MPI Leipzig Summer School in Algebraic Combinatorics

The Koszul property in Algebraic Combinatorics

Vic Reiner (Univ. of Minnesota)

Lectures

1: Motivation, definition of Koszul algebras
- the monomial case

2: Methods for proving Koszulity
and more examples

3: Bar complex, topology,
and inequalities

4: Group actions

1: Motivation, definition of Koszul algebras – the monomial case

Motivation: Roughly Koszul algebras are

- certain special (associative) $\mathbb{N}$-graded $\Bbbk$-algebras $A = \bigoplus_{d=0}^{\infty} A_d$ defined by quadratic relations

- always traveling in pairs with their Koszul dual algebra $A^!$

- satisfying nice generatingfunctionology for their Hilbert series

$$\mathrm{Hilb}(A,t) := \sum_{d=0}^{\infty} \dim_{\Bbbk}(A_d) \cdot t^d$$

namely
$$\mathrm{Hilb}(A^!, -t) = \frac{1}{\mathrm{Hilb}(A,t)}$$

and they give us an excuse to discuss...

- resolutions
- (quadratic) Gröbner bases
- Lots of fun combinatorial rings:
  - ▷ Stanley-Reisner rings (of flag complexes)
  - ▷ Orlik-Solomon algebras (of supersolvable matroids)
  - ▷ Affine semigroup rings (whose poset is Cohen-Macaulay)
  - ▷ Matroid Chow rings (with maximal building set)
  - ▷ Matroid/polymatroid basis rings (maybe all)
  - ▷ Dual braid monoid algebras
  - ▷ Veronese and Segre rings

EXAMPLES Rings from walks in digraphs

$D \subset [n] \times [n]$ a binary relation or a digraph on vertices $[n] := \{1, 2, \dots, n\}$

↓

$$\text{algebra } A_D := \underbrace{\Bbbk\langle x_1, x_2, \dots, x_n\rangle}_{\substack{\text{free associative}\\ \text{algebra on}\\ x_1, \dots, x_n}} \Big/ \overbrace{\left(x_i x_j \text{ if } i \rightarrow j \text{ is not in } D\right)}^{\substack{\text{2-sided}\\ \text{ideal}\\ \text{gen'd by}}}$$

$$= \operatorname{span}_{\Bbbk}\left\{ x_{i_1} x_{i_2} \cdots x_{i_\ell} : \begin{array}{l} \text{walks } i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_\ell \\ \text{along arcs in } D \end{array} \right\}$$

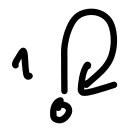
Let's examine their

ℕ-graded Hilbert series

$$\mathrm{Hilb}(A_D, t) := \sum_{d=0}^{\infty} \dim_{\Bbbk} (A_D)_d \cdot t^d$$

$$= \sum_{\substack{\text{walks}\\ w \text{ in } D}} t^{\#\text{vertices visited by } w}$$

EXAMPLES

$D =$ 

$$A_D = \Bbbk\langle x_1\rangle = \mathrm{span}_{\Bbbk}\{\overset{\text{empty walk!}}{1}, \underset{\text{lazy walk at 1}}{x_1}, x_1^2, x_1^3, \dots\}$$

$$\mathrm{Hilb}(A_D, t) = 1 + t + t^2 + t^3 + \dots$$

$$= \frac{1}{1-t}$$

The complement digraph to D

$D^c =$ 

$$A_{D^c} = \Bbbk\langle x_1\rangle / (x_1^2) = \mathrm{span}_{\Bbbk}\{1, x_1\}$$

$$\mathrm{Hilb}(A_{D^c}, t) = 1 + t$$

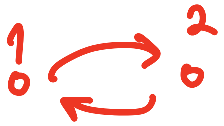
$$\left[= \frac{1}{\mathrm{Hilb}(A, -t)} \right]$$

We could even keep track of an $\mathbb{N}^n$-graded

$$\mathrm{Hilb}(A_D, t, \underline{q}) := \sum_{\substack{\text{walks } \omega \\ \text{in } D}} t^{\#\text{stops}} \, q_1^{\#\text{stops at } 1} \cdots q_n^{\#\text{stops at } n}$$

EXAMPLES

$D =$



$$A_D = \Bbbk\langle x_1, x_2\rangle / (x_1^2, x_2^2)$$

$$= \mathrm{span}_{\Bbbk}\left\{ \begin{array}{lllll} 1, & x_1, & x_1x_2, & x_1x_2x_1, & x_1x_2x_1x_2, \; \cdots \\ & x_2, & x_2x_1, & x_2x_1x_2, & x_2x_1x_2x_1, \; \cdots \end{array} \right\}$$

