EXERCISE SESSION 2

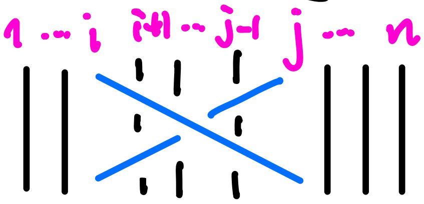
Problem 5: Let $A := \Bbbk\langle x_1, \dots, x_n \rangle$.

(a) Show $\mathrm{Hilb}(A,t) = \dfrac{1}{1-nt}$.

(b) Explain why A is Koszul.

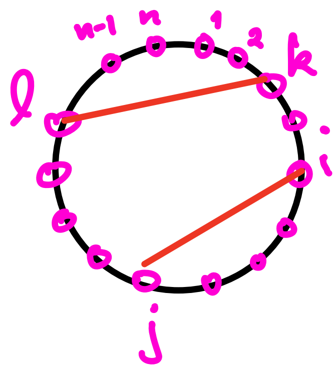
(c) What is $A^!$ here, and $\mathrm{Hilb}(A^!,t)$?

Problem 6: Recall that the Birman-Ko-Lee dual presentation of the braid group Br_n uses the generators $\left\{ \sigma_{ij} = \right.$



$\left. : 1 \le i < j \le n \right\}$.

Explain why $\sigma_{13}\sigma_{24} \neq \sigma_{24}\sigma_{13}$ in Br_4, and hence why their presentation only includes commutation relations $\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij}$ for i,j,k,l distinct when the arcs ij, kl are noncrossing on the circle:



Problem 7:

The goal here is to illustrate a bit of the subtlety of quadratic algebras, Koszulity and Gröbner bases, even when working with principal ideals, but over exterior algebras.

Consider the exterior and polynomial algebras

$$\Lambda := \Lambda_{\Bbbk}\langle x_1, x_2, x_3, x_4\rangle,$$
$$S := \Bbbk[x_1, x_2, x_3, x_4].$$

(a) Show $\mathrm{Hilb}(S,t) = \frac{1}{(1-t)^4}$

and $\mathrm{Hilb}(\Lambda,t) = \mathrm{Hilb}(S/(x_1^2, x_2^2, x_3^2, x_4^2), t)$

$$= (1+t)^4 = 1+4t+6t^2+4t^3+t^4$$

Define these quadratically generated ideals:

$J_1 = (x_1x_2) \subset \Lambda$

$J_2 = (x_1x_2 + x_3x_4) \subset \Lambda$

$I_1 := (x_1x_2, x_1^2, x_2^2, x_3^2, x_4^2) \subset S$

$I_2 := (x_1x_2 + x_3x_4, x_1^2, x_2^2, x_3^2, x_4^2) \subset S$

$I_3 := (x_1x_2) \subset S$

$I_4 := (x_1x_2 + x_3x_4) \subset S$

(b) Show $\mathrm{Hilb}(S/I_1, t) = \mathrm{Hilb}(\Lambda/J_1, t) = 1 + 4t + 5t^2 + 2t^3$

$\mathrm{Hilb}(S/I_2, t) = \mathrm{Hilb}(\Lambda/J_2, t) = 1 + 4t + 5t^2$

$\mathrm{Hilb}(S/I_3, t) = \mathrm{Hilb}(S/I_4, t) = \dfrac{1-t^2}{(1-t)^4}$

[Hint for the last part:

can you show, for any homogeneous polynomial $f(\underline{x})$ of degree d in $\Bbbk[x_1, \ldots, x_n]$, that $\mathrm{Hilb}(\Bbbk[x_1, \ldots, x_n]/(f), t) = \dfrac{1-t^d}{(1-t)^n}$?]

$J_1 = (x_1x_2) \subset \Lambda$

$J_2 = (x_1x_2 + x_3x_4) \subset \Lambda$

$I_1 := (x_1x_2, x_1^2, x_2^2, x_3^2, x_4^2) \subset S$

$I_2 := (x_1x_2 + x_3x_4, x_1^2, x_2^2, x_3^2, x_4^2) \subset S$

$I_3 := (x_1x_2) \subset S$

$I_4 := (x_1x_2 + x_3x_4) \subset S$

(c) Show that the listed generators for I_1, I_3, I_4, J_1 are (quadratic) Gröbner bases for those ideals, and hence
S/I_1, S/I_3, S/I_4, Λ/J_1 are all Koszul.

(d) Show that, regardless of the choice of monomial order $\prec$ on S or Λ, the ideals I_2, J_2 have no quadratic Gröbner bases.

(e) Check (e.g. in Wolfram Alpha) that

$$\frac{1}{1-4t+5t^2} = 1+4t+11t^2+24t^3+41t^4+44t^5-29t^6-\ldots$$

and explain why this shows S/I_2, Λ/J_2 are not Koszul.

