{ij}) has a quadratic Gröbner basis, cf. [St1], so $k[x_{ij}]/I$ is Koszul.

If A is any commutative graded algebra then $A^{(d)}$ has a quadratic Gröbner basis if $d >> 0$, cf. [ERT]. This improves the result in [B3] mentioned above. The result in [ERT] is further generalized to a larger class of algebras (not necessarily generated in degree one) in [BGT].

Let $A = k[x_1, \ldots, x_n]$. The subring of $A^{(d)}$ generated by the monomials $\{x_1^{i_1} \cdots x_n^{i_n} \ ; \ i_1 + \cdots + i_n = d, 0 \le i_1 \le s_1, \ldots, 0 \le i_n \le s_n\}$ is called an algebra of *Veronese type.* (If $s_1 = \cdots = s_n = d$ we get $A^{(d)}$.) It is shown in [St1] that the defining ideal of an algebra of Veronese type has a Gröbner basis (in a certain ordering) which is not only quadratic but also squarefree. This has interesting combinatorial consequences, cf. [St1].

5 Operations on Koszul algebras

The class of Koszul algebras is closed under a number of operations. The *coproduct* of two graded algebras A and B over k is the pushout $A \coprod B$ of A and B of $A \longleftarrow k \longrightarrow B$. The *(fibre) product* $A \prod B$ over k is the pullback of $A \longrightarrow k \longleftarrow B$. The following results are proved in [BF1].

Theorem 2 *i) A is Koszul if and only if the Veronese subalgebra $A^{(d)}$ is Koszul for all d.*
ii) If A and B are Koszul then the Segre product $A \circ B$ is Koszul.
iii) $A \coprod B$ is Koszul if and only if A and B are both Koszul.
iv) $A \prod B$ is Koszul if and only if A and B are both Koszul.
v) $A \otimes_k B$ is Koszul if and only if A and B are both Koszul.
vi) If f is strongly free of degree one or two, then A is Koszul if and only if $A/(f)$ is Koszul.
vii) If f is a nonzerodivisor of degree one or two in a commutative algebra A, then A is Koszul if and only if $A/(f)$ is Koszul.

6 More Examples and some Counterexamples

In this section we consider only quadratic algebras.

If $A = k\langle \mathbf{x} \rangle / I$ and I is principal, then A is Koszul, cf. [B2], but there are counterexamples already when I is generated by two elements. The ideals generated by two elements in $k\langle \mathbf{x} \rangle$ are classified up to isomorphisms in [B1], and all possible Hilbert and Poincaré series are determined. There is a small number of exceptions to Koszulness.

On the commutative side, probably the first counterexample to Koszulness (due to C. Lech) is the following. Let $f_1, \ldots, f_5$ be "generic" quadratic forms in $k[x_1, \ldots, x_4]$. (A concrete example is $(f_1, \ldots, f_5) = (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 + x_3x_4)$.) It is not so hard to see that if $A = k[x_1, \ldots, x_4]/(f_1, \ldots, f_5)$ then $A(z) = 1 + 4z + 5z^2$, so $1/A(-z) = 1 + 4z + 11z^2 + 24z^3 + 41z^4 + 44z^5 - 29z^6 - \cdots$. Since this series has negative coefficients, A is not Koszul.

In [Ro1], J.-E. Roos constructed an example of an algebra A for which $\operatorname{Ext}_A(k, k)$ is not finitely generated, and in [An1] D. Anick gave an example of an algebra with non-rational Poicaré series.

In embedding dimension two all commutative algebras are Koszul. In [BF2] all quadratic ideals in $k[x_1, x_2, x_3]$ generated by three elements and which are not complete intersections are classified up to isomorphisms and their Hilbert and double Poincaré series are determined. There are two exceptions to Koszulness (the ideals $(x_1^2, x_2x_3, x_1x_3 + x_2^2)$ and $(x_1x_2, (x_1 + x_2)x_3, x_1^2 + x_1x_3 + x_2^2)$) if $\operatorname{char}(k) \neq 3$ and one further (namely $(x_1^2, x_2x_3, x_1x_2 + x_1x_3 + x_2^2)$) if $\operatorname{char}(k) = 3$. It is shown in [BF2] that these are the only counterexamples in embedding dimension three to Koszulness.

J.-E. Roos has made an extensive study of homological properties of quadratic commutative algebras. He has found 83 different double Poincaré series in embedding dimension four and more than 4500 in embedding dimension five. Among the examples in embedding dimension four there are 37 Koszul algebras. In embedding dimension five there are about ten per cent Koszul algebras. There are lots of examples of the homological behaviour of commutative quadratic algebras given in [Ro4], [Ro5], [Ro6], and [Ro7].

If A is commutative and $\dim_k A_2 \leq 2$, then A is Koszul, cf. [B1], and the examples in [BF2] show that there are counterexamples if $\dim_k A_2 = 3$.

There are commutative Koszul algebras which do not have a quadratic Gröbner basis in any ordering, even after any linear change of coordinates. An example from [ERT]: Take three generic forms (of degree two) f_1, f_2, f_3 in $k[x_1, x_2, x_3]$. Since $k[x_1, x_2, x_3]/(f_1, f_2, f_3)$ is a complete intersection it is Koszul. If $I = (f_1, f_2, f_3)$ had a quadratic Gröbner basis, then $\operatorname{in}(I) = (x_1^2, x_2^2, x_3^2)$ since this is the only quadratic monomial ideal with correct Hilbert series. But if $x_1 > x_2 > x_3$, say, and $x_3^2 \in \operatorname{in}(I)$ then $x_3^2 \in I$. But it is easily seen that I does not contain any square. A concrete example is $I = (x_1^2 + x_1x_2, x_2^2 + x_2x_3, x_3^2 + x_1x_3)$, cf. [ERT].