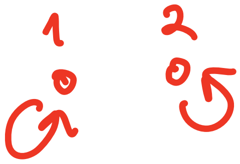
$$\mathrm{Hilb}(A_D, t, q_1, q_2)$$

$$= 1 + t(q_1+q_2) + 2t^2 q_1q_2 + t^3(q_1^2q_2 + q_1q_2^2) + 2t^4 q_1^2q_2^2 + \cdots$$

$$= 1 + \frac{t(q_1+q_2) + 2t^2 q_1q_2}{1 - t^2 q_1q_2}$$

$$= \frac{(1+tq_1)(1+tq_2)}{1-t^2q_1q_2}$$

a little algebra

$D^c =$ 

$$A_{D^c} = \mathbb{k}\langle x_1, x_2\rangle / (x_1x_2, x_2x_1)$$

$$= \mathrm{span}_{\mathbb{k}} \left\{ \begin{matrix} 1, & x_1, & x_1^2, & x_1^3, & \dots \\ & x_2, & x_2^2, & x_2^3, & \dots \end{matrix} \right\}$$

$$\mathrm{Hilb}(A_{D^c}, t, q_1, q_2)$$

$$= 1 + tq_1 + t^2q_1^2 + t^3q_1^3 + \dots$$
$$+ tq_2 + t^2q_2^2 + t^3q_2^3 + \dots$$

$$= 1 + \frac{tq_1}{1-tq_1} + \frac{tq_2}{1-tq_2}$$

a little algebra

$$= \frac{1-t^2q_1q_2}{(1-tq_1)(1-tq_2)}$$

Recall
$$\mathrm{Hilb}(A_D, t, q_1, q_2) = \frac{(1+tq_1)(1+tq_2)}{1-t^2q_1q_2}$$

$$\left[= \frac{1}{\mathrm{Hilb}(A_D, -t, q_1, q_2)} \right]$$

ASIDE:

Why were $\mathrm{Hilb}(A_D,t) \in \mathbb{Q}(t)$

$\mathrm{Hilb}(A_D,t,\underline{q}) \in \mathbb{Q}(t,\underline{q})$?

EXERCISE 1 recalls Transfer-Matrix Method to show

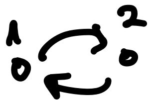
$$\mathrm{Hilb}(A_D,t) = 1 + t\cdot \mathbb{1}_n^T\left(I_n - tT_D\right)^{-1}\mathbb{1}_n$$

$$\Big\uparrow\ q_i=1$$

$$\mathrm{Hilb}(A_D,t,\underline{q}) = 1 + t\cdot[q_1,\dots,q_n]\left(I_n - tT_D(\underline{q})\right)^{-1}\cdot \mathbb{1}_n$$

where $T_D, T_D(\underline{q})$ are Transfer-Matrices for D

EXAMPLE

$D=$ 

$$T_D = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1 & 0 & 1 \\ x_2 & 1 & 0 \end{array} \qquad T_D(\underline{q}) = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1 & 0 & q_2 \\ x_2 & q_1 & 0 \end{array}$$

$$\mathrm{Hilb}(A_D,t,\underline{q}) = 1 + t[q_1\ q_2]\frac{\begin{bmatrix}1 & tq_1\\ tq_2 & 1\end{bmatrix}}{\det\begin{bmatrix}1 & -tq_2\\ -tq_1 & 1\end{bmatrix}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{(1+tq_1)(1+tq_2)}{1-t^2q_1q_2}$$

PROPOSITION: $\mathrm{Hilb}(A_{D^c}, t, \underline{q}) = \dfrac{1}{\mathrm{Hilb}(A_D, -t, \underline{q})}$

that is, $\mathrm{Hilb}(A_{D^c}, -t, \underline{q}) \cdot \mathrm{Hilb}(A_D, t, \underline{q}) = 1.$

proof: Left side equals

$$\sum t^{|\underline{x}|} \overset{q_{j_1} q_{j_2} \cdots q_{j_m}}{q_{\underline{x}}} \quad \cdot \quad \sum (-t)^{|\underline{y}|} \overset{q_{i_1} q_{i_2} \cdots q_{i_\ell}}{q_{\underline{y}}}$$

words $\underline{x} = x_{j_1} \cdots x_{j_m}$ only allowing $x_i x_j$ adjacent if $i \to j$ in D

words $\underline{y} = y_{i_1} \cdots y_{i_\ell}$ only allowing $y_i y_j$ adjacent if $i \to j$ in D^c

$$= \sum_{(\underline{x}, \underline{y}):} (-1)^{|\underline{y}|} t^{|\underline{y}| + |\underline{x}|} q_{\underline{x}} \cdot q_{\underline{y}}$$

$x_i x_j$ adjacent $\Rightarrow i \to j$ in D

$y_i y_j$ adjacent $\Rightarrow i \to j$ in D^c

$$\overset{?}{=} 1$$

massive cancellation ??