Problem 8: Recall the reflection arrangements

Type A_{n-1} : $\{H^+_{ij} : 1\leq i<j\leq n\}$ where $H^+_{ij}=\{x_i=+x_j\}$

Type B_n : $\{H^+_{ij}, H^-_{ij} : 1\leq i<j\leq n\}$ where $H^-_{ij}=\{x_i=-x_j\}$
$\cup \{H^-_{ii} : 1\leq i\leq n\}$ $H^-_{ii}=\{x_i=0\}$

Type D_n : $\{H^+_{ij}, H^-_{ij} : 1\leq i<j\leq n\}$

Also recall an arrangement $\mathcal{H}$ is supersolvable if it has an ordered decomposition $\mathcal{H}=\mathcal{H}_1 \sqcup \ldots \sqcup \mathcal{H}_d$ such that

- $\forall H, H' \in \mathcal{H}_j$ with $H\neq H'$, $\exists H'' \in \mathcal{H}_1 \sqcup \ldots \sqcup \mathcal{H}_{j-1}$ with $\{H, H', H''\}$ a circuit
- each initial segment $\mathcal{H}_1 \sqcup \mathcal{H}_2 \sqcup \ldots \sqcup \mathcal{H}_j$ is a flat: $X=\bigcap_{H\in\mathcal{H}_1\sqcup\ldots\sqcup\mathcal{H}_j} H$ lies in no other hyperplanes from $\mathcal{H}_{j+1}\sqcup\ldots\sqcup\mathcal{H}_d$

(a) Show A_{n-1} is supersolvable via decomposition

$$\{H^+_{12}\} \sqcup \begin{Bmatrix} H^+_{13} \\ H^+_{23} \end{Bmatrix} \sqcup \begin{Bmatrix} H^+_{14} \\ H^+_{24} \\ H^+_{34} \end{Bmatrix} \sqcup \cdots \sqcup \begin{Bmatrix} H^+_{1,n} \\ H^+_{2,n} \\ \vdots \\ H^+_{n-2,n} \\ H^+_{n-1,n} \end{Bmatrix}$$

(b) Show B_n is supersolvable via decomposition

$$\{H^-_{11}\} \sqcup \left\{\begin{matrix} H^+_{12}\\ H^-_{22}\\ H^-_{12}\end{matrix}\right\} \sqcup \left\{\begin{matrix} H^+_{13}\\ H^+_{23}\\ H^-_{33}\\ H^-_{23}\\ H^-_{13}\end{matrix}\right\} \sqcup \cdots \sqcup \left\{\begin{matrix} H^+_{1,n}\\ \vdots \\ H^+_{n-1,n}\\ H^-_{n,n}\\ H^-_{n-1,n}\\ \vdots\\ H^-_{1,n}\end{matrix}\right\}$$

(c) Explain why this decomposition for D_n

$$\left\{\begin{matrix} H^+_{12}\\ H^-_{12}\end{matrix}\right\} \sqcup \left\{\begin{matrix} H^+_{13}\\ H^+_{23}\\ H^-_{23}\\ H^-_{13}\end{matrix}\right\} \sqcup \cdots \sqcup \left\{\begin{matrix} H^+_{1,n}\\ \vdots \\ H^+_{n-2,n}\\ H^+_{n-1,n}\\ H^-_{n-1,n}\\ H^-_{n-2,n}\\ \vdots\\ H^-_{1,n}\end{matrix}\right\}$$

fails the conditions for supersolvability.

(d) Find a decomposition $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2 \sqcup \mathcal{H}_3$ for

$$D_3 = \{ H^+_{12}, H^-_{12}, H^+_{13}, H^-_{13}, H^+_{23}, H^-_{23} \}$$

which shows that it is supersolvable.

Problem 9:

(a) Given a chain complex $C_\bullet$ (of abelian groups)

$$\cdots \xrightarrow{d} C_{i+1} \xrightarrow{d} C_i \xrightarrow{d} C_{i-1} \xrightarrow{d} \cdots$$

so $d^2=0$, that is, $\mathrm{im}(d_{i+1}) \subset \ker(d_i)$,

show that if one has backward maps

$$\cdots \xleftarrow{D} C_{i+1} \xleftarrow{D} C_i \xleftarrow{D} C_{i-1} \xleftarrow{D} \cdots$$

satisfying $dD + Dd = 1_{C_i}$

(called a chain contraction or contracting homotopy),

then $C_\bullet$ is exact, i.e. $\mathrm{im}(d_{i+1}) = \ker(d_i)$.

(b) For a graded $\Bbbk$-algebra $A = \bigoplus_{d=0}^{\infty} A_d$,

recall the bar complex $\mathcal{B}_\bullet$ resolving $\Bbbk$ is

$$\cdots \longrightarrow \underset{\substack{\parallel \\ F_2}}{A \otimes A_+ \otimes A_+} \xrightarrow{d} \underset{\substack{\parallel \\ F_1}}{A \otimes A_+} \xrightarrow{d} \underset{\substack{\parallel \\ F_0}}{A} \xrightarrow{d} \Bbbk \to 0$$

with $F_i := A \otimes \underbrace{A_+ \otimes A_+ \otimes \cdots \otimes A_+}_{i \text{ factors}}$

and $d : F_i \longrightarrow F_{i-1}$ defined A-linearly via

$$d\,[a_1|a_2|\cdots|a_i] := a_1[a_2|\cdots|a_i] + \sum_{j=1}^{i} (-1)^j [a_1|\cdots|a_{j-1}|a_j a_{j+1}|a_{j+2}|\cdots|a_i]$$

Show that the backward maps $D : F_i \to F_{i+1}$ defined $\Bbbk$-linearly (but not A-linearly) by

$$D\big(a_0[a_1|a_2|\cdots|a_i]\big) := \begin{cases} [a_0|a_1|a_2|\cdots|a_i] & \text{if } a_0 \in A_+ \\ 0 & \text{if } a_0 \in \Bbbk = A_0 \end{cases}$$

satisfy $dD + Dd = 1_{F_i}$,

and hence the bar complex $\mathcal{B}_\bullet$ is exact.