MPI Leipzig Summer School
in Algebraic Combinatorics

The Koszul property
in Algebraic Combinatorics

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Lectures

1: Motivation, definition of Koszul algebras
- the monomial case

2: Methods for proving Koszulity
and more examples

3: Bar complex, topology,
and inequalities

4: Group actions

Methods for proving Koszulity (and more examples!)

The best math concepts have many definitions...

THEOREM: For a standard graded $\Bbbk$-algebra

$$A = \bigoplus_{d=0}^{\infty} A_d = \Bbbk\langle x_1, \ldots, x_n\rangle / I$$

the following are equivalent definitions of Koszulity

(a) There exists a linear A-free resolution of $\Bbbk = A/A^+$

$$\cdots \to A(-3)^{\beta_3} \to A(-2)^{\beta_2} \to A(-1)^{\beta_1} \to A \to \Bbbk \to 0$$

(b) Priddy's complex on $A \otimes (A^!)^*$ is exact.

(c) (Backelin 1981) $\forall d \geq 0$, inside $\Bbbk\langle \underline{x}\rangle_d$, the $\Bbbk$-subspaces

$$\left\{ \begin{array}{c} I_2 \otimes \Bbbk\langle \underline{x}\rangle_{d-2} \\ \Bbbk\langle \underline{x}\rangle_1 \otimes I_2 \otimes \Bbbk\langle \underline{x}\rangle_{d-3} \\ \Bbbk\langle \underline{x}\rangle_2 \otimes I_2 \otimes \Bbbk\langle \underline{x}\rangle_{d-4} \\ \vdots \\ \Bbbk\langle \underline{x}\rangle_{d-2} \otimes I_2 \end{array} \right\}$$

generate a distributive sublattice under $+$, $\cap$

(d) $\mathrm{Tor}_i^A(\Bbbk, \Bbbk)_j = 0$ for $j \neq i$. ($\rightsquigarrow$ MORE METHODS FOR KOSZULITY)

As methods for proving A is Koszul, this one...

(a) There exists a linear A-free resolution of $\Bbbk = A/A^+$

$$\cdots \to A(-3)^{\beta_3} \to A(-2)^{\beta_2} \to A(-1)^{\beta_1} \to A \to \Bbbk \to 0$$

... may be the hardest to implement.

Nevertheless, it's a method that worked for ...

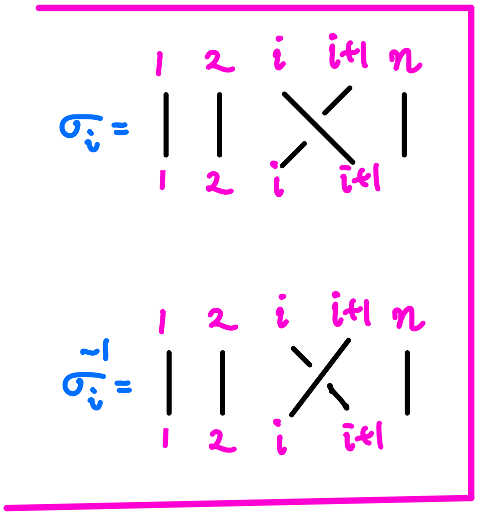
- digraph algebras $A_D = \Bbbk\langle x_1, \dots, x_n \rangle / (x_i x_j : i \to j \text{ not in } D)$
- $\underbrace{\Bbbk[x_1, \dots, x_n]}_{\text{commutative polynomials}}$, $\underbrace{\Lambda_{\Bbbk}\langle x_1, \dots, x_n \rangle}_{\text{exterior algebras}}$
- More generally, the partial commutation/ anticommutation/ annihilation algebras (Fröberg 1975)
- Dual braid monoid algebras via positive cluster complexes (!) (Josuat-Vergès & Nadeau 2021)

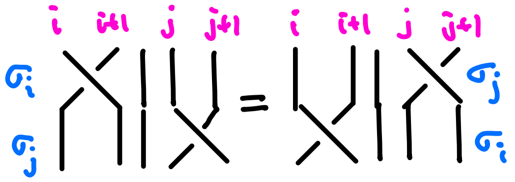
EXAMPLE: Dual braid monoid algebras

E. Artin (1925) gave a simple presentation

for the braid group on n strands Br_n as

$$Br_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_n \mid \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2$$

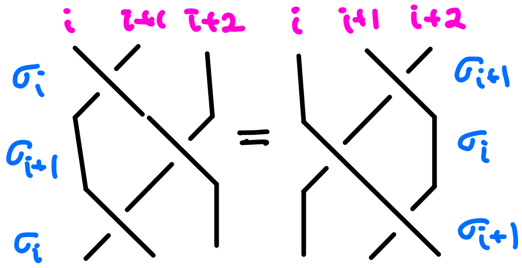




quadratic

cubic

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$



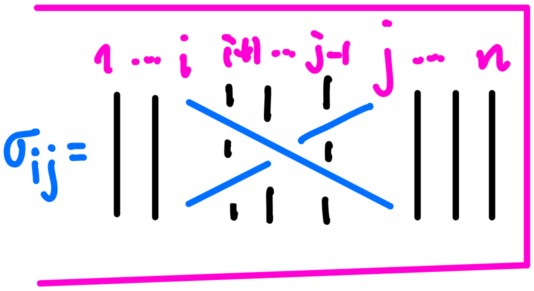
$$\rangle$$

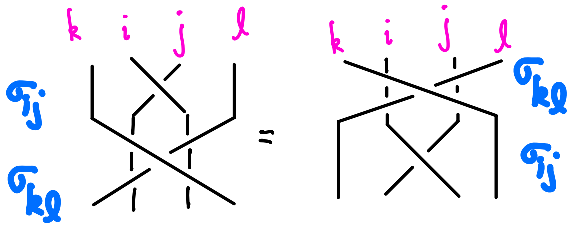
which has both quadratic and cubic relations.