YES.

MASSIVE CANCELLATION = SIGN-REVERSING INVOLUTION

$$\sum_{\substack{(\underline{x},\underline{y}):\\ x_i x_j \text{ adjacent} \Rightarrow i\to j \text{ in } D \\ y_i y_j \text{ adjacent} \Rightarrow i\to j \text{ in } D^c}} (-1)^{|\underline{y}|}\, t^{|\underline{y}|+|\underline{x}|}\, g_{\underline{x}} \cdot g_{\underline{y}} = +1$$

because one can cancel terms via this involution:

$$\begin{tikzcd}[column sep=small] x_{j_1} \arrow[r, bend left, "\text{in } D"] & x_{j_2} \arrow[r, bend left, "\text{in } D"] & \cdots \arrow[r, bend left, "\text{in } D"] & x_{j_{m-1}} \arrow[r, bend left, "\text{in } D"] & x_{j_m} \arrow[r, bend right, "{\text{in } D \text{ or in } D^c ?}"'] & \cdot\, y_{i_1} \arrow[r, bend left, "\text{in } D^c"] & y_{i_2} \arrow[r, bend left, "\text{in } D^c"] & \cdots \arrow[r, bend left, "\text{in } D^c"] & y_{i_{\ell-1}} \arrow[r, bend left, "\text{in } D^c"] & y_{i_\ell} \end{tikzcd}$$

$$\longmapsto \begin{cases} x_{j_1}x_{j_2}\cdots x_{j_{m-1}}x_{j_m}\cdot x_{i_1}\cdot y_{i_2}\cdots y_{i_{\ell-1}}y_{i_\ell} & \text{if } j_m \to i_1 \text{ in } D \\ x_{j_1}x_{j_2}\cdots x_{j_{m-1}}y_{j_m}\cdot y_{i_1}\cdot y_{i_2}\cdots y_{i_{\ell-1}}y_{i_\ell} & \text{if } j_m \to i_1 \text{ in } D^c \end{cases}$$

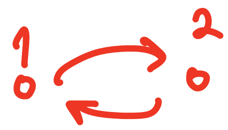
It really is an involution, and matches terms with opposite signs $(-1)^{|\underline{y}|}$, so it cancels down to the pair $(\underline{x},\underline{y}) = (\emptyset,\emptyset)$, giving $+1$ ▨

In fact, this sign-reversing involution hides an $\mathbb{N}$-graded free (left-) A_D-module resolution of the trivial module

$$\Bbbk = A_D/(x_1,\dots,x_n) = A_D/(A_D)_+$$

which is linear in the sense that all the maps have only degree one entries.

EXAMPLE

$D=$ 

$$A_D = \Bbbk\langle x_1,x_2\rangle/(x_1^2,x_2^2)$$

$$\begin{tikzcd} & A_D/(x_1,x_2) \arrow[d, equal] & \\ A_D \arrow[r] & \Bbbk \arrow[r] & 0 \end{tikzcd}$$

$$x_1 \longmapsto 0$$
$$x_2 \longmapsto 0$$

EXAMPLE

$$\begin{tikzcd} D = \overset{1}{\circ} \arrow[r, bend left] & \overset{2}{\circ} \arrow[l, bend left] \end{tikzcd}$$

$$A_D = \Bbbk\langle x_1, x_2 \rangle / (x_1^2, x_2^2)$$

$$\begin{array}{c c c} & y_1 & y_2 \\ 1 & [\, x_1 & x_2 \,] \end{array}$$

$$\begin{tikzcd} & & A_D/(x_1,x_2) \arrow[d, phantom, "\parallel"] & \\ A_D y_1 \oplus A_D y_2 \arrow[r] & A_D \arrow[r] & \Bbbk \arrow[r] & 0 \end{tikzcd}$$

