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KOSZUL ALGEBRAS AND HYPERPLANE ARRANGEMENTS

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This is a survey to apply theory from noncommutative graded algebras to questions about the holonomy algebra and the Orlik-Solomon algebra of a hyperplane arrangement. We first recall the main properties of Koszul algebras and hyperplane arrangements. Then, we focus our interest on the class of hypersolvable arrangements which includes both the fiber-type and the generic arrangements. For these hypersolvable arrangements, the holonomy algebra is Koszul and koszulness of the Orlik-Solomon algebra characterizes the subclass of fiber-type's.

1. Koszul Duality and Koszul Algebras (overall)

Let k be an arbitrary field and let V be an n-dimensional k-vector space ($V \cong k^n$) and let $T(V) = \bigoplus_{n\geq 0} T_n$ be the k-tensor algebra over V where $T_0 \cong k$, $T_1 \cong V$. Then $T(V) \cong k < x_1, \ldots, x_n >$, the free associative k-algebra. Let A be a k-graded algebra, $A = \sum_{p\geq 0} A_p$. Assume that A is connected, i.e. $A_0 = k$ and is generated by A_1. A is "naturally" represented as the factor of the tensor algebra $T(A_1)$ by a homogeneous ideal $I = \sum_{p\geq 2} I_p$.

$$A \cong T(A_1)/I$$

Definition 1.1. A is said to be quadratic if I is generated by $I_2 \subset A_1 \bigotimes A_1$.

Therefore, a quadratic algebra A is determined by a vector space of generators $V{=}A_1$ and a subspace of quadratic relations $I_2 \subset V \bigotimes V$. Such a *quadratic* algebra is denoted as $A = \{V; I\}$.

Definition 1.2. Let $A = \{V; I\}$ be a quadratic algebra. The quadratic dual or Koszul dual algebra of A is defined by $A^! = \{V^*; I^\perp\}$, where V^* is the dual of V, $I^\perp \subset V^* \bigotimes V^*$ is the orthogonal complement to I with respect to the natural pairing: $\langle v \otimes v', v^* \otimes v'^* \rangle = \langle v, v^* \rangle \langle v', v'^* \rangle$ between $V \bigotimes V$ and $V^* \bigotimes V^*$.

Remark 1.1. $(A^!)^! = A$.

Example 1.1.

$$\begin{aligned} A = &\ k[x_1, \dots, x_n] \quad \text{(commutative polynomial algebra)} \\ = &\ k\langle x_1, \dots, x_n \rangle / \langle x_i x_j - x_j x_i \rangle \text{ for } i < j. \end{aligned}$$

Then $A \cong \mathcal{S}_n$ which is the symmetric algebra. A is a quadratic algebra.

$$\begin{aligned} A^! = &\ k\langle y_1, \dots, y_n \rangle / \langle y_k^2;\ y_i y_j - y_j y_i \rangle \text{ (for i < j)} \\ = &\ \bigwedge(y_1, \dots, y_n) \quad \text{(exterior algebra)}. \end{aligned}$$

Definition 1.3. Let A be a quadratric algebra,and ${}_Ak$ be the trivial graded left A-module A/A^+ where A^+ is the augmentation ideal $\bigoplus\limits_{p>0} A_p$. A is said to be Koszul if ${}_Ak$ admits a free graded resolution:

$$\cdots \to P^i \to P^{i-1} \to \cdots \to P^1 \to P^0 \to_A k \to 0,$$

where P^i is generated by its components of degree i.

Let denote the following objects:
$E(A) := \mathrm{Ext}_A^*({}_Ak, {}_Ak)$ the graded cohomology algebra of the trivial graded A-module ${}_Ak$.

The *Hilbert series* $H(A,t) := \sum\limits_{n\geq 0} \dim(A_n) t^n$.

The *Koszul complex* of A:

$$\cdots \to K_i \overset{d_i}{\to} K_{i-1} \to \cdots \to K_1 \to K_0 \to_A k \to 0,$$

where K_i free A-modules, $K_i = \mathrm{Hom}_k(A_i^!, A)$ and d_i is defined as $d_i f(a) = \sum\limits_{k=1}^{n} f(x_i a) e_i$, $a \in A_{i-1}^!$, $(x_1, \dots, x_n)$ is the basis of $A_1^!$, dual basis of the basis $(e_1, \dots, e_n)$ of A_1.