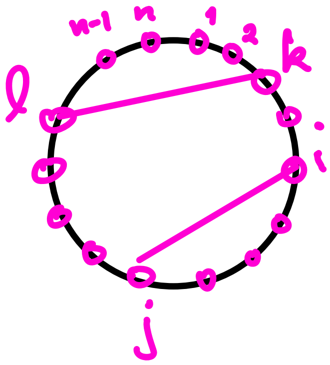
[And it generalizes to braid groups Br_W for real reflection groups W.]

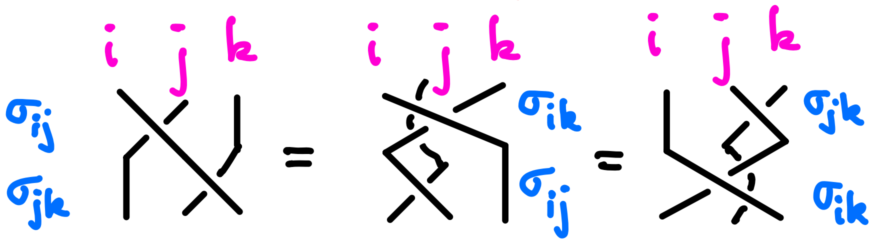
On the other hand, Birman, Ko, Lee (1998) introduced what is now called the dual braid presentation

$$Br_n = \left\langle \begin{matrix} \sigma_{ij} \\ 1\le i<j\le n \end{matrix} \;\middle|\; \begin{matrix} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \text{ if } i,j,k,l \text{ distinct and } ij,\, kl \text{ are noncrossing:} \\ \sigma_{ij}\sigma_{jk} = \sigma_{ik}\sigma_{ij} = \sigma_{jk}\sigma_{ik} \text{ if } 1\le i<j<k\le n \end{matrix} \right\rangle$$









with only quadratic relations.

[And Brady & Watt 2002, Bessis 2003 generalized it to braid groups Br_W for real reflection groups W, using Garside (1969) theory.]

THEOREM

(Josuat-Vergès & Nadeau 2021)

The dual braid monoid algebra

$$A := \Bbbk \left\langle \begin{matrix} x_{ij} \\ 1\leq i<j\leq n \end{matrix} \;\middle|\; \begin{matrix} x_{ij}x_{kl} = x_{kl}x_{ij} & ij,kl \text{ noncrossing} \\ x_{ij}x_{jk} = x_{ik}x_{ij} = x_{jk}x_{ik} & 1\leq i<j<k\leq n \end{matrix} \right\rangle$$

- is Koszul,
- with an explicit linear A-resolution of $\Bbbk$
- "built on" the positive part Δ_+ of the cluster complex.

[- generalizing to all real reflection groups W]

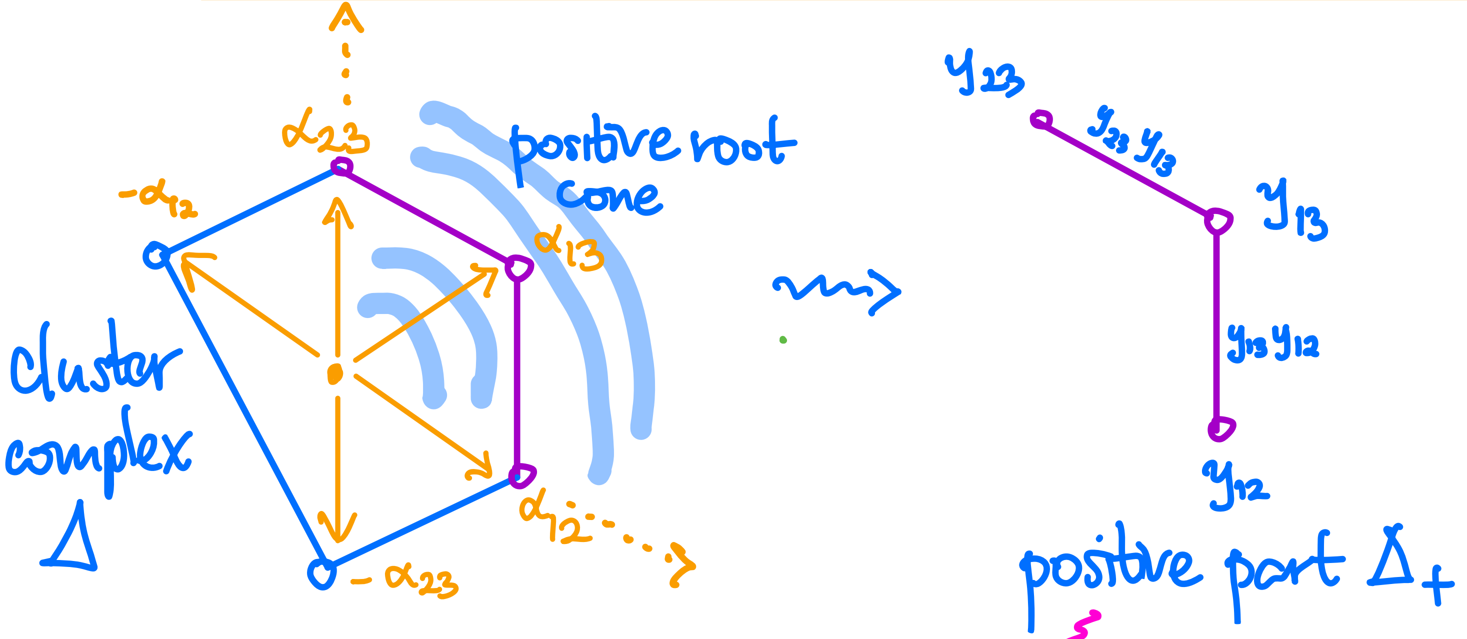
EXAMPLE $n=3$

$$A=\Bbbk\langle x_{12},x_{13},x_{23}\rangle \Big/ \underbrace{(x_{12}x_{23}=x_{13}x_{12}=x_{23}x_{13})}$$

$$=\begin{pmatrix} x_{12}x_{23}-x_{13}x_{12} \\ x_{12}x_{23}-x_{23}x_{13}\end{pmatrix}$$