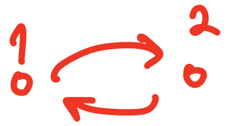
$$y_1 \longmapsto x_1 \longmapsto 0$$

$$y_2 \longmapsto x_2 \longmapsto 0$$

$$x_1 y_1 \longmapsto x_1^2 = 0$$

$$x_2 y_2 \longmapsto x_2^2 = 0$$

EXAMPLE

$D =$ 

$A_D = \Bbbk\langle x_1, x_2\rangle/(x_1^2, x_2^2)$

$$\begin{tikzcd} & & & A_D/(x_1,x_2) \arrow[d, equal] & \\ A_D y_1^2 \oplus A_D y_2^2 \arrow[r, "{\begin{array}{c|cc} & y_1^2 & y_2^2 \\ \hline y_1 & x_1 & 0 \\ y_2 & 0 & x_2 \end{array}}"] & A_D y_1 \oplus A_D y_2 \arrow[r, "{\begin{array}{c|cc} & y_1 & y_2 \\ \hline 1 & x_1 & x_2 \end{array}}"] & A_D \arrow[r] & \Bbbk \arrow[r] & 0 \end{tikzcd}$$

$y_1 \longmapsto x_1 \longmapsto 0$

$y_2 \longmapsto x_2 \longmapsto 0$

$y_1^2 \longmapsto x_1 y_1 \longmapsto x_1^2 = 0$

$y_2^2 \longmapsto x_2 y_2 \longmapsto x_2^2 = 0$

$x_1 y_1^2 \longmapsto x_1^2 y_1 = 0$

$x_2 y_2^2 \longmapsto x_2^2 y_2 = 0$

EXAMPLE

$D = \overset{1}{\circ} \rightleftarrows \overset{2}{\circ}$ $\qquad A_D = \Bbbk\langle x_1, x_2 \rangle / (x_1^2, x_2^2)$

$$\begin{tikzcd}[column sep=large]
&&&&& A_D/(x_1,x_2) \arrow[d, equal] & \\
\cdots \arrow[r] & A_D y_1^3 \oplus A_D y_2^3 \arrow[r, "{\begin{array}{ccc} & y_1^3 & y_2^3 \\ y_1^2 & x_1 & 0 \\ y_2^2 & 0 & x_2 \end{array}}"] & A_D y_1^2 \oplus A_D y_2^2 \arrow[r, "{\begin{array}{ccc} & y_1^2 & y_2^2 \\ y_1 & x_1 & 0 \\ y_2 & 0 & x_2 \end{array}}"] & A_D y_1 \oplus A_D y_2 \arrow[r, "{\begin{array}{ccc} & y_1 & y_2 \\ 1 & x_1 & x_2 \end{array}}"] & A_D \arrow[r] & \Bbbk \arrow[r] & 0
\end{tikzcd}$$

$y_1 \longmapsto x_1 \longmapsto 0$

$y_2 \longmapsto x_2 \longmapsto 0$

$y_1^2 \longmapsto x_1 y_1 \longmapsto x_1^2 = 0$

$y_2^2 \longmapsto x_2 y_2 \longmapsto x_2^2 = 0$

$y_1^3 \longmapsto x_1 y_1^2 \longmapsto x_1^2 y_1 = 0$

$y_2^3 \longmapsto x_2 y_2^2 \longmapsto x_2^2 y_2 = 0$

EXAMPLE

$$\begin{tikzcd} D = \overset{1}{\circ} \arrow[r, bend left] & \overset{2}{\circ} \arrow[l, bend left] \end{tikzcd}$$

$$A_D = \Bbbk\langle x_1, x_2 \rangle / (x_1^2, x_2^2)$$

This resolution

$$\begin{tikzcd} & & & & & A_D/(x_1,x_2) \arrow[d, equal] & \\ \cdots \arrow[r] & \begin{matrix} A_D y_1^3 \\ \oplus \\ A_D y_2^3 \end{matrix} \arrow[r, "{\begin{array}{ccc} & y_1^3 & y_2^3 \\ y_1^2 & x_1 & 0 \\ y_2^2 & 0 & x_2 \end{array}}"] & \begin{matrix} A_D y_1^2 \\ \oplus \\ A_D y_2^2 \end{matrix} \arrow[r, "{\begin{array}{ccc} & y_1^2 & y_2^2 \\ y_1 & x_1 & 0 \\ y_2 & 0 & x_2 \end{array}}"] & \begin{matrix} A_D y_1 \\ \oplus \\ A_D y_2 \end{matrix} \arrow[r, "{\begin{array}{ccc} & y_1 & y_2 \\ 1 & x_1 & x_2 \end{array}}"] & A_D \arrow[r] & \Bbbk \arrow[r] & 0 \end{tikzcd}$$