Theorem 1.1. *Let A be a quadratic algebra. Then the following assertions are equivalent:*

(1) A is Koszul;

(2) $A^!$ *is Koszul;*
(3) $E(A) = A^!$*;*
(4) The Koszul complex of A *is acyclic;*
(5) $H(A,t).H(E(A),-t) = 1$.

Corollary 1.1. A *is Koszul iff* $H(A,t).H(A^!,-t) = 1$.

Example 1.2. $A = k[x_1, \dots, x_n]$ is Koszul and $H(A,t) = \frac{1}{(1-t)^n}$, $A^! = \bigwedge(y_1, \dots, y_n)$ is Koszul and $H(A^!,t) = (1+t)^n$.

2. Hyperplane Arrangements

We refer the reader to [6] as a general reference on arrangements.
Let $\mathcal{A}$ be an arrangement of hyperplanes over $\mathbf{C}$ i.e. $\mathcal{A} = \{H_1, \dots, H_n\}$, where H_i are linear hyperplanes of $\mathbf{C}^l$. Define the complement $M(\mathcal{A}) = \mathbf{C}^l - \bigcup_{i=1}^{n} H_i$ and $L(\mathcal{A})$ the geometric lattice intersection of hyperplanes with reverse order

$$X \leq Y \text{ if } Y \subseteq X.$$

Notice that $\mathrm{rk}(X) = \mathrm{codim}(X)$.

Orlik-Solomon algebra (combinatorially defined)

$$A_k^*(\mathcal{A}) := \bigwedge(e_1, \dots, e_n)/\mathcal{J} \cong H^*(M(\mathcal{A}); k),$$

where $\mathcal{J}$ ideal generated by the relations of the form:

$$\sum_{j=1}^{s} (-1)^{i-1} e_{i_1} \dots \widehat{e}_{i_j} \dots e_{i_s}$$

for all $1 \leq i_1 < \dots < i_s \leq n$ such that $\mathrm{rk}(H_{i_1} \cap \dots \cap H_{i_s}) < s$.
$A_k^*(\mathcal{A})$ is not necessary quadratic.

Poincaré polynomial

$$\begin{aligned} P(\mathcal{A},t) &:= \sum_i \mathrm{dim} A_k^i(\mathcal{A}) t^i \\ &= \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\mathrm{codim}(X)} \\ &= P(A_k^*(\mathcal{A}), t) \end{aligned}$$

Quadratic Orlik-Solomon algebra

$$\overline{A_k^*}(\mathcal{A}) := \bigwedge(e_1, \dots, e_n)/\overline{\mathcal{J}}$$

where $\overline{\mathcal{J}}$ ideal generated by:

$$e_ie_j - e_ie_k + e_je_k, \ 1 \leq i < j < k \leq n,$$

and

$$\mathrm{rk}(H_i \cap H_j \cap H_k) = 2.$$

$\overline{A_k^*}(\mathcal{A})$ only depends on $L_2(\mathcal{A})$, the elements of codimension 2 of $L(\mathcal{A})$.

Quadratic Poincaré polynomial

$$\overline{P}(\mathcal{A}, t) := P(\overline{A_k^*}(\mathcal{A}), t)$$

Example 2.1. Braid arrangements in $\mathbf{C}^l$.

$\mathcal{A}_l = \{H_{ij} \mid 1 \leq i < j \leq l\}$, where $H_{ij} = \ker(z_i - z_j)$.

Notice that the fundamental group of the complement is isomorphic to the Pure braid group P_l.

Moreover, $A_k^*(\mathcal{A}_l) = \overline{A_k^*}(\mathcal{A}_l)$, the Orlik-Solomon algebra of a braid arrangement is quadratic.