$$A^{!}=\Bbbk\langle y_{12},y_{13},y_{23}\rangle \Big/ \begin{pmatrix} y_{12}^2, y_{13}^2, y_{23}^2, y_{23}y_{12}, y_{12}y_{13}, y_{13}y_{23} \\ y_{12}y_{23}+y_{13}y_{12}+y_{23}y_{13}\end{pmatrix}$$

$$=\operatorname{span}_{\Bbbk}\{1,\ y_{12},y_{13},y_{23},\ y_{13}y_{12},\ y_{23}y_{13}\}$$



$$0 \to \begin{array}{c} Ay_{13}y_{12} \\ \oplus \\ Ay_{23}y_{13}\end{array} \xrightarrow{\begin{array}{c|cc} & y_{13}y_{12} & y_{23}y_{13} \\ \hline y_{12} & x_{13} & 0 \\ y_{13} & -x_{12} & x_{23} \\ y_{23} & 0 & -x_{13}\end{array}} \begin{array}{c} Ay_{12} \\ \oplus \\ Ay_{13} \\ \oplus \\ Ay_{23}\end{array} \xrightarrow{\begin{array}{ccc} y_{12} & y_{13} & y_{23} \\ \left[x_{12}\right. & x_{13} & \left.x_{23}\right]\end{array}} A\cdot 1 \to \Bbbk \to 0$$

What about (c) Backelin's criterion ?

THEOREM (Backelin 1981) $A = \Bbbk\langle x_1, \ldots, x_n\rangle / I$ is Koszul

$\iff$ $\forall d \geq 0$, inside $\Bbbk\langle \underline{x}\rangle_d$, the $\Bbbk$-subspaces

$$\left\{ \begin{array}{c} I_2 \otimes \Bbbk\langle \underline{x}\rangle_{d-2} \\ \Bbbk\langle \underline{x}\rangle_1 \otimes I_2 \otimes \Bbbk\langle \underline{x}\rangle_{d-3} \\ \Bbbk\langle \underline{x}\rangle_2 \otimes I_2 \otimes \Bbbk\langle \underline{x}\rangle_{d-4} \\ \vdots \\ \Bbbk\langle \underline{x}\rangle_{d-2} \otimes I_2 \end{array} \right\}$$

generate a distributive sublattice under $+$, $\cap$

It gets used mainly to develop more theory, e.g.,

COROLLARY If A, B are Koszul then so are...

- the r^{th} Veronese subalgebra of A

$$A^{(r)} := \Bbbk \oplus A_r \oplus A_{2r} \oplus A_{3r} \oplus \cdots = \bigoplus_{d=0}^{\infty} A_{dr}$$

- their Segre product

$$A \circ B := \Bbbk \oplus A_1 \otimes B_1 \oplus A_2 \otimes B_2 \oplus \cdots = \bigoplus_{d=0}^{\infty} A_d \otimes B_d$$

(proofs omitted - see Polishchuk & Positselski Ch. 3)

Tor ??

(d) A Koszul $\Leftrightarrow$ $\mathrm{Tor}_i^A(\Bbbk,\Bbbk)_j = 0$ for $j \neq i$.

- Let's recall what $\mathrm{Tor}_\cdot^A(M,N)$ is,
- relate it to resolutions,
- see how it gives more methods to prove Koszulity:
  - ▷ Koszul filtrations
  - ▷ quadratic Gröbner bases
  - ▷ Bar complex and topology (in Lecture 3)

What was $\mathrm{Tor}_i^A(M,N)$ again,
for a (right) A-module M,
(left) A-module N ?

Pick any resolution $\mathcal{F}_\bullet$ of M
by projective (e.g. free) right A-modules

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to M \to 0$$

- remove M,
- tensor $(-)\otimes_A N$,
- compute homology

$$\cdots \to F_2\otimes_A N \to F_1 \otimes_A N \to F_0\otimes_A N \to 0$$

That is, $\mathrm{Tor}_i^A(M,N) := H_i(\mathcal{F}_\bullet \otimes_A N)$

If A, M, N are $\mathbb{N}$-graded, then one can choose $\mathcal{F}_\bullet$ to be $\mathbb{N}$-graded with homogeneous maps d_i

$$\begin{tikzcd}
\cdots \arrow[r] & F_i \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r] & M \arrow[r] & 0 \\
& \bigoplus_{j \in \mathbb{N}} A(-j)^{\beta_{ij}} & & & & & &
\end{tikzcd}$$

so that $\mathcal{F}_\bullet \otimes_A N$ is also $\mathbb{N}$-graded

as is $\mathrm{Tor}_i^A(M,N) = \bigoplus_{j \in \mathbb{N}} \mathrm{Tor}_i^A(M,N)_j$

Choosing d_i in $\mathcal{F}_\bullet$ to hit minimal generators of $\ker(d_{i-1})$ ensures the d_i have all entries in A_+.

EXAMPLE $A = \Bbbk[x_1, x_2] = \Bbbk\langle x_1, x_2 \rangle / (x_1x_2 - x_2x_1)$

$M = \Bbbk$

$$\begin{tikzcd}[column sep=huge]
0 \arrow[r] & Az_{112} \arrow[r, "{\begin{array}{cc} & z_{112} \\ y_{12} & x_1 \\ y_{112} & \boxed{-1} \end{array}}"] & \begin{matrix} Ay_{12} \\ \oplus \\ Ay_{112} \end{matrix} \arrow[r, "{\begin{array}{ccc} & y_{12} & \overset{\text{redundant}}{\boxed{y_{112}}} \\ y_1 & -x_2 & -x_1x_2 \\ y_2 & x_1 & x_1^2 \end{array}}"] & \begin{matrix} Ay_1 \\ \oplus \\ Ay_2 \end{matrix} \arrow[r, "{\begin{array}{ccc} & y_1 & y_2 \\ 1 & x_1 & x_2 \end{array}}"] & A \arrow[r] & \Bbbk \arrow[r] & 0
\end{tikzcd}$$