$y_1 \longmapsto x_1$

$y_2 \longmapsto x_2$

$y_1^2 \longmapsto x_1 y_1$

$y_2^2 \longmapsto x_2 y_2$

$y_1^3 \longmapsto x_1 y_1^2$

$y_2^3 \longmapsto x_2 y_2^2$

e.g. $x_2 x_1 x_2 x_1 \cdot y_2 y_2 y_2$

has monomial $\Bbbk$-basis

$$\underline{x} \cdot \underline{y} = x_{j_1} x_{j_2} \cdots x_{j_{m-1}} x_{j_m} \cdot y_{i_1} y_{i_2} \cdots y_{i_{l-1}} y_{i_l} = \hat{\underline{x}}\, x_{j_m} y_{i_1} \hat{\underline{y}}$$

bijecting with the terms canceled in the resolution:

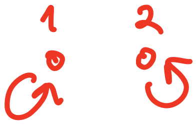
boundaries = cycles

$$\hat{\underline{x}}\, y_{j_m} y_{i_1} \hat{\underline{y}} \longmapsto \hat{\underline{x}}\, x_{j_m} y_{i_1} \hat{\underline{y}}$$

non-boundaries = non-cycles

$$\hat{\underline{x}}\, x_{j_m} y_{i_1} \hat{\underline{y}} \longmapsto \hat{\underline{x}}\, x_{j_m} x_{i_1} \hat{\underline{y}}$$

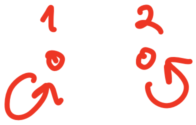
EXAMPLE: $D^c =$



$A_{D^c} = \mathbb{k}\langle x_1, x_2\rangle / (x_1x_2, x_2x_1)$

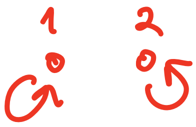
$$\begin{tikzcd}
 & & A_{D^c}/(x_1,x_2) & \\
A_D y_1 \oplus A_D y_2 \arrow[r, "{\begin{array}{cc} & \begin{array}{cc} y_1 & y_2 \end{array} \\ 1 & \begin{bmatrix} x_1 & x_2 \end{bmatrix} \end{array}}"] & A_{D^c} \arrow[r] & \mathbb{k} \arrow[u, equal] \arrow[r] & 0 \\
y_1 \arrow[r] & x_1 \arrow[r, mapsto] & 0 & \\
y_2 \arrow[r] & x_2 \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

EXAMPLE: $D^c =$



$A_{D^c} = \Bbbk\langle x_1, x_2\rangle / (x_1x_2, x_2x_1)$

$$\begin{tikzcd}
 & & & A_{D^c}/(x_1,x_2) \arrow[d, equal] & \\
A_D y_2y_1 \oplus A_D y_1y_2 \arrow[r, "{\begin{array}{ccc} & y_2y_1 & y_1y_2 \\ y_1 & x_2 & 0 \\ y_2 & 0 & x_1 \end{array}}"] & A_D y_1 \oplus A_D y_2 \arrow[r, "{\begin{array}{cc} y_1 & y_2 \\ x_1 & x_2 \end{array}}"] & A_{D^c} \arrow[r] & \Bbbk \arrow[r] & 0 \\
 & y_1 \arrow[r, mapsto] & x_1 \arrow[r, mapsto] & 0 & \\
 & y_2 \arrow[r, mapsto] & x_2 \arrow[r, mapsto] & 0 & \\
y_2y_1 \arrow[r, mapsto] & x_2y_1 \arrow[r, mapsto] & x_2x_1=0 & & \\
y_1y_2 \arrow[r, mapsto] & x_1y_2 \arrow[r, mapsto] & x_1x_2=0 & &
\end{tikzcd}$$

EXAMPLE: $D^c =$ 

$A_{D^c} = \Bbbk\langle x_1, x_2\rangle / (x_1x_2, x_2x_1)$

$$\begin{tikzcd}
\cdots \arrow[r] & {\begin{array}{c} A_D\, y_1y_2y_1 \\ \oplus \\ A_D\, y_2y_1y_2 \end{array}} \arrow[r, "{\begin{array}{c|cc} & y_1y_2y_1 & y_2y_1y_2 \\ \hline y_2y_1 & x_1 & 0 \\ y_1y_2 & 0 & x_2 \end{array}}"] & {\begin{array}{c} A_D\, y_2y_1 \\ \oplus \\ A_D\, y_1y_2 \end{array}} \arrow[r, "{\begin{array}{c|cc} & y_2y_1 & y_1y_2 \\ \hline y_1 & x_2 & 0 \\ y_2 & 0 & x_1 \end{array}}"] & {\begin{array}{c} A_D\, y_1 \\ \oplus \\ A_D\, y_2 \end{array}} \arrow[r, "{\begin{array}{c|cc} & y_1 & y_2 \\ \hline 1 & x_1 & x_2 \end{array}}"] & A_{D^c} \arrow[r] & {\begin{array}{c} A_{D^c}/(x_1,x_2) \\ \shortparallel \\ \Bbbk \end{array}} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
y_1 \arrow[r] & x_1 \arrow[r, mapsto] & 0 \\
y_2 \arrow[r] & x_2 \arrow[r, mapsto] & 0 \\
y_2y_1 \arrow[r, mapsto] & x_2y_1 \arrow[r, mapsto] & x_2x_1 = 0 \\
y_1y_2 \arrow[r, mapsto] & x_1y_2 \arrow[r, mapsto] & x_1x_2 = 0 \\
x_1y_2y_1 \arrow[r, mapsto] & x_1x_2y_1 = 0 & \\
x_2y_1y_2 \arrow[r, mapsto] & x_2x_1y_2 = 0 &
\end{tikzcd}$$