Remark 2.1. There is a linear fibration given by forgetting the last coordinates:

$$\mathbf{C} - \{(l-1) \text{ points}\} \hookrightarrow M(\mathcal{A}_l) \longrightarrow M(\mathcal{A}_{l-1})$$

where $M(\mathcal{A}_l)$ is the complement of the braid arrangement in $\mathbf{C}^l$.

$$\begin{array}{ccc} \mathbf{C}^l & \longrightarrow & \mathbf{C}^{l-1} \\ \cup & & \cup \\ M(\mathcal{A}_l) & \longrightarrow & M(\mathcal{A}_{l-1}) \end{array}$$

Remark 2.2. The Coxeter arrangements $\mathcal{D}_l$, $l \geq 4$ in $\mathbf{C}^l$ are defined by $\{(x_i - x_j), (x_i + x_j), \ 1 \leq i < j \leq n\}$. Then $A_k^*(\mathcal{D}_l) \neq \overline{A_k^*}(\mathcal{D}_l)$, and the Orlik-Solomon algebra is not quadratic.

As a "natural" generalization of braid arrangements, we define the **fiber-type** arrangements.

Definition 2.1 (Falk,Randell). *[1] $\mathcal{A} = \{0\}$ is a fiber-type arrangement in $\mathbf{C}$. The arrangement $\mathcal{A}$ in $\mathbf{C}^l$ is fiber-type if it is strictly linearly fibered with base $M(\mathcal{B})$ the complement of the fiber-type arrangement $\mathcal{B}$ in $\mathbf{C}^{l-1}$.*

Then $\mathcal{A}$ is a fiber-type arrangement iff there is a composition series

$$\mathcal{A}_1 \subset \ldots \subset \mathcal{A}_i \subset \mathcal{A}_{i+1} \subset \ldots \subset \mathcal{A}_l = \mathcal{A}$$

where $\mathrm{rk}\mathcal{A}_1 = 1$ and $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ defines a linear fibration

$$\mathbf{C} - \{|\mathcal{A}_{i+1} - \mathcal{A}_i| \text{ points}\} \hookrightarrow M(\mathcal{A}_{i+1}) \longrightarrow M(\mathcal{A}_i).$$

$\mathcal{A}$ is fiber-type iff the lattice $L(\mathcal{A})$ is *supersolvable.*

Proposition 2.1. *Let $\mathcal{A}$ be a fiber-type arrangement. Then*

$$A_k^*(\mathcal{A}) = \overline{A_k^*}(\mathcal{A}).$$

2.1. *Holonomy Lie Algebra*

Let $\mathrm{Lib}_k(\mathcal{A})$ be the k-graded free Lie algebra over $\{x_1, \ldots, x_n\}$.

Definition 2.2 (Kohno). *[4] The* ***holonomy Lie*** *algebra of $\mathcal{A}$ is denoted $\mathcal{G}_k(\mathcal{A})$:*

$$\mathcal{G}_k(\mathcal{A}) := \mathrm{Lib}_k(\mathcal{A})/\mathcal{N},$$

where $\mathcal{N}$ ideal generated by $\left[x_{i_k}, \sum_{j=1}^{s} x_{i_j}\right]$ for $k = 1, \cdots, s$, $1 \leq i_1 < \cdots < i_s \leq n$ such that rk $\bigcap_{j=1}^{s} H_{i_j} = 2$ and it is maximal with this property.

Remark 2.3. Let $H_*\big(M(\mathcal{A})\big)$ be the homology coalgebra with coefficients in k and comultiplication dual to the cup product,

$$\mathcal{G}_k(\mathcal{A}) \cong \mathrm{Lib}_k\big(H_1(M(\mathcal{A}))/\ker\Big(H_2(M(\mathcal{A})) \to \wedge^2 H_1\big(M(\mathcal{A})\big)\Big).$$

Definition 2.3. The **holonomy algebra** of $\mathcal{A}$ denoted $\mathcal{U}_k(\mathcal{A})$ is the universal enveloping algebra of $\mathcal{G}_k(\mathcal{A})$.

A holonomy algebra is a quadratic algebra.