$y_1 \longmapsto x_1$

$y_2 \longmapsto x_2$

$y_{12} \longmapsto x_1y_2 - x_2y_1$

$y_{112} \longmapsto x_1^2y_2 - x_1x_2y_1$

PROPOSITION: For a graded $\Bbbk$-algebra A, and graded A-free resolution of M, if all entries of d_i lie in A_+, then

$$\operatorname{Tor}_i^A(M,\Bbbk)_j = \Bbbk^{\beta_{ij}}, \text{ that is, } \beta_{ij} = \dim_{\Bbbk} \operatorname{Tor}_i^A(M,\Bbbk)_j$$

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proof:

$$\begin{tikzcd}
\cdots \arrow[r] & F_i \arrow[r, "d_i"] \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r] & M \arrow[r] & 0 \\
& \bigoplus_j A(-j)^{\beta_{ij}} & & & & &
\end{tikzcd}$$

$\rightsquigarrow$ remove M, then $(-)\otimes_A \Bbbk$

$$\cdots \longrightarrow \bigoplus_j \Bbbk(-j)^{\beta_{ij}} \xrightarrow{0} \cdots \xrightarrow{0} \bigoplus_j \Bbbk(-j)^{\beta_{1j}} \xrightarrow{0} \bigoplus_j \Bbbk(-j)^{\beta_{0j}} \to 0$$

All maps becomes 0 after tensoring with $\Bbbk = A/A_+$, since all d_i had entries in A_+. ▨

COROLLARY: A is Koszul

$\iff$ $\Bbbk$ has a linear A-free resolution

that is, with $\beta_{ij} = 0 \ \forall j \neq i$

$\iff$ $\operatorname{Tor}_i^A(\Bbbk,\Bbbk)_j = 0$ for $j \neq i$

How might $\mathrm{Tor}_i^A(\Bbbk, \Bbbk)_j = 0$ for $j \neq i$ help?

One usual homological story:

$$0 \to M \to M' \to M'' \to 0$$

short exact

$\rightsquigarrow$

long exact

$$\cdots \to \mathrm{Tor}_i^A(M, \Bbbk) \to \mathrm{Tor}_i^A(M', \Bbbk) \to \mathrm{Tor}_i^A(M'', \Bbbk) \to \cdots$$

METHOD: Koszul filtration (Conca, Trung, Valla 2001)

Want $\mathrm{Tor}_i^A(\underset{A/(x_1,\ldots,x_n)}{\underset{\|}{\Bbbk}}, \Bbbk)_j = 0$ for $j \neq i$.

Know $\mathrm{Tor}_i^A(\underset{A/(0)}{\underset{\|}{A}}, \Bbbk)_j = 0$ for $j \neq i$

(it's 0 for $i \neq 0$)

PROPOSITION: Suppose $I = (0)$ and $I = (x_1, \ldots, x_n)$ lie in a family $\mathcal{I} = \{I\}$ where $\forall I \in \mathcal{I} \setminus (0)$ $\exists J \in \mathcal{I}$ with

- $I = J + (\ell)$, ℓ a linear form
- $(J : I) = \{a \in A : aI \subseteq J\}$ is also in $\mathcal{I}$

Then A is Koszul because all $I \in \mathcal{I}$ have $\mathrm{Tor}_i(A/I, \Bbbk)_j = 0$ $\forall j \neq i$.

proof by EXAMPLE for quadratic monomial quotients:

$$A = \Bbbk[x,y,z]/(x^2, yz)$$

$$\text{Pick } \underset{\sim}{I} = \left\{ \begin{array}{cccc} & (x) & (x,y)=(J:I) & \\ (0) & (y) & \boxed{(\overset{\ell}{\overset{\shortparallel}{x}}, z) = I} & (x,y,z) \\ & J=(z) & (y,z) & \end{array} \right\}$$

Induct on i and inclusion $J \subset I$ to show $\operatorname{Tor}_i^A(A/I, \Bbbk)_j = 0$ if $j \neq i$

$$0 \to \overset{I/J}{(x,z)/(z)} \to \overset{A/J}{A/(z)} \longrightarrow \overset{A/I}{A/(x,z)} \longrightarrow 0$$

where $\overset{I/J}{(x,z)/(z)} \cong \overset{A/(J:I)\ (-1)}{A/(x,y)(-1)}$ due to

$$(x,y) = \overset{(J:I)}{(z):(x,z)} = \ker\left(\begin{array}{ccc} A(-1) & \longrightarrow & \overset{I/J}{(x,z)/(z)} \\ 1 & \longmapsto & x = \ell \end{array} \right)$$

Long exact sequence:

$$\underset{\underset{0}{\shortparallel}}{\operatorname{Tor}_i(\overset{A/J}{A/(z)}, \Bbbk)_j} \to \operatorname{Tor}_i(\overset{A/I}{A/(x,z)}, \Bbbk)_j \to \underset{\underset{\operatorname{Tor}_{i-1}(A/(x,y), \Bbbk)_{j-1}}{\shortparallel}}{\operatorname{Tor}_{i-1}(\overset{A/(J:I)}{A/(x,y)(-1)}, \Bbbk)_j}$$

for $j \neq i$ by induction on $J \subsetneq I$

UPSHOT: must vanish for $j \neq i$

$\operatorname{Tor}_{i-1}(A/(x,y), \Bbbk)_{j-1} = 0$ by induction on i

Koszul filtration is the method that has worked to prove Koszulity in the hard cases ($\overset{\text{DEF'N}}{:=}$ where quadratic Gröbner bases are not known!)

e.g.

THEOREM (Maestroni & McCullough 2022)

For any (simple) matroid M, the Chow ring A_M (with respect to the maximal building set) is Koszul:

Chow ring

$$A_M := \mathbb{k}[x_F : \text{flats } F \neq \emptyset] \Big/ I_M$$

(Feichtner & Yuzvinsky 2004)

where

$$I_M = \underset{\substack{\uparrow\\ \text{quadratic}}}{\left(x_F x_G : F \not\subseteq G,\ G \not\subseteq F\right)} + \underset{\substack{\uparrow\\ \text{linear}}}{\left(\sum_{\substack{\text{flats } F:\\ F \ni a}} x_F : \begin{matrix}\text{atoms}\\ a \text{ in the}\\ \text{lattice of}\\ \text{flats}\end{matrix} \right)}$$

REMARK: A_M is the Chow ring for the Bergman fan of M, finely subdivided.

