EXERCISE SESSION 1

Problem 1: The goal is to show that for any digraph/binary relation $D \subseteq [n] \times [n]$, and $A_D = \Bbbk\langle x_1, \ldots, x_n\rangle / (x_i x_j : i \to j \text{ not in } D)$, that $\mathrm{Hilb}(A_D, t)$, $\mathrm{Hilb}(A_D, t, q)$ are rational,

where
$$\mathrm{Hilb}(A_D, t) := \sum_{\substack{\text{walks} \\ i_1 \to \cdots \to i_\ell \\ \text{in } D}} t^\ell$$

$$\mathrm{Hilb}(A_D, t, q) := \sum_{\substack{\text{walks} \\ i_1 \to \cdots \to i_\ell \\ \text{in } D}} t^\ell q_{i_1} \cdots q_{i_\ell}$$

Define $n \times n$ transfer matrices T_D, $T_D(q)$ by

$$(T_D)_{i,j} := \begin{cases} 1 & \text{if } i \to j \text{ in } D \\ 0 & \text{otherwise.} \end{cases} \qquad (T_D(q))_{i,j} := \begin{cases} q_j & \text{if } i \to j \text{ in } D \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show $\left(T_D^{m-1}\right)_{ij} = \#\{\text{walks } i = i_1 \to \cdots \to i_m = j \text{ in } D\}$

and $q_i\left(T_D(q)^{m-1}\right)_{ij} = \sum_{\substack{\text{walks} \\ i = i_1 \to \cdots \to i_m = j \text{ in } D}} q_{i_1} \cdots q_{i_m}$

(b) Show $t \cdot \left(I_n - t T_D\right)^{-1}_{ij} = \sum_{\substack{\text{walks} \\ i = i_1 \to \cdots \to i_m = j \\ \text{in } D}} t^m$

and $t q_i \cdot \left(I_n - t T_D(q)\right)^{-1}_{ij} = \sum_{\substack{\text{walks} \\ i = i_1 \to \cdots \to i_m = j \text{ in } D}} t^m q_{i_1} \cdots q_{i_m}$

(c) Show $\mathrm{Hilb}(A_D, t) = 1 + t\, \mathbb{1}_n^T \left(I_n - t T_D\right)^{-1} \mathbb{1}_n$

where $\mathbb{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$

and $\mathrm{Hilb}(A_D, t, q) = 1 + t\,[q_1 \cdots q_n]\left(I_n - t T_D(q)\right)^{-1} \mathbb{1}_n$.

(d) Explain why (c) implies

$\mathrm{Hilb}(A_D, t)$, $\mathrm{Hilb}(A_D, t, q)$ lie in $\mathbb{Q}(t)$, $\mathbb{Q}(t, q_1, \ldots, q_n)$.

Problem 2: (Euler-Poincaré)

(a) Explain why a short exact sequence of (finite-dimensional) $\mathbb{k}$-vector spaces

$$0 \to V^2 \to V^1 \to V^0 \to 0$$

implies

$$\dim_{\mathbb{k}} V^0 - \dim_{\mathbb{k}} V^1 + \dim_{\mathbb{k}} V^2 = 0$$

(b) Explain why more generally, any exact sequence

$$0 \to V^\ell \to V^{\ell-1} \to \cdots \to V^2 \to V^1 \to V^0 \to 0$$

implies $$\sum_{i=0}^{\ell} (-1)^i \dim_{\mathbb{k}} V^i = 0$$

(c) Explain why an exact sequence of $\mathbb{N}$-graded vector spaces $V^i = \bigoplus_{d=0}^{\infty} (V^i)_d$ and homogeneous maps

$$\cdots \to V^2 \to V^1 \to V^0 \to 0 \quad \text{with } (V^i)_j = 0 \text{ for } j < i$$

implies $$\sum_{i=0}^{\infty} (-1)^i \operatorname{Hilb}(V^i, t) = 0$$

where $\operatorname{Hilb}(V, t) := \sum_{d=0}^{\infty} \dim_{\mathbb{k}}(V_d) \cdot t^d$ as usual.

Problem 3: Let A be a Koszul algebra.

(a) Explain why exactness of Priddy's resolution of $\Bbbk$ built on $A \otimes (A^!)^*$

$$\cdots \to A \otimes (A^!_2)^* \to A \otimes (A^!_1)^* \to A \otimes (A^!_0)^* \to \Bbbk \to 0$$

implies $\mathrm{Hilb}(A,t) \cdot \mathrm{Hilb}(A^!, -t) = 1$.

(b) Defining $a_d := \dim_{\Bbbk}(A_d)$, $a^!_d := \dim_{\Bbbk}(A^!_d)$

show that $a_0 = a^!_0 = 1$ and

$a_1 = a^!_1 = n$ (= # of $x_1, \ldots, x_n$ or $y_1, \ldots, y_n$)

and $\forall d \geq 1$, $a^!_d - a_1 a^!_{d-1} + a_2 a^!_{d-2} - a_3 a^!_{d-3} + \cdots \pm a_d = 0$

that is, $\displaystyle\sum_{i=0}^{d} (-1)^i a_i \cdot a^!_{d-i} = 0$

(c) Show $a^!_2 = a_1^2 - a_2 = \det \begin{bmatrix} a_1 & a_2 \\ 1 & a_1 \end{bmatrix}$

$a^!_3 = a_1^3 - 2a_1 a_2 + a_3 = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & a_1 & a_2 \\ 0 & 1 & a_1 \end{bmatrix}$

and $a^!_d = \det \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_d \\ 1 & a_1 & a_2 & a_3 & \vdots \\ 0 & 1 & a_1 & \ddots & a_3 \\ \vdots & \ddots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & 1 & a_1 \end{bmatrix}$

Problem 4:

(a) Prove that $A = \Bbbk\langle x\rangle/(x^3)$ has no linear graded free A-resolution of $\Bbbk$, but does have this nice (nonlinear) periodic one:

$$\cdots \to \underset{F_5}{\underset{\|}{A(-7)}} \xrightarrow{[x]} \underset{F_4}{\underset{\|}{A(-6)}} \xrightarrow{[x^2]} \underset{F_3}{\underset{\|}{A(-4)}} \xrightarrow{[x]} \underset{F_2}{\underset{\|}{A(-3)}} \xrightarrow{[x^2]} \underset{F_1}{\underset{\|}{A(-1)}} \xrightarrow{[x]} \underset{F_0}{\underset{\|}{A}} \to \Bbbk \to 0$$

(b) When A has an $\mathbb{N}$-graded free A-resolution of $\Bbbk$

$$\cdots \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow \Bbbk \to 0$$

with $F_i = \bigoplus_{j \geq 0} A(-j)^{\beta_{ij}}$, and all d_i entries in A_+, define its Poincaré series $\mathrm{Poin}(A,t,u) := \sum_{i,j=0}^{\infty} \beta_{ij} u^i t^j \in \mathbb{Z}[[u,t]]$

and show $\mathrm{Hilb}(A,t) \cdot \mathrm{Poin}(t,-1) = 1$.

(c) Show $A = \Bbbk\langle x\rangle/(x^3)$ has

$$\mathrm{Hilb}(A,t) = \frac{1-t^3}{1-t} \quad \text{and} \quad \mathrm{Poin}(A,t,u) = \frac{1+ut}{1-u^2t^3}$$

and verify the equation $\mathrm{Hilb}(A,t) \cdot \mathrm{Poin}(t,-1) = 1$