Lemma 2.1. $\mathcal{U}_k(\mathcal{A}) \cong \big(\overline{A_k^*}(\mathcal{A})\big)^!.$

In the following, we will study a "large" class of arrangements for which $\mathcal{U}_k(\mathcal{A})$ is Koszul. However, let us give an example showing that this result is not always true.

Example 2.2. Let $\mathcal{A}$ be the arrangement defined by the linear forms $x, y, z, x+y, x+z, y+z$. Then $P(\mathcal{A}, t) = 1 + 6t + 12t^2 + 7t^3$ and $\overline{P}(\mathcal{A}, t) = 1 + 6t + 12t^2 + 8t^3 + t^4$. The holonomy algebra $\mathcal{U}_k(\mathcal{A})$ is not Koszul because $(1 - 6t + 12t^2 - 8t^3 + t^4)^{-1}$ has some negative coefficients (eg t^{13}).

Remark 2.4. Let $\left(\Gamma_n\pi_1(M(\mathcal{A}))\right)_{n\geq 1}$ be the Lower Central Series of the fundamental group, defined as follows:

(1) $\Gamma_1\pi_1(M(\mathcal{A})) = \pi_1(M(\mathcal{A}))$,
(2) $\Gamma_{n+1}\pi_1(M(\mathcal{A})) = [\pi_1(M(\mathcal{A})), \Gamma_n\pi_1(M(\mathcal{A}))]$,

$$\mathrm{gr}^*_\Gamma\left(\pi_1(M(\mathcal{A}))\right) := \bigoplus_{i\geq 1} \mathrm{gr}^i_\Gamma\left(\pi_1(M(\mathcal{A}))\right),$$

where $\mathrm{gr}^i_\Gamma\left(\pi_1(M(\mathcal{A}))\right) = \Gamma_{i+1}\pi_1(M(\mathcal{A}))/\Gamma_i\pi_1(M(\mathcal{A}))$.

Then as graded Lie algebras

$$\mathcal{G}^*_{\mathbf{Q}}(\mathcal{A}) \cong \mathrm{gr}^*_\Gamma(\pi_1(X)) \bigotimes \mathbf{Q}.$$

2.2. *Hypersolvable Arrangements*

This is a "large" class of arrangements containing both, the fiber-type ones (whose the complement is a $K[\pi, 1]$-space), and the generic ones (whose the complement is never a $K[\pi, 1]$-space).

Let $\mathcal{B}$ be a subarrangement of $\mathcal{A}$, denote $\overline{\mathcal{B}} = \mathcal{A} - \mathcal{B}$. In the following definition, we denote $\mathrm{rk}(\alpha, \beta)$ as $\mathrm{rk}(H_\alpha \cap H_\beta)$, where $H_\alpha = \ker\alpha$ and $H_\beta = \ker\beta$.

Definition 2.4. [2] $(\mathcal{A}, \mathcal{B})$ is said to be a **solvable extension** if

(1) For any $H_\alpha, H_\beta \in \mathcal{B}$ and any $H_a \in \overline{\mathcal{B}}$, then $\mathrm{rk}(\alpha, \beta, a) = 3$.
(2) Given $H_a, H_b \in \mathcal{B}$, $a \neq b$, there exists $H_\gamma \in \mathcal{B}$ such that $\mathrm{rk}(a, b, \gamma) = 2$. Denote $\gamma = f(a, b)$.
(3) Given distinct elements $H_a, H_b, H_c \in \overline{\mathcal{B}}$, then
$$\mathrm{rk}\big(f(a,b), f(b,c), f(c,a)\big) = 2.$$

Then we can distinguish 2 cases:

(1) There is a fibration:

$$\mathbf{C} - \{|\mathcal{A} - \mathcal{B}| \text{ points}\} \hookrightarrow M(\mathcal{A}) \longrightarrow M(\mathcal{B})$$

called the *fibered case.*
(2) $\mathrm{rk}\mathcal{B} = \mathrm{rk}\mathcal{A}$, called the *singular case.*

In case 2, there exists a *deformation* such that we eliminate singular case in order to get a fibration as in case 1.