The usual method for proving Koszulity is:

Quadratic Gröbner bases

FIRST: Recall Gröbner bases in

$$\Bbbk\langle \underline{x}\rangle = \Bbbk\langle x_1,\dots,x_n\rangle$$
$$\Bbbk[\underline{x}] = \Bbbk[x_1,\dots,x_n]$$
$$\Lambda_{\Bbbk}(\underline{x}) = \Lambda_{\Bbbk}\langle x_1,\dots,x_n\rangle$$

with respect to a monomial order $\prec$, that is,

- a total order on the monomials
- with no ∞ descending chains, and
- $m \prec m' \Rightarrow amb \prec am'b$

DEF'N: $\mathrm{in}_\prec(f) :=$ $\prec$-largest monomial m with nonzero coefficient in f

i.e. $f = c_m \cdot \underbrace{m}_{\substack{\shortparallel \\ \mathrm{in}_\prec(f)}} + \sum\limits_{m' : m' \prec m} c_{m'} \cdot m'$ with $c_m \neq 0$

Given an ideal

$$I \subset R = \Bbbk\langle\underline{x}\rangle \text{ or } \Bbbk[\underline{x}] \text{ or } \Lambda_{\Bbbk}\langle\underline{x}\rangle,$$

its initial ideal $\mathrm{in}_\prec(I) := (\,\mathrm{in}_\prec(f) : f \in I\,)$

DEF'N: A subset $\mathcal{G} = \{g\} \subset I$ is called a Gröbner basis (GB) for I with respect to $<$ if any of the following equivalent conditions hold:

(a) $\text{in}_<(I) = (\text{in}_<(g) : g \in \mathcal{G}) =: (\text{in}_<(\mathcal{G}))$

(b) the $\mathcal{G}$-standard monomials

$$\left\{ \begin{matrix}\text{monomials} \\ m \in R\end{matrix} : \forall g \in \mathcal{G} \ \text{in}_<(g) \text{ does not divide } m \right\}$$

have their images $\bar{m}$ in R/I giving a $\Bbbk$-basis.

(Only when I homogeneous)

(c) $\text{Hilb}(R/I, t) = \text{Hilb}(R/(\text{in}_<(\mathcal{G})), t)$

PROPOSITION:

Any Gröbner basis $\mathcal{G}$ for I generates I as an ideal.

Warning: Although ideals $I \subset \Bbbk[\underline{x}]$ or $\Lambda_{\Bbbk}\langle \underline{x} \rangle$ always have finite GB's, computable in Macaulay, ideals $I \subset \Bbbk\langle x \rangle$ often have no finite GB.

(CAUTIONARY) EXAMPLES

- In $\Bbbk[x_1,\dots,x_n]$, a principal ideal $I = (f(\underline{x}))$ always has $\mathcal{G} = \{f(\underline{x})\}$ as a GB for any $\prec$.

e.g. $$\Bbbk[x_1,x_2,x_3,x_4]/(\underbrace{x_1x_2+x_3x_4}_{\text{a GB for } I})$$

$$\Bbbk[x_1,x_2,x_3]/(\overbrace{x_1x_2x_3})$$

$$\cong$$

- $\Bbbk\langle x_1,x_2,x_3 \rangle/(\underbrace{x_1x_2-x_2x_1,\ x_1x_3-x_3x_1,\ x_2x_3-x_3x_2,\ x_1x_2x_3}_{\text{has } \underline{\text{no}}\ \underline{\text{finite}} \text{ GB for all } \prec})$ (Eisenbud-Peeva-Sturmfels 1998)

see Problem 7

- $\Lambda_{\Bbbk}\langle x_1,x_2,x_3,x_4 \rangle/(\underbrace{x_1x_2+x_3x_4}_{\underline{\text{not}} \text{ a GB for all } \prec})$
- $\Bbbk[x_1,x_2,x_3,x_4]/(\overbrace{x_1^2,x_2^2,x_3^2,x_4^2,\ x_1x_2+x_3x_4})$

Replacing $A = R/I \rightsquigarrow R/(\underbrace{\mathrm{in}_<(I)}_{\text{initial ideal}})$

is often called a Gröbner deformation.

(FOLKLORE) THEOREM:

Let I be a homogeneous ideal in $R = \Bbbk[\underline{x}]$, $\Lambda_{\Bbbk}\langle \underline{x}\rangle$ or $\Bbbk\langle \underline{x}\rangle$, and $<$ any monomial ordering on R.

Then
$$\dim_{\Bbbk} \mathrm{Tor}_i^{R/I}(\Bbbk,\Bbbk)_j \leq \dim_{\Bbbk} \mathrm{Tor}_i^{R/\mathrm{in}_< I}(\Bbbk,\Bbbk)_j$$

Why "Folklore"?

- For $R = \Bbbk[x_1,\ldots,x_n]$, it goes back to results by Robbiano 1982, Anick 1986, Kempf 1990; full proof in Peeva's book "Graded Syzygies" Thm 22.9
- For $R = \Lambda_{\Bbbk}\langle x_1,\ldots,x_n\rangle$, the proof is similar, but seems written down nowhere, as far as we know.
- For $R = \Bbbk\langle x_1,\ldots,x_n\rangle$, it is proven for certain kinds of monomial orders by Priddy 1970, Jollenbeck-Welker 2005; general proof written out recently by Backelin.

COROLLARY: Let I be a (homogeneous) ideal in $R = \Bbbk[\underline{x}]$, $\Lambda_{\Bbbk}\langle \underline{x}\rangle$ or $\Bbbk\langle \underline{x}\rangle$.