This resolution also has monomial $\Bbbk$-basis

e.g. $x_1x_1x_1 \cdot y_2y_1y_2$

$$\underline{x} \cdot \underline{y}$$

$\Bbbk$-basis element of A_{D^c}

$\Bbbk$-basis element of A_D

bijecting with terms canceled via involution.

But now the roles of $\underline{x}$ & $\underline{y}$ are swapped.

These digraph algebras $\mathring{A}_D$ are examples of ...

DEF'N (Priddy 1970) A Koszul algebra A is an

- associative $\Bbbk$-algebra ($\Bbbk$ a field)
- finitely generated: $A = \Bbbk\langle x_1, \dots, x_n\rangle / I$

some 2-sided ideal

- standard $\mathbb{N}$-graded, connected:

$$A = \underbrace{A_0}_{\cong \Bbbk} \oplus \underset{\underset{\operatorname{span}_{\Bbbk}\{x_1,\dots,x_n\}}{\shortparallel}}{A_1} \oplus A_2 \oplus \cdots$$

so $\deg(x_i) = 1$ $\forall i$

and I homogeneous: $I = I_2 \oplus I_3 \oplus I_4 \oplus \cdots$

DEF'N of standard graded $\Bbbk$-algebra

- with some linear free (left-) A-module resolution of the trivial module $\Bbbk = A/A_+ = A/(x_1, \dots, x_n)$

$$\to A(-3)^{\beta_3} \xrightarrow{\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}} A(-2)^{\beta_2} \xrightarrow{\begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}} A(-1)^{n} \xrightarrow{[x_1 \cdots x_n]} A \to \Bbbk \to 0$$

all linear entries

$x_i \mapsto 0$

$A(-m) := A$ with 1 in degree m

MORE EXAMPLES, NON-EXAMPLES

- Koszulity implies A is a quadratic algebra, i.e.

  $$A = \Bbbk\langle x_1, \dots, x_n\rangle / I$$

  with I generated by I_2 as 2-sided ideal

NON-EXAMPLE (EXERCISE 4)

$A = \Bbbk\langle x\rangle / (x^3)$ has a simple, periodic graded free A-resolution of $\Bbbk$, starting like this:

$$\begin{tikzcd}
\cdots \arrow[r] & \overset{F_3}{A(-4)^1} \arrow[r, "\substack{e_4 \\ e_3{[x]}}"] & \overset{F_2}{A(-3)^1} \arrow[r, "\substack{e_3 \\ e_1{[x^2]}}"] & \overset{F_1}{A(-1)^1} \arrow[r, "\substack{e_1 \\ 1{[x]}}"] & \overset{F_0}{A^1} \arrow[r] & \Bbbk \arrow[r] & 0
\end{tikzcd}$$

$$x \longmapsto 0$$

$$e_1 \longmapsto x$$

$$\boxed{e_3 \longmapsto x^2 e_1} \longmapsto x^3 = 0$$

$$x e_3 \longmapsto x^3 e_1 = 0$$

the cubic relation $x^3=0$ in A leads to degree 3 syzygy in F_2

EXAMPLE Koszul algebras get their name from the Koszul resolution of $\Bbbk = A/A_+$ over

$$A = \underbrace{\Bbbk[x_1,\dots,x_n]}_{\substack{\text{commutative}\\ \text{polynomial ring}}} = k\langle x_1,\dots,x_n\rangle \Big/ \left(x_ix_j - x_jx_i : 1\le i<j\le n \right)$$

$n=3$:

$\Bbbk$-basis y_1, y_2, y_3

$$0 \to A\otimes\Lambda^3\Bbbk^3 \xrightarrow{\begin{array}{c|c} & y_1\wedge y_2\wedge y_3 \\ \hline y_1\wedge y_2 & x_3 \\ y_1\wedge y_3 & -x_2 \\ y_2\wedge y_3 & x_1 \end{array}} A\otimes\Lambda^2\Bbbk^3 \xrightarrow{\begin{array}{c|ccc} & y_1\wedge y_2 & y_1\wedge y_3 & y_2\wedge y_3 \\ \hline y_1 & x_2 & x_3 & 0 \\ y_2 & -x_1 & 0 & x_3 \\ y_3 & 0 & -x_1 & -x_2 \end{array}} A\otimes\Bbbk^3 \xrightarrow{\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline 1 & x_1 & x_2 & x_3 \end{array}} A \to \Bbbk \to 0$$

$$e_1 \longmapsto x_1 \longmapsto 0$$
$$e_2 \longmapsto x_2 \longmapsto 0$$
$$e_3 \longmapsto x_3 \longmapsto 0$$

$$y_1\wedge y_2 \longmapsto x_2y_1 - x_1y_2$$
$$y_1\wedge y_3 \longmapsto x_3y_1 - x_1y_3$$
$$y_2\wedge y_3 \longmapsto x_3y_2 - x_2y_3$$

$$y_1\wedge y_2\wedge y_3 \longmapsto x_3\, y_1\wedge y_2 - x_2\, y_1\wedge y_3 + x_1\, y_2\wedge y_3$$

So $A = \Bbbk[x_1,\dots,x_n]$ is a Koszul algebra

Priddy's Resolution

He proved something amazing and beautiful about the structure of these linear A-resolutions of $\Bbbk$ when A is Koszul

$$\begin{tikzcd}
\cdots \arrow[r] & A(-3)^{\beta_3} \arrow[r] \arrow[d, equal] & A(-2)^{\beta_2} \arrow[r] \arrow[d, equal] & A(-1)^{n} \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & \Bbbk \arrow[r] & 0 \\
& F_3 & F_2 & F_1 & F_0 & &
\end{tikzcd}$$

They can always be built with $F_i := (A^!_i)^*$ where $A^!$ is the quadratic dual algebra to A.

DEF'N: If $A = \Bbbk\langle x_1, \ldots, x_n\rangle / (I_2)$ is a quadratic algebra

then $A^! := \Bbbk\langle y_1, \ldots, y_n\rangle / (I_2^\perp)$ quadratic dual of A

where perp $I_2 \rightsquigarrow I_2^\perp$ is with respect to $\Bbbk$-bilinear pairing

$$\Bbbk\langle \underline{x}\rangle_2 \times \Bbbk\langle \underline{y}\rangle_2 \longrightarrow \Bbbk$$

$$(\underset{i \le j}{x_i x_j}, \underset{k \le l}{y_k y_l}) := \begin{cases} 1 & \text{if } (i,j) = (k,l) \\ 0 & \text{otherwise} \end{cases}$$

THEOREM (Priddy 1970) For any Koszul algebra $A = \Bbbk\langle \underline{x}\rangle/(I_2)$ one has an explicit linear free A-resolution of $\Bbbk$

$$\begin{tikzcd}
\cdots \arrow[r] & A(-3)^{\beta_3} \arrow[r, "d_3"] \arrow[d, equal] & A(-2)^{\beta_2} \arrow[r, "d_2"] \arrow[d, equal] & A(-1)^{n} \arrow[r, "d_1"] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & \Bbbk \arrow[r] & 0 \\
& F_3 \arrow[d, equal] & F_2 \arrow[d, equal] & F_1 \arrow[d, equal] & F_0 \arrow[d, equal] & & \\
& A\otimes(A^!_3)^* & A\otimes(A^!_2)^* & A\otimes(A^!_1)^* & A & &
\end{tikzcd}$$

- built on $A\otimes(A^!)^*$ where $A^! = \Bbbk\langle \underline{y}\rangle / (I_2^\perp)$

- with a differential

$$\begin{aligned} d_i : A\otimes(A^!_i)^* &\longrightarrow A\otimes(A^!_{i-1})^* \\ a\otimes\varphi &\longmapsto \sum_{j=1}^{n} a x_j \otimes \varphi.y_j \end{aligned}$$

$(\varphi.y_j)(b) := \varphi(y_j b)$

COROLLARIES to Priddy's Theorem/resolution

COROLLARY 1: Taking $\Bbbk$-duals turns the (linear) A-free resolution $A \otimes (A^!)^*$ of $\Bbbk$

$$\Big\downarrow \quad \Bbbk\text{-dual, i.e. } \mathrm{Hom}_{\Bbbk}(-,\Bbbk) \text{ degree by degree}$$

into an (linear) $A^!$-free resolution $A^! \otimes A^*$ of $\Bbbk$,

so $\boxed{A \text{ Koszul} \iff A^! \text{ Koszul.}}$

COROLLARY 2: (EXERCISES 2,3) Exactness of $A \otimes (A^!)^*$ resolving $\Bbbk$

$$\Rightarrow \boxed{\begin{gathered}\mathrm{Hilb}(A,t) \cdot \mathrm{Hilb}(A^!,-t) = 1 \\ \text{for } A \text{ Koszul}\end{gathered}}$$

That's a mouthful, but one can check ...