Definition 2.5. [2,3] $\mathcal{A}$ is said to be **hypersolvable** if there is a composition series

$$\mathcal{A}_1 \subset \ldots \subset \mathcal{A}_i \subset \mathcal{A}_{i+1} \subset \ldots \subset \mathcal{A}_{l(\mathcal{A})} = \mathcal{A}$$

with $\mathrm{rk}(\mathcal{A}_1) = 1$ and $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ is a solvable extension.

Then, after "enough" deformations, we can eliminate all the fibered cases and we get $\widetilde{\mathcal{A}}$ which is a fiber-type arrangement with the same lattice up to rank 2.

Remark that both fiber-type and generic arrangements are hypersolvable. The arrangement defined (in the Example 2.2) by the defining equation:

$xyz(x+y)(x+z)(y+z) = 0$ is not hypersolvable.

The arrangement defined by the following equation:

$(x+y)(x-y)(z+2y)(z+y)(z-y)(z-2y)z = 0$ is hypersolvable but neither fiber-type nor generic.

The Orlik-Solomon algebra of a hypersolvable arrangement is not necessary quadratic.

Theorem 2.1 (Jambu, Papadima). *[2] Let $\mathcal{A}$ be a hypersolvable arrangement; then $\overline{A_k^*}(\mathcal{A})$ is Koszul, for any field k; therefore $\mathcal{U}_k(\mathcal{A})$ is Koszul.*

Corollary 2.1 (Jambu, Papadima). *[2] Let $\mathcal{A}$ be a fiber-type arrangement; then $A_k^*(\mathcal{A})$ is Koszul.*

Sketch of the (algebraic) proof of the theorem:

(1) $H^*(\bigvee_m S^1; k) \cong T(V)$ is Koszul where $V \cong k^m$.

(2) Suppose $(\mathcal{A}, \mathcal{B})$ solvable, then as $A_k^*(\mathcal{B})$-modules

$$\overline{A_k^*}(\mathcal{A}) \cong \overline{A_k^*}(\mathcal{B}) \otimes H^*(\bigvee_{|\overline{\mathcal{B}}|} S^1; k).$$

(3) Recall that a graded subalgebra B^* of A^* is *normal* if $AB^+ = B^+A$. Then there is a canonical graded algebra projection $\pi : A \to F = A/AB^+$ $(B \hookrightarrow A \to F)$.

(4) **Lemma :** *Suppose B normal subalgebra of A such that A is free as a right B-module and all algebras are quadratic. If B and F are Koszul, then A is Koszul.*

(5) Let point out that $\overline{A_k^*}(\mathcal{B})$ is normal in $\overline{A_k^*}(\mathcal{A})$ which is a free right $\overline{A_k^*}(\mathcal{B})$-module, the quotient $F \cong H^*(\bigvee_{|\overline{\mathcal{B}}|} S^1; k)$ is Koszul. Then $\overline{A_k^*}(\mathcal{A})$ is Koszul if $\overline{A_k^*}(\mathcal{B})$ is Koszul.

Another proof is given using Shelton and Yusvinsky's result [8] saying that the Orlik-Solomon algebra of a fiber-type arrangement is Koszul altogether with the deformations from $\mathcal{A}$ to a fiber-type arrangement $\widetilde{\mathcal{A}}$.

Theorem 2.2 (Jambu, Papadima). *[2](Generalized LCS Formula) Let $\mathcal{A}$ be a hypersolvable arrangement; then for any field k:*

$$\overline{P}(\mathcal{A},-t)=\prod_{i=1}^{\infty}(1-t^i)^{\dim \mathcal{G}_k^i(\mathcal{A})}.$$

Proof. Recall that for all arrangements $\mathcal{U}_k(\mathcal{A})\cong\left(\overline{A_k^*}(\mathcal{A})\right)^!$.