If there exists $<$ a monomial order for which

I has a quadratic Gröbner basis $\mathcal{G}$,

then $A = R/I$ is Koszul.

Proof:

$$\dim_{\Bbbk} \operatorname{Tor}_i^{A}(\Bbbk,\Bbbk)_j$$

$$\leq \dim_{\Bbbk} \operatorname{Tor}_i^{R/\mathrm{in}_<(I)}(\Bbbk,\Bbbk)_j$$

$$= \dim_{\Bbbk} \operatorname{Tor}_i^{R/(\mathrm{in}_<(\mathcal{G}))}(\Bbbk,\Bbbk)_j = 0 \text{ if } j \neq i$$

since

$R/(\underbrace{\mathrm{in}_<(\mathcal{G})}_{\text{quadratic monomials}})$ is Koszul

by Fröberg's Theorem

▨

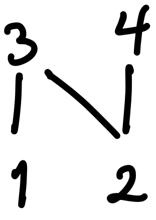
EXAMPLE 1: Hibi rings A_P
(1985)

P a poset on $[n]=\{1,2,\ldots,n\}$ ⇝ $J(P)$ = distributive lattice of all order ideals $I \subseteq P$ ⇝ affine semigroup/toric ring

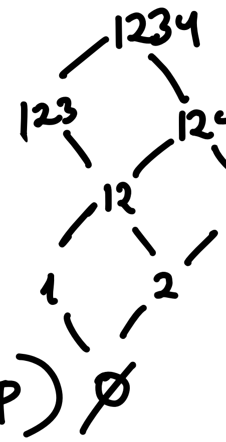
$$A_P \subseteq \Bbbk[t_0,t_1,\ldots,t_n]$$

$$\begin{tikzcd} \Bbbk[y_I : \text{order ideals } I] \,/\, (x_I x_J - x_{I\cap J}\, x_{I\cup J} : I \not\subseteq J,\ J \not\subseteq I) \arrow[r, "\sim"] & A_P \overset{..}{=} \Bbbk\left[t_0 \prod_{i\in I} t_i : \text{order ideals } I\right] \end{tikzcd}$$

EXAMPLE



P ⇝



$J(P)$ ⇝

$$A_P = \Bbbk[t_0,\ t_0t_1,\ t_0t_2,\ t_0t_1t_2,\ t_0t_2t_4,\ t_0t_1t_2t_3,\ t_0t_1t_2t_4,\ t_0t_1t_2t_3t_4]$$

$$\begin{tikzcd} \Bbbk[x_\emptyset,\ x_1, x_2,\ x_{12}, x_{24},\ x_{123}, x_{124},\ x_{1234}] \,/\, \begin{pmatrix} x_1x_2 - x_\emptyset x_{12} \\ x_1x_{24} - x_\emptyset x_{124} \\ x_{12}x_{24} - x_2x_{124} \\ x_{24}x_{123} - x_2x_{1234} \\ x_{124}x_{123} - x_{12}x_{1234} \end{pmatrix} \arrow[r, "\sim"] & A_P \end{tikzcd}$$

THEOREM The presentation
(Hibi 1985)

$$A_P = \Bbbk[x_I : \text{order ideals } I] \Big/ \big(\underline{x_I x_J} - x_{I\cap J}\, x_{I\cup J} : I \not\subseteq J,\ J \not\subseteq I\big)$$

has the generators shown as quadratic GB for any monomial order $\prec$ that makes the underlined term $x_I x_J = \mathrm{in}_\prec(\underline{x_I x_J} - x_{I\cap J} x_{I\cup J})$.

(e.g. a lexicographic order with $x_J > x_I$ for $J \supset I$)

COROLLARY: Hibi rings A_P always Koszul.

REMARK: These are special cases of (homogeneous) ASL's defined by DeConcini-Eisenbud-Procesi (1982)

algebras with straightening law

motivated by coordinate rings of Grassmannians.

ASL's will have similar-looking quadratic GB presentations, and hence all are Koszul.

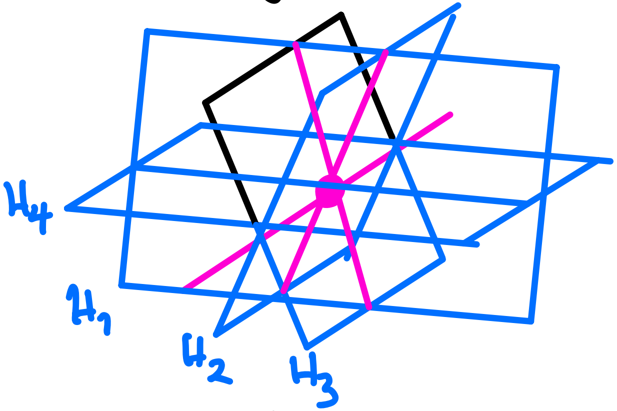
EXAMPLE(S): Orlik-Solomon (1980) algebras and (graded) Varchenko-Gelfand (1987) rings

---

$\mathcal{H} = \{H_1, H_2, \dots, H_n\} \subset \mathbb{R}^d$ a (central) arrangement of linear hyperplanes

with (choice of) normal vectors $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$

↓



Orlik-Solomon algebra

$$OS(\mathcal{H}) := \Lambda_{\mathbb{k}}\langle x_1, \dots, x_n \rangle \Big/ \left(\partial(x_C) : \text{circuits } C\right)$$

graded Varchenko-Gelfand ring

$$VG(\mathcal{H}) := \mathbb{k}[x_1, \dots, x_n] \Big/ \left(x_1^2, \dots, x_n^2,\ \partial_{\pm}(x_C) : \text{circuits } C\right)$$

where circuits $C = \{i_1, \dots, i_k\}$ index inclusion-minimal linearly dependent sets $c_1\alpha_{i_1} + c_2\alpha_{i_2} + \dots + c_k\alpha_{i_k} = \underline{0}$

and $x_C := x_{i_1} x_{i_2} \cdots x_{i_k}$

$$\partial(x_C) := \sum_{j=1}^{k} (-1)^{j-1}\, x_{i_1} \cdots \hat{x}_{i_j} \cdots x_{i_k}$$

$$\partial_{\pm}(x_C) := \sum_{j=1}^{k} \mathrm{sgn}(c_i) \cdot x_{i_1} \cdots \hat{x}_{i_j} \cdots x_{i_k}$$