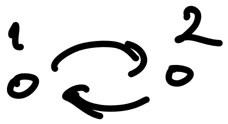
EXAMPLE

For a digraph D

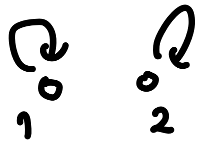
$$A_D = \Bbbk\langle x_1, \dots, x_n\rangle / (\overbrace{x_i x_j}^{(I_2)})_{i \to j \text{ not in } D}$$

has

$$A_D^! = A_{D^c} = \Bbbk\langle y_1, \dots, y_n\rangle / (\underbrace{y_i y_j}_{(I_2^\perp)})_{i \to j \text{ not in } D^c}$$

D :


$$A_D = \Bbbk\langle x_1, x_2\rangle / (x_1^2, x_2^2)$$

D^c :


$$A_{D^c} = \Bbbk\langle y_1, y_2\rangle / (y_1 y_2, y_2 y_1)$$

and before we were essentially writing down Priddy's resolution of $\Bbbk$ over A_D and A_{D^c}.

EXAMPLE

$$A = \mathbb{k}[x_1, \ldots, x_n] = \mathbb{k}\langle x_1, \ldots, x_n\rangle \Big/ \left(x_ix_j - x_jx_i : 1 \le i < j \le n \right)$$

commutative polynomial algebra

has

$$A^! = \Lambda_{\mathbb{k}}\langle y_1, \ldots, y_n\rangle = \mathbb{k}\langle y_1, \ldots, y_n\rangle \Big/ \left(\begin{array}{l} y_iy_j + y_jy_i : 1 \le i < j \le n \\ y_i^2 : 1 \le i \le n \end{array} \right)$$

exterior algebra

and Priddy's resolution built on $\mathbb{k}[\underline{x}] \otimes \left(\Lambda_{\mathbb{k}}\langle \underline{y}\rangle\right)^*$

$=$ Koszul complex resolving $\mathbb{k}$ over $\mathbb{k}[\underline{x}]$

REMARK: Taking duals gives the Cartan complex built on $\Lambda_{\mathbb{k}}\langle \underline{y}\rangle \otimes \left(\mathbb{k}[\underline{x}]\right)^*$

divided power algebra

resolving $\mathbb{k}$ over exterior algebra $\Lambda_{\mathbb{k}}\langle \underline{y}\rangle$

EXAMPLE: Fröberg (1975) essentially wrote down Priddy's resolution and checked its exactness for these partial $\left\{\begin{array}{l}\text{commutation}\\ \text{anticommutation}\\ \text{annihilation}\end{array}\right\}$ quadratic algebras:

$$A := \Bbbk\langle x_1,\dots,x_n\rangle / I$$

where $I = (x_i^2)_{i\in S}$ swap S for $[n]\setminus S$ in $A^!$

$$+\left(\begin{array}{l}\text{for each}\\ 1\le i<j\le n\\ \text{either}\end{array}\left\{\begin{array}{ll} x_ix_j + c_{ij}\,x_jx_i = 0 \text{ for } c_{ij}\in\Bbbk^\times & \text{swap } c_{ij} \text{ for } -c_{ij}^{-1} \text{ in } A^! \\ \text{— or —} & \\ x_ix_j = x_jx_i = 0 & \text{swap in } A^! \\ \text{— or —} & \\ \text{no relation on } x_ix_j,\, x_jx_i & \\ \text{— or —} & \\ x_ix_j = 0 & \text{swap in } A^! \\ \text{— or —} & \\ x_jx_i = 0 & \end{array}\right\}\right)$$

[Also studied by Cartier & Foata 1969
Kobayashi 1990]

COROLLARY (Froberg 1975)

All partial $\left\{\begin{array}{c}\text{commutation}\\ \text{anticommutation}\\ \text{annihilation}\end{array}\right\}$ algebras are Koszul.

In particular,

all quadratic monomial ideal quotients

- $\Bbbk[x_1,\dots,x_n]/J$ of commutative polynomials
- $\Lambda_{\Bbbk}\langle x_1,\dots,x_n\rangle/J$ of exterior algebras

are Koszul.

[$\overset{\text{(next time)}}{\rightsquigarrow}$ quadratic Gröbner deformations ...]