Compute the inverse of the Hilbert series of $\mathcal{U}_k(\mathcal{A})$ by the well-known Poincaré-Birkhoff-Witt theorem:

$$H(\mathcal{U}_k(\mathcal{A}),t)^{-1}=\prod_{i=1}^{\infty}(1-t^i)^{\dim \mathcal{G}_k^i(\mathcal{A})}.$$

$\overline{A_k^*}(\mathcal{A})$ is Koszul, then $H(\overline{A_k^*}(\mathcal{A}),t).H\left((\overline{A_k^*}(\mathcal{A}))^!,-t\right)=+1.$ □

Corollary 2.2. *(LCS Formula) Let $\mathcal{A}$ be a fiber-type arrangement. Then*

$$P(\mathcal{A},-t)=\prod_{i=1}^{\infty}(1-t^i)^{\dim \mathcal{G}_k^i(\mathcal{A})}.$$

Remark 2.5. Kohno obtained this result for braid arrangements.

Theorem 2.3 (Jambu, Papadima). *[2] Let $\mathcal{A}$ be a hypersolvable arrangement. Then*

$$\mathcal{G}_{\mathbf{Z}}^*(\mathcal{A})\cong \mathrm{gr}_\Gamma^*\Big(\pi_1\big(M(\mathcal{A})\big)\Big)$$

as graded Lie algebras.

The main point of the proof is to show that $\mathcal{G}_{\mathbf{Z}}^*(\mathcal{A})$ is torsion-free as a graded abelian group.

Definition 2.6. $\mathcal{A}$ is said to be a **rational** $\mathrm{K}[\pi,1]$-arrangement if the $\mathbf{Q}$-completion of $M(\mathcal{A})$, denoted $\mathbf{Q}_\infty\big(M(\mathcal{A})\big)$, is aspheric.

Equivalently, $\mathcal{A}$ is rational $\mathrm{K}[\pi,1]$ iff the 1-minimal model $\mathcal{M}$ of $M(\mathcal{A})$ satisfies $f^*:H^*(\mathcal{M})\longrightarrow H^*(M(\mathcal{A}),\mathbf{Q})$ is an isomorphism.

Theorem 2.4 (Papadima, Yuzvinsky). *[7] $\mathcal{A}$ is rational $K[\pi,1]$ iff $H^*\big(M(\mathcal{A});\mathbf{Q}\big),(\cong A_{\mathbf{Q}}^*(\mathcal{A}))$ is a Koszul algebra and the LCS formula holds.*

Theorem 2.5 (Jambu, Papadima). *[2] Let $\mathcal{A}$ be a hypersolvable arrangement. Then the following assertions are equivalent:*

(1) $\mathcal{A}$ is fiber-type.
(2) $\mathbf{Q}_\infty(M(\mathcal{A}))$ is aspheric. (ie $A^*_{\mathbf{Q}}(\mathcal{A})$ is Koszul)
(3) The LCS formula holds.

Let us remark that if $\mathcal{A}$ is hypersolvable, then $A^*(\mathcal{A})$ is quadratic iff $\mathcal{A}$ is fiber-type. Therefore for the hypersolvable arrangements, quadraticity of $A^*_{\mathbf{Q}}(\mathcal{A})$ is equivalent to being fiber-type so is equivalent to being rational $K[\pi, 1]$.
Question : Is quadraticity of $A^*_{\mathbf{Q}}(\mathcal{A})$ sufficient for $M(\mathcal{A})$ being rational $K[\pi, 1]$?

Example 2.3. $\mathcal{A}$ in $\mathbf{C}^3$ given by the following linear forms $(x, y, z, x+y, x-z, y-z, x+y-2z)$. $A^*(\mathcal{A})$ is quadratic (but $\mathcal{A}$ is not hypersolvable and therefore not fiber-type). $P(\mathcal{A}, t) = (1+t)(1+6t+10t^2)$. If $\mathcal{A}$ is rational $K[\pi, 1]$, then $A^*(\mathcal{A})$ is Koszul and $P(\mathcal{A}, -t)^{-1} = H(\mathcal{U}(\mathcal{A}), t)$, therefore $(1-6t+10t^2)^{-1}$ as an infinite formal series has its coefficients integer and nonnegative which implies that $1-6t+10t^2$ has a real root $r \in (0, 1]$ (interesting exercise for undergraduate students following a course on Complex Analysis). Hence we obtain a contradiction.

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