EXAMPLE:

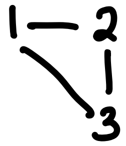
$\mathcal{H} =$ type A_{n-1} reflection arrangement (symmetric group $\mathfrak{S}_n$)

$= \{ H_{ij} = \{x_i = x_j\} : 1 \le i < j \le n \} \subset \mathbb{R}^n$

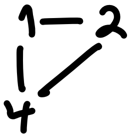
with choice of normal vectors (positive roots) $\{ \alpha_{ij} = e_i - e_j : 1 \le i < j \le n \}$

Circuits C:

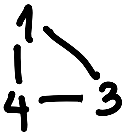
12, 23, 31



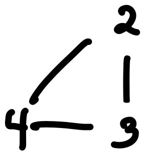
12, 24, 41



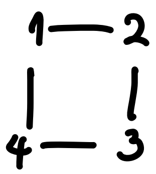
13, 34, 41



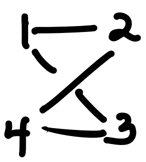
23, 34, 42



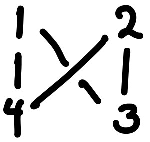
12, 23, 34, 41



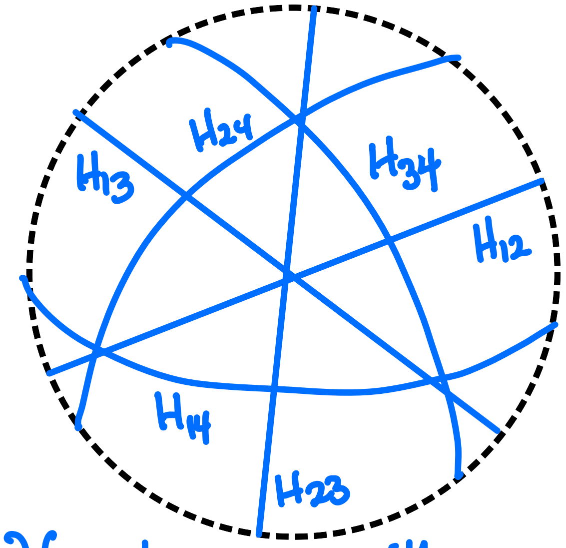
12, 24, 43, 31



14, 42, 23, 31



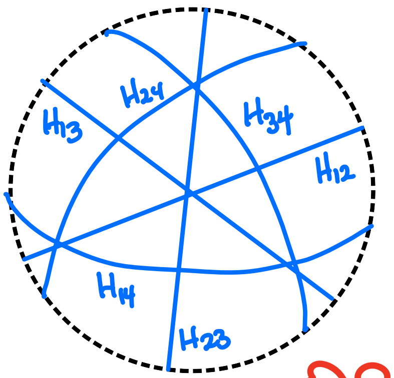
$n=4$:

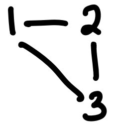
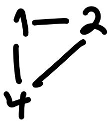
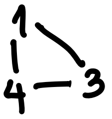
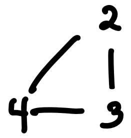
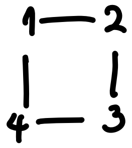
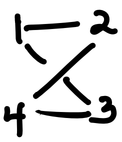
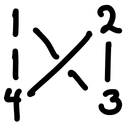


$\mathcal{H}$ intersected with unit sphere $\mathbb{S}^2$ inside $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}^\perp \subset \mathbb{R}^4$

EXAMPLE: $\mathcal{H} = \{H_{12}, H_{13}, H_{14}, H_{23}, H_{24}, H_{34}\} \subset \mathbb{R}^4$

$\{e_1-e_2,\ e_1-e_3,\ e_1-e_4,\ e_2-e_3,\ e_2-e_4,\ e_3-e_4\}$



	$OS(\mathcal{H}) = \bigwedge_{\Bbbk}\langle x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}\rangle$ modulo	$VG(\mathcal{H}) = \Bbbk[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]/(x_{ij}^2)$ modulo
	$x_{12}x_{13} - x_{12}x_{23} + x_{13}x_{23}$	$x_{12}x_{13} - x_{12}x_{23} + x_{13}x_{23}$
	$x_{12}x_{14} - x_{12}x_{24} + x_{14}x_{24}$	$x_{12}x_{14} - x_{12}x_{24} + x_{14}x_{24}$
	$x_{13}x_{14} - x_{13}x_{34} + x_{14}x_{34}$	$x_{13}x_{14} - x_{13}x_{34} + x_{14}x_{34}$
	$x_{23}x_{24} - x_{23}x_{34} + x_{24}x_{34}$	$x_{23}x_{24} - x_{23}x_{34} + x_{24}x_{34}$
	$x_{12}x_{23}x_{34} - x_{12}x_{23}x_{14} + x_{12}x_{34}x_{14} - x_{23}x_{34}x_{14}$	$+ x_{12}x_{23}x_{34} - x_{12}x_{23}x_{14} - x_{12}x_{34}x_{14} - x_{23}x_{34}x_{14}$
	$x_{12}x_{24}x_{34} - x_{12}x_{24}x_{14} + x_{12}x_{34}x_{14} - x_{24}x_{34}x_{14}$	$+ x_{12}x_{24}x_{34} + x_{12}x_{24}x_{14} - x_{12}x_{34}x_{14} - x_{24}x_{34}x_{14}$
	$x_{14}x_{24}x_{23} - x_{14}x_{24}x_{13} + x_{14}x_{23}x_{13} - x_{24}x_{23}x_{13}$	$- x_{14}x_{24}x_{23} + x_{14}x_{24}x_{13} - x_{14}x_{23}x_{13} + x_{24}x_{23}x_{13}$