Given the embedding dimension n of A it is natural to ask for a bound $N(n)$ such that if $\operatorname{Tor}_i^A(k, k) = (\operatorname{Tor}_i^A(k, k))_i$ for $i \leq N(n)$ then A is Koszul. That such a bound does not exist for noncommutative algebras follows from a result in [FLo]. As an example, if $A = k\langle x_1, x_2, x_3, x_4 \rangle/(x_1x_2 - x_1x_3, x_2x_3 - x_3x_2 - \lambda x_3^2, x_2x_4)$, $\operatorname{char}(k) = 0, \lambda^{-1} = l \in \mathbb{N}$, then $\operatorname{Tor}_i^B(k, k) = (\operatorname{Tor}_i^B(k, k))_i$ if $i \leq l+2$ and $\operatorname{Tor}_{l+3}^B(k, k) \neq (\operatorname{Tor}_{l+3}^B(k, k))_{l+3}$, where $B = A^!$. More surprising is perhaps that not even for commutative algebras such a bound exist. An example is given in [Ro2], $k[x_1, \ldots, x_6]/I$, where $I = (x_1^2, x_1x_2, x_2x_3, x_3^2, x_3x_4, x_4^2, x_4x_5, x_5x_6, x_6^2,$

$x_1x_3 + \lambda x_3x_6 - x_4x_6, x_3x_6 + x_1x_4 + (\lambda - 2)x_4x_6), \lambda \in \mathbb{N}$. Over this algebra k has a linear resolution up to and including degree λ, but not in degree $\lambda + 1$.

It has been conjectured that the equality $A(z)A^!(-z) = 1$ should imply that A is Koszul. Counterexamples to this is independently given in [Po] (noncommutative algebras) and [Ro3] (commutative algebras). One example in [Ro3] is $k[x_1, \ldots, x_5]/(x_1^2, x_1x_2 + x_3^2, x_3^2 + x_4x_5, x_1x_3, x_2x_3 + x_1x_4, x_3x_4 + x_2x_5, x_3x_5, x_5^2)$.

7 Local Rings

If (A, m) is a local ring and the associated graded algebra $g(A) = \oplus_{i \geq 0} m^i/m^{i+1}$ is a Koszul algebra, then A is called generalized Koszul. For a generalized Koszul algebra it is true that $g(A)(z)P_A(-z) = 1$, cf. [F3] or [HRW]. Concrete examples of generalized Koszul algebras are local CM rings of maximal embedding dimension. Another example is local Gorenstein rings (A, m) of maximal embedding dimension $e(A) + \dim A - 2$, cf. [Sc] and [F2].

8 Semigroup rings

Let S be a subsemigroup of $\mathbb{N}^d$ and let $k[S] = k[\mathbf{x}]/I(S)$ be its semigroup ring. In [HRW] a topological condition for the initial ideal $\mathrm{in}(I(S))$ to be quadratic is given.

In [PRS] it is shown that all normal subsemigroups of $\mathbb{N}^2$ give quadratic initial ideals (in some ordering) and thus are (generalized) Koszul algebras. This result is also proved in [HRW] in another way.

It has been asked if all monomial projective curves with quadratic defining ideal were Koszul, and even if they had a quadratic Gröbner basis. Sturmfels has recently shown that for projective monomial curves in $P^n, n \leq 4$ this is true, and for $n = 5$ there is only one exception (and its symmetric), namely $k[s^{11}, s^8t^3, s^6t^5, s^5t^6, s^4t^7, t^{11}]$, which is not Koszul, cf. [St2]. All Koszul algebras have quadratic Gröbner bases for $n = 5$.

In [RS] projective monomial curves in $P^n, n > 5$ are studied. It is shown that all such Koszul algebras have quadratic Gröbner bases if $n = 6$ and that there is a counterexample to this if $n = 7$. This counterexample is $k[t^{22}, t^{19}s^3, t^{18}s^4, t^{17}s^5, t^{16}s^6, t^{15}s^7, t^{11}s^{11}, s^{22}] \simeq k[x_1, \ldots, x_8]/(x_2^2 - x_1x_5, x_2x_3 - x_1x_6, x_3^2 - x_2x_4, x_3x_4 - x_2x_5, x_2x_6 - x_3x_5, x_4^2 - x_3x_5, x_1x_7 - x_4x_5, x_3x_6 - x_4x_5, x_4x_6 - x_5^2, x_2x_7 - x_6^2, x_1x_8 - x_7^2)$, cf. [RS]. There is also an example of a monomial curve in P^6 for which the Ext-algebra of its coordinate ring is not finitely generated, and an example in P^8 with non-rational Poincaré series.

9 Generalizations

The concept of Koszul algebras has been generalized (starting with [Ma]) to other tensor categories than k-algebras. A good general reference is [BGSo].

We will not discuss this subject, but only give some references: [Ar], [BGSc], [BG], [Be], [GK], [GM], [H], [M-V], [Pl], [PV], [PS], and [Rs].

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