THEOREM (Orlik-Solomon, Varchenko-Gelfand) For any monomial order $\prec$ on $\Lambda_{\Bbbk}\langle x_1,\ldots,x_n\rangle$ or $\Bbbk[x_1,\ldots,x_n]$, assuming $x_1 \prec \cdots \prec x_n$, then the above generators $\mathcal{G}$ are Gröbner bases for their ideals, with $\prec$-initial terms underlined here: if the circuit $C=\{i_1<i_2<\cdots<i_k\}$, then

$$\partial(x_C) = \underline{x_{i_2}x_{i_3}\cdots x_{i_k}} + \sum_{j=2}^{k}(-1)^{j-1}\, x_{i_1}\cdots\hat{x}_{i_j}\cdots x_{i_k} \quad \text{in } \Lambda_{\Bbbk}\langle x\rangle$$

$$\partial_{\pm}(x_C) = \underline{x_{i_2}x_{i_3}\cdots x_{i_k}} + \sum_{j=2}^{k}\operatorname{sgn}(c_i)\cdot x_{i_1}\cdots\hat{x}_{i_j}\cdots x_{i_k}.$$

ALSO: The $\mathcal{G}$-standard monomial bases are the NBC-monomials $\{x_I\}_{I \text{ contains no broken circuit}}$

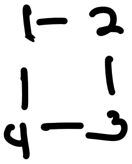
no broken circuit

$C-\{i_1\}=\{i_2,\ldots,i_k\}$
with $i_1<i_2<\cdots<i_k$

In fact, in the Gröbner bases, one really only needs the $\partial(x_C)$ or $\partial_\pm(x_C)$ whose broken circuits $C-\{i_1\}$ are inclusion-minimal

EXAMPLE: If $\prec$ has $x_{12} < x_{13} < x_{14} < x_{23} < x_{24} < x_{34}$

then you can omit

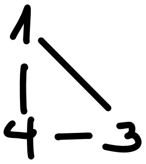


$$x_{12}x_{23}x_{34} - x_{12}x_{23}x_{14} + x_{12}x_{34}x_{14} - \underline{x_{23}x_{34}x_{14}}$$

from the Gröbner basis $\mathcal{G}$,

divides

since you have



$$x_{13}x_{14} - x_{13}x_{34} + \underline{x_{14}x_{34}}$$

So it would be nice to know when all inclusion-minimal broken circuits are size 2

so $\partial(x_C)$, $\partial_\pm(x_C)$ are quadratic

THEOREM: (Björner & Ziegler 1991)

The hyperplane arrangement $\mathcal{H}$ has all minimal broken circuits of size 2 for $x_1 \prec \dots \prec x_n$

$$\Updownarrow$$

$\mathcal{H}$ is supersolvable

DEF'N

one can decompose

$$\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2 \sqcup \dots \sqcup \mathcal{H}_{j-1} \sqcup \mathcal{H}_j \sqcup \dots \sqcup \mathcal{H}_d$$

such that

- $\mathcal{H}_1 \prec \mathcal{H}_2 \prec \dots \prec \mathcal{H}_d$
- each initial segment $\mathcal{H}_1 \sqcup \mathcal{H}_2 \sqcup \dots \sqcup \mathcal{H}_j$ is a flat: $X = \bigcap_{H \in \mathcal{H}_1 \sqcup \dots \sqcup \mathcal{H}_j} H$ lies in no other hyperplanes from $\mathcal{H}_{j+1} \sqcup \dots \sqcup \mathcal{H}_d$
- $\forall H, H' \in \mathcal{H}_j$ $\exists H'' \in \mathcal{H}_1 \sqcup \dots \sqcup \mathcal{H}_{j-1}$ with $\{H, H', H''\}$ a circuit

---

One calls the sequence

$$\mathcal{H}_1,\quad \mathcal{H}_1 \sqcup \mathcal{H}_2,\quad \mathcal{H}_1 \sqcup \mathcal{H}_2 \sqcup \mathcal{H}_3,\quad \vdots$$

an M-chain for $\mathcal{H}$.

COROLLARY: For supersolvable $\mathcal{H}$,

(Shelton & Yuzvinsky 1995
Peeva 2003
Dorpalen-Barry 2023)

$OS(\mathcal{H})$ and $VG(\mathcal{H})$ have quadratic GB presentations

and hence are Koszul algebras.

QUESTION (Yuzvinsky) Does the converse hold,

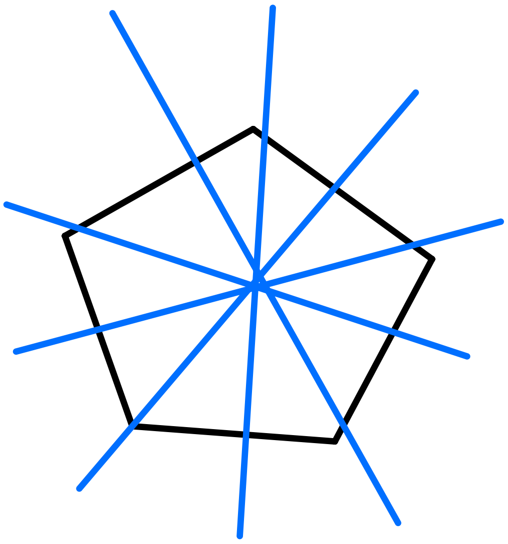
that is, does $OS(\mathcal{H})$ Koszul imply $\mathcal{H}$ is supersolvable?

EXAMPLES:

Among reflection hyperplane arrangements the supersolvables are rare:

see Problem 8

- Type A_{n-1} $\{x_i = x_j : 1 \le i < j \le n\}$
- Type B_n $\{x_i = \pm x_j : 1 \le i < j \le n\}$ $\dot{\cup}$ $\{x_i : 1 \le i \le n\}$
- Dihedral = type $I_2(m)$



Those not supersolvable include

- type D_n for $n \ge 4$ $\{x_i = \pm x_j : 1 \le i < j \le